

 Open access • Journal Article • DOI:10.1080/0309192031000108698

A formal theory for vortex rossby waves and vortex evolution — [Source link](#)

James C. McWilliams, Lee Paul Graves, Michael T. Montgomery

Institutions: University of California, Los Angeles, Colorado State University

Published on: 01 Aug 2003 - Geophysical and Astrophysical Fluid Dynamics (Taylor & Francis Group)

Topics: Vortex ring, Vortex, Starting vortex, Horseshoe vortex and Burgers vortex

Related papers:

- [A theory for vortex rossby-waves and its application to spiral bands and intensity changes in hurricanes](#)
- [A Three-Dimensional Balance Theory for Rapidly Rotating Vortices](#)
- [Three-Dimensional Alignment and Corotation of Weak, TC-like Vortices via Linear Vortex Rossby Waves](#)
- [Polygonal Eyewalls, Asymmetric Eye Contraction, and Potential Vorticity Mixing in Hurricanes](#)
- [Tropical Cyclone Evolution via Potential Vorticity Anomalies in a Three-Dimensional Balance Model](#)

Share this paper:    

View more about this paper here: <https://typeset.io/papers/a-formal-theory-for-vortex-rossby-waves-and-vortex-evolution-39yip1zaki>

A FORMAL THEORY FOR VORTEX ROSSBY WAVES AND VORTEX EVOLUTION

JAMES C. McWILLIAMS^{a,*}, LEE P. GRAVES^a and
MICHAEL T. MONTGOMERY^b

^a*Department of Atmospheric Sciences and Institute of Geophysics and Planetary Physics,
University of California, Los Angeles, Los Angeles, CA 90095-1565;*

^b*Department of Atmospheric Science, Colorado State University, Fort Collins, CO 80523*

(Received 1 May 2002; In final form 7 February 2003)

A formal theory is presented for the balanced evolution of a small-amplitude, small-scale wave field in the presence of an axisymmetric vortex initially in gradient-wind balance and the accompanying changes induced in the vortex by the azimuthally averaged wave fluxes. The theory is a multi-parameter, asymptotic perturbation expansion for the conservative, rotating, f -plane, shallow-water equations. It extends previous work on Rossby-wave dynamics in vortices and more generally provides a new perspective on wave/mean-flow interaction in finite Rossby-number regimes. Some illustrative solutions are presented for a perturbed vortex undergoing axisymmetrization.

Keywords: Vortex dynamics; Wave-mean flow interaction; Vortex Rossby waves

1 INTRODUCTION

Vortices commonly arise and often persist for lengthy intervals in the atmosphere and ocean, especially in circumstances influenced by Earth's rotation and stable density stratification. A central component in the dynamical theory of vortices is the fact that an axisymmetric azimuthal circulation in hydrostatic, gradient-wind momentum balance with the radial and vertical pressure-gradient and density fields is an exact, stationary solution of the conservative fluid-dynamical equations (McWilliams, 1989). Another element is the fact that perturbed vortices with smooth and approximately monotonic radial vorticity profiles tend to relax back towards a stationary state (i.e., axisymmetrize), with a decay of the asymmetric components and accompanying changes in the radial and vertical profiles of the azimuthal vortical flow. This relaxation process is an essential feature of the robustness (emergence, persistence) of vortices. It can be advectively nonlinear for sufficiently large initial perturbations, and it is ultimately dissipative as the asymmetric components

*Corresponding author. Tel.: 310-206-2829. Fax: 310-206-5219. E-mail: jcm@atmos.ucla.edu

irreversibly evolve towards very small scales. Nevertheless, useful insights into how relaxation occurs should come from linearized, inviscid analyses of intermediate-time evolution.

In this article we derive an asymptotic perturbation theory for the evolution of a weak-amplitude, small-scale wave field in the presence of an axisymmetric vortex and for the azimuthally averaged changes in the initial vortex profile caused by rectified wave fluxes of mass, momentum, and potential vorticity. The theory generalizes a previous vortex Rossby-wave theory in Montgomery and Kallenbach (1997) – hereafter MK97 – and Moller and Montgomery (1999) (see Appendix B). We derive it for the simple dynamics of a rotating, shallow-water fluid layer on an f -plane (i.e., constant Coriolis frequency), though we envision that it may later be extended to a fully three-dimensional, rotating, stratified fluid. We make the assumption of inviscid, adiabatic dynamics, since nonconservative influences are often either weak in magnitude or occur only at late times in large-scale flows with large Reynolds number. These assumptions can later be relaxed where physically required (e.g., internal heating or surface drag). Finally, we assume that the evolution of both the waves and the vortex satisfies the dynamical constraints of *balance* (i.e., the [total horizontal] velocity in the plane perpendicular to the [vertical] vortex axis is weakly divergent). Balance constraints exclude fast wave motions of several types from the class of solutions: acoustic, surface and internal gravitational, and inertial.¹ These constraints are typically well satisfied by large-scale flows in the atmosphere and ocean either due to the influences of Earth's rotation and a stable density stratification with small or modest values of Rossby and Froude numbers (e.g., the Balance Equations, BE; McWilliams *et al.*, 1998) or due to the rapid rotation of the vortex core (e.g., Asymmetric Balance, AB; Shapiro and Montgomery, 1993). Quite a large variety of approximate, balanced models have been previously proposed (McWilliams and Gent, 1980), but experience comparing alternatives has shown that they often yield rather similar results (e.g., Allen *et al.*, 1990).

For the asymptotic theory of vortex dynamics, the desirable properties are the following:

- Validity in vortices with finite Rossby number.
- Validity in vortices whose horizontal scale can be either large or small compared to the internal deformation radius.
- Validity for vortex wave fluctuations whose horizontal divergence is not necessarily small compared to the vertical vorticity.
- Simple conservation laws for both the wave and vortex evolution equations.
- Governing equations for only a single dependent variable, rather than for a multi-variate system, for both the wave and vortex evolutions.

The assumptions of the theory are that the lateral scale and amplitude of the vortex perturbation are small compared to the vortex profile. Thus the theory is in the category of quasi-linear waves (i.e., the only retained nonlinearity is for wave/mean-flow interaction) in a slowly varying medium (the vortex). The scale-separation

¹Unbalanced instabilities are also possible; see, e.g. Ford (1994) for a Rankine vortex or Molemaker *et al.* (2001) for a Taylor-Couette flow. However, the growth rates for such instabilities are typically small for moderate or small Rossby number, and they are not our present focus.

and vortex-profile assumptions exclude two other linear, vortex-wave phenomena: vortex shear instability (e.g., Flierl, 1988) and large-scale, quasi-mode perturbation decay (Schecter *et al.*, 2000; Balmforth *et al.*, 2001). We make no attempt here to define the regime boundaries among these phenomena. The scale separation assumption also allows an escape from the singular spatial structure of the continuous spectrum of neutral normal modes in a shear flow (Drazin and Howard, 1966).

The motivations for having such a theory are manifold. Among the phenomena that may be addressed by applications of this theory are the following: horizontal vortex axisymmetrization (McCalpin, 1987; Melander *et al.*, 1987; Sutyrin, 1989; MK97; Bassom and Gilbert, 1998; Brunet and Montgomery, 2002); vortex spiral evolution (Lundgren, 1982; Moffat, 1986; Gilbert, 1988); vertical alignment (i.e., relaxation of perturbations that tilt the vortex axis away from the vertical Sutyrin *et al.*, 1998; Polvani and Saravanan, 2000; Reasor and Montgomery, 2001; Schecter *et al.*, 2002); evolutionary parity selection of either anticyclonic vortices away from boundaries (Cushman-Roisin and Tang, 1990; Polvani *et al.*, 1994; Arai and Yamagata, 1994; Yavneh *et al.*, 1997; Stegner and Dritschel, 2000) or cyclonic vortices adjacent to solid horizontal boundaries (Simmons and Hoskins, 1978; Snyder *et al.*, 1991; Rotunno *et al.*, 2000; Hakim *et al.*, 2002), both due to their greater robustness to perturbations at finite Rossby number; conservative vortex dynamics in shearing or straining flows (Marcus, 1990; Bassom and Gilbert, 1999); tropical cyclone development and potential vorticity redistribution (Guinn and Schubert, 1993; Montgomery and Enagonio, 1998; Schubert *et al.*, 1999; Moller and Montgomery, 1999, 2000); and astrophysical accretion and protoplanetary disks (Bracco *et al.*, 1999; Mayer *et al.*, 2002; Nauta, 1999). It is not our present purpose to report particular solutions of the formal theory required for these various applications. However, we do include an illustration of axisymmetrization (Section 8).

The derived wave/mean-flow theory is accurate to second order in the wave amplitude. The balanced-vortex evolution is a generalization of Eliassen's (1951) classic model. The vortex theory possesses a "nonacceleration" theorem for stationary disturbances (Andrews and McIntyre, 1978; Andrews *et al.*, 1987), although the nature of vortex Rossby waves is such that the stationarity assumption is generally not satisfied for freely evolving waves except for strictly neutral normal modes or at long times for sheared waves when the geopotential wave amplitude decays to zero. The equation for the vortex tangential velocity tendency exhibits two distinct eddy forces: the first is related to the tendency of angular "pseudo-momentum density" (e.g., Held, 1985), and the second involves the contribution from the balanced waves to the eddy fluxes of radial momentum. Thus, in this vortex-evolution theory the "pseudo-momentum rule" does not hold generally (cf., McIntyre, 1981).

The plan of the article is as follows. Section 2 presents the basic equations and their stationary vortex solutions. Section 3 introduces balanced approximations for wave and vortex evolution. Section 4 is a nondimensionalization and identification of the important parameters. Section 5 derives the slowly varying wave theory, and Section 6 derives the quasi-linear vortex-evolution theory. Section 7 discusses some special circumstances implicated in these theories, and Section 8 analyzes some illustrative solutions for typical vortex Rossby waves. Section 9 is a summary and discussion. Appendix A records several asymptotic limits of the theory, and Appendix B describes the relation between the present theory and a previous one (MK97).

2 SHALLOW-WATER EQUATIONS AND THE MEAN VORTEX

The conservative, rotating shallow-water equations with a free top surface and flat solid bottom surface in cylindrical coordinates (r, λ, t) are the following

$$\begin{aligned} \frac{D\hat{u}}{Dt} - \frac{\hat{v}^2}{r} - f\hat{v} &= -\frac{\partial\hat{\phi}}{\partial r}, \\ \frac{D\hat{v}}{Dt} + \frac{\hat{u}\hat{v}}{r} + f\hat{u} &= -\frac{1}{r}\frac{\partial\hat{\phi}}{\partial\lambda}, \\ \frac{D\hat{\phi}}{Dt} + g\hat{h}\nabla\cdot\hat{\mathbf{u}} &= 0, \end{aligned} \quad (1)$$

where $\hat{\mathbf{u}} = (\hat{u}, \hat{v})$ is the horizontal=(radial, azimuthal) velocity; $\hat{\phi}$ is the dynamic pressure; $\hat{h} = H_o + g^{-1}\hat{\phi}$ is the layer thickness (with H_o the mean thickness and g the vertical gravitational acceleration); f is the constant Coriolis frequency; $\nabla = (\partial_r, r^{-1}\partial_\lambda)$ is the horizontal gradient operator; and $D/Dt = \partial_t + \hat{\mathbf{u}}\cdot\nabla$ is the substantial time derivative. The first two equations in (1) are horizontal momentum balances, and the last one is mass conservation, or continuity. A linear differential combination of (1) yields the equation for the advective conservation of potential vorticity,

$$\frac{D\hat{q}}{Dt} = 0, \quad \hat{q} = \frac{f + \hat{\xi}}{g\hat{h}}, \quad (2)$$

where $\hat{\xi} = r^{-1}\partial_r(r\hat{v}) - r^{-1}\partial_\lambda\hat{u}$ is the vertical component of vorticity. The shallow-water equations conserve the area integrals of energy and enstrophy densities,

$$\mathcal{E}_{\text{swe}} = \frac{1}{2} \left[\hat{h}(\hat{u}^2 + \hat{v}^2) + \frac{1}{g}\hat{\phi}^2 \right], \quad \mathcal{V}_{\text{swe}} = \frac{1}{2}\hat{h}\hat{q}^2. \quad (3)$$

We denote a mean vortex solution to (1) by

$$\hat{u} = 0, \quad \hat{v} = \bar{v}(r), \quad \partial_r\hat{\phi} = \partial_r\bar{\phi}(r) = f\bar{v} + \frac{\bar{v}^2}{r}, \quad (4)$$

where the final relation is referred to as axisymmetric gradient-wind balance. This solution is a dynamically stationary state that may or may not be stable to weak perturbations. It has the following auxiliary variables:

$$\begin{aligned} \bar{h} &= H_o + \frac{1}{g}\bar{\phi}, \quad \bar{\Omega} = \frac{\bar{v}}{r}, \quad \bar{\xi} = \frac{1}{r}\partial_r(r\bar{v}), \\ \bar{\delta} &\equiv \nabla\cdot\bar{\mathbf{u}} = 0, \quad \bar{q} = \frac{f + \bar{\xi}}{g\bar{h}}, \end{aligned} \quad (5)$$

which are, respectively, the mean layer thickness, angular velocity, vertical vorticity, horizontal divergence, and potential vorticity.

To obtain equations for the evolution of deviations from the mean vortex, we substitute

$$\hat{u} = u(r, \lambda, t), \quad \hat{v} = \bar{v}(r) + v(r, \lambda, t), \quad \hat{\phi} = \bar{\phi}(r) + \phi(r, \lambda, t) \quad (6)$$

into (1). The result is

$$\begin{aligned} \bar{D}u - \bar{\xi}v &= -\partial_r \phi - N^u, \\ \bar{D}v + \bar{\eta}u &= -\frac{1}{r} \partial_\lambda \phi - N^v, \\ \bar{D}\phi + g\bar{h}\delta + \partial_r \bar{\phi}u &= -N^\phi, \end{aligned} \quad (7)$$

where $\delta = r^{-1} \partial_r(ru) + r^{-1} \partial_\lambda v$ is the horizontal divergence; $\bar{D} = \partial_t + \bar{\Omega} \partial_\lambda$ is the substantial derivative due to the mean vortex; $\bar{\xi} = f + 2\bar{\Omega}$ is the modified Coriolis frequency; $\bar{\eta} = f + \bar{\zeta}$ is the mean absolute vorticity; and the nonlinear terms are defined by

$$\begin{aligned} N^u &= u \partial_r u + \frac{1}{r} v \partial_\lambda u - \frac{v^2}{r}, \\ N^v &= u \partial_r v + \frac{1}{r} v \partial_\lambda v + \frac{uv}{r}, \\ N^\phi &= \frac{1}{r} \partial_r(ru\phi) + \frac{1}{r} \partial_\lambda(v\phi). \end{aligned} \quad (8)$$

The associated potential-vorticity principle for (7) is

$$\begin{aligned} \bar{D}q + \partial_r(\ln \bar{q})u &= \frac{1}{r\bar{\eta}} [\partial_\lambda N^u - \partial_r(rN^v)] + \frac{1}{g\bar{h}} N^\phi, \\ q &= \frac{1}{r\bar{\eta}} [\partial_r(rv) - \partial_\lambda u] - \frac{1}{g\bar{h}} \phi. \end{aligned} \quad (9)$$

(Note the difference in units compared to \hat{q} in (2).) The fluctuation amplitudes may be assigned quadratic energy and enstrophy norms, similar in form to (3), viz., the area integrals of

$$\mathcal{E}_f = \frac{1}{2} \left[\bar{h}(u^2 + v^2) + \frac{1}{g} \phi^2 \right], \quad \mathcal{V}_f = \frac{1}{2} \bar{h} q^2. \quad (10)$$

However, these quantities are not conserved due to exchanges with the mean vortex and cubic nonlinearities.

