Ann. Inst. Statist. Math.
Vol. 44, No. 1, 97–106 (1992)

A FORM OF MULTIVARIATE GAMMA DISTRIBUTION

A. M. MATHAI[1] AND P. G. MOSCHOPOULOS[2]

[1] *Department of Mathematics and Statistics, McGill University, Montreal, Canada H3A 2K6*
[2] *Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968-0514, U.S.A.*

(Received July 30, 1990; revised February 14, 1991)

Abstract. Let V_i, $i = 1, \dots, k$, be independent gamma random variables with shape α_i, scale β, and location parameter γ_i, and consider the partial sums $Z_1 = V_1, Z_2 = V_1 + V_2, \dots, Z_k = V_1 + \cdots + V_k$. When the scale parameters are all equal, each partial sum is again distributed as gamma, and hence the joint distribution of the partial sums may be called a multivariate gamma. This distribution, whose marginals are positively correlated has several interesting properties and has potential applications in stochastic processes and reliability. In this paper we study this distribution as a multivariate extension of the three-parameter gamma and give several properties that relate to ratios and conditional distributions of partial sums. The general density, as well as special cases are considered.

Key words and phrases: Multivariate gamma model, cumulative sums, moments, cumulants, multiple correlation, exact density, conditional density.

1. Introduction

The three-parameter gamma with the density

$$f(x; \alpha, \beta, \gamma) = \frac{(x-\gamma)^{\alpha-1} \exp\left(-\frac{x-\gamma}{\beta}\right)}{\beta^{\alpha}\Gamma(\alpha)}, \quad x > \gamma, \quad \alpha > 0, \quad \beta > 0 \tag{1.1}$$

stands central in the multivariate gamma distribution of this paper.

Multivariate extensions of gamma distributions such that all the marginals are again gamma are the most common in the literature. Such extensions involve the standard gamma ($\beta = 1$, $\gamma = 0$), or the exponential ($\alpha = 1$), see Johnson and Kotz (1972). Other extensions include the multivariate chi-square (Miller *et al.* (1958), Krishnaiah and Rao (1961) and Krishnaiah *et al.* (1963)), while the particular case of a bivariate gamma received special consideration, see Kibble (1941), Eagleson (1964), Ghirtis (1967), Moran (1967, 1969) and Sarmanov (1970). The bivariate gamma is particularly useful in modeling the lifetimes of two parallel systems, see Freund (1961), Becker and Roux (1981), Lingappaiah (1984) and Steel and le Roux (1987). Other applications include uses of the distribution in rainmaking experiments (Moran (1970)).

In this paper we consider a new form of a multivariate gamma that has potential applications in situations in which partial sums of independent, positive random variables are of interest. Such situations appear in the area of reliability and stochastic processes. Let V_i, $i = 1, \ldots, k$ be the times between successive occurrences of a phenomenon, and let $Z_i = Z_{i-1} + V_i$, $i = 1, \ldots, k$, with $Z_0 = 0$. Then, Z_i is the total time required until the i-th occurrence. In stochastic processes, it is usually assumed that the times V_i are also identically distributed. In this case, the occurrence time Z_i, $i \in N$ can be viewed as a renewal process, and the times V_i can be called renewal times, see for example Çinlar (1975). Note that Z_i is the i-th partial sum of the V_i's. In actual practice the V_i's may be times between arrivals, or, for example, time delays of an airplane at several airports. Then, Z_i is the total waiting time for the i-th occurrence, or the total delay at the i-th airport.

As another example, consider the following application from reliability. An item is installed at time $Z_0 = 0$ and when it fails, it is replaced by an identical (or different) item. Then, when the new item fails it is replaced again by another item and the process continues. In this case $Z_i = Z_{i-1} + V_i$ where V_i is the time of operation of the i-th item, and Z_i is the time at which the i-th replacement is needed. Z_k denotes the time interval in which a total of k items need replacement.

The applications mentioned above, motivate the new form of a multivariate gamma that we consider in this paper. This is given in the form of a theorem.

