A FORMALISM FOR THE CONSISTENT DESCRIPTION OF NON-LINEAR ELASTICITY OF ANISOTROPIC MEDIA

K. HELBIG[1]

FORMALISME POUR UNE DESCRIPTION COHÉRENTE DE L'ÉLASTICITÉ NON LINÉAIRE DES MILIEUX ANISOTROPES

La propagation des ondes élastiques est généralement traitée sous quatre hypothèses :

– le milieu est isotrope,
– le milieu est homogène,
– il y a une relation biunivoque entre la tension et la déformation,
– les tensions sont reliées d'une façon linéaire aux déformations (et de manière équivalente, les déformations sont reliées d'une façon linéaire aux tensions).

En général au moins une de ces hypothèses — et souvent toutes — n'est pas vérifiée dans les milieux réels. Une description théorique valide de la propagation des ondes dans les milieux réels dépend ainsi de la description à la fois qualitative et quantitative de l'hétérogénéité, de l'anisotropie et de la non-linéarité : soit on doit supposer (ou montrer) que l'écart par rapport à l'hypothèse de départ peut être — pour le problème considéré — négligé, soit on doit développer une description théorique, valide même en présence de ces écarts. Alors que l'effet d'un seul écart par rapport à un état idéal est relativement bien connu, les difficultés surviennent quand on veut combiner plusieurs de ces écarts.

Les propriétés élastiques non linéaires d'échantillons de roche anisotropes (tricliniques) ont été étudiées, par P. Rasolofosaon et H. Yin au 6e IWSA à Trondheim (Rasolofosaon et Yin, 1996). L'élasticité anisotrope non linéaire est importante seulement pour les amplitudes « non infinitésimales », c'est-à-dire dans un certain voisinage de la source. L'étendue de ce voisinage dépend de la précision de l'observation et de l'interprétation que l'on tente de maintenir, de l'intensité de la source, et du degré de non-linéarité. Cet article traite du dernier aspect, c'est-à-dire de la signification des nombres au-delà du fait qu'ils sont le résultat de mesures.

Pour la mesure de la non-linéarité des matériaux, on peut utiliser le seuil de déformation au niveau duquel le tenseur de rigidité effective s'écarte sensiblement du tenseur de rigidité à déformation nulle. Il est particulièrement utile de prendre en compte le système propre du tenseur de rigidité (six rigidités propres et six déformations propres) : les déformations propres fournissent des « types de déformation » adaptés au calcul du tenseur de rigidité effective, et la perturbation peut être exprimée par le changement relatif des rigidités propres et par la variation des directions propres associées aux déformations propres (exprimées en tant que vecteurs dans un espace à six dimensions).

(1) Kiebtzrain 84,
3000 Hannover 51 - Germany

La méthode suggérée est appliquée aux deux matériaux étudiés par Rasolofosaon et Yin (1996). Les résultats permettent une évaluation heuristique de la signification de la « déformation de référence », définie comme la racine carrée du rapport des normes des tenseurs de rigidité du quatrième et du sixième rang.

Il est à signaler qu'il ne s'agit pas d'une nouvelle théorie de la non-linéarité, mais d'une nouvelle approche de la théorie existante et des résultats.

A FORMALISM FOR THE CONSISTENT DESCRIPTION OF NON-LINEAR ELASTICITY OF ANISOTROPIC MEDIA

The propagation of elastic waves is generally treated under four assumptions:

– that the medium is isotropic,
– that the medium is homogeneous,
– that there is a one-to-one relationship between stress and strain,
– that stresses are linearly related to strains (equivalently, that strains are linearly related to stresses).

Real media generally violate at least some—and often all—of these assumptions. A valid theoretical description of wave propagation in real media thus depends on the qualitative and quantitative description of the relevant inhomogeneity, anisotropy, and non-linearity: one either has to assume (or show) that the deviation from the assumption can—for the problem at hand—be neglected, or develop a theoretical description that is valid even under the deviation. While the effect of a single deviation from the ideal state is rather well understood, difficulties arise in the combination of several such deviations.

Non-linear elasticity of anisotropic (triclinic) rock samples has been reported, e.g. by P. Rasolofosaon and H. Yin at the 6th IWSA in Trondheim (Rasolofosaon and Yin, 1996). Non-linear anisotropic elasticity matters only for "non-infinitesimal" amplitudes, i.e., at least in the vicinity of the source. How large this vicinity is depends on the accuracy of observation and interpretation one tries to maintain, on the source intensity, and on the level of non-linearity. This paper is concerned with the last aspect, i.e., with the meaning of the numbers beyond the fact that they are the results of measurements.