We define a vortex as a balanced, axisymmetric azimuthal flow. Obviously, the mean vortex, \bar{v} in (4), has these attributes, but so also does the vortex change, $\langle v \rangle$ (with angle brackets denoting an azimuthal average), that may develop from (7) to (8) due to the presence of wave fluctuations. (We complementarily define a wave as a

fluctuation with zero azimuthal mean.) We are interested in azimuthally averaged fluctuation dynamics, both for the shallow-water equations,

$$\begin{aligned}\partial_t \langle u \rangle - \bar{\xi} \langle v \rangle &= -\partial_r \langle \phi \rangle - \langle N^u \rangle, \\ \partial_t \langle v \rangle + \bar{\eta} \langle u \rangle &= -\langle N^v \rangle, \\ \partial_t \langle \phi \rangle + g\bar{h} \langle \delta \rangle + \partial_r \bar{\phi} \langle u \rangle &= -\langle N^\phi \rangle,\end{aligned}\tag{11}$$

where $\langle \delta \rangle = r^{-1} \partial_r(r \langle u \rangle)$, and the associated potential-vorticity principle,

$$\begin{aligned}\partial_t \langle q \rangle + \partial_r(\ln \bar{q}) \langle u \rangle &= -\frac{1}{r\bar{\eta}} \partial_r(r \langle N^v \rangle) + \frac{1}{g\bar{h}} \langle N^\phi \rangle, \\ \langle q \rangle &= \frac{1}{r\bar{\eta}} \partial_r(r \langle v \rangle) - \frac{1}{g\bar{h}} \langle \phi \rangle.\end{aligned}\tag{12}$$

These simpler formulas, compared to those above, are a consequence of $\langle \partial_\lambda Q \rangle = 0$ for any Q .

3 BALANCE MODELS

The mean vortex (4) is an exact solution to (1). It also satisfies balanced constraints (i.e., gradient-wind balance), with $\bar{\delta} = 0$ and no manifestation of fast-wave behavior. However, both the wave equations (7)–(9) and the vortex-evolution equations (11)–(12) admit balanced and unbalanced solutions in general. Since our focus is on balanced evolution, we make approximations in this section that introduce balance constraints separately for the waves and vortex.

One procedure for excluding unbalanced motions was proposed by Shapiro and Montgomery (1993), viz., Asymmetric Balance (AB). For our purposes we view AB as an iterative procedure in a small parameter (defined in Section 4), where \mathbf{u} is successively approximated as a functional of ϕ by substitutions in the right-hand side of the rewritten momentum equations from (7),

$$\begin{aligned}u &= -\frac{1}{\bar{\eta}} \left(\frac{1}{r} \partial_\lambda \phi + \bar{D}v + N^v \right) \\ v &= \frac{1}{\bar{\xi}} \left(\partial_r \phi + \bar{D}u + N^u \right),\end{aligned}\tag{13}$$

followed by their substitution into either the continuity relation in (7) or the potential-vorticity equation (9), which then yields a single (balanced) equation for ϕ . So there are varieties of AB models, depending upon the number of iterations and the substitution path followed, as well as other balanced varieties, e.g., the Balance Equations.²

²As is commonly true for asymptotic analyses, an approximation at any given order in an expansion is in principle nonunique with respect to differences at higher order. In all cases here the balanced approximations are accurate through two orders in the appropriate expansion parameter. Furthermore, we provisionally adopt the view that a more inclusive approximation is preferable, although this must be verified with solutions in particular situations.

3.1 Balanced Waves

For the linear wave theory (Section 5), we neglect the nonlinear terms in (13) and perform a single iteration. The result is

$$\begin{aligned} u &= -\frac{1}{\bar{\eta}r} \partial_\lambda \phi - \frac{1}{\bar{\eta}\bar{\xi}} \bar{D}(\partial_r \phi), \\ v &= \frac{1}{\bar{\xi}} \partial_r \phi - \frac{1}{r\bar{\eta}\bar{\xi}} \bar{D}(\partial_\lambda \phi). \end{aligned} \quad (14)$$

Following the path in Shapiro and Montgomery (1993), we substitute this into the continuity relation in (7) with $N = 0$ to yield

$$\bar{D}(\tilde{\nabla}^2 \phi - \bar{\gamma}^2 \phi) - \frac{1}{r} \bar{G} \partial_\lambda \phi = 0. \quad (15)$$

This is the linear evolution equation for the AB model, first derived in MK97. The inverse square of a local deformation radius,

$$\bar{\gamma}^2(r) = \frac{\bar{\xi}\bar{\eta}}{gh}, \quad (16)$$

enters in the vortex-stretching term in the potential vorticity;

$$\bar{G}(r) = \bar{\xi} \partial_r (\ln \bar{q}) \quad (17)$$

is a normalized mean-vortex potential-vorticity gradient; and

$$\tilde{\nabla}^2 = \frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{\gamma}^2} \partial_r \right) + \frac{\bar{\varphi}^2}{r^2} \partial_\lambda^2 \quad (18)$$

is a normalized Laplacian operator, which comprises the relative-vorticity term in the potential vorticity, where

$$\bar{\varphi}^2 = \bar{\varphi}_{AB,c}^2 \equiv 1. \quad (19)$$

The subscript denotes the formula for $\bar{\varphi}$ in the AB model derived by substitution into the continuity relation.

When the procedure outlined above is replaced by substituting (14) into the potential vorticity equation in (9), instead of (7), then the outcome (at the same order in the small parameter used to justify (14) as a first approximation) can still be expressed as (15), but with

$$\bar{\varphi}^2 = \bar{\varphi}_{AB,p}^2 \equiv \frac{2\bar{\xi} - \bar{\eta}}{\bar{\eta}} \quad (20)$$

instead of (19). The form for $\bar{\varphi}_{AB,p}$ is the more general one since it contains additional terms that can be shown by the nondimensionalization in Section 4 to be of $O(R)$ compared to $\bar{\varphi}_{AB,c}$, where R is the Rossby number for the mean vortex, not assumed small. Therefore, the AB theory for the waves is nonunique by the choice of the derivation path, and the consequences of its variant forms will be explored in later solutions. In contrast to a BE theory for the waves, however, AB yields a single evolutionary equation for ϕ , which simplifies the computational solution and its interpretation.

The operand of \bar{D} in (15),

$$q^{ab} = \tilde{\nabla}^2 \phi - \bar{\gamma}^2 \phi, \quad (21)$$

is an appropriate definition of potential vorticity for the linear AB model (as in Ren, 1999) because it appears in (15) and in the vortex-evolution model (25) below. Alternatively, we can define the pseudo-potential vorticity,

$$q_\xi^{ab} = \frac{1}{gh} \left[\frac{1}{r} \partial_r \left(\frac{r}{\xi} \partial_r \phi \right) + \frac{\bar{\varphi}^2}{\xi r^2} \partial_\lambda^2 \phi - \bar{q} \phi \right], \quad (22)$$

that can be shown to satisfy a rewritten form of (15) [analogous to (9) with $N = 0$],

$$\bar{D}(q_\xi^{ab}) + u \partial_r \bar{q} = 0,$$

with u defined by (14), for either the continuity or potential-vorticity derivation paths (MK97). Equation (15) can be recognized as the AB generalization of the more familiar quasigeostrophic (QG) balance model that approximates $\bar{\xi}$ and $\bar{\eta}$ by f and $\bar{\gamma}^2$ by $1/L_d^2$ (where $L_d = \sqrt{gH_o}/f$ is the global deformation radius) in the zeroth iterate in (13):

$$\bar{D} \left[\frac{1}{r} \partial_r (r \partial_r \phi) + \frac{1}{r^2} \partial_\lambda^2 \phi - \frac{1}{L_d^2} \phi \right] - \frac{1}{r} \left(\partial_r \bar{\xi} - \frac{1}{f L_d^2} \partial_r \bar{\phi} \right) \partial_\lambda \phi = 0. \quad (23)$$

Here the operand of \bar{D} , $q^{qg} = \nabla^2 \phi - (1/L_d^2) \phi$, is the QG potential vorticity.

3.2 Balanced Vortex Evolution

The AB vortex-evolution theory requires nonlinear contributions from the azimuthally averaged wave fluxes, although for consistency with (14) we retain only the leading-order nonlinear contributions in wave amplitude. With again only a single iteration on (13), we obtain

$$\begin{aligned} \langle u \rangle &= -\frac{1}{\bar{\eta}} \left(\frac{1}{\xi} \partial_r \partial_r \langle \phi \rangle + \langle N^v \rangle \right), \\ \langle v \rangle &= \frac{1}{\xi} (\partial_r \langle \phi \rangle + \langle N^u \rangle), \end{aligned} \quad (24)$$

where the nonlinear terms, $\langle N \rangle$, are functionals of ϕ that will be described more fully for the case of sheared waves in Section 6. Substituting (24) into (12) yields

$$\begin{aligned} \frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{\gamma}^2} \partial_r \partial_t \langle \phi \rangle \right) - \bar{\gamma}^2 \partial_t \langle \phi \rangle &= \bar{\gamma}^2 \langle N^\phi \rangle - \frac{\bar{\xi}}{r} \partial_r \left(r \langle N^v \rangle + \frac{r}{\bar{\xi}} \partial_t \langle N^u \rangle \right) + \bar{G} \langle N^v \rangle \\ &= \bar{\gamma}^2 \langle N^\phi \rangle - \frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{q}} \langle N^v \rangle \right) - \bar{\Gamma}(\partial_t \langle N^u \rangle), \end{aligned} \quad (25)$$

where

$$\bar{\Gamma}(Q) = \bar{\Gamma}_{AB,p}(Q) \equiv \frac{\bar{\xi}}{r} \partial_r \left(\frac{r}{\bar{\xi}} Q \right) \quad (26)$$

is an operator with argument Q . Not surprisingly, given the nonuniqueness of (15), this AB model for $\langle \phi \rangle$ is also nonunique. Following the alternative path of substituting (24) into the continuity relation in (11), we again obtain (25), except with an alternatively defined operator,

$$\bar{\Gamma}(Q) = \bar{\Gamma}_{AB,c}(Q) \equiv 0. \quad (27)$$

As with the AB wave theory (Section 3.1), the potential-vorticity AB vortex-evolution theory is the more general one, with added terms in (25) of $O(R)$.

Unlike for the wave theory, however, it is possible to derive a single BE equation for $\langle \phi \rangle$. This is because $\langle \delta \rangle$ and $\langle \zeta \rangle$ are differential functions only of $\langle u \rangle$ and $\langle v \rangle$, respectively. So the balance approximation, $\langle \delta \rangle \ll \langle \zeta \rangle$ (usually justified by Ro and/or $Fr = V/\sqrt{gH_o}$ values being somewhat small), may be applied directly to the momentum equations in (11) merely by dropping $\partial_t \langle u \rangle$ in the first one. In the vortex-evolution context, this is a less restrictive approximation than the usual one for BE. Instead of (24), the result is

$$\begin{aligned} \langle u \rangle &= -\frac{1}{\bar{\eta}} (\partial_t \langle v \rangle + \langle N^v \rangle), \\ \langle v \rangle &= \frac{1}{\bar{\xi}} (\partial_r \langle \phi \rangle + \langle N^u \rangle), \end{aligned} \quad (28)$$

which when substituted into either (12) or the continuity relation in (11) again yields (25) except with the alternative operator,

$$\bar{\Gamma}(Q) = \bar{\Gamma}_{BE}(Q) \equiv \frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{\gamma}^2} Q \right). \quad (29)$$

The BE theory for the vortex evolution (25) is therefore unique, unlike AB. Furthermore, the BE form $\bar{\Gamma}_{BE}$ is more general than the AB forms, containing all the AB terms plus additional ones of $O(R)$.

In the QG approximation described above, (25) takes a simpler and more familiar form,

$$\partial_t \left[\frac{1}{r} \partial_r (r \partial_r \langle \phi \rangle) - \frac{1}{L_d^2} \langle \phi \rangle \right] = \frac{1}{fr} \partial_r \left[\frac{1}{r} \partial_r \left(r \langle \partial_r \phi \partial_\lambda \phi \rangle \right) \right], \quad (30)$$

where the vortex QG potential-vorticity tendency is due to the second-order derivative of the geostrophic Reynolds stress from the waves.

3.3 Wave/Mean-flow Interaction

Equations (25) and (29), together with a specification of the nonlinear terms (Section 6), comprise a closed system for vortex evolution. From these we can derive some further relations for the implied wave/mean-flow interaction before focusing on the details of the wave evolution.

An Eliassen-like equation for $\langle u \rangle$ forced by the balanced wave fluctuations is derived from (28) by eliminating $\partial_t \langle v \rangle$ and using the continuity equation in (11). The result is

$$\partial_r \left[\frac{1}{r} \partial_r (r g \bar{h} \langle u \rangle) \right] - \bar{\eta} \bar{\xi} \langle u \rangle = \bar{\xi} \langle N^v \rangle - \partial_r \langle N^\phi \rangle + \partial_t \langle N^u \rangle. \quad (31)$$

Apart from the last term on the right-hand side, (31) is the shallow-water analogue of Eliassen's (1951) equation for the transverse (secondary) circulation driven by "heat" and "momentum" sources within the vortex. $\langle N^\phi \rangle$ is analogous to the heat source, and $\langle N^v \rangle$ is analogous to the momentum source. The last term, $\partial_t \langle N^u \rangle$, represents the contribution from the balanced waves to the eddy flux of radial momentum as a departure from gradient-wind balance.

An explicit equation for the acceleration of the vortex can also be derived. First, we condense (25) with (29) by recalling the definitions of N^v and N^ϕ in Section 2. The result is

$$\begin{aligned} \frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{\gamma}^2} \partial_r \partial_t \langle \phi \rangle \right) - \bar{\gamma}^2 \partial_t \langle \phi \rangle &= -\frac{\bar{\gamma}^2}{r} \partial_r (g \bar{h} r \langle u q \rangle) - \frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{\gamma}^2} \partial_t \langle N^u \rangle \right) \\ &= \frac{\bar{\gamma}^2}{r} \left[\partial_r \left(\bar{\eta} \partial_t \frac{r \langle q^2 \rangle}{2 \partial_r \bar{q}} \right) - \partial_r \left(\frac{r}{\bar{\gamma}^2} \partial_t \langle N^u \rangle \right) \right]. \end{aligned} \quad (32)$$

The last equality assumes that $\partial_r \bar{q}$ is nonzero and uses (9) with $N = 0$ to obtain the eddy potential-vorticity radial flux at second order in wave amplitude. (The limiting case of $\partial_r \bar{q} = 0$ is discussed in Appendix A.2.) If we now combine (32) with the time derivative of $\langle v \rangle$ in (24), we obtain a single equation for $\partial_t \langle v \rangle$ forced by the balanced waves:

$$\partial_r \left[\frac{1}{r} \partial_r \left(\frac{r}{\bar{q}} \partial_t \langle v \rangle \right) \right] - \bar{\xi} \partial_t \langle v \rangle = \partial_r \left[\frac{1}{r} \partial_r \left(\bar{\eta} \partial_t \frac{r \langle q^2 \rangle}{2 \partial_r \bar{q}} \right) \right] - \partial_t \langle N^u \rangle. \quad (33)$$

As in (31) and (32), this equation, together with the boundary conditions $\partial_t \langle v \rangle = 0$ at $r = 0$ and $r \rightarrow \infty$, defines a well-posed elliptic boundary value problem for $\partial_t \langle v \rangle$ provided $\bar{q}\bar{\xi} = \bar{\gamma}^2$ is everywhere positive. The wave forcing in (33) has a form similar to the generalized Eliassen–Palm relations obtained in Andrews and McIntyre (1978). The wave forcing has two distinct terms, one from the perturbation potential enstrophy (a divergence of the Eliassen–Palm flux, defined below) and the second from the departure from gradient-wind balance due to balanced waves. If the waves are steady in time, both terms vanish; hence, given zero forcing with zero boundary conditions, the ellipticity of the differential operator implies that $\partial_t \langle v \rangle = 0$. This is the “non-acceleration” theorem for this theory. Because of the transient behavior of sheared vortex Rossby waves (Section 8), only rarely will freely evolving wave disturbances satisfy this condition, except possibly for strictly neutral waves or for sheared waves at late time when the fluctuation geopotential amplitude decays to zero. Therefore, only for the unlikely situation of $N = 0$ is there a valid nonacceleration theorem in this theory.