THEOREM 1.1. *Suppose $V_1, \ldots, V_k$ are mutually independent where $V_i \sim G(\alpha_i, \beta, \gamma_i)$, $i = 1, \ldots, k$ (same β). Let*

$$Z_1 = V_1, Z_2 = V_1 + V_2, \ldots, Z_k = V_1 + \cdots + V_k.$$

Then, the joint distribution of $\boldsymbol{Z} = (Z_1, \ldots, Z_k)'$ is a multivariate gamma with density function

$$f(z_1, \ldots, z_k) = \frac{(z_1 - \gamma_1)^{\alpha_1 - 1}}{\beta^{\alpha_k^*} \prod_{i=1}^k \Gamma(\alpha_i)} \cdot (z_2 - z_1 - \gamma_2)^{\alpha_2 - 1} \cdots (z_k - z_{k-1} - \gamma_k)^{\alpha_k - 1} \cdot e^{-(z_k - (\gamma_1 + \cdots + \gamma_k))/\beta} \tag{1.2}$$

for $\alpha_i > 0$, $\beta > 0$, γ_i real, $z_{i-1} + \gamma_i < z_i$, $i = 2, \ldots, k$, $z_k < \infty$, $\gamma_1 < z_1$, $\alpha_k^* = \alpha_1 + \cdots + \alpha_k$, and zero elsewhere.

Many applications exist in which the V_i's represent independent and identically distributed times. Here, they are assumed to be distributed as in (1.1), thus allowing maximum flexibility in shape, scale and location.

The requirement of equal scale parameter β is to ensure that the marginals are of the same form. Basic properties of the distribution of $\boldsymbol{Z}$, including the moment generating function, means, variances, properties of the covariance matrix and the reproductive property are given in Section 2. In Section 3 we give the moments and cumulants, and in Section 4 we discuss conditional distributions and special cases.

Before discussing the properties of the model in (1.2) a brief description of the various methods of construction of multivariate gamma distributions will be given here.

After giving a brief sketch of the historical development Dussauchoy and Berland (1974) define a multivariate gamma random variable $\boldsymbol{Z} = (Z_1, \dots, Z_n)'$ in terms of the characteristic function defined by

$$\psi_z(u_1, \dots, u_n) = \prod_{j=1}^{n} \frac{\psi_{z_j}\left(u_j + \sum_{k=j+1}^{n} \beta_{jk} u_k\right)}{\psi_{z_j}\left(\sum_{k=j+1}^{n} \beta_{jk} u_k\right)} \tag{1.3}$$

where

$$\psi_{z_j}(u_j) = (1 - iu_j/a_j)^{-e_j}, \quad j = 1, \dots, n, \ i = \sqrt{-1}, \ \beta_{jk} \geq 0,$$
$$a_j \geq \beta_{jk} a_k > 0, \quad j < k = 1, \dots, n, \ 0 < e_1 \leq e_2 \leq \dots \leq e_n.$$

Various properties are studied with the help of (1.3) but explicit form of the density is not evaluated except for the bivariate case.

Gaver (1970) considered a mixture of gamma variables with negative binomial weights and came up with a multivariate vector $\boldsymbol{Z} = (Z_1, \dots, Z_m)'$ as that one with the Laplace transform of the density given by

$$L_z(s_1, \dots, s_m) = \left\{ \frac{\alpha}{(1+\alpha)\prod_{j=1}^{m}(s_j + 1) - 1} \right\}^k \tag{1.4}$$

for $k > 0$, $\alpha > 0$. He also looked at the possibility of generating a multivariate gamma as a mixture with Poisson weights.

Kowalczyk and Tyrcha (1989) start with the three-parameter gamma in (1.1), denoting the random variable by $\Gamma(\alpha, \beta, \gamma)$. They call the joint distribution of $Z_i = [\sigma_i(V_0 + V_i - \alpha_i)/\sqrt{\alpha_i}] + \mu_i$, $i = 1, \dots, k$ as the multivariate gamma, where $V_0 = \Gamma(\theta_0, 1, 0)$, $V_i = \Gamma(\alpha_i - \theta_0, 1, 0)$, $0 \leq \theta_0 \leq \min(\alpha_1, \dots, \alpha_k)$, $\sigma_i > 0$, μ_i a real number, $i = 1, \dots, k$ and $V_0, V_1, \dots, V_k$ are assumed to be mutually independently distributed. They look at some properties including convergence to a multivariate normal and estimation problems.