As a measure of the non-linearity of the material, one can use the strain level at which the effective stiffness tensor deviates significantly from the zero-strain stiffness tensor. Particularly useful for this evaluation is the eigensystem (six eigenstiffnesses and six eigenstrains) of the stiffness tensor: the eigenstrains provide suitable "strain types" for the calculation of the effective stiffness tensor, and the deviation can be expressed by the relative change of the eigenstiffnesses and by the variation in the direction of the eigenstrains (expressed as vectors in six-dimensional strain space).

The suggested procedure is applied to the two materials discussed by Rasolofosaon and Yin (1996). The results allow a heuristic evaluation of the meaning of the "reference strain", the square root of the ratio of the norms of the fourth-rank and sixth-rank stiffness tensors.

It is stressed that this is not a new theory of non-linearity, but only a different way of viewing the existing theory and results.

FORMALIZACIÓN PARA LA DESCRIPCIÓN CONSISTENTE DE LA ELASTICIDAD NO LINEAL EN MEDIOS ANISOTRÓPICOS

La propagación de las ondas elásticas es tratada en general suponiendo que :

– el medio es isotrópico,
– el medio es homogéneo,
– existe una relación uno a uno entre tensión de compresión y tensión de dilatación,
– los tensiones de compresión están linealmente relacionadas con las tensiones de dilatación (y, de modo recíproco, que las tensiones de dilatación están linealmente relacionadas con las tensiones de compresión).

Los medios reales violan en general algunas, y a menudo todas, estas premisas. Una descripción teórica válida de la propagación de ondas en medios reales depende por lo tanto de la descripción cualitativa y cuantitativa de la no-homogeneidad, anisotropía y no linealidad relevantes. Debe suponerse (o demostrarse) que la desviación con respecto a la premisa puede, en el caso del problema tratado, considerarse insignificante, o bien puede desarrollarse una descripción teórica que sea válida incluso en condiciones de desviación. Mientras que el efecto de una desviación única a partir del estado ideal ha sido bastante bien comprendido, las dificultades surgen cuando se trata de combinaciones de varias desviaciones de este tipo.

La elasticidad no lineal de muestras de rocas anisotrópicas (triclínicas) fue mencionada, por ejemplo, por P. Rasolofosaon y H. Yin en la 6ª IWSA en Trondheim (Rasolofosaon y Yin, 1996). La elasticidad anisotrópica no lineal tienen importancia sólo para las amplitudes "no-infinitesimales", es decir, por lo menos en la cercanía de la fuente. La amplitud de esta cercanía depende de la precisión de la observación y de la interpretación que se trata de mantener, así como de la intensidad de la fuente y del nivel de no-linealidad. Este artículo se refiere al último aspecto, es decir, al significado de los números más allá del hecho de que sean el resultado de mediciones.

Como medición de la no-linealidad del material se puede utilizar el nivel de tensión de dilatación al cual el tensor de rigidez efectiva desvía significativamente el tensor de rigidez con respecto a la tensión cero. Para esta evaluación es particularmente útil el sistema eigen o de valores específicos (seis rigideces eigen y seis tensiones eigen) del tensor de rigidez : las tensiones eigen proporcionan "tipos de tensión de dilatación" adecuados para el cálculo del tensor de rigidez efectivo, y la desviación puede ser expresada por el cambio relativo de las rigideces eigen así como por la variación en la dirección de las tensiones eigen (expresadas como vectores en el espacio hexadimensional de la tensión de dilatación).

El procedimiento propuesto se aplica a dos materiales discutidos por Rasolofosaon y Yin (1996). Los resultados permiten una evaluación heurística del significado de la "tensión de dilatación de referencia", la raíz cuadrada de la relación de las normas de los tensores de rigidez de cuarto orden y de sexto orden.

Se enfatiza que ésta no es una nueva teoría de no linealidad, sino un enfoque nuevo de la teoría y de los resultados ya existentes.

INTRODUCTION

The standard description of non-linear elasticity is based on the expression of the six components of the stress tensor as a Taylor series in the components of the strain tensor, or alternatively on the expression of the strain energy as a Taylor series in the six components of the strain tensor. Except for some subtle differences between the two treatments, the two descriptions are equivalent: the coefficients of the first terms in the series are the components of the "standard" elastic tensor of rank four (of which at most 21 are independent), and the coefficients of the second terms in the series are the components of a tensor of rank six (of which at most 56 are independent). Higher terms are not taken into account. Different authors refer to the components of the fourth and sixth-rank tensors under different—and unfortunately conflicting—names: either the components of the fourth and sixth order tensor are referred to as "first order" and "second order" stiffnesses, respectively, or as "second order" and "third order" stiffnesses, respectively (standard tables of crystal physics refer to the non-linear terms as "third order"). Both expressions are justified, depending on the quantities one expresses by the Taylor series. Rasolofosaon and Yin (1996) use terms corresponding to "linear stiffnesses" and "non-linear stiffnesses". Note that an equally valid description would be obtained by developing either the six components of the strain tensor or the elastic energy into Taylor series in terms of the six components of the stress tensor. The coefficients should be given names similar to those quoted, with "compliance" instead of "stiffness".