The first term on the right-hand side of (33) can be rendered more familiar by considering two limits. The first limit is QG. The second term is $O(R)$ relative to the first (Section 4) and can be dropped, leaving

$$\partial_r \left[\frac{1}{r} \partial_r (r \partial_t \langle v_g \rangle) \right] - \frac{1}{L_d^2} \partial_t \langle v_g \rangle = \frac{1}{f} \partial_r \left[\frac{1}{r} \partial_r \left(\partial_t \frac{r \langle (q^{qg})^2 \rangle}{2 \partial_r \bar{q}^{qg}} \right) \right], \quad (34)$$

where $\langle v_g \rangle$ is the azimuthally averaged, geostrophic, tangential velocity, and q^{qg} is the quasigeostrophic potential vorticity defined following (23). It is evident that (34) is the radial derivative of (30). Therefore, apart from a divergence-free vector field, the term within the large parentheses on the right-hand side of (34) is the time derivative of the angular pseudo-momentum (or wave activity) density for the linearized QG shallow-water model (Andrews *et al.*, 1987). The second limiting case is the nondivergent limit ($L_d, gH_0 \rightarrow \infty$). In this limit the second terms on the left-hand side and right-hand side of (33) both vanish. Assuming $\partial_t \langle v \rangle$ vanishes at $r = 0$ and $r \rightarrow \infty$, the limiting equation may be integrated to obtain

$$\partial_t r \langle v \rangle = \partial_t \frac{r \langle \zeta^2 \rangle}{2 \partial_r \bar{\xi}}. \quad (35)$$

Again apart from a divergence-free vector field, the right-hand side term inside the time derivative in (35) is the angular pseudo-momentum density for the barotropic (nondivergent) model in curvilinear flow (Held and Phillips, 1987).

In the QG and nondivergent limits, the acceleration of the vortex is governed solely by the first right-hand side term in (33). In these cases the change in angular momentum due to the waves is proportional to the change in angular pseudo-momentum. This property is called the “pseudo-momentum rule” (McIntyre, 1981; Grimshaw, 1984). As is evident from (33), the pseudo-momentum rule does not apply in general for finite R and finite gH_0 . Moreover, the quantity inside the parentheses in the first term on the right-hand side of (33) is only one of two terms in the angular pseudo-momentum density. For the shallow-water equations the angular pseudo-momentum density is defined such that its time derivative equals the flux divergence of azimuthally

averaged eddy angular momentum. Specifically, in cylindrical coordinates (cf., Eq. (2.20) of Held, 1985),

$$\partial_t \left(\bar{\eta}^2 \frac{r \langle q^2 \rangle}{2 \partial_r \bar{q}} + \langle r v \phi \rangle \right) = -\frac{1}{r} \partial_r (r \bar{\phi} r \langle u v \rangle), \quad (36)$$

which, in combination with (33), has a more general structure than $\partial_t \langle v \rangle \sim \partial_t \langle q^2 \rangle \sim \partial_r \langle u v \rangle$. (On rearranging (36), the quantity inside the parentheses in the first term on the right-hand side of (33) defines the space-time (pseudo) divergence of the Eliassen-Palm flux.)

4 NONDIMENSIONALIZATION AND PARAMETERS

In order to clarify various approximations and asymptotic relations, we now transform our variables and equations to nondimensional forms. The vortex environment is characterized by g , f , and H_o , from which we define the global deformation radius, $L_d = \sqrt{g H_o} / f$. The mean vortex itself is characterized by an azimuthal velocity scale, V ; a length scale for the velocity profile, L ; and another length scale, $L_{\bar{q}}$, for the mean-vortex potential-vorticity gradient, \bar{G} , as it appears in (15). From these we construct the following nondimensionalizing factors for mean-vortex quantities:

$$\begin{aligned} \bar{\phi} &\sim V f L, & \bar{h} &\sim H_o, & \bar{\Omega}, \bar{\xi} &\sim \frac{V}{L}, & \bar{\eta}, \bar{\xi} &\sim f, & \bar{q} &\sim \frac{f}{g H_o}, \\ \partial_r \bar{q} &\sim \frac{V}{g H_o L L_{\bar{q}}}, & \bar{\gamma} &\sim L_d^{-1}, & \bar{G} &\sim \frac{V}{L L_{\bar{q}}} \equiv \frac{V}{L^2} \rho. \end{aligned} \quad (37)$$

With these we derive from (4) to (5) the following nondimensional relations:

$$\begin{aligned} \partial_r \bar{\phi}(r) &= \bar{v} + R \frac{\bar{v}^2}{r}, & \bar{h} &= 1 + R \left(\frac{L}{L_d} \right)^2 \bar{\phi}, & \bar{\Omega} &= \frac{\bar{v}}{r}, \\ \bar{\xi} &= \frac{1}{r} \partial_r (r \bar{v}), & \bar{\xi} &= 1 + 2 R \bar{\Omega}, & \bar{\eta} &= 1 + R \bar{\xi}, & \bar{\gamma}^2 &= \frac{\bar{\xi} \bar{\eta}}{h}, \\ \bar{q} &= \frac{\bar{\eta}}{h}, & \bar{G} &= \frac{\bar{\xi}}{\bar{q}} \partial_r \bar{q} = \frac{L_{\bar{q}}}{L} \left[\frac{\bar{\xi}}{\bar{\eta}} \partial_r \bar{\xi} - \left(\frac{L}{L_d} \right)^2 \frac{\bar{\xi}}{h} \partial_r \bar{\phi} \right], \end{aligned} \quad (38)$$

where

$$R = \frac{V}{f L} \leq O(1) \quad (39)$$

is the Rossby number of the mean vortex and

$$\rho = \frac{L}{L_{\bar{q}}} \geq O(1) \quad (40)$$

is a measure of how steep the potential-vorticity gradient is for the mean vortex compared to the velocity profile. In (38) and henceforth, all variables are

nondimensional (n.b., we have not changed the symbols from Sections 2–3, where they represent dimensional quantities).

The asymmetric, wave-like fluctuations about the mean vortex are characterized by a velocity amplitude, V' , oscillatory length scale, L' , and intrinsic frequency, Ω' (i.e., the rate associated with the operator \bar{D} in (7)).³ The associated geopotential amplitude is $V'fL'$. These characteristic scales permit us to define several other important parameters and to state the conditions for the formal validity of the theory:

$$\epsilon = \frac{L'}{L} \ll 1, \quad (41)$$

the small ratio between wave and vortex spatial scales (i.e., assuming ascale separation);

$$\nu = \frac{L'}{L_d} = \epsilon \frac{L}{L_d} \leq O(1), \quad (42)$$

the measure of the relative importance of free-surface deformations through the vortex stretching term in q on the wave scale (n.b., $\nu = 0$ implies rigid-surface, barotropic dynamics)⁴; and

$$R' = \frac{V'}{fL'} = \frac{V'}{V} \frac{1}{\epsilon} R \ll 1, \quad (43)$$

a wave Rossby number that is a measure of the relative importance of nonlinearity in the wave dynamics.^{5,6}

Two other pertinent parameters are

$$\Delta' = \frac{\Omega'}{f} \ll 1, \quad (44)$$

a ratio of the intrinsic wave frequency and the Coriolis frequency, a measure of the departure from geostrophic momentum balances in (7) when $R' \ll \Delta'^7$, and

$$\mu = \frac{L'^2}{LL\bar{q}} = \epsilon^2 \rho \leq O(1), \quad (45)$$

³This is not the resting-frame rate associated with mean azimuthal advection, $L'/V(= 1/T')$, that is generally larger than $M\Omega'$ since balanced waves tend to have a significant Doppler shift in the azimuthal direction (Eq. (60)).

⁴From (38) and (42), we have $\bar{h} = 1 + R(\nu/\epsilon)^2 \bar{\phi}$. This shows that the shallow-water physical consistency condition, $\bar{h} > 0$, implies a constraint on the parameters, $R(\nu/\epsilon)^2 \leq O(1)$, for a cyclonic vortex with a minimum depth at its center.

⁵This can be rewritten as $V'/V \ll \epsilon/R \ll 1$, given (39)–(41). This indicates the weakness of the waves compared to the mean vortex.

⁶The smallness of R' depends not only on the initial amplitude and the initial length scales, but also on the subsequent evolution. Recent theoretical work suggests that linear barotropic dynamics for the interior region of monopolar vortices on the f -plane should remain uniformly valid at long times (thereby furnishing an accurate approximation to the vortex evolution via the rectified wave fluxes at small but finite amplitude) provided the mean, radial potential-vorticity gradient is single-signed and has a sufficient magnitude relative to the shear of the mean vortex (Brunet and Montgomery, 2002).

⁷This inequality is required for consistent linearization in (14). The square of R'/Δ' serves as the AB wave-field expansion parameter (see Section 3). More generally, the AB expansion can be justified for rapidly rotating, cyclonic vortices even when $R > O(1)$ because $\bar{\eta}$ and $\bar{\xi}$ become large and thereby act to make the AB iteration in Section 3 convergent (Shapiro and Montgomery, 1993).

a wave-dispersion parameter proportional to the magnitude of the mean-vortex potential-vorticity gradient.

4.1 Wave Dynamics

With these parameters we can write the nondimensional linear wave equation (15) for $\phi(\mathbf{x}', t')$ as

$$\begin{aligned}\overline{D}'(q^{ab}) - \frac{\mu}{r} \overline{G} \partial'_\lambda \phi &= 0, \\ q^{ab} &= \tilde{\nabla}'^2 \phi - v^2 \overline{\gamma}^2 \phi \\ \overline{D}' &= \partial'_t + \overline{\Omega} \partial'_\lambda.\end{aligned}\tag{46}$$

Here we use primes for coordinates and derivatives defined on the wave scales, L' and $T' = L'/V$. When such a “fast” operator is applied to a quantity varying only on the scale of the mean vortex, then it is reinterpreted as the appropriate “slow” derivative; e.g., $\partial'_r \bar{\zeta}(r) = \epsilon \partial_r \bar{\zeta}(r)$. A fast wave solution to (46) will also have slow coordinate dependences within the region of the mean vortex because the coefficients do (Section 5). Since the separate terms in \overline{D}' have been made nondimensional by the wave advection rate, $1/T' = V/L'$, rather than the intrinsic frequency, Ω' , we can reinterpret (46) to provide an estimate of the latter: $\Omega' T' = \mu$, or $\Delta' = R\epsilon\rho \ll 1$ (cf., (44)).

The relation (46) is the governing relation for wave evolution. As may be anticipated from the appearance of ϵ in the definition of μ in (45), μ will often be small except where ρ is large because \bar{q} has a sharp shoulder.⁸ The implication of this in (46) is that often the dominant wave tendency is to develop spiraling, nondispersive phase patterns due to differential azimuthal swirling by $\overline{\Omega}(r)$. Nevertheless, vortex Rossby waves also are known to exhibit radial propagation and dispersion (MK97), which we shall see only occurs if $\overline{D}'(q^{ab}) \neq 0$; therefore, we formally treat μ as $O(1)$ in our asymptotic expansions.

We derive an AB wave-energy principle from (46) by multiplying it by $-\phi/\overline{\gamma}^2$ and integrating over all space. The result is

$$\frac{d}{dt} \int \int r dr d\lambda \mathcal{E}_w = \int \int r dr d\lambda \frac{1}{\overline{\gamma}^2} \partial_r \overline{\Omega} \partial'_r \phi \partial'_\lambda \phi,\tag{47}$$

where \mathcal{E}_w is the wave-energy density,

$$\mathcal{E}_w = \frac{1}{2\overline{\gamma}^2} \left[(\partial'_r \phi)^2 + \frac{\overline{\varphi}^2}{r^2} (\partial'_\lambda \phi)^2 + v^2 \overline{\gamma}^2 \phi^2 \right].\tag{48}$$

The interpretation of (47) is that wave energy can grow or decay through a spatial correlation between the mean-vortex strain rate, $r\partial_r \overline{\Omega}$, and a balanced approximation to the horizontal Reynolds stress, $-(r\overline{\gamma}^2)^{-1} \partial'_r \phi \partial'_\lambda \phi$. The wave-energy

⁸There is a long practice of idealizing vortices with a sharp shoulder in vorticity or potential vorticity (e.g., a Rankine vortex).

density is a positive quadratic functional that is the sum of the kinetic energy – the first two terms in (48) – and potential energy – the last term. The only new feature of the AB wave-energy principle, different from its QG counterpart, is the weighting factor, $1/\bar{\gamma}^2(r)$, representing local variation of the deformation radius.

An AB wave-*enstrophy* principle can be derived similarly by multiplying (46) by $rq^{ab}/\mu\bar{G}\bar{\gamma}^2$ and integrating over all space. The result is

$$\frac{d}{dt} \int \int r dr d\lambda \mathcal{V}_w = 0, \quad \mathcal{V}_w = \frac{1}{2} \frac{r}{\mu\bar{G}\bar{\gamma}^2} (q^{ab})^2 \equiv \mathcal{A}^*, \quad (49)$$

where \mathcal{V}_w is a conserved quadratic functional of the wavefield, often referred to as the wave activity, \mathcal{A}^* . This principle is equivalent to the conservation of angular pseudo-momentum (Ren, 1999). A further integral relation combines \mathcal{E}_w and \mathcal{V}_w into a pseudo-energy conservation law (Ren, 1999):

$$\frac{d}{dt} \int \int r dr d\lambda (\mathcal{E}_w + \bar{\Omega} \mathcal{V}_w) = 0. \quad (50)$$

4.2 Vortex Dynamics

We posit that the azimuthal averaging operation eliminates dependence on the fast coordinates. Therefore, the mean-vortex coordinate scales, L and $T = L/V$, are appropriate ones for the vortex evolution dynamics.⁹ In addition, we define a velocity scale for changes in the vortex, $\langle V \rangle$, and its accompanying geopotential scale, $\langle V \rangle fL$. We choose as nondimensionalizing factors,

$$\langle N^u \rangle, \langle N^v \rangle \sim \frac{V'^2}{L}, \quad \langle N^\phi \rangle \sim fV'^2, \quad (51)$$

consistent with the QG relation (30). From (25) and (29), the resulting nondimensional, BE, vortex-evolution equation is

$$\begin{aligned} & \frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{\gamma}^2} \partial_r \partial_t \langle \phi \rangle \right) - \left(\frac{v}{\epsilon} \right)^2 \bar{\gamma}^2 \partial_t \langle \phi \rangle \\ & = \left(\frac{v}{\epsilon} \right)^2 \bar{\gamma}^2 \langle N^\phi \rangle - \frac{\bar{\xi}}{r} \partial_r (r \langle N^v \rangle) + R \rho \bar{G} \langle N^v \rangle - R \bar{\Gamma}_{BE} (\partial_t \langle N^u \rangle), \end{aligned} \quad (52)$$

for the choice of the scaling amplitude relation

$$\frac{\langle V \rangle}{V} = \left(\frac{V'}{V} \right)^2 \ll 1. \quad (53)$$

⁹As evident in (25), it is the vortex time derivative that is proportional to the averaged wave fluxes. While the dimensional scale ratio, $\langle V \rangle / T$, is well determined here, these separate choices for T and $\langle V \rangle$ are somewhat arbitrary. Nevertheless, they do seem apt for the solutions in Section 8.