Mathai and Moschopoulos (1991) start with (1.1) and look at the joint distribution of $Z_i = (\beta_i/\beta_0)V_0 + V_i$, where V_i, $i = 0, \dots, k$ are mutually independently distributed as in (1.1) with different parameters. They look at the explicit form of the multivariate density and study various properties.

None of the multivariate gamma models discussed above or the ones studied by others falls in the category of (1.2) defined for the present study. Hence, we will look at some properties of (1.2) here.

2. Properties

Several properties of the distribution can be obtained from the definition, while others will follow from the moment generating function (m.g.f.). The m.g.f. of V_i is

$$\begin{aligned} M_{V_i}(t) &= E(e^{tV_i}) \\ &= [\beta^{\alpha_i}\Gamma(\alpha_i)]^{-1}\int_{\gamma_i}^{\infty}(v_i-\gamma_i)^{\alpha_i-1}\exp\left(-\frac{v_i-\gamma_i}{\beta}\right)\exp(tv_i)dv_i \\ &= \frac{e^{\gamma_i t}}{(1-\beta t)^{\alpha_i}}. \end{aligned} \tag{2.1}$$

From this we get the m.g.f. of $\boldsymbol{Z}$ as follows.

$$\begin{aligned} M_{\boldsymbol{z}}(\boldsymbol{t}) &= M_{\boldsymbol{z}}(t_1,\ldots,t_k) = E(e^{t_1Z_1+\cdots+t_kZ_k}) \\ &= \frac{e^{\gamma_1(t_1+\cdots+t_k)}}{[1-\beta(t_1+\cdots+t_k)]^{\alpha_1}}\frac{e^{\gamma_2(t_2+\cdots+t_k)}}{[1-\beta(t_2+\cdots+t_k)]^{\alpha_2}}\cdots\frac{e^{\gamma_k t_k}}{(1-\beta t_k)^{\alpha_k}}. \end{aligned} \tag{2.2}$$

The m.g.f. exists if $|t_i+t_{i+1}+\cdots+t_k|<1/\beta$ for $i=1,\ldots,k$. From the definition directly or from the m.g.f. above we obtain the following properties:

(i) The marginal distribution of Z_i is gamma,

$$Z_i \sim G(\alpha_i^*,\beta,\gamma_i^*), \qquad i=1,\ldots,k \tag{2.3}$$

where $\alpha_i^*=\alpha_1+\cdots+\alpha_i$, $\gamma_i^*=\gamma_1+\cdots+\gamma_i$.

(ii) The mean and variance of Z_i are given by

$$E(Z_i)=\beta\alpha_i^*+\gamma_i^*, \tag{2.4}$$

$$\mathrm{Var}(Z_i)=\beta^2\alpha_i^*. \tag{2.5}$$

(iii) Z_i and Z_j are correlated. For $i<j$ we have

$$\mathrm{Cov}(Z_i,Z_j)=\mathrm{Cov}(Z_i,Z_i+V_{i+1}+\cdots+V_j)=\mathrm{Var}(Z_i)=\beta^2\alpha_i^*, \tag{2.6}$$

$$\rho=\mathrm{Corr}(Z_i,Z_j)=\sqrt{\frac{\alpha_i^*}{\alpha_j^*}}. \tag{2.7}$$

Clearly, the correlation is always positive. Now, the covariance matrix of $\boldsymbol{Z}$ is given by

$$\Sigma=\begin{pmatrix} \sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 \\ \sigma_1^2 & \sigma_1^2+\sigma_2^2 & \cdots & \sigma_1^2+\sigma_2^2 \\ \vdots & \vdots & & \vdots \\ \sigma_1^2 & \sigma_1^2+\sigma_2^2 & \cdots & \sigma_1^2+\cdots+\sigma_k^2 \end{pmatrix}, \tag{2.8}$$

where $\sigma_i^2=\alpha_i\beta^2$. A matrix of the above structure has several interesting properties that hold regardless of the distribution of the V_i's. The determinant of Σ is the product of σ_i^2, $i=1,\ldots,k$, i.e.