A quantification of non-linearity might be attempted by comparing the magnitude of (some of) the non-linear stiffnesses with the linear stiffnesses. There is some misgiving about the comparison of the components of tensors of different rank, but at least the dimension of all coefficients involved are the same (Pa). However, even this is only apparently correct: the similarity of the dimensions is due to the fact that the independent variable in the series (strain) has no physical dimension, and thus products of the strain components have apparently the same dimension (i.e. none). The fallacy of this argument is illustrated by an attempt to compare plane angles with solid angles, or by looking at the components of the fourth- and sixth order compliance tensors: they have dimensions of Pa^{-1} and Pa^{-2}, respectively, and thus cannot be compared.

For plane and solids angles one gets around the quandary by using the "pseudo units" rad and rad^2. For the strain one could use "proc" (for Procrustes, who inflicted strain in his victims to make them fit his bed). Sometimes the term "strain" is used as the unit of strain (e.g. "0.5 millistrain" for $\varepsilon = 5 \times 10^{-4}$). This criticism does not imply that the parameters of non-linearity based on the ration of the components of the sixth- and fourth-rank tensors (Johnson and Rasolofosaon, 1996) are meaningless, only that a quantitative interpretation of such parameters is not easy.

The difficulty is completely avoided by expressing the stiffness tensor formally as a linear function of the components of the strain tensor with strain-dependent coefficients, i.e., by using:

$$\begin{aligned}\sigma_{ij} &= c^{(1)}_{ijkl}\,\varepsilon_{kl} + c^{(2)}_{ijklmn}\,\varepsilon_{kl}\,\varepsilon_{mn} + \dots \\ &= \left(c^{(1)}_{ijkl} + c^{(2)}_{ijklmn}\,\varepsilon_{mn} + \dots\right)\varepsilon_{kl} = C_{eff}(\varepsilon)\,\varepsilon\end{aligned} \tag{1}$$

All terms in the braces—the components of the effective stiffness tensor—are now of equal rank and equal (pseudo-) dimensions and can be compared without constraint.

1 THEORY

The basic ideas put forward in this paper apply equally to other types of non-linear "constitutive equations", thus a few tensors beyond the elastic tensor are mentioned. The later development is restricted to elasticity. The tensor relations under discussion are represented by tensors of rank $2n$ connecting two tensors of rank n. Examples are Hooke's Law, where the "elastic" fourth-rank tensor c_{ijkl} connects the second rank stress-tensor σ_{ij} with the second-rank strain-tensor ε_{kl}, and the second-rank tensor of magnetic susceptibility that connects the vector of magnetic flux density with the magnetic field vector.

$$\begin{aligned}\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} &\Leftrightarrow \varepsilon_{ij} = S_{ijkl}\,\sigma_{kl} \\ B_i = \mu_{ij}\,H_j &\Leftrightarrow H_i = S_{ij}\,B_j\end{aligned} \tag{2}$$

While the formal tools can be applied to any non-linear tensor relationship, the current discussion is primarily concerned with situations as in the two examples above, where the tensor of rank $2n$ describes material properties and the tensors of rank n are canonically related field tensors. Field tensors are canonically related if their scalar product can be interpreted as an *energy density:*

$$W = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij} = \frac{1}{2}c_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} = \frac{1}{2}S_{ijkl}\,\sigma_{ij}\,\sigma_{kl}$$
$$W = \frac{1}{2}B_i\,H_i = \frac{1}{2}\mu_{ij}\,H_i\,H_j = \frac{1}{2}M_{ij}\,B_i\,B_j \qquad (3)$$

We shall assume an equilibrium state where vanishing of one field tensor implies the vanishing of the other, and further assume that the energy density vanishes if both field tensors vanish.

The "material tensors" in (2) and (3) are often assumed to be constant, but there is no *a priori* reason that they should be independent of the field tensors. For *large* field vectors (large deformations, strong magnetic fields), the tensor relationship may be highly non-linear and completely beyond the domain of the linear theory. For *sufficiently* small field vectors it can be *linearized.* i.e., approximated by the standard linear relationship.

Most physical theories deal with small field vectors, i.e., they are *linearized theories.* The terms "sufficiently small" and "large" are, at this moment, defined loosely by the validity (respectively, the non-validity) of the linearized theory. One of the aims of this paper is to develop a quantitative norm for these terms.

2 TERMINOLOGY OF NON-LINEARITY

Linear theories can be generalized to accept "large" field tensors by regarding the material tensor of rank $2n$ as the first coefficient in a Taylor series in terms of the components of the field tensor. The higher terms are then tensors of rank $3n$, $4n$, ... For moderate magnitude of the field tensor one truncates the series after the first new term. If one additional term in the Taylor series is not sufficient, the problem may not be tractable as a "deviation from linearity". In this discussion it is assumed that fields are of moderate magnitude and thus one additional term in the material tensor is sufficient.