The rationale for the scaling choices, (51) and (53), is that they make the dominant terms in the barotropic QG balance (i.e., (30) with large L_d) have $O(1)$ parametric coefficients in (52). We designate nondimensional, vortex-evolution energy and enstrophy norms, based on (10), as the area integrals of

$$\mathcal{E}_{(v)} = \frac{1}{2} \left[\bar{h}(\langle u \rangle^2 + \langle v \rangle^2) + \left(\frac{v}{\epsilon} \right)^2 \langle \phi \rangle^2 \right], \quad \mathcal{V}_{(v)} = \frac{1}{2} \bar{h} \langle q \rangle^2 \quad (54)$$

[with dimensional scales of $H_o \langle V \rangle^2$ and $H_o (\langle V \rangle / fL)^2$, respectively], where

$$\langle q \rangle = \frac{1}{r\bar{\eta}} \partial_r(r\langle v \rangle) - \left(\frac{v}{\epsilon} \right)^2 \frac{1}{\bar{h}} \langle \phi \rangle \quad (55)$$

(after factoring by $\langle V \rangle / fL$). These norms are not conserved.

In summary, for small-scale, weak-amplitude, balanced wave fluctuations (i.e., $\epsilon \ll 1$ and $R' \ll \Delta' \ll 1$), the important nondimensional parameters in (46)–(52) are R , the Rossby number of the mean vortex; v and v/ϵ , the scales of the waves and vortex compared to the deformation radius; μ , a measure of the dispersiveness of the waves; and ρ , a measure of the steepness of the mean-vortex potential-vorticity profile, associated with both wave dispersion and a non-QG contribution to the averaged wave-flux forcing.

5 SLOWLY VARYING LINEAR WAVE DYNAMICS

We seek solutions of (46) in the usual form for fast waves in the slowly varying medium provided by the mean vortex:

$$\phi(r', \lambda', t'; r, \lambda, t) = \mathcal{Re} \left[A(r, \lambda, t) e^{i\beta(r, \lambda, t)/\epsilon} \right] + O(\epsilon), \quad (56)$$

where \mathcal{Re} denotes the real part, $\epsilon \ll 1$ is the scale separation parameter (41), A is the wave amplitude function, and β is the wave phase function. The coordinate dependences for A and β are slow ones on the mean-vortex scale, and the fast oscillatory behavior in ϕ occurs as a result of the ϵ factor in the exponential function. Because (46) is linear, any number of wave components (56) can be combined.

We insert $A \exp(i\beta/\epsilon)$ into (46) after changing the fast derivatives into slow ones (e.g., $\nabla' = \epsilon \nabla$) to operate on A and β . The balance at $O(1)$ is

$$\left(\check{\nabla} \beta \cdot \check{\nabla} \beta + v^2 \check{\nabla}^2 \right) \overline{D}(\beta) + \mu \frac{\overline{G}}{r} \partial_\lambda \beta = 0 \quad (57)$$

after factoring out $iA \exp(i\beta/\epsilon)$, where $\check{\nabla} = (\partial_r, r^{-1} \partial_\lambda)$. The balance at $O(\epsilon)$ has terms multiplying $e^{i\beta/\epsilon}$ whose coefficients must collectively vanish to prevent forcing algebraic growth in the fast coordinates for the higher-order corrections in (56), viz.,

$$\overline{D} \left[\left(\check{\nabla} \beta \cdot \check{\nabla} \beta + v^2 \check{\nabla}^2 \right) A \right] + \left(2 \check{\nabla} \beta \cdot \check{\nabla} A + A \check{\nabla}^2 \beta \right) \overline{D}(\beta) + \mu \frac{\overline{G}}{r} \partial_\lambda A = 0. \quad (58)$$

The equations (57) and (58) govern the evolution of β and A , respectively. Since both equations have wholly real coefficients, β and A may be taken to be purely real functions, so that $\phi = A \cos(\beta/\epsilon) + O(\epsilon)$. There is no exponential growth or decay of the waves on the fast scale (i.e., they are neutral waves in contrast to quasi-modes or unstable modes).

In group-velocity theory (Lighthill, 1978), (57) is interpreted as the dispersion relation. We define a local frequency, $\omega(r, \lambda, t)$, and wavenumber vector, $\mathbf{k}(r, \lambda, t) = (k, n/r)$, by

$$\omega = -\partial_t \beta, \quad k = \partial_r \beta, \quad n = \partial_\lambda \beta, \quad (59)$$

and rewrite (57) as the dispersion relation,

$$\omega = \bar{\Omega}n + \mu \frac{\bar{G}n}{rK^2} \equiv W(k, n; r), \quad (60)$$

where

$$K^2 = k^2 + \left(\frac{\bar{\varphi}n}{r} \right)^2 + v^2 \bar{\gamma}^2$$

is the inverse square of a composite length scale for the waves. We immediately recognize (60) as the local dispersion relation for sheared vortex Rossby waves (MK97; Section 3b). (In the general case W may depend upon λ and t as well, but the mean-vortex problem does not have such dependences.) Since (59)–(60) are uniformly valid in (r, λ, t) , we can differentiate them to obtain the phase-consistency condition,

$$\nabla \omega + \partial_t \mathbf{k} = \mathbf{0},$$

and the “ray” equations for the evolution of (ω, \mathbf{k}) along characteristics defined by the group velocity, $\mathbf{C}^g(r, \lambda, t)$,

$$\begin{aligned} D^g(\omega) &= \partial_t W = 0, \\ D^g(k) &= -\partial_r W \\ &= -n \left[\partial_r \bar{\Omega} + \mu \frac{1}{K^2} \partial_r \left(\frac{\bar{G}}{r} \right) + \mu \frac{\bar{G}}{rK^4} \left(\frac{2n^2}{r^3} \bar{\varphi}^2 - \frac{n^2}{r^2} \partial_r [\bar{\varphi}^2] - v^2 \partial_r \bar{\gamma}^2 \right) \right] \\ D^g(n) &= -\partial_\lambda W = 0, \end{aligned} \quad (61)$$

with

$$\begin{aligned} D^g &= \partial_t + \mathbf{C}^g \cdot \nabla \\ \mathbf{C}^g &= (\partial_k W, r \partial_n W) \\ &= \left(-\mu \frac{2\bar{G}}{rK^4} kn, r \left\{ \bar{\Omega} + \mu \frac{\bar{G}}{rK^4} [k^2 - (\bar{\varphi}n/r)^2 + v^2 \bar{\gamma}^2] \right\} \right). \end{aligned} \quad (62)$$

In slowly varying wave theory, the wave-amplitude evolution equation (58) is interpreted as conservation of wave action, $\mathcal{A}c$ (Bretherton and Garrett, 1968), or wave activity, \mathcal{A}^* (Andrews *et al.*, 1987). Wave action is defined as the ratio of the wave energy density and the intrinsic frequency, $\omega - \overline{\Omega}n$. By using \mathcal{E} from (48), ϕ from (56), and the dispersion relation (60), then averaging over a fast oscillation period (denoted by $\text{avg}[\]$), we define

$$\mathcal{A}c = \frac{\text{avg}[\mathcal{E}]}{(\omega - \overline{\Omega}n)} = \frac{1}{4\mu} \frac{rK^4}{\overline{\gamma^2 G}} A^2. \quad (63)$$

After some rearrangement, (58) can be shown to take the form,

$$\partial_t \mathcal{A}c + \frac{1}{r} \partial_r (r C^{g,r} \mathcal{A}c) + \frac{1}{r} \partial_\lambda (C^{g,\lambda} \mathcal{A}c) = D^g(\ln \mathcal{A}c) + \nabla \cdot \mathbf{C}^g = 0. \quad (64)$$

This is the expected wave-action conservation principle along ray paths.¹⁰ For balanced dynamics, wave activity, \mathcal{A}^* , is defined as the ratio of the wave-*enstrophy* density and the mean-vortex potential-vorticity gradient. Guided by the wave-*enstrophy* principle (49) and following the procedure for determining $\mathcal{A}c$ above, we define

$$\mathcal{A}^* = \frac{r}{\mu \overline{\gamma^2 G}} \frac{1}{2} \text{avg}[(q^{ab})^2] = \frac{1}{4\mu} \frac{rK^4}{\overline{\gamma^2 G}} A^2. \quad (65)$$

Comparing with (63), we see that $\mathcal{A}^* = n\mathcal{A}c$.¹¹ Since $D^g[n] = 0$ from (61), wave-activity conservation follows directly from (64):

$$D^g(\ln \mathcal{A}^*) + \nabla \cdot \mathbf{C}^g = 0. \quad (66)$$

Referring to (47)–(49), a useful norm for wave evolution is the energy norm,

$$E_w(t) = \frac{1}{4} \int \int r \, dr \, d\lambda \, \frac{K^2}{\overline{\gamma^2}} A^2, \quad (67)$$

while the analogous *enstrophy* norm,

$$V_w = \frac{1}{4\mu} \int \int r \, dr \, d\lambda \, \frac{rK^4}{\overline{\gamma^2 G}} A^2, \quad (68)$$

is conserved.

¹⁰Eqs. (58) and (64) are equivalent to a local wave-energy principle that has the same form as the integrands in (47) after averaging over a fast oscillation period.

¹¹cf. Eq. (4A.12) of Andrews *et al.* (1987).

6 VORTEX EVOLUTION BY WAVE FLUXES

The wave field evolves according to (60)–(61) for β , or equivalently (ω, \mathbf{k}) , and (64) or (66) for A , both under the influence of the mean vortex, $\bar{v}(r)$. Changes in the vortex, $\partial_t \langle v \rangle$, occur according to (52) for $\partial_t \langle \phi \rangle$ in combination with the time derivative of the nondimensional form of the second relation in (28),

$$\langle v \rangle = \frac{1}{\xi} (\partial_r \langle \phi \rangle + R \langle N^u \rangle). \quad (69)$$

The norms for this evolution are defined in (54). In addition, we can diagnose the vortex radial velocity that occurs during the wave-induced adjustment by the nondimensional form of the first relation in (28),

$$\langle u \rangle = -\frac{R}{\eta} (\partial_t \langle v \rangle + \langle N^v \rangle), \quad (70)$$

that vanishes whenever $\langle N^u \rangle$ and $\langle N^v \rangle$ do [see Eq. (31)]. All that remains to complete the specification of the vortex-evolution theory is to relate the $\langle N \rangle$ in (52) and (69)–(70) to (β, A) .

These relations are somewhat laborious to derive, although the procedure is a straightforward one to describe. First, an azimuthal average and nondimensionalization of (8) using (51) yields relations between $\langle N \rangle$ and (u, v, ϕ) :

$$\begin{aligned} \langle N^u \rangle &= \frac{1}{2} \partial_r (\langle u^2 \rangle) - \frac{1}{r} \langle v^2 \rangle + \frac{1}{\epsilon} \frac{1}{r} \langle v \partial'_\lambda u \rangle, \\ \langle N^v \rangle &= \frac{1}{r} \partial_r (r \langle uv \rangle) - \frac{1}{\epsilon} \langle v \partial'_r u \rangle = \frac{1}{\epsilon r} \langle u \partial'_r (rv) \rangle, \\ \langle N^\phi \rangle &= \frac{\epsilon}{r} \partial_r (r \langle u \phi \rangle). \end{aligned} \quad (71)$$

Next, we insert the wave solution form (56) for ϕ into the nondimensional AB velocity relations (14), keeping terms through $O(\epsilon)$ while treating $(R\rho/\mu)\bar{D}$ as an $O(1)$ operator:

$$\begin{aligned} u &= -\frac{1}{r\eta} \operatorname{Re} [(iA\partial_\lambda \beta + \epsilon\partial_\lambda A)e^{i\beta(r, \lambda, t)/\epsilon}] + \epsilon \frac{R\rho}{\mu} \frac{1}{r\eta\xi} \operatorname{Re} [(A\bar{D}[\beta]\partial_r \beta)e^{i\beta(r, \lambda, t)/\epsilon}] \\ v &= \frac{1}{\xi} \operatorname{Re} [(iA\partial_r \beta + \epsilon\partial_r A)e^{i\beta(r, \lambda, t)/\epsilon}] + \epsilon \frac{R\rho}{\mu} \frac{1}{r\eta\xi} \operatorname{Re} [(A\bar{D}[\beta]\partial_\lambda \beta)e^{i\beta(r, \lambda, t)/\epsilon}]. \end{aligned} \quad (72)$$

We next insert u , v , and ϕ into (71). One consequence of taking the average of a quadratic product is that only the coefficient of $e^{i0\beta/\epsilon}$ survives, while those of $e^{\pm i2\beta/\epsilon}$ do not. After further substituting for the derivatives of β using (59)–(40) and retaining only leading-order terms in ϵ while treating ρR as an $O(1)$ quantity – equivalent to saying Δ'/ϵ is $O(1)$ and the first and last terms in (57) are comparable – we get the

desired relations for the nonlinear terms that appear in (52) and (69)–(70):

$$\begin{aligned}
\langle N^u \rangle &= \partial_r \left(\frac{1}{4\bar{\eta}^2 r^2} \langle n^2 A^2 \rangle \right) - \frac{1}{2\bar{\xi}^2 r} \langle k^2 A^2 \rangle + \frac{1}{2\bar{\eta}\bar{\xi}r^2} \left[\langle nA(n\partial_r A - k\partial_\lambda A) - kA\partial_\lambda[nA] \rangle \right] \\
&\quad - R\rho \frac{\bar{G}}{2\bar{\eta}\bar{\xi}} \left\langle \left(\frac{k^2}{\bar{\xi}} + \frac{n^2}{\bar{\eta}r^2} \right) \frac{n^2 A^2}{r^2 K^2} \right\rangle, \\
\langle N^v \rangle &= -\frac{1}{2\bar{\eta}r^2} \langle n\partial_r \left(\frac{r}{\bar{\xi}} k A^2 \right) \rangle + \frac{1}{2\bar{\eta}\bar{\xi}r} \langle k^2 A \partial_\lambda A \rangle + R\rho \frac{\bar{G}}{2\bar{\eta}\bar{\xi}} \left\langle \left(\frac{k^2}{\bar{\xi}} + \frac{n^2}{\bar{\eta}r^2} \right) \frac{kn A^2}{r K^2} \right\rangle, \\
\langle N^\phi \rangle &= -\epsilon^2 R\rho \frac{1}{2r} \partial_r \left(\frac{\bar{G}}{\bar{\eta}\bar{\xi}} \langle \frac{kn}{K^2} A^2 \rangle \right). \tag{73}
\end{aligned}$$

Note in particular that $\langle N^\phi \rangle$ appears to be small here by $O(\epsilon^2)$; this implies that it may often be negligible compared to the other right-hand side terms in forcing the vortex evolution in (52) (but see Appendix A.4). Since these expressions and their assemblage in (52) are quite complicated in their general dependences on the parameters $(R, v, v/\epsilon, \mu, \rho)$, we identify several simpler limiting cases in Appendix A.