(iv)

$$|\Sigma| = \sigma_1^2\sigma_2^2\cdots\sigma_k^2. \tag{2.9}$$

This is easily seen by adding (-1) times the first row to all other rows, then (-1) times the second row to all the following rows etc. It should be noted that the eigenvalues of Σ are not equal to σ_i^2, $i = 1,\ldots,k$. Now if we let $\|\Sigma\| = \max_j \sum_{i=1}^k |\sigma_{ij}|$, where σ_{ij} denotes the (ij)-th element of Σ, then we have:

(v)

$$\|\Sigma\| = \mathrm{tr}(\Sigma). \tag{2.10}$$

In general $\|A\|$ need not be equal to the trace of A. Next, consider the determinants of the principal minors, starting from the last. These are as follows:

$$|\Sigma|_{kk} = \sigma_1^2 + \cdots + \sigma_k^2, \tag{2.11}$$

$$\begin{aligned}|\Sigma|_{k-1,k-1} &= \det\begin{pmatrix}\sigma_1^2+\cdots+\sigma_{k-1}^2 & \sigma_1^2+\cdots+\sigma_{k-1}^2\\ \sigma_1^2+\cdots+\sigma_{k-1}^2 & \sigma_1^2+\cdots+\sigma_k^2\end{pmatrix}\\ &= (\sigma_1^2+\cdots+\sigma_{k-1}^2)\sigma_k^2,\end{aligned} \tag{2.12}$$

$$|\Sigma|_{22} = (\sigma_1^2+\sigma_2^2)\sigma_3^2\cdots\sigma_k^2, \tag{2.13}$$

$$|\Sigma|_{11} = |\Sigma| = \sigma_1^2\sigma_2^2\cdots\sigma_k^2. \tag{2.14}$$

Now, consider the partitioning

$$\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}, \quad \text{where} \quad \Sigma_{11} = \sigma_{11} = \sigma_1^2.$$

Then,

$$|\Sigma| = |\Sigma_{22}|\left[\sigma_1^2 - \sigma_1^2(1,\ldots,1)\Sigma_{22}^{-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\sigma_1^2\right] \quad \text{or,}$$

$$\sigma_1^2\sigma_2^2\cdots\sigma_k^2 = (\sigma_1^2+\sigma_2^2)\sigma_3^2\cdots\sigma_k^2[\sigma_1^2(1-\sigma_1^2\vec{1}'\Sigma_{22}^{-1}\vec{1})] \quad \text{or,}$$

(vi)

$$\vec{1}'\Sigma_{22}^{-1}\vec{1} = \frac{1}{\sigma_1^2+\sigma_2^2}.$$

Note that the sum of the elements of Σ_{22}^{-1} is free of $\sigma_3^2,\ldots,\sigma_k^2$.

From (vi) one can also get the multiple correlation of Z_1 on $Z_2,\ldots,Z_k$ in a nice form. Using the standard notation

$$R_{1(2\ldots k)}^2 = \frac{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}{\sigma_{11}}, \quad \sigma_{11} = \sigma_1^2, \quad \Sigma_{12} = \sigma_1^2(1,\ldots,1),$$

one has from (vi)

(vii)

$$R^2_{1(2\ldots k)} = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}.$$

The partial correlation between Z_1 and Z_2 given $Z_3, \ldots, Z_k$, denoted by $\rho_{12\cdot(3\ldots k)}$ can be seen to have a nice form. Using standard notations