For most material tensors the first new term is a tensor of rank $3n$. This is the case if there is a physical difference between the two algebraic signs of the field tensors. For instance, a material might get stiffer under compression and more compliant under extension. However, there are situations where the deviation from linearity must be independent of the sign of the field vector (an example is the relationship between the magnetic field and the magnetic flux density in a diamagnetic material). In such case the tensor of rank $3n$ vanishes and the first new term in the Taylor series is of rank $4n$.

The Taylor series can be established either for the field equation, or for the energy density. For an example, see (4) at the bottom of this page.

The non-linearity tensor c_{ijklmn} is called sometimes the "second order stiffness tensor", because in the expression (4.1) its components are the factors of second order products of components of the strain tensor, but with equal justification the "third order stiffness tensor", because in the expression (4.3) its components are the factors of third order products of components of the strain tensor. Similarly, μ_{ijkl} is called either the third order susceptibility tensor or the fourth order susceptibility tensor. We shall avoid these terms and use, where necessary, the unambiguous rank of the different tensors.

Note that, in general, the different coefficient tensors in the Taylor series (3) have different dimensions:

$$\begin{array}{llll} [\mu_{ij}] = \mathrm{V\,s\,A^{-1}\,m^{-1}} & [M_{ij}] = \mathrm{A\,m\,V^{-1}\,s^{-1}} \\ [\mu_{ijkl}] = \mathrm{V\,s\,A^{-3}\,m^{-3}} & [M_{ijkl}] = \mathrm{A\,m\,V^{-3}\,s^{-3}} \\ [c_{ijkl}] = \mathrm{Pa} & [S_{ijkl}] = \mathrm{Pa^{-1}} \\ [c_{ijklmn}] = \mathrm{Pa} & [S_{ijklmn}] = \mathrm{Pa^{-2}} \end{array} \qquad (5)$$

The exception are the stiffness tensors C of rank 4, 6, 8, ...This is so because the "independent" field tensor in the stress-strain relation—the strain $\varepsilon_{ij} \approx (u_{i,j} + u_{j,i})/2$ —is dimensionless. This is only an apparent simplification that might even lead to confusion: in the inverse strain-stress relation the coefficients (the compliance tensors S) have again different dimensions.

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} + c_{ijklmn}\,\varepsilon_{kl}\,\varepsilon_{mn} + \ldots \Leftrightarrow \varepsilon_{ij} = S_{ijkl}\,\sigma_{kl} + S_{ijklmn}\,\sigma_{kl}\,\sigma_{mn} + \ldots$$
$$B_i = \mu_{ij}\,H_j + \mu_{ijkl}\,H_j\,H_k\,H_l + \ldots \Leftrightarrow H_i = M_{ij}\,B_j + M_{ijkl}\,B_j\,B_k\,B_l + \ldots$$
$$W = \frac{1}{2}c_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} + \frac{1}{6}c_{ijklmn}\,\varepsilon_{ij}\,\varepsilon_{kl}\,\varepsilon_{mn} + \ldots = \frac{1}{2}S_{ijkl}\,\sigma_{ij}\,\sigma_{kl} + \frac{1}{6}S_{ijklmn}\,\sigma_{ij}\,\sigma_{kl}\,\sigma_{mn} + \ldots \qquad (4)$$
$$W = \frac{1}{2}\mu_{ij}\,H_i\,H_j + \frac{1}{24}\mu_{ijkl}\,H_i\,H_j\,H_k\,H_l + \ldots = \frac{1}{2}M_{ij}\,B_i\,B_j + \frac{1}{24}M_{ijkl}\,B_i\,B_j\,B_k\,B_l + \ldots$$

In similar cases one introduces pseudo-dimensions like "rad" for the dimensionless planar angle and rad² for the equally dimensionless solid angle. For the strain we might use "Proc" (named after Procrustes, who forcibly shortened or extended his guests to fit his bed). The corresponding dimensions in (4) would then be Pa/Proc, Pa/Proc², Proc/Pa, and Proc/Pa², in full agreement with other field relationships.