7 SPECIAL SITUATIONS

Before considering a specific problem in vortex Rossby wave dynamics, we briefly remark on several special situations implicated in the preceding sections.

Critical Radius A critical radius is defined as a place where $\omega = n\bar{\Omega}(r)$; i.e., the intrinsic frequency is zero. For this relation to also satisfy the dispersion relation (60) requires that $\bar{G}n$ vanish at this place, which will not generally be true (but see below). This implies that wave energy initially not located at a critical radius cannot propagate following the group velocity to reach or cross a critical radius, since (61) requires such propagation to preserve the values of ω and n .¹²

Vanishing \bar{G} If $\bar{G} = 0$, then the waves are locally also at a critical radius and have vanishing $C^{g,r}$ persistently in time. Appendix A.2 has an analysis of where this condition holds over a finite region.

Vanishing k If $k = 0$ at a point, then (62) requires $C^{g,r} = 0$ there. This implies that wave energy is not moving either inward or outward at this particular place and time. Furthermore, if the waves are azimuthally periodic (Appendix A.6), then by (64),

$$\partial_t \mathcal{A}c = -\mathcal{A}c \partial_r [C^{g,r}].$$

Thus, $\mathcal{A}c$ is locally growing or decaying exponentially with time, depending upon whether $C^{g,r}$ is locally divergent or convergent. However, note that $k \rightarrow 0$ is formally

¹²The validity of a WKB theory may become questionable near a critical radius if the spatial scale of k , etc, shrinks to violate the assumption $\epsilon \ll 1$ or the amplitude increases to violate the quasi-linear assumption. We do not address these issues here.

inconsistent with our assumption of a scale separation between the waves and the vortex so implications from this theory should be viewed cautiously.

Large r At large radius, then we can assume that $\bar{G} \rightarrow 0$, essentially as a definition of a mean vortex as having only a compact central region of nonuniform potential vorticity. Again, Appendix A.2 is relevant.

8 ILLUSTRATIVE SOLUTIONS

Here we demonstrate asymptotic convergence of our theory to the shallow-water solutions for a case of an initially perturbed, balanced vortex that relaxes back to an altered axisymmetric, steady state. We illustrate typical vortex Rossby-wave behavior near the nondivergent, quasigeostrophic, azimuthally homogeneous limits (cf., Appendices A.1, A.4, and A.6).

Consider a mean cyclonic vortex with a monotonic vorticity profile ($\bar{\zeta} \propto e^{-r^3}$; Fig. 1). We choose a maximum velocity of $V = 1$ m/s at a radius of maximum winds (r_m) of 200 km so that the Rossby number is $R = 0.05$. The ambient deformation radius is chosen much larger, $L_d = 2200$ km ($H_0 = 5000$ m), so the flow is nearly barotropic. For simplicity we choose initial perturbations that are azimuthally homogeneous, such that $\beta(r, \lambda, t) = \tilde{\beta}(r, t) + n_o \lambda$ and $A = A(r, t)$. This simplifies the wave action and activity equations as well as the wave-mean equation through the nonlinear terms (Appendix A.6). The nondimensional parameter choices for a basic case are $n_o = 1$ and $\rho = 1$, with $\epsilon = 0.25$, which is large enough to allow for more than just azimuthal propagation in the dispersion relation (60) but small enough that the scale separation assumption is valid. We choose an initially uniform nondimensional radial wavenumber $k(r) = 1.75$ (i.e., a dimensional value of $3.5 \times 10^{-5} \text{ m}^{-1}$ when multiplied by $1/\epsilon L$) and a perturbation pseudo-potential vorticity amplitude, $|q_{\xi}|(r) = q_{\xi 0} \sin^4\{\pi[(r/r_m) - (1/4)]\}$, $0 \leq r \leq r_m$, with $q_{\xi 0} (\sim V'/gH_0\epsilon L) = 1.24 \times 10^{-11} \text{ s/m}^2$ chosen so that V' is slightly less than one percent of the maximum mean velocity V

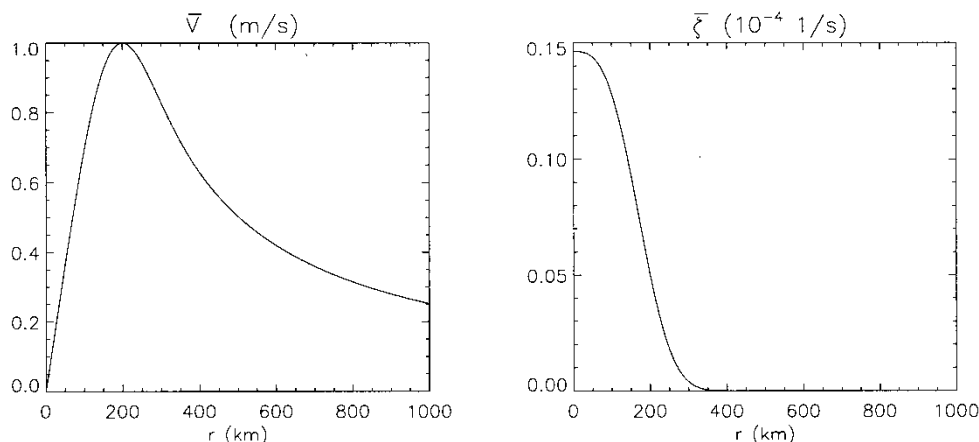


FIGURE 1 The mean vortex profiles for $\bar{v}(r)$ and $\bar{\zeta}(r)$.

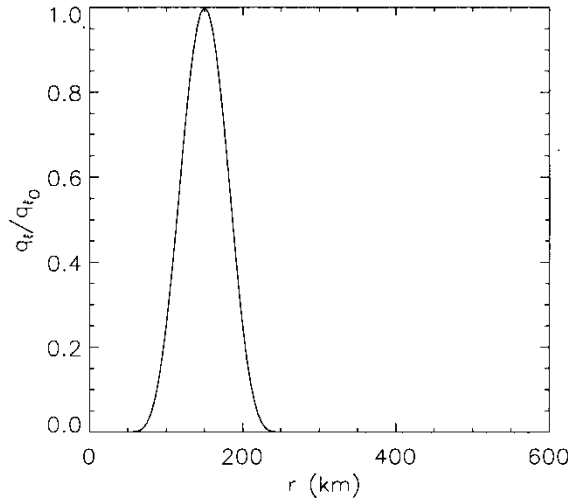


FIGURE 2 Initial profile of the perturbation q_ξ amplitude, normalized by its maximum value.

(Fig. 2),¹³ which is multiplied by an exponential phase function as in (56). The geopotential $\phi(r, \lambda)$ is obtained by inverting q_ξ in (22). We use high-resolution, first-order, upwind, finite-difference discretization schemes in both space and time to solve for k and A in (60) and (91), and we use second-order, centered, spatial finite-differences and first-order, forward-Euler time-steps to solve the wave/mean-flow equation (52) for $\langle \phi \rangle$. The boundary conditions are $\partial_t[k, A] = 0$ at $r = 0, r_{\max}$; $\langle v \rangle = 0$ at $r = 0$; and $(\partial_r + v\bar{\gamma}/\epsilon)\langle v \rangle = 0$ at $r = r_{\max}$ (i.e., a decaying exponential solution for (52) with vanishing right-hand side), for large but finite r_{\max} . We solve the nondimensional equations then transform the solutions to dimensional quantities for plotting purposes.

Figure 3 shows q_ξ at different times. There are four azimuthal oscillations in $q_\xi(r, \lambda)$ around the vortex since $n_o/\epsilon = 1/0.25 = 4$. As time progresses, the spiral arms rotate following the mean angular velocity $\bar{\Omega}$ and begin to wrap up. We see a growth in the radial wavenumber k (Fig. 4) since $k \sim n_o t d\bar{\Omega}/dr$ is the dominant effect from the dispersion relation (60). This behavior is essentially similar to the long familiar progressive tilting of phase lines in a parallel shear flow and associated wave/mean-flow interaction (Orr, 1907; Yamagata, 1976). Figure 5 shows plots of $|q_\xi|(r, t)$ and $A(r, t)$ to expose the radial propagation and amplitude changes. The $|q_\xi|$ maximum is initially at $r = 150$ km and moves radially outward to a stagnation radius of ≈ 165 km while its magnitude decreases somewhat. Since $q_\xi \propto \sqrt{Ac}$ through (49) and the radial propagation of $\partial_r Ac$ is primarily due to $C^{g,r} \partial_r Ac$ in (91), the stagnation radius is determined by the temporal decay of the radial group velocity (Fig. 4). Note how the radial group velocity decreases in magnitude and moves inward away from the location of the $|q_\xi|(r)$ maximum, so that there is soon no further radial motion or decay in q_ξ .

¹³We determined that the quasi-linear assumption of our theory is accurate for the fluctuation evolution at this amplitude since full shallow-water solutions exhibit significant nonlinear effects only for initial amplitudes about an order of magnitude larger.

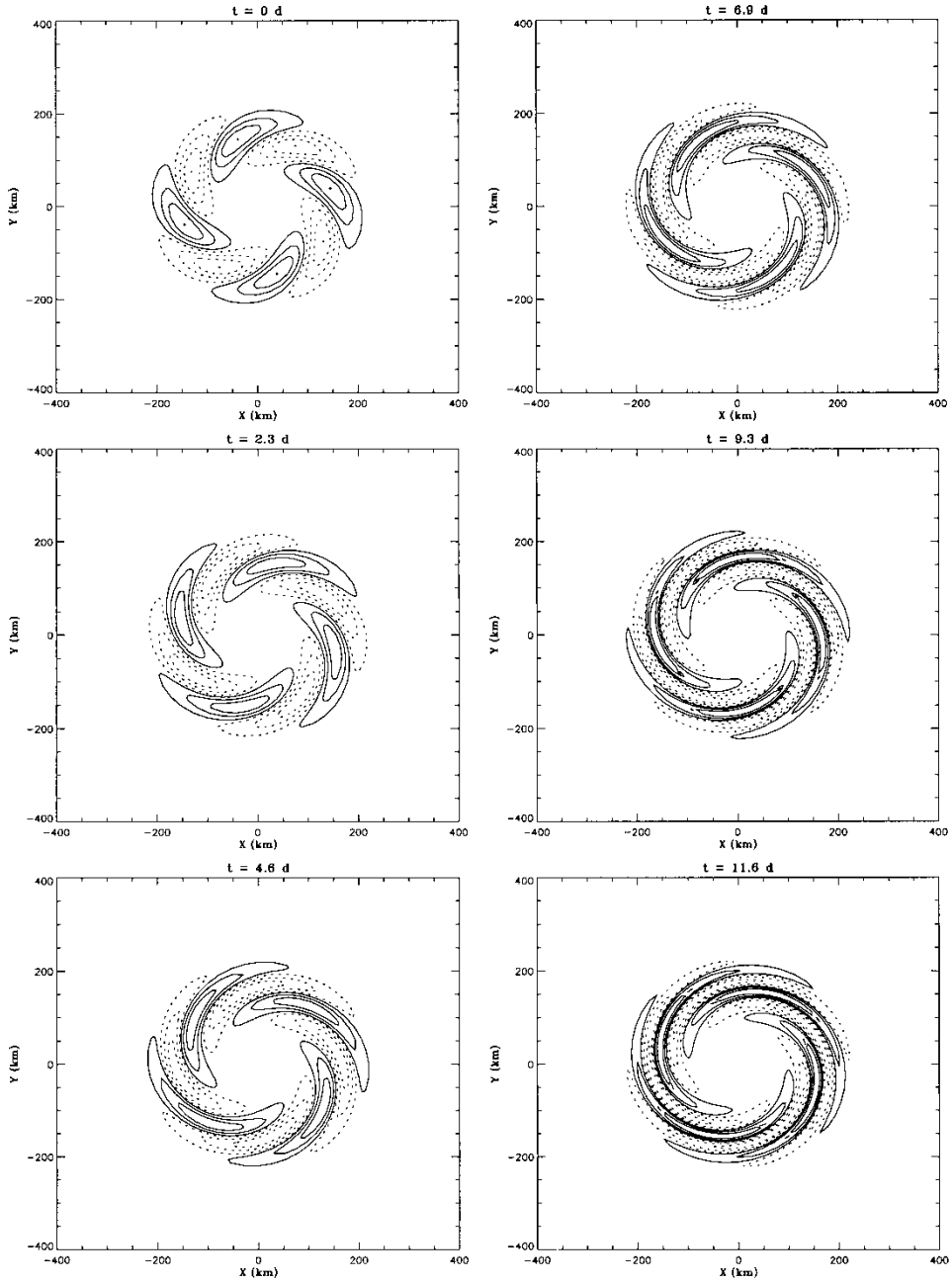


FIGURE 3 Horizontal plots of normalized q_ε at increasing times (from the asymptotic theory). Contour values are $(\pm 0.1, \pm 0.4, \pm 0.7, \pm 1)$, where solid lines are positive values and dashed ones negative.

Even with this radial propagation, however, the main effect on ϕ is a consequence of approximate action conservation, implying $\mathcal{A} \sim (AK^2)^2 \sim t^0$. Hence there is a systematic decay of A in time (Fig. 5), since $A \sim K^{-2} \sim k^{-2} \sim t^{-2}$. By $t = 10$ days, the A maximum has already decreased by a factor of about 6.

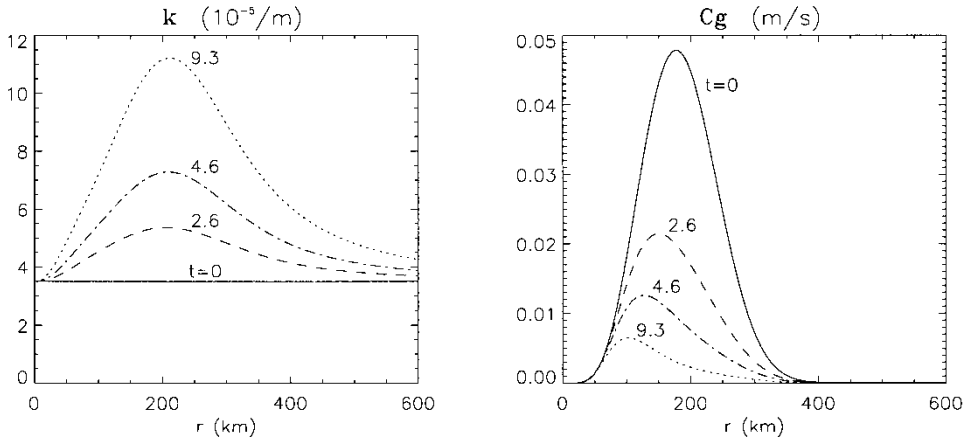


FIGURE 4 Profiles of radial wavenumber $k(r, t)$ and radial group velocity $C_g^r(r, t)$ from the asymptotic theory at $t = 0$ (solid), 2.6 (dashed), 4.6 (dash/dot), and 9.3 (dotted) days.