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \Sigma_{13} \\ \sigma_{21} & \sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{pmatrix},$$

$$\rho^2_{12\cdot(3\ldots k)} = \frac{(\sigma_{12}-\Sigma_{13}\Sigma_{33}^{-1}\Sigma_{32})^2}{(\sigma_{11}-\Sigma_{13}\Sigma_{33}^{-1}\Sigma_{31})(\sigma_{22}-\Sigma_{23}\Sigma_{33}^{-1}\Sigma_{32})}$$

where $\sigma_{11}=\sigma_1^2$, $\sigma_{12}=\sigma_1^2$, $\sigma_{22}=\sigma_1^2+\sigma_2^2$ one has

$$\sigma_{11}-\Sigma_{13}\Sigma_{33}^{-1}\Sigma_{31} = \sigma_1^2[1-\sigma_1^2\vec{1}'\Sigma_{33}^{-1}\vec{1}],$$
$$\sigma_{22}-\Sigma_{23}\Sigma_{33}^{-1}\Sigma_{32} = (\sigma_1^2+\sigma_2^2)[1-(\sigma_1^2+\sigma_2^2)\vec{1}'\Sigma_{33}^{-1}\vec{1}],$$
$$\sigma_{12}-\Sigma_{13}\Sigma_{33}^{-1}\Sigma_{32} = \sigma_1^2[1-(\sigma_1^2+\sigma_2^2)\vec{1}'\Sigma_{33}^{-1}\vec{1}].$$

But it is easy to note that $\vec{1}'\Sigma_{33}^{-1}\vec{1} = (\sigma_1^2+\sigma_2^2+\sigma_3^2)^{-1}$. Thus one has

(viii)

$$\rho^2_{12\cdot(3\ldots k)} = \frac{\sigma_1^2\sigma_3^2}{(\sigma_1^2+\sigma_2^2)(\sigma_2^2+\sigma_3^2)}.$$

(ix) (Reproductive property) Let W_1 be a multivariate gamma with parameters α_i, β, γ_i, $i=1,\ldots,k$, and W_2 independently distributed as a multivariate gamma with parameters α_i', β, γ_i', $i=1,\ldots,k$. Then, it is clear from the m.g.f. in (2.1) that W_1+W_2 is also distributed as a multivariate gamma with parameters $\alpha_i+\alpha_i'$, β, $\gamma_i+\gamma_i'$, $i=1,\ldots,k$.

3. Moments and cumulants

The cumulant generating function of $\boldsymbol{Z}$ is the logarithm of the m.g.f. in (2.2) and is given by

$$\text{(3.1)}\quad K_z(\boldsymbol{t}) = \gamma_1\sum_{i=1}^k t_i + \gamma_2\sum_{i=2}^k t_i + \cdots + \gamma_k t_k - \alpha_1 \ln\left[1-\beta\sum_{i=1}^k t_i\right] - \alpha_2\ln\left[1-\beta\sum_{i=2}^k t_i\right] - \alpha_k \ln[1-\beta t_k].$$

Thus, the m-th cumulant of Z_i and the (m_1,m_2)-th product cumulant of Z_i and Z_j are given by

$$\text{(3.2)}\qquad K_m = \frac{\partial^m}{\partial t_i^m}\ln M_z(\boldsymbol{t})\lfloor_{t=0} = \begin{cases} \gamma_i^* + \beta\alpha_i^*, & \text{if } m=1 \\ (m-1)!\beta^m\alpha_i^*, & \text{if } m\geq 2, \end{cases}$$

$$\text{(3.3)}\qquad K_{m_1,m_2} = \frac{\partial^{m_1+m_2}}{\partial t_i^{m_1}\partial t_j^{m_2}}\ln M_z(\boldsymbol{t})\lfloor_{t=0} = (m_1+m_2-1)!\beta^{m_1+m_2}\alpha_r^*,$$

where $r = \min(i, j)$. Next, we obtain the moments of Z_i. These are easier to get from the moments of V_i.

$$\frac{d^m M_{v_i}(t)}{dt^m} = \sum_{k_1=0}^{m} \binom{m}{k_1} \left(\left\{ \frac{d^{k_1}}{dt^{k_1}} (1-\beta t)^{-\alpha_i} \right\} \left\{ \frac{d^{m-k_1}}{dt^{m-k_1}} e^{(\gamma_i t)} \right\} \right).$$