In this context, the stress-strain relationship—Hooke's law—is exceptional also in another sense. The full definition of (Lagrangian) strain is:

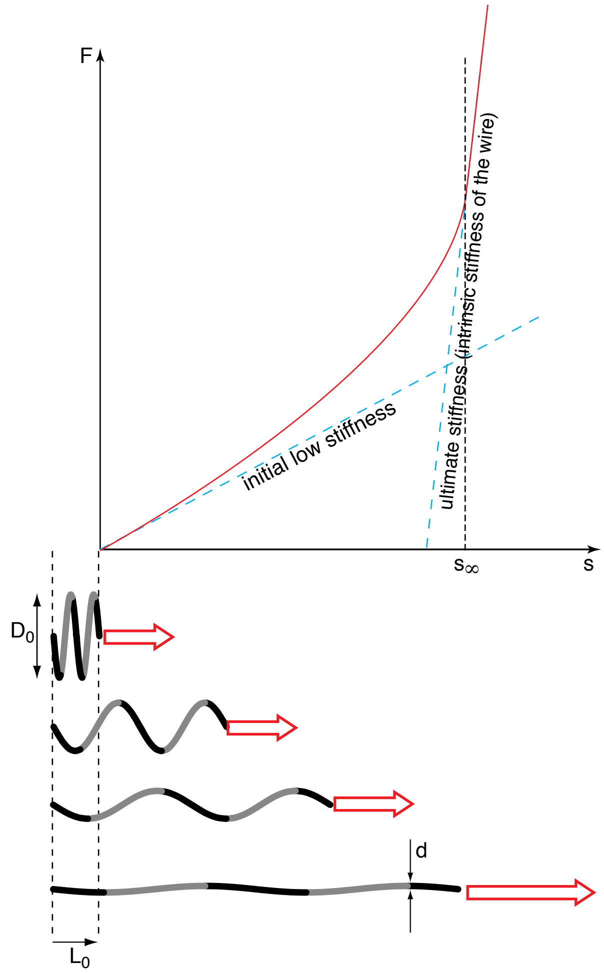
$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i} + u_{k,j}\right) \tag{6}$$

The third term is quadratic in the components of the displacement gradient and thus neglected in the linear theory, which assumes infinitesimal strain. For moderate and large strain this term cannot be neglected, thus for the stress-strain relationship there is—in addition to any "material non-linearity" also a "geometric non-linearity". Even the hypothetical ideal elastic material shows this geometrical non-linearity. To compound matters, for *limited samples* of a material there is a third type of non-linearity, the "structural non-linearity". This has to do with the change of the geometric relationship between different parts of the sample. As an example I refer to the stiffness of a helical spring (Fig. 1). Under extension the diameter of the helix decreases and the pitch increases. For very large extensions, the wire in the spring is fully stretched. For small extension, the stiffness (change of length divided by change of load) of the spring is controlled by the (extension dependent) geometric parameters of the helix, the shear stiffness μ of the wire, and the axial "moment of inertia" of the cross section of the wire. At very large extensions, the stiffness is controlled entirely by the dimensions of the "wire" and by the corresponding Young modulus (Fig. 1).

For discrete pieces of matter —rods, cantilevers, shells, geophone springs— structural non-linearity is the most significant type of non-linearity. It plays hardly any role in the extended medium. Thus it can be neglected for the description of wave propagation, though one has to take into account that laboratory measurements are invariably done on small samples.

In numerical modelling by, e.g. Finite Element (FE) Algorithms, one comes across still another type of non-linearity in the program manuals: in a linear FE simulation of stress-strain behaviour, one calls the calculation "linear" if the initial stiffness parameters are maintained throughout the calculation. If the parameters are updated at intermediate stages, one calls the calculation "non-linear". To distinguish this type of non-linearity, one might call it "computational non-linearity".

The following discussion is not concerned with structural and computational non-linearity. No distinction is made between material and geometric non-linearity.



small extension: $\Delta F = \frac{\mu\, d^4}{8\, L\, D^3}\,\Delta s$

large extension: $\Delta F = \frac{E\, d^2}{8\, n\, D_0}\,\Delta s$

Figure 1

Extension-load curve for a helical spring as example for "structural non-linearity". This spring stiffens under extension. It also stiffens under compression once the turns make contact (not shown).

3 QUANTIFICATION OF NON-LINEARITY

A quantification of non-linearity is not a simple matter. One would prefer a dimensionless number that allows to compare different materials, ideally even different kinds of field tensors. A direct comparison of the different coefficient tensor of the Taylor series is—except for Hooke's law—not possible since they have different dimensions. For Hooke's law this difficulty does not seem to exist, thus it has been suggested to use the ratio of the second term to the first term. Of course, the quotient of two tensors of different rank is defined only in special cases. Even then, this quotient is, in general, itself a tensor. If:

$$a_{ij} = b_{ijklmn}\, c_{klmn} \tag{7}$$

one might say with some justification that:

$$\frac{a_{ij}}{c_{klmn}} = b_{ijklmn}, \quad \text{or} \quad \frac{a_{ij}}{b_{ijklmn}} = c_{klmn} \tag{8}$$

but such a relationship does not exist between the coefficient tensors in the Taylor series.