The wave/mean-flow interaction appears in $\langle v \rangle(r, t)$. The asymptotic-theory prediction is shown as the solid line in Fig. 6 at a relatively late time ($t = 11.6$ days, about 5 turnover times of the mean vortex, $2\pi r_m/V$). At even later times there is little further change due to the continuing decay of A . An increase in mean velocity occurs just to the inside of the initial q_ξ maximum, and a decrease occurs outside. More is gained by the mean than lost; i.e., the total energy and enstrophy of the azimuthally averaged part of the flow (the vortex) increase steadily with time until A becomes small. This is an expected behavior for waves whose phase lines are progressively tilting due to vortex shear. In the quasigeostrophic limit ($R \rightarrow 0$), $\bar{\xi}, \bar{\eta}$, and $\bar{\gamma}^2 \rightarrow 1$, and in the barotropic limit ($\nu \rightarrow 0$), (52) and (92) imply that $\partial_r \langle v \rangle \sim (n_o/2r^2) \partial_r (rkA^2)$. Since $R = 0.05$, and $\nu \approx \epsilon/10$ here, this approximation is apt. Because k and A have the shapes shown in Figs. 4 and 8 respectively, then rkA^2 has a positive convex shape in r , hence its radial derivative matches the shape of $\langle v \rangle(r)$ in Fig. 6. When added to the mean profile, $\bar{v}(r)$, this shape implies a shrinking of the size of the vortex and a sharpening of the profile at the edge of the vorticity-containing core (cf., Fig. 1). In this example with its small R , there is no significant asymmetry in the evolution of disturbed cyclones and anticyclones.

In order to validate our theory, we make comparisons with the shallow-water, asymmetric-balance model used in Moller and Montgomery (1999). For the present problem with its small R we are quite confident that the asymmetric-balance approximation is accurate, based on past experience of comparisons with Primitive-Equation solutions (e.g., Schechter and Montgomery, 2002). To expose the asymptotic convergence behavior, we calculate shallow-water solutions for different values of ϵ , while holding fixed the vortex properties (L, V, R, L_d) as well as the wave properties [$V', n_o, k(0, t), \rho$]. This experimental path does imply changes in (L', ν, μ) (Section 4), hence in the \bar{G} term in both the dispersion relation (60) and the wave equation (46); the net effect is that the wave behavior approaches purely azimuthal propagation due to $\bar{\Omega}$ for small ϵ . In the wave-mean flow equation (52), the $\langle N^\phi \rangle$ term is significantly smaller, but the $\langle N'' \rangle$ and $\langle N' \rangle$ terms in (73) are only slightly affected through ν in K^2 .

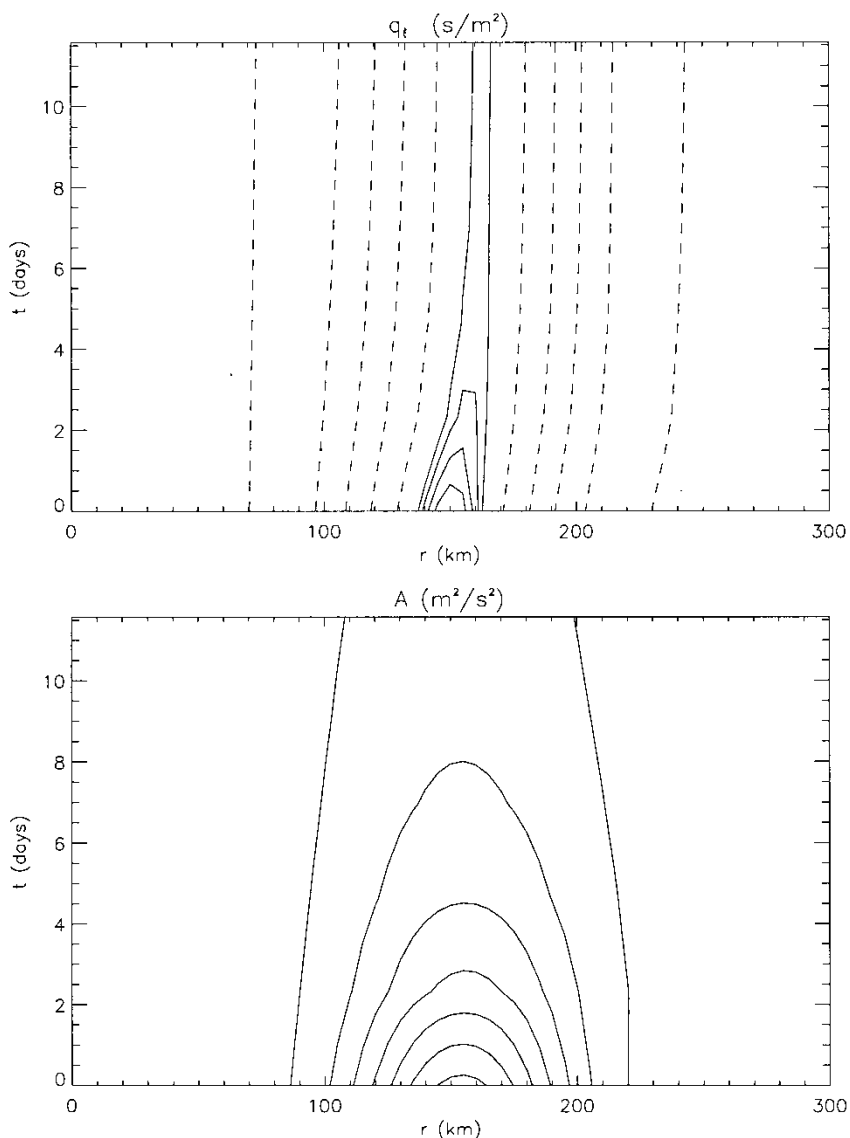


FIGURE 5 Hovmöller plots of normalized $|q_\xi|(r, t)$ and normalized $A(r, t)$ from the asymptotic theory. In the upper panel, dashed contour values are $(0.2, \dots, 0.8)$ and the solid contour values are $(0.92, 0.94, 0.96, 0.98)$. In the lower panel, the contour values are $(0.05, 0.2, 0.35, \dots, 0.95)$.

We calculate a shallow-water solution with the same initial condition for $q_\xi(r, \lambda)$ as in the asymptotic-theory solution, then numerically solve for $\phi(r, \lambda)$ from (23).¹⁴

Figure 6 shows late-time $\langle v \rangle(r)$ for the asymptotic theory (solid) and shallow-water models (dashed) for $\epsilon = 0.25$ and 0.5 respectively. For $\epsilon = 0.25$, the two models match very closely, and they do so even for the rather large value of $\epsilon = 0.5$, where

¹⁴We could, of course, match initial ϕ fields instead. In this alternative ϵ sequence, the modest solution differences at large ϵ differ in detail, but the converged solution at small ϵ is the same.

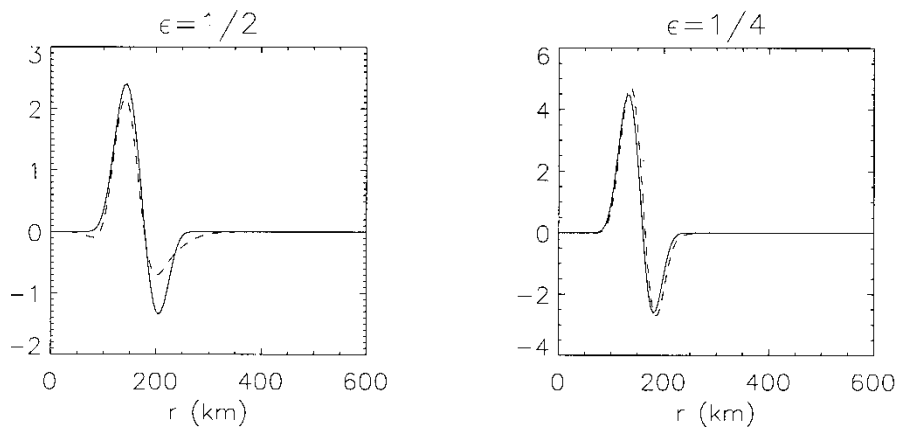


FIGURE 6 Profiles of $\langle v \rangle (10^{-4} \text{ m/s})$, the wave/mean-flow interaction velocity, at $t = 11.6$ days, for $\epsilon = 0.25$ and 0.5 . The solid lines are from the asymptotic theory, and the dashed ones are from the shallow-water model.

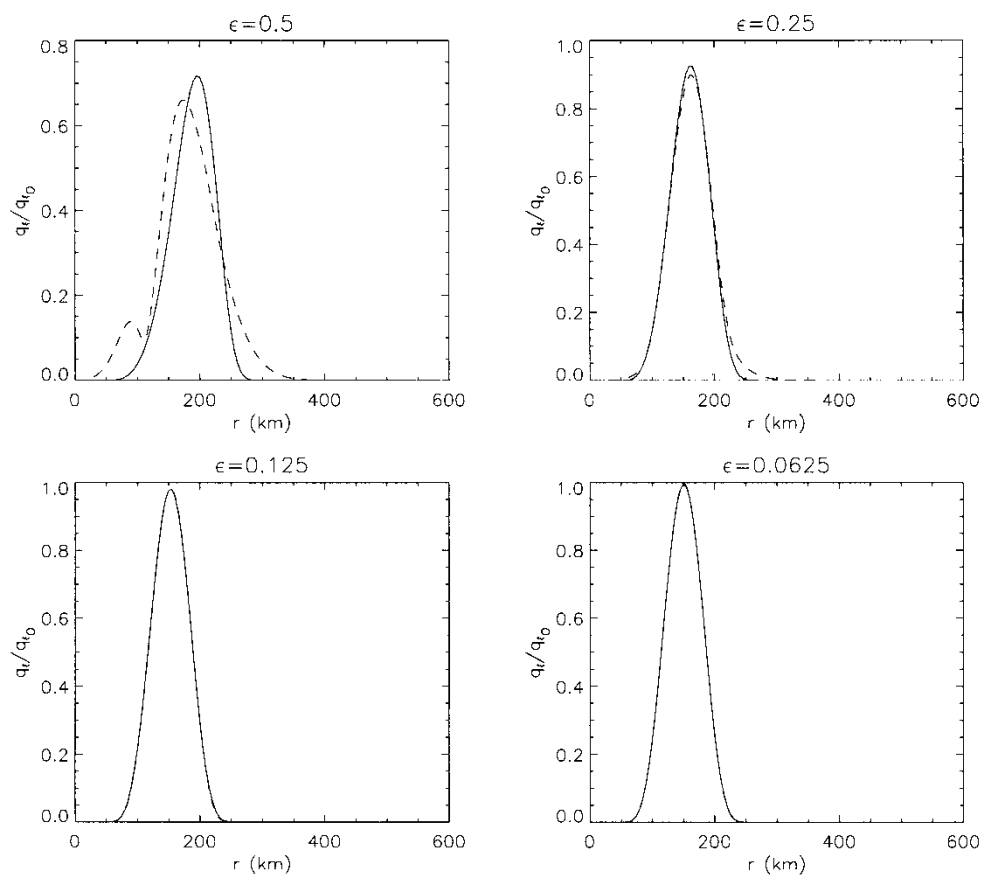


FIGURE 7 Profiles of normalized $|q_e|(r)$ at $t = 9.3$ days for several ϵ values. Solid lines are asymptotic-theory results, and the dashed lines are from the shallow-water model.

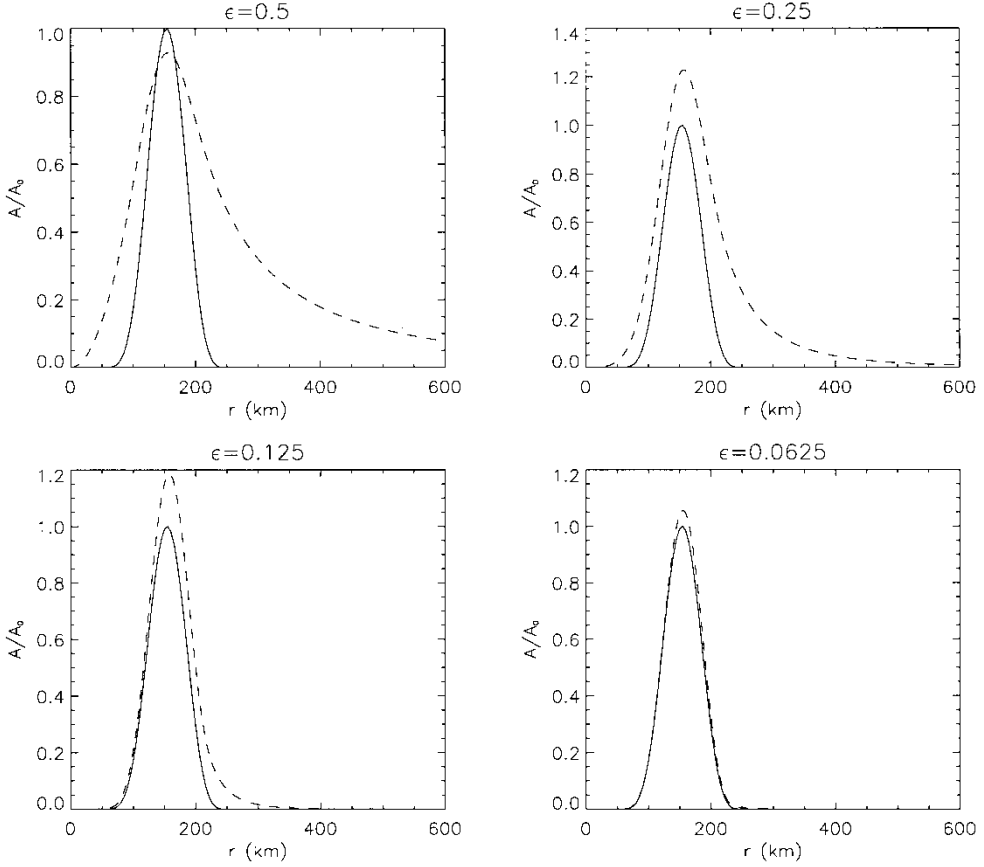


FIGURE 8 Profiles of $A(r, 0)$ for the asymptotic theory (solid lines) and shallow-water model (dashed), normalized by the maximum initial value of the asymptotic-theory's A for each value of ϵ .

the wave fields visibly differ (Figs. 7 and 9). The asymptotic theory still gives a reasonable approximation to the wave/mean-flow interaction since the dominant term in (52) is $\langle N^v \rangle$ for all ϵ , and $\langle N^v \rangle$ is only weakly dependent on ϵ (above). Figure 7 shows $|q_\xi|(r, t)$ for both the asymptotic theory and the shallow-water model at $t = 9.3$ days (i.e., about $4r_m/V$), and Figs. 8 and 9 show $A(r, t)$ at $t = 0$ and 9.3 days, respectively, for ϵ values ranging between 0.0625 and 0.5. Recall that the initial q_ξ is the same for both models (Fig. 2). For $\epsilon = 0.5$ we see some discrepancy in the location of the q_ξ peak and its magnitude. The shallow-water model develops a small inner peak that we suspect is indicative of aquasi-mode component in the initial conditions (Schecter *et al.*, 2000) that is absent in the asymptotic theory and occurs in a shallow-water initialization due to the nonasymptotic inversion of (22). However, as ϵ drops even to 0.25, the solutions match very closely, becoming visually indistinguishable at $\epsilon = 0.125$ and 0.0625. The ϕ solutions differ more substantially with ϵ largely due to the fact that we initialize with equal q_ξ fields. The asymptotic theory drops higher order terms in ϵ when solving for the initial ϕ from q_ξ with (22), whereas the shallow-water solution is exact. For example, the initial shallow-water ϕ has a somewhat different peak magnitude and an extended reach in r on either side of the peak