Hence, putting $t = 0$ we get the m-th moment of V_i:

$$\text{(3.4)} \quad M_i^{(m)} = E(V_i^m) = \sum_{k_1=0}^{m} \binom{m}{k_1} \alpha_i(\alpha_i+1)\cdots(\alpha_i+k_1-1)\beta^{k_1}\gamma_i^{m-k_1}$$
$$= \sum_{k_1=0}^{m} \binom{m}{k_1} (\alpha_i)_{k_1} \beta^{k_1} \gamma_i^{m-k_1}$$

where $(\alpha)_r = \alpha(\alpha+1)\cdots(\alpha+r-1)$, $(\alpha)_0 = 1$. Using the above, we now have,

$$\text{(3.5)} \quad E(Z_i^m) = E(V_1 + \cdots + V_i)^m = \sum_{\kappa(r_1,\ldots,r_i,m)} \frac{m!}{r_1!\cdots r_i!} \prod_{l=1}^{i} E(V_l^{r_l})$$
$$= \sum_{\kappa(r_1,\ldots,r_i,m)} \frac{m!}{r_1!\cdots r_i!} \prod_{l=1}^{i} \{M_l^{(r_l)}\}$$

where $M_l^{(r_l)}$ is given in (3.4),

$$\kappa(r_1,\ldots,r_k,m) = \{(r_1,\ldots,r_k) \in N_+^k \mid r_1 + \cdots + r_k = m\}$$

and N_+ is the set of non-negative integers. Similarly,

$$\text{(3.6)} \quad E(Z_i^{k_1} Z_j^{k_2}) = \sum_{\kappa(r_1,\ldots,r_i,k_1)} \sum_{\kappa(s_1,\ldots,s_j,k_2)} \frac{k_1!}{r_1!\cdots r_i!} \frac{k_2!}{s_1!\cdots s_j!} Q$$

where

$$Q = \begin{cases} E(V_1)^{r_1+s_1} \cdots E(V_i)^{r_i+s_i} E(V_{i+1})^{s_{i+1}} \cdots E(V_j)^{s_j} & \text{if } j > i \\ E(V_1)^{r_1+s_1} \cdots E(V_j)^{r_j+s_j} E(V_{j+1})^{r_{j+1}} \cdots E(V_i)^{r_i} & \text{if } j < i \\ E(Z_i)^{k_1+k_2} & \text{if } j = i. \end{cases}$$

4. Densities

From the joint distribution in (1.2) we can obtain the distribution of subsets of $Z_1, \ldots, Z_k$. First, consider the density of $(Z_1, \ldots, Z_{k-1})$. This is easily obtained by integrating out Z_k. Integration over Z_k leads to the following integral

$$\text{(4.1)} \quad \int_{z_{k-1}+\gamma_k}^{\infty} (z_k - z_{k-1} - \gamma_k)^{\alpha_k - 1} e^{-(z_k - (\gamma_1 + \cdots + \gamma_k))/\beta} dz_k$$
$$= \int_0^{\infty} u^{\alpha_k - 1} e^{-(u + z_{k-1} + \gamma_k - (\gamma_1 + \cdots + \gamma_k))/\beta} du$$
$$= \beta^{\alpha_k} \Gamma(\alpha_k) e^{-(z_{k-1} - (\gamma_1 + \cdots + \gamma_{k-1}))/\beta}.$$

Hence, the joint density of $Z_1, \ldots, Z_{k-1}$ is of the same form as the density of $Z_1, \ldots, Z_k$. This is also clear from the definition of the Z_i's.