The difficulty of comparing tensors of different rank might be overcome by forming the ratio of the norms of two tensors or the rms values of the components of the two tensors. Such a quantity is still not dimensionless (for the stiffness tensor it has the pseudo-dimension Proc, but in most other cases it has dimensions Pa, A^2 m^2, V^2 s^2, ... Such a ratio might be a meaningful guide, but it is not the unambiguous measure we are looking for.

From a physical standpoint one should rather describe the problem as "strain-dependent stiffness" and "field-dependent susceptibility", respectively. This leads immediately to:

$$\begin{aligned}
\sigma_{ij} &= \left(c_{ijkl} + c_{ijklmn}\, \varepsilon_{mn} + \dots\right) \varepsilon_{kl} \\
B_i &= \left(\mu_{ij} + \mu_{ijkl}\, H_k\, H_l\right) H_j \\
W &= \frac{1}{2}\left(c_{ijkl} + \frac{1}{3}\, c_{ijklmn}\, \varepsilon_{mn} + \dots\right) \varepsilon_{ij}\, \varepsilon_{kl} \\
W &= \frac{1}{2}\left(\mu_{ij} + \frac{1}{12}\, \mu_{ijkl}\, H_k\, H_l + \dots\right) H_i\, H_j
\end{aligned} \tag{9}$$

In the parentheses one compares tensors of equal rank. One can compare component for component, or globally via the magnitude of the two tensors.

The magnitude of a tensor is the root of its norm, i.e., the root of the sum of the squares of all components. The magnitude of the field tensor for which the two tensors are of equal magnitude (or at which the magnitude of the correction term is a given fraction of the magnitude of the small-field term) might be used as a "measure" for the non-linearity. Even the "magnitude of the field tensor at which the correction term is a given fraction of the small-field term" cannot be used in this simple form, since the correction term does not depend on the magnitude of the field tensor alone. If the field tensor is a vector (e.g. the magnetic field H_k), one could let the end point of a unit field vector range over the entire unit sphere and then either average the corresponding corrections or plot them in a 3D plot.

If the field tensor is of rank 2 (e.g. the strain ε_{ij}), matters are even more complicated. A tensor of rank 2 in n dimensions can be mapped on a vector (tensor of rank 1) in n (n – 1)/2 dimensions. A 3D strain tensor thus becomes a 6D strain vector.

We then either have to average the correction over a 6D unit sphere, or somehow visualize the distribution of the correction in 6-space.

We leave this point for a moment and look at the much simpler case of the non-linearity of a scalar, e.g. the spring as in Figure 1. The load-deflection relationship with a load-dependent stiffness can be written as a two-term Taylor series:

$$\begin{aligned}
\boldsymbol{F} = \boldsymbol{k(s)s} = \boldsymbol{k}_1\, \boldsymbol{s} &\begin{cases} +\boldsymbol{k}_3 \boldsymbol{(s)}^3 + \dots \\ -\boldsymbol{k}_3 \boldsymbol{(s)}^3 + \dots \end{cases} \\
&= \begin{cases} \boldsymbol{k}_1\left(1 + \dfrac{\boldsymbol{s}^2}{2\boldsymbol{s}_3^2} + \dots\right) \boldsymbol{s} \\ \boldsymbol{k}_1\left(1 - \dfrac{\boldsymbol{s}^2}{4\boldsymbol{s}_3^2} + \dots\right) \boldsymbol{s} \end{cases}
\end{aligned} \tag{10}$$

The upper line refers to a spring that stiffens under load, the lower one to a spring that softens under load. Note that the change of the stiffness depends on the sign of the deflection: if the spring hardens under extensions, it softens under compression (and vice versa). The reference-extension s_2 is defined through:

$$\left.\frac{\partial F}{\partial s}\right|_{S=s_2} = \begin{Bmatrix} 2 \\ \dfrac{1}{2} \end{Bmatrix} \left.\frac{\partial F}{\partial s}\right|_{S=0}$$

or

$$\left|\log\left(\left.\frac{\partial F}{\partial s}\right|_{S=s_2} \middle/ \left.\frac{\partial F}{\partial s}\right|_{S=0} \right)\right| = 0.30103\dots \tag{11}$$

Instead of the reference extension s_2 one can use $s_{\sqrt{2}}$ or $s_{1.1}$ or any other convenient definition.

Equation (12) is the corresponding expression for a spring with symmetric load-deflection relationship, i.e., one which stiffens (or softens, as the case may be) for any deflection, i.e., for both compression and extension. A typical example is the ideal geophone spring (prestressed to be flat at rest):

$$F = k(s)s = k_1\, s \begin{cases} +k_3(s)^3 + \dots \\ -k_3(s)^3 + \dots \end{cases}$$

$$= \begin{cases} k_1\left(1 + \dfrac{s^2}{3s_3^2} + \dots\right) s \\ k_1\left(1 - \dfrac{s^2}{6s_3^2} + \dots\right) s \end{cases} \tag{12}$$

with reference extensions defined in the same way as above. The corresponding load-deflection curves are shown in Figure 2.