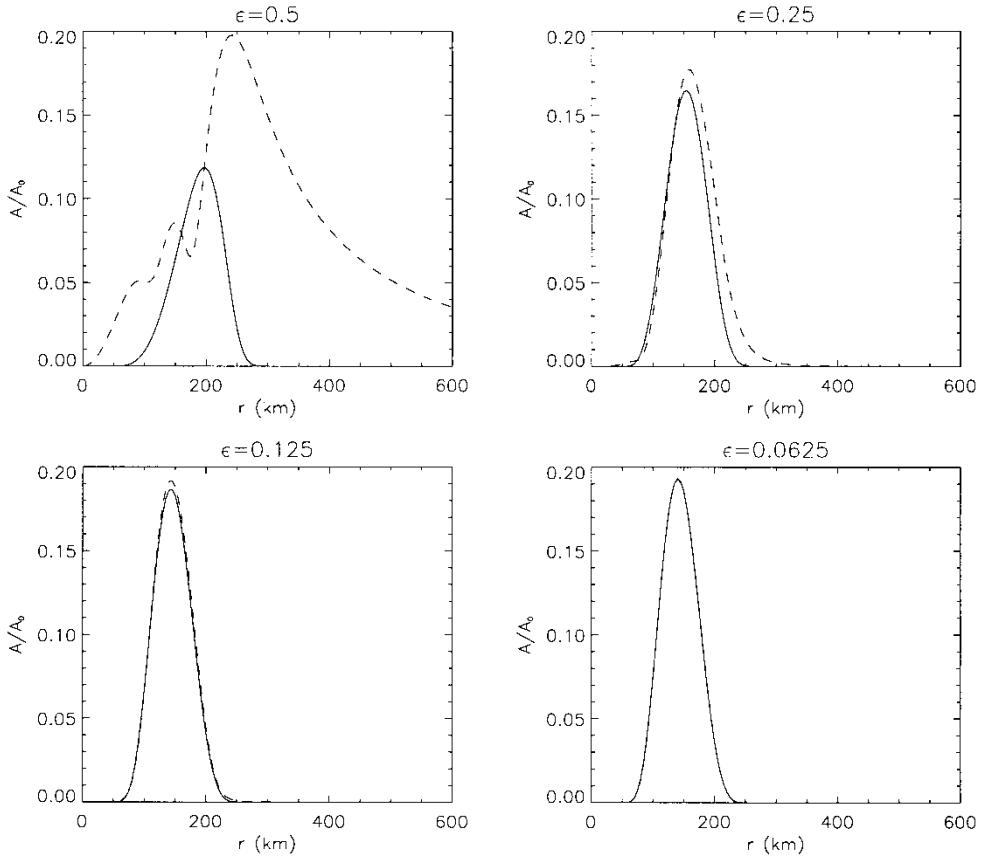


FIGURE 9 Profiles of $A(r)$ at $t = 9.3$ days for the asymptotic theory (solid lines) and shallow-water model (dashed), normalized by the maximum initial value of the asymptotic-theory's A for each value of ϵ .

for larger ϵ values, but as ϵ decreases the initial ϕ profiles approach each other quite accurately (Fig. 8). We see that as time increases the ϕ fields in the two models converge, with the exception of $\epsilon = 0.5$. For $\epsilon = 0.5$, the quasi-mode decays differently than the main peak of ϕ so the two models' solutions are even more different.

In summary, the principal behaviors of these vortex Rossby waves are k growth; ϕ decay and modest radial propagation; q_ξ radial propagation and only weak decay; $\langle v \rangle(r)$ shrinking and sharpening the vortex profile through wave-averaged angular-momentum flux; and vortex strengthening in energy and enstrophy through fluctuation decay. The asymptotic theory accurately captures the main features of these shallow-water behaviors even for fairly large values of ϵ , especially the wave-mean flow interaction that describes the outcome of the vortex-relaxation process.

9 SUMMARY AND DISCUSSION

In this article we present a formal asymptotic theory for the quasi-linear evolution of weak, small-scale disturbances of ageostrophic, balanced vortices, using the rotating shallow-water equations as a testing ground. Our goal is to create a theoretical

framework for examining the many different parameter regimes in vortex strength and shape, varieties of balanced dynamics (including generalizations of the usual AB and BE models; Section 3), stratification and vortex size, and fluctuation shape. The wave dynamics in the theory have group-velocity and action-conservation laws, as in many previous instances of waves in a slowly varying medium. The wave/mean-flow interaction has substantial similarity with, but does not fully match, the paradigm of Andrews and McIntyre (1978) due to the non-Hamiltonian nature of our theory. The momentum-balanced structure of the vortex evolution dynamics through Section 4 is more general, with respect to the wave dynamics, than its particular form in Section 6 based on the scale-separation (WKB) assumption for the waves. We analyze solutions near the quasigeostrophic, barotropic, and azimuthally homogeneous limits to illustrate typical vortex Rossby-wave and vortex-relaxation behaviors and to demonstrate the asymptotic convergence of the theory in the scale-separation parameter ϵ .

Many interesting issues remain to be further explored in future applications of this theory, among which are the following: vortex-profile and fluctuation initial-condition dependences, stratification effects, finite- Ro effects in both the balanced-wave and vortex evolution, cyclone/anticyclone asymmetries, and critical-layer encounters by the wave packets.

Acknowledgments

JCM and LPG gratefully acknowledge support from the Office of Naval Research (N00014-95-1-0316, N00014-98-1-0165). MTM gratefully acknowledges support from the Office of Naval Research (N00014-93-1-0456) and the National Science Foundation (ATM-9732678, ATM-0101781).

References

- Allen, J.S., Barth, J.A. and Newberger, P.A., "On intermediate models for barotropic continental shelf and slope flow fields. Part III: comparison of numerical model solutions in periodic channels", *J. Phys. Ocean.* **20**, 1949–1973 (1990).
- Andrews, D.G. and McIntyre, M.E., "Generalized Eliassen-Palm and Charney-Drazin theorems for waves on axisymmetric mean flows incompressible atmospheres", *J. Atmos. Sci.* **35**, 175–185 (1978).
- Andrews, D.G., Holton, J.R. and Leovy, C.B., *Middle Atmosphere Dynamics*, Academic Press, Inc. (1987).
- Arai, M. and Yamagata, T., "Asymmetric evolution of eddies in rotating shallow water", *Chaos* **4**, 163–173 (1994).
- Balmforth, N.J., Llewellyn Smith, S.G. and Young, W.R., "Disturbing vortices", *J. Fluid Mech.* **426**, 95–133 (2001).
- Bassom, A.P. and Gilbert, A.D., "The spiral wind-up of vorticity in an inviscid planar vortex", *J. Fluid Mech.* **371**, 109–140 (1998).
- Bassom, A.P. and Gilbert, A.D., "The spiral wind-up and dissipation of vorticity and a passive scalar in a strained planar vortex", *J. Fluid Mech.* **398**, 245–270 (1999).
- Bracco, A., Chavanis, H., Provenzale, A. and Spiegel, E.A., "Particle aggregation in a turbulent Keplerian flow", *Phys. Fluids* **11**, 2280–2287 (1999).
- Bretherton, F. and Garrett, C., "Wave trains in inhomogeneous moving media", *Proc. Roy. Soc. A* **302**, 529–554 (1968).
- Brunet, G. and Montgomery, M.T., "Vortex Rossby waves on smooth circular vortices. Part I: theory", *Dyns. of Atmos. and Oceans* **35**, 153–177 (2002).
- Cushman-Roisin, B. and Tang, B., "Geostrophic turbulence and emergence of eddies beyond the radius of deformation", *J. Phys. Ocean.* **20**, 97–113 (1990).
- Drazin, P.B. and Howard, L.N., "Hydrodynamic instability of parallel flow of inviscid fluid", *Adv. in Appl. Mech.* **9**, 1–89 (1966).
- Eliassen, A., "Slow thermally or frictionally controlled meridional circulation in a circular vortex", *Astrophys. Norv.* **5**, 19–60 (1951).

- Flierl, G.R., "On the instability of geostrophic vortices", *J. Fluid Mech.* **197**, 349–388 (1988).
- Ford, R., "The instability of an axisymmetric vortex with monotonic potential vorticity in rotating, shallow water", *Phys. Fluids* **280**, 303–334 (1994).
- Gilbert, A., "Spiral structures and spectra in two-dimensional turbulence", *J. Fluid Mech.* **93**, 475–497 (1988).
- Grimshaw, R., "Wave action and wave-mean flow interaction, with application to stratified shear flows", *Ann. Rev. Fluid Mech.* **16**, 11–44 (1984).
- Guinn, T.A. and Schubert, W.H., "Hurricane spiral bands", *J. Atmos. Sci.* **50**, 3380–3403 (1993).
- Hakim, G., Snyder, C. and Muraki, D., "A new surface model for cyclone-anticyclone asymmetry", *J. Atmos. Sci.* **59**, 2405–2420 (2002).
- Held, I.M., "Pseudomomentum and the orthogonality of modes in shear flows", *J. Atmos. Sci.* **42**, 2280–2288 (1985).
- Held, I.M. and Phillips, P.J., "Linear and nonlinear barotropic decay on the sphere", *J. Atmos. Sci.* **44**, 200–207 (1987).
- Lighthill, M.J., *Waves in Fluids*, Cambridge University Press, Cambridge, England (1978).
- Lundgren, T.S., "Strained spiral vortex model for turbulent fine structure", *Phys. Fluids* **25**, 2193–2203 (1982).
- Marcus, P.S., "Vortex dynamics in a shearing zonal flow", *J. Fluid Mech.* **215**, 393–430 (1990).
- Mayer, L., Quin, T., Wadsley, J. and Stadel, J., "Formation of giant planets by fragmentation of protoplanetary disks", *Science* **298**, 1756–1759 (2002).
- McCalpin, J.D., "On the adjustment of azimuthally perturbed vortices", *J. Geophys. Res.* **92**, 8213–8225 (1987).
- McIntyre, M.E., "On the 'wave momentum' myth", *J. Fluid Mech.* **106**, 331–347 (1981).
- McWilliams, J.C., "Geostrophic vortices", *Proc. Int. School of Physics 'Enrico Fermi'*, pp. 5–50, Course 109, Italian Physical Society, (1989).
- McWilliams, J.C. and Gent, P.R., "Intermediate models of planetary circulations in the atmosphere and ocean", *J. Atmos. Sci.* **37**, 1657–1678 (1980).
- McWilliams, J.C., Yavneh, I., Cullen, M.J.P. and Gent, P.R., "The breakdown of large-scale flows in rotating, stratified fluids", *Phys. Fluids* **10**, 3178–3184 (1998).
- Melander, M.V., McWilliams, J.C. and Zabusky, N.J., "Axisymmetrization and vorticity gradient intensification of an isolated two-dimensional vortex", *J. Fluid Mech.* **178**, 137–159 (1987).
- Moffat, H.K., "Geophysical and astrophysical turbulence", In: *Advances in Turbulence* (Compte-Ballot, G. and Mathieu, J. (Eds.)), pp. 228–244, Springer-Verlag, New York, Berlin (1987).
- Molemaker, M.J., McWilliams, J.C. and Yavneh, I., "Instability and equilibration of centrifugally-stable stratified Taylor-Couette flow", *Phys. Rev. Lett.* **23**, 5270–5273 (2001).
- Moller, J.D. and Montgomery, M.T., "Vortex Rossby waves and hurricane intensification in a barotropic model", *J. Atmos. Sci.* **56**, 1674–1687 (1999).
- Moller, J.D. and Montgomery, M.T., "Tropical cyclone evolution via PV anomalies in a three-dimensional balance model", *J. Atmos. Sci.* **57**, 3366–3387 (2000).
- Montgomery, M.T. and Enagonio, J., "Tropical cyclogenesis via convectively forced vortex Rossby waves in a three-dimensional quasigeostrophic model", *J. Atmos. Sci.* **55**, 3176–3207 (1998).
- Montgomery, M.T. and Kallenbach, R.K., "A theory for vortex Rossby waves and its application to spiral bands and intensity changes in hurricanes", *Quart. J. Roy. Meteor. Soc.* **123**, 435–465 (1997).
- Nauta, D.N., *Two-dimensional vortices and accretion disks*, 137 pp., Ph.D. thesis, Universiteit Utrecht, ISBN 90–393–2300–3 (1999).
- Orr, W.M., "Stability or instability of the steady motions of a perfect liquid", *Proc. Roy. Irish Acad.*, 9–69 (1907).
- Polvani, L.M. and Saravanan, R., "The three-dimensional structure of breaking Rossby waves in the polar winter time stratosphere", *J. Atmos. Sci.* **57**, 3663–3685 (2000).
- Polvani, L.M., McWilliams, J.C., Spall, M.A. and Ford, R., "The coherent structures of shallow-water turbulence: deformation-radius effects, cyclone/anti-cyclone asymmetry, and gravity-wave generation", *Chaos* **4**, 177–186 (1994).
- Reasor, P. and Montgomery, M.T., "Three-dimensional alignment and co-rotation of weak, TC like vortices via linear vortex Rossby waves", *J. Atmos. Sci.* **58**, 2306–2330 (2001).
- Ren, S., "Further results on the stability of rapidly rotating vortices in the asymmetric balance formulation", *J. Atmos. Sci.* **56**, 475–482 (1999).
- Rotunno, R., Muraki, D.J. and Snyder, C., "Unstable baroclinic waves beyond quasigeostrophic theory", *J. Atmos. Sci.* **57**, 3285–3295 (2000).
- Schecter, D.A. and Montgomery, M.T., "On the symmetrization of an intense geophysical vortex", *Dyn. Atmos. Oceans*, **37**, 55–88 (2002).
- Schecter, D.A., Montgomery, M.T. and Reasor, P.D., "A theory for the vertical alignment of a quasigeostrophic vortex", *J. Atmos. Sci.* **59**, 150–168 (2002).
- Schecter, D.A., Dubin, D.H.E., Cass, A.C., Driscoll, C.F., Lansky, I.M. and O'Neil, T.M., "Inviscid damping of asymmetries on a two-dimensional vortex", *Phys. Fluids* **12**, 2397–2412 (2000).
- Schubert, W.H., Montgomery, M.T., Taft, R.K., Guinn, T.A., Fulton, S.R., Kossin, J.P. and Edwards, J.P., "Polygonal eyewalls, asymmetric eye contraction, and potential vorticity mixing in hurricanes", *J. Atmos. Sci.* **56**, 1197–1223 (1999).

- Shapiro, L.J. and Montgomery, M.T., “A three-dimensional balance theory for rapidly rotating vortices”, *J. Atmos. Sci.* **50**, 3322–3335 (1993).
- Simmons, A.J. and Hoskins, B.J., “The life cycles of some nonlinear baroclinic waves”, *J. Atmos. Sci.* **35**, 414–432 (1978).
- Snyder, C., Skamarock, W.C. and Rotunno, R., “A comparison of primitive-equation and semigeostrophic simulations of baroclinic waves”, *J. Atmos. Sci.* **48**, 2179–2194 (1991).
- Stegner, A. and Dritschel, D.G., “A numerical investigation of the stability of isolated shallow-water vortices”, *J. Phys. Ocean.* **30**, 2562–2573 (2000).
- Sutyrin, G.G., “Azimuthal waves and symmetrization of an intense vortex”, *Sov. Phys. Dokl.* **34**, 104–106 (1989).
- Sutyrin, G.G., McWilliams, J.C. and Saravanan, R., “Co-rotating stationary states and vertical alignment of geostrophic vortices with thin cores”, *J. Fluid Mech.* **357**, 321–349 (1998).
- Tung, K.K., “Initial-value problems for Rossby waves in a shear flow with critical level”, *J. Fluid Mech.* **133**, 443–469 (1983).
- Yamagata, T., “On trajectories of Rossby wave-packets released in a lateral shear flow”, *J. Oceanogr. Soc. Japan* **32**, 162–168 (1976).
- Yavneh, I., Shchepetkin, A.F., McWilliams, J.C. and Graves, L.P., “Multigrid solution of rotating, stably stratified flows: the balance equations and their turbulent dynamics”, *J. Comp. Phys.* **136**, 245–262 (1997).