Next, consider the joint density of $Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_k$. This is obtained by integrating out Z_i, which leads to the following integral:

$$(4.2)\quad \int_{z_{i-1}+\gamma_i}^{z_{i+1}-\gamma_{i+1}} (z_i - z_{i-1} - \gamma_i)^{\alpha_i - 1}(z_{i+1} - z_i - \gamma_{i+1})^{\alpha_{i+1}-1} dz_i$$
$$= \int_0^{z_{i+1}-\gamma_{i+1}-z_{i-1}-\gamma_i} u^{\alpha_i - 1}(z_{i+1} - \gamma_{i+1} - z_{i-1} - \gamma_i - u)^{\alpha_{i+1}-1} du$$
$$= \frac{\Gamma(\alpha_i)\Gamma(\alpha_{i+1})}{\Gamma(\alpha_i + \alpha_{i+1})}(z_{i+1} - \gamma_{i+1} - z_{i-1} - \gamma_i)^{\alpha_i + \alpha_{i+1} - 1}.$$

Note that the location parameter of $z_{i+1} - z_{i-1}$ is $\gamma_1 + \cdots + \gamma_{i+1} - (\gamma_1 + \cdots + \gamma_{i-1}) = \gamma_i + \gamma_{i+1}$ and the shape parameter is $\alpha_i + \alpha_{i+1}$; hence, the joint distribution of the subset $Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_k$ is also of the same form as the density of the $Z_1, \ldots, Z_k$.

The above discussion shows that the joint density of *all subsets* of $Z_1, \ldots, Z_k$ is of the same functional form. We now establish several interesting results concerning conditional densities and densities of ratios of the Z_i's.

(a) The conditional density of Z_{i+1} given $Z_i = z_i$ is evidently a gamma with parameters α_{i+1}, β, $z_i + \gamma_{i+1}$, i.e. $Z_{i+1} \mid Z_i \sim G(\alpha_{i+1}, \beta, z_i + \gamma_{i+1})$. We note that for $j > i$ we have

$$\text{(b)}\qquad \frac{Z_i - \gamma_i^*}{Z_j - \gamma_j^*} \sim \text{Beta}(\alpha_i^*, \alpha_j^* - \alpha_i^*) \text{ type-1.}$$

$$\text{(c)}\qquad \frac{Z_i - \gamma_i^*}{Z_j - Z_i + \gamma_i^* - \gamma_j^*} \sim \text{Beta}(\alpha_i^*, \alpha_j^* - \alpha_i^*) \text{ type-2.}$$

Also, since $(V_i - \gamma_i)/\beta \sim G(\alpha_i, 1, 0)$, as a consequence of a well known result (see, for example Wilks (1962)), we have the following:

$$\text{(d)}\qquad Y_1 = \frac{Z_1 - \gamma_1}{Z_k - \gamma_k^*}, Y_2 = \frac{Z_2 - Z_1 - \gamma_2}{Z_k - \gamma_k^*}, \ldots, Y_{k-1} = \frac{Z_k - Z_{k-1} - \gamma_k}{Z_k - \gamma_k^*}$$

jointly have the *Dirichlet* density with parameters $\alpha_1, \ldots, \alpha_k$ and they are independent of Z_k. The *Dirichlet* density is

$$h(y_1, y_2, \ldots, y_{k-1}) = \frac{\Gamma(\alpha_k^*)}{\prod_{l=1}^k \Gamma(\alpha_l)} \left\{ \prod_{l=1}^{k-1} y_l^{\alpha_l - 1} \right\} (1 - y_{k-1}^*)^{\alpha_k - 1}$$

where $y_i^* = y_1 + \cdots + y_i \geq 0$, $i = 1, \ldots, k-1$ and $\sum_{i=1}^{k-1} y_i \leq 1$. Clearly, each Y_i is a Beta type-1 (see (b)).

Finally we note the following results that concern the special case in which each of the V_i's is exponential ($\alpha_i = 1$, $\beta = 1$, $\gamma_i = 0$). From the density above we have:

$$\text{(e)} \qquad h(y_1, y_2, \ldots, y_{k-1}) = (k-1)!$$

and hence $Y_1, \ldots, Y_{k-1}$ are distributed like the order statistic from the uniform $U(0,1)$-distribution. In this case, the joint density of $Z_1, \ldots, Z_k$ reduces to

$$f(z_1, \ldots, z_k) = e^{-z_k}, \qquad 0 < z_1 < z_2 < \cdots < z_{k-1}, \qquad 0 < z_k < \infty.$$