These scalar concepts can be applied to tensors in general—and to the elastic tensor in particular—in different ways. The simplest is to use as reference strain any finite strain that changes the original stiffness norm by a pre-set amount. The set of all such reference strains defines a surface in 6-space that is indicative of the magnitude and directional dependence of the non-linearity. There are a few drawbacks:

– Every real medium is, of course, stable, and it is likely that it remains stable at any realizable strain below the level of permanent deformation. However, this does not mean that the two-term Taylor representation remains stable too. In a later example, the strains have been increased until the two-term approximation became unstable. This would be another unambiguous measure of non-linearity, if there were always a strain level to cause instability. However, if the medium stiffens under strain, this does not necessarily happen.
– The set of all reference strains is conceptually simple, but difficult to obtain even at moderate sampling of the 6D unit sphere.
– Meaningful visualization of a 6D surface is a difficult matter.

The first of these drawbacks can be remedied by using a “modest” definition of the reference strain. The other two can be mitigated somewhat by using the eigensystem of the medium. This eigensystem consists of six mutually perpendicular eigenstrains and six corresponding eigenstiffnesses.

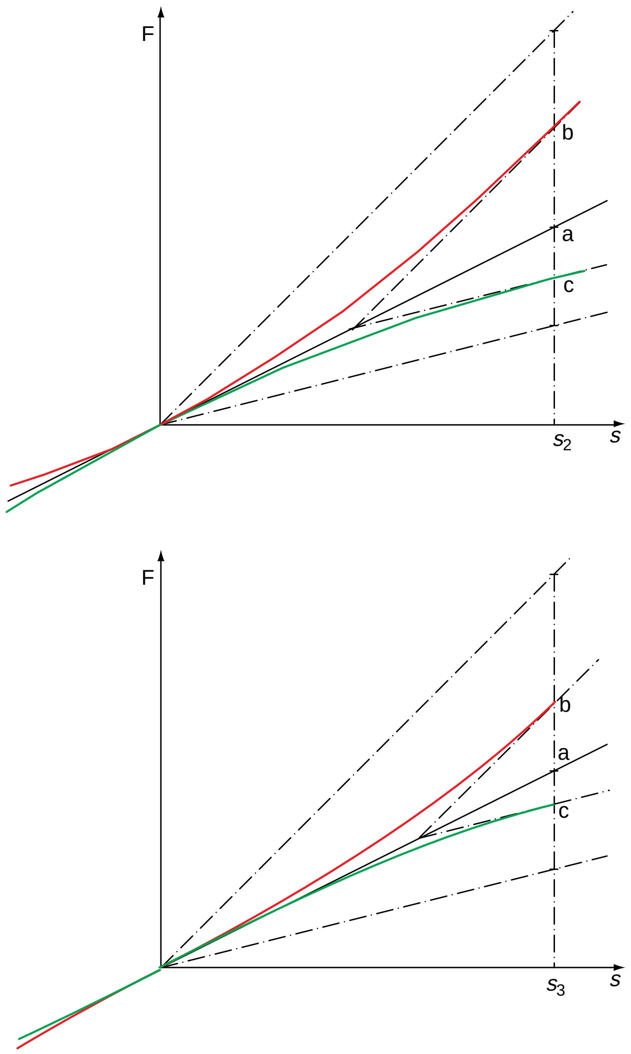


Figure 2

Two deflection-load curves as examples of non-linearity. Top: unsymmetric non-linearity. Bottom: symmetric non-linearity. At the reference deflections s_2 and s_3 the slope of the curve has changed by a factor 2.

The eigenstrains provide a preferred coordinate system, and it might be sufficient to use as test strains only (positive *and* negative) multiples of the eigenstrains. This reduces the high number of test strains to twelve.

4 STABILITY AND THE EIGENSYSTEM OF A MATERIAL TENSOR

For material tensors that connect canonically related field tensors the concept of *stability* is meaningful: a material tensor is stable if and only if any deviation from the equilibrium requires energy. This is the case if all eigenvalues of the material tensor are positive, or—equivalently—if the matrix representing the tensor is positive definite. A square matrix is positive definite if all its leading principal minors are positive.

A second rank tensor can be regarded as its own matrix representation. To establish the stability of the tensor then requires only the evaluation of the principle minors. For tensors of higher rank it is convenient to determine the eigensystem (eigen-values and eigenvectors or their equivalents).

Since in the coordinate system of the eigenvectors the material tensor is represented by a diagonal matrix, the condition of stability reduces to the requirement that all six eigenvalues ("eigenstiffnesses") be positive.