APPENDIX A

Limiting Cases

Here we identify different vortex dynamical regimes associated with the magnitude of the Rossby number, the horizontal scale of the vortex and waves compared to the deformation radius, the scale separation between the vortex and waves, the strength of the mean-vortex potential vorticity gradient, and the special case of azimuthally homogeneous waves.

A.1 Quasigeostrophy

The QG limit formally neglects all contributions proportional to R , R' , and Δ' in the wave-evolution equation (46), the vortex-evolution equation (52), and the balance relations for velocity (72) used to evaluate the nonlinear wave-flux forcing [i.e., this is equivalent to also setting ρ to zero in (73)]. As a consequence all the relations for the mean vortex become linear, and there is a parity symmetry between the dynamics of cyclonic and anticyclonic vortices such that the transformation

$$\begin{aligned}
 &(\bar{v}, \dots; [\omega, k, n]; [r, \lambda, t]; \langle \phi \rangle, \langle v \rangle, \langle N^v \rangle, \dots) \\
 &\longleftrightarrow (-\bar{v}, \dots; [-\omega, -k, n]; [r, -\lambda, t]; -\langle \phi \rangle, -\langle v \rangle, -\langle N^v \rangle, \dots)
 \end{aligned} \tag{74}$$

remains a valid solution. Otherwise the evolutionary equations for the waves and vortex are only modestly reduced in their contributing terms, although the r dependences of the coefficients are much simplified. Some of the nondimensional, mean-vortex relations in (38) do have the following simpler forms:

$$\begin{aligned}
 \bar{h} = \bar{\xi} = \bar{\eta} = \bar{\gamma}^2 = \bar{q} = 1, \\
 \partial_r \bar{\phi}(r) = \bar{v}, \quad \bar{G} = \frac{L\bar{q}}{L} \left[\partial_r \bar{\xi} - \left(\frac{v}{\epsilon} \right)^2 \partial_r \bar{\phi} \right].
 \end{aligned} \tag{75}$$

The nondimensional QG wave equation (cf., (23)) is

$$\overline{D}' \left[\frac{1}{r} \partial_r' (r \partial_r' \phi) + \frac{1}{r^2} \partial_\lambda^2 \phi - v^2 \phi \right] - \mu \frac{\overline{G}}{r} \partial_\lambda' \phi = 0. \quad (76)$$

In addition, for the vortex-evolution equation (34), all but one of the forcing terms are absent [assuming $v^2 \mu$ is not large; see (42) and (45)]:

$$\frac{1}{r} \partial_r [r \partial_r \partial_t \langle \phi \rangle] - \left(\frac{v}{\epsilon} \right)^2 \partial_t \langle \phi \rangle = -\frac{1}{r} \partial_r [r \langle N^v \rangle], \quad (77)$$

where $\langle v \rangle = \partial_r \langle \phi \rangle$ and

$$\langle N^v \rangle = -\frac{1}{2r^2} \langle n \partial_r [rkA^2] \rangle + \frac{1}{2r} \langle k^2 A \partial_\lambda A \rangle. \quad (78)$$

In the QG limit, N^u and N^ϕ are not needed.

An oddity of this limit, from the perspective of the more general theory, is that terms of $O(R\rho)$ are neglected, which might not be appropriate if ρ is large, so that μ is not small, even as $R, R', \Delta' \rightarrow 0$. This oddity arises because the purely QG wave fluxes ascend in their contributing order after averaging so that nominally higher order terms in (u, v) may actually contribute to the averaged fluxes at comparable order. Including these latter contributions, the generalized QG expressions for the vortex evolution are the following:

$$\begin{aligned} \frac{1}{r} \partial_r [r \partial_r \partial_t \langle \phi \rangle] - \left(\frac{v}{\epsilon} \right)^2 \partial_t \langle \phi \rangle &= -\frac{1}{r} \partial_r [r \langle N^v \rangle] + R\rho \overline{G} \langle N^v \rangle, \\ \langle N^v \rangle &= -\frac{1}{2r^2} \langle n \partial_r [rkA^2] \rangle + \frac{1}{2r} \langle k^2 A \partial_\lambda A \rangle + R\rho \frac{\overline{G}}{2} \left(\left(k^2 + \frac{n^2}{r^2} \right) \frac{knA^2}{rK^2} \right). \end{aligned} \quad (79)$$

This wave-flux forcing breaks the parity symmetry in (74) above for $(\langle v \rangle, \langle N^v \rangle, \dots)$, so the evolutions of cyclones and anticyclones differ with finite $R\rho$.

A.2 Vortices with Weak Potential Vorticity Gradient

If we discard all terms proportional to μ or ρ , then \overline{G} effects are absent. The wave dynamics becomes nondispersive with only azimuthal propagation:

$$\begin{aligned} \omega &= \overline{\Omega} n \equiv W(k, n; r), \quad D^g \omega = 0, \\ D^g k &= -n \partial_r \overline{\Omega}, \quad D^g n = -\partial_\lambda W = 0, \\ \mathbf{C}^g &= (0, r \overline{\Omega}). \end{aligned} \quad (80)$$

This implies a linear proportionality, $k \propto t$, following azimuthal ray paths. The amplitude evolution (58) has the simple form,

$$D^g [K^2 A] = 0, \quad (81)$$

indicating that the amplitude changes inversely with the total wavenumber squared, $\propto k^2$. Finally, the associated vortex-evolution equations are

$$\begin{aligned}
 & \frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{\gamma}^2} \partial_r \partial_t \langle \phi \rangle \right) - \left(\frac{\nu}{\epsilon} \right)^2 \bar{\gamma}^2 \partial_t \langle \phi \rangle \\
 &= \left(\frac{\nu}{\epsilon} \right)^2 \bar{\gamma}^2 \langle N^\phi \rangle - \frac{\bar{\xi}}{r} \partial_r \left(r \langle N^v \rangle + R \frac{r}{\bar{\xi}} \partial_t \langle N^u \rangle \right), \\
 & \langle v \rangle = \frac{1}{\bar{\xi}} (\partial_r \langle \phi \rangle + R \langle N^u \rangle), \\
 & \langle N^u \rangle = \partial_r \left(\frac{1}{4\bar{\eta}^2 r^2} \langle n^2 A^2 \rangle \right) - \frac{1}{2\bar{\xi}^2 r} \langle k^2 A^2 \rangle \\
 & \quad + \frac{1}{2\bar{\eta}\bar{\xi}r^2} \left[\langle nA(n\partial_r A - k\partial_\lambda A) - kA\partial_\lambda[nA] \rangle \right], \\
 & \langle N^v \rangle = -\frac{1}{2\bar{\eta}r^2} \left\langle n\partial_r \left(\frac{r}{\bar{\xi}} kA^2 \right) \right\rangle + \frac{1}{2\bar{\eta}\bar{\xi}r} \langle k^2 A \partial_\lambda A \rangle, \\
 & \langle N^\phi \rangle = 0.
 \end{aligned} \tag{82}$$

A.3 Ageostrophic Vortices with Strong Potential-vorticity Gradient

When $L_{\bar{q}} \ll L$ (i.e., $\rho \gg 1$ from (40)) and $R \sim 1$, then the vortex-evolution dynamics simplifies greatly. The amplitude of the vortex change is larger than in (53):

$$\frac{\langle V \rangle}{V} = \left(R\rho \frac{V'}{V} \right)^2, \tag{83}$$

and the rescaled vortex-evolution equation (52) becomes

$$\frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{\gamma}^2} \partial_r \partial_t \langle \phi \rangle \right) - \left(\frac{\nu}{\epsilon} \right)^2 \bar{\gamma}^2 \partial_t \langle \phi \rangle = \frac{\bar{G}^2}{2\bar{\eta}\bar{\xi}} \left\langle \left(\frac{k^2}{\bar{\xi}} + \frac{n^2}{\bar{\eta}r^2} \right) \frac{knA^2}{rK^2} \right\rangle. \tag{84}$$

A.4 Small-scale Barotropic Vortices

In Sections 4–6, the strict barotropic limit occurs for $\nu \ll \epsilon$ (i.e., $L \ll L_d$). The mean-vortex quantities are

$$\bar{h} = 1, \quad \bar{q} = \bar{\eta}, \quad \bar{\gamma}^2 = \bar{\xi}\bar{\eta}, \quad \bar{G} = \frac{1}{\rho} \frac{\bar{\xi}}{\bar{\eta}} \partial_r \bar{\xi}; \tag{85}$$

the composite wavenumber for the waves is simplified to $K^2 = k^2 + n^2/r^2$; and the vortex-evolution equations (52) and (69) become

$$\begin{aligned} \frac{\bar{\gamma}^2}{r} \partial_r \left(\frac{r}{\bar{\gamma}^2} \partial_r \partial_t \langle \phi \rangle \right) &= -\frac{\bar{\xi}}{r} \partial_r (r \langle N^v \rangle) - R \frac{\bar{\xi} \bar{\eta}}{r} \partial_r \left(\frac{r}{\bar{\xi} \bar{\eta}} \partial_t \langle N^u \rangle \right) + R \rho \bar{G} \langle N^v \rangle, \\ \langle v \rangle &= \frac{1}{\bar{\xi}} [\partial_r \langle \phi \rangle + R \langle N^u \rangle]. \end{aligned} \quad (86)$$

When $\nu = \epsilon$ (i.e., $L = L_d$), there is no simplification in the mean-vortex quantities compared to the general case, but the wave dynamics has the barotropic composite wavenumber, and the vortex-evolution equation (52) is simplified by the neglect of $\langle N^\phi \rangle$ forcing (because of the factor ϵ^2 in (73)).

A.5 Large-scale Baroclinic Vortices

When $\nu \geq 1$ (i.e., $L_d \leq L' \ll L$), the vortex stretching dominates over relative vorticity in \bar{q} :

$$\bar{h} = 1 + R \left(\frac{\nu}{\epsilon} \right)^2 \bar{\phi}, \quad \bar{G} = -\frac{1}{\rho \bar{h}} \frac{\bar{\xi}}{\bar{\eta}} \left(\frac{\nu}{\epsilon} \right)^2 \partial_r \bar{\phi}. \quad (87)$$

The wave dynamics now involves the general form of the composite wavenumber. In the vortex-evolution equation, the relative vorticity effects are negligible on the left-hand side of (52):

$$\begin{aligned} \partial_t \langle \phi \rangle &= -\epsilon^2 \left(\frac{1}{\epsilon^2} \langle N^\phi \rangle \right) \\ &+ \epsilon^2 \frac{1}{\nu^2} \left(\frac{\bar{\xi}}{\bar{\gamma}^2 r} \partial_r [r \langle N^v \rangle] + \frac{R}{r} \partial_r \left[\frac{r}{\bar{\gamma}^2} \partial_t \langle N^u \rangle \right] - \frac{R \rho}{\bar{\gamma}^2} \bar{G} \langle N^v \rangle \right). \end{aligned} \quad (88)$$

The vortex tendency, $\partial_t \langle \phi \rangle$, is smaller by a factor of ϵ^2 , indicating that an appropriate rescaling of (53) would be

$$\frac{\langle V \rangle}{V} = \epsilon^2 \left(\frac{V'}{V} \right)^2, \quad (89)$$

which would allow us to drop the leading ϵ^2 factors in the right-hand side of (88). Furthermore, the $\langle N^\phi \rangle$ wave-flux forcing now competes with the other wave-flux forcing terms. When $\nu \gg 1$, $\langle N^\phi \rangle$ dominates the forcing in (88).

A.6 Azimuthally Homogeneous Waves

When the vortex waves are strictly periodic on their oscillation scale, then $n(r, \lambda, t) = n_o$, a constant. We can write

$$\beta(r, \lambda, t) = \tilde{\beta}(r, t) + n_o \lambda \quad \text{and} \quad A = A(r, t), \quad (90)$$

and all of the other wave quantities $(\omega, k, \mathcal{A}c)$ depend only on (r, t) , not λ , yielding a dynamical system with a reduced order of dimensionality.

The dispersion relation and ray equations, (60)–(62), and the definitions of wave action and activity, (63)–(65), retain the same forms, but the action and activity conservation equations simplify by dropping ∂_λ terms; e.g., (64) becomes

$$\partial_t \mathcal{A}c + \frac{1}{r} \partial_r (r C^{g,r} \mathcal{A}c) = 0. \quad (91)$$

Similarly, the vortex-evolution equation, (52), is unchanged, but the wave-flux forcing terms in (73) are somewhat simpler:

$$\begin{aligned} \langle N^u \rangle &= \frac{n_o^2}{4} \partial_r \left[\frac{1}{\bar{\eta}^2 r^2} A^2 \right] - \frac{1}{2\bar{\xi}^2 r} k^2 A^2 + \frac{n_o^2}{2\bar{\eta}\bar{\xi}r^2} A \partial_r A - R\rho \frac{\bar{G}n_o^2}{2\bar{\eta}\bar{\xi}r^2} \left(\frac{k^2}{\bar{\xi}} + \frac{n_o^2}{\bar{\eta}r^2} \right) \frac{A^2}{K^2}, \\ \langle N^v \rangle &= -\frac{n_o}{2\bar{\eta}r^2} \partial_r \left[\frac{r}{\bar{\xi}} k A^2 \right] + R\rho \frac{\bar{G}n_o}{2\bar{\eta}\bar{\xi}r} \left(\frac{k^2}{\bar{\xi}} + \frac{n_o^2}{\bar{\eta}r^2} \right) \frac{k A^2}{K^2}, \\ \langle N^\phi \rangle &= -\epsilon^2 R\rho \frac{n_o}{2r} \partial_r \left[\frac{\bar{G}}{\bar{\eta}\bar{\xi}} \frac{k}{K^2} A^2 \right]. \end{aligned} \quad (92)$$

We have dropped the angle brackets on the right-hand side since there is no longer a dependence on λ .

APPENDIX B

Relation to a Previous Vortex Rossby Wave Theory

The wave propagation characteristics illustrated in linear initial-value solutions – similar to those in Section 8 for sheared waves – were interpreted in MK97 with a local WKB theory that neglected certain spatial dependences in the evolution of the wave phase and amplitude. While locally valid in (r, t) for the vortex Rossby wave kinematics, this theory does not account accurately for finite radial displacements of the wave field, nor account at all for the vortex evolution. The present theory is more general in these aspects.

For the wave phase, the primary difference between the present theory and MK97 is in the azimuthal group velocity. Following Tung (1983), the time-dependent radial wavenumber $k(t)$ in the local dispersion relation was regarded in MK97 as a function of the initial radial wavenumber $k(0)$ and azimuthal wavenumber n . Here, $k(t)$ and n are treated as independent variables.

For the wave amplitude, the present theory and MK97 differ even more. In MK97 the wave amplitude for azimuthal wavenumber n was governed by the time invariance of perturbation potential-vorticity amplitude (i.e., the passive scalar limit). Strictly speaking, this is only valid for sheared waves in the limit of vanishing μ (as in (81)), or the limit of long times for $\mu \sim O(1)$ when the influence of the mean potential-vorticity gradient is overcome by the straining from the mean vortex. When $\mu \sim O(1)$, potential-vorticity gradient effects in (63)–(64) are important at short-to-intermediate times.