The transformation

$$W_1 = \frac{Z_1}{Z_k}, W_2 = \frac{Z_2}{Z_k}, \ldots, W_{k-1} = \frac{Z_{k-1}}{Z_k}, W_k = Z_k$$

is one-to-one with Jacobian $J(Z \to W) = W_k^{k-1}$. Thus, the joint density of $W_1, \ldots, W_{k-1}$ is

$$g(w_1, \ldots, w_{k-1}) = \int_0^\infty w_k^{k-1} e^{-w_k} dw_k = (k-1)!.$$

Hence, $W_1, \ldots, W_{k-1}$ are also distributed like the order statistic from the uniform $U(0,1)$-distribution.

Acknowledgements

The authors would like to thank the referees for the constructive suggestions which enabled them to improve the manuscript. The first author would like to thank the Natural Sciences and Engineering Research Council of Canada for financial support.

REFERENCES

Becker, P. J. and Roux, J. J. J. (1981). A bivariate extension of the gamma distribution, *South African Statist. J.*, **15**, 1–12.

Çinlar, E. (1975). *Introduction to Stochastic Processes*, Prentice Hall, New Jersey.

Dussauchoy, A. and Berland, R. (1974). A multivariate gamma type distribution whose marginal laws are gamma, and which has a property similar to a characteristic property of the normal case, *Statistical Distributions in Scientific Works*, Vol. 1 (eds. G. P. Patil, S. Kotz and J. K. Ord), 319–328, Reidel, Dordrecht.

Eagleson, G. K. (1964). Polynomial expansions of bivariate distributions, *Ann. Math. Statist.*, **35**, 1208–1215.

Freund, J. E. (1961). A bivariate extension of the exponential distribution, *J. Amer. Statist. Assoc.*, **56**, 971–977.

Gaver, D. P., Jr. (1970). Multivariate gamma distributions generated by mixture, *Sankhyā Ser. A*, **32**, 123–126.

Ghirtis, G. C. (1967). Some problems of statistical inference relating to double gamma distribution, *Trabajos de Estadistica*, **18**, 67–87.

Johnson, N. L. and Kotz, S. (1972). *Distributions in Statistics*, Vol. 4, Houghton Mifflin, Boston.

Kibble, W. F. (1941). A two-variate gamma distribution, *Sankhyā*, **5**, 137–150.

Kowalczyk, T. and Tyrcha, J. (1989). Multivariate gamma distributions-properties and shape estimation, *Statistics*, **20**, 465–474.

Krishnaiah, P. R. and Rao, M. M. (1961). Remarks on a multivariate gamma distribution, *Amer. Math. Monthly*, **68**, 342–346.

Krishnaiah, P. R., Hagis, P. and Steinberg, L. (1963). A note on the bivariate chi-distribution, *SIAM Rev.*, **5**, 140–144.

Lingappaiah, G. S. (1984). Bivariate gamma distribution as a life test model, *Applikace Matematiky*, **29**, 182–188.

Mathai, A. M. and Moschopoulos, P. G. (1991). On a multivariate gamma, *J. Multivariate Anal.*, **39**, 135–153.

Miller, K. S., Bernstein, R. I. and Blumenson, L. E. (1958). Generalized Rayleigh processes, *Quart. Appl Math.*, **16**, 137–145 (Correction: ibid. (1963). **20**, p. 395).

Moran, P. A. P. (1967). Testing for correlation between non-negative variates, *Biometrika*, **54**, 385–394.

Moran, P. A. P. (1969). Statistical inference with bivariate gamma distributions, *Biometrika*, **56**, 627–634.

Moran, P. A. P. (1970). The methodology of rain making experiments, *Review of the International Statistical Institute*, **38**, 105–115.

Sarmanov, I. O. (1970). Gamma correlation process and its properties, *Dokl. Akad. Nauk SSSR*, **191**, 30–32 (in Russian).

Steel, S. J. and le Roux, N. J. (1987). A reparameterisation of a bivariate gamma extension, *Comm. Statist. Theory Methods*, **16**, 293–305.

Wilks, S. S. (1962). *Mathematical Statistics*, Wiley, New York.