The eigensystem is formally obtained by solving, e.g.:

$$(c_{ijkl} - \delta_{ik}\,\delta_{jl}\,\Lambda)\,\varepsilon_{kl} = 0 \qquad (13)$$

with subscripts i, j, k, l assuming all values between 1 and 3. Equation (13) is a system of homogeneous equations in second rank tensors, which has non-trivial solutions if and only if the determinant of the coefficient matrix vanishes. This conditions leads to a polynomial in Λ. The six solution of this system of equations are the eigenvalues (eigenstiffnesses, eigencompliances, eigensusceptibilities, etc.).

There is a shortcut to the determination of the determinant of the system (13): one can map the tensors in (13) isomorphically—i.e., without loss of any physical significant information—on a relationship between first-rank tensors (vectors) in 6-spaces specified by a second-rank tensor. This changes (13) to:

$$(C_{pq} - \delta_{pq}\,\Lambda)\,\varepsilon_q = 0 \qquad (14)$$

with subscripts p and q assuming all values between 1 and 6.

The mapping is accomplished by the Kelvin mapping (Kelvin, 1856; Helbig, 1994):

$$\sigma_{ij} \to \sigma_p^{(K)} \qquad \varepsilon_{kl} \to \varepsilon_q^{(K)} \qquad c_{ijkl} \to c_{pq}^{(K)}$$

with
$$p = \delta_{ij} i + (1 - \delta_{ij})\,(9 - i - j)$$
$$q = \delta_{kl} k + (1 - \delta_{kl})\,(9 - k - l)$$

$$\sigma_p^{(K)} = \alpha\,\sigma_{ij}, \qquad \alpha = \delta_{ij} + (1 - \delta_{ij})\,\sqrt{2} \qquad (15)$$
$$\varepsilon_q^{(K)} = \beta\,\varepsilon_{kl}, \qquad \beta = \delta_{kl} + (1 - \delta_{kl})\,\sqrt{2}$$
$$c_{pq}^{(K)} = \alpha\,\beta c_{ijkl} = \alpha\,\beta c_{pq},$$

where c_{pq} is the customary Voigt matrix representation of the elastic tensor. The Kelvin mapping assures that the norms of the tensors and the eigensystem of the stiffness tensor are preserved.

In Kelvin notation the system of six linear equations (14) is an ordinary eigenvalue-eigenvector problem which leads, in general, to a sextic in Λ. Though a general sextic cannot be solved analytically, numerical solution poses no difficulty. The same mapping process maps c_{ijklmn} on C_{pqr} which reduces (9.1) to:

$$\sigma_p = (C_{pq} + C_{pqr}\,\varepsilon_r + \ldots)\,\varepsilon_q \qquad (16)$$

A search algorithm to find the reference strains according to any of these conditions is easily programmed. A few examples based on a Mathematica implementation show the application of these concepts.

5 CAUSES OF NON-LINEARITY

There are several causes that can render the stress-strain relationship non-linear:

- material non-linearity refers to effects that have their root ultimately in microstructure (at any level below grain size);
- geometric non-linearity is due to the linearization in the definition of strain: the full definition of (Lagrangian) strain is:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i} + u_{k,j}\right) \qquad (17)$$

The third term is quadratic in the components of the displacement gradient and thus neglected in the linear theory, which assumes infinitesimal strain. For moderate and large strain this term cannot be neglected, thus for the stress-strain relationship there is —in addition to any "material non-linearity" also a "geometric non-linearity".

Even the hypothetical ideal elastic material shows this geometrical non-linearity.

6 TWO EXAMPLES

The concepts are tested on the elastic data of a calcite single crystal (taken from Landoldt-Börnstein, 1966) and on the elastic data of a marble sample obtained by P. Rasolofosaon and H. Yin in the rock physics laboratory of the *IFP* (Rasolofosaon and Yin, 1996).

The calcite crystal is trigonal, as indicated by the characteristic surfaces of the (linear) rank four stiffness tensor and the (non-linear) rank six stiffness tensor (Fig. 3), i.e. the graphic representation of the homogeneous forms:

$$C_{ijkl}\, x_i\, x_j\, x_k\, x_l = 1 \text{ and } C_{ijklmn}\, x_i\, x_j\, x_k\, x_l\, x_m\, x_n = 1 \qquad (18)$$

Similarly, the corresponding tensors for the marble sample both show no symmetry, i.e., they are triclinic. The components of the four tensors are listed in (16) and (17), respectively.

For both media the "standard reference strain", the square root of the ratio of the norms of the fourth-rank and sixth-rank tensors, is given. Note that the standard reference strain is used as a qualitative indication for the starting level at which the effects of non-linearity have reached a certain level. This reference strain is the inverse of the commonly used "non-linearity coefficient".

$$\beta_{NL} = \frac{\|C_{linear}\|}{\|C_{nonlinear}\|} = \frac{1}{\varepsilon_{ref}} \qquad (21)$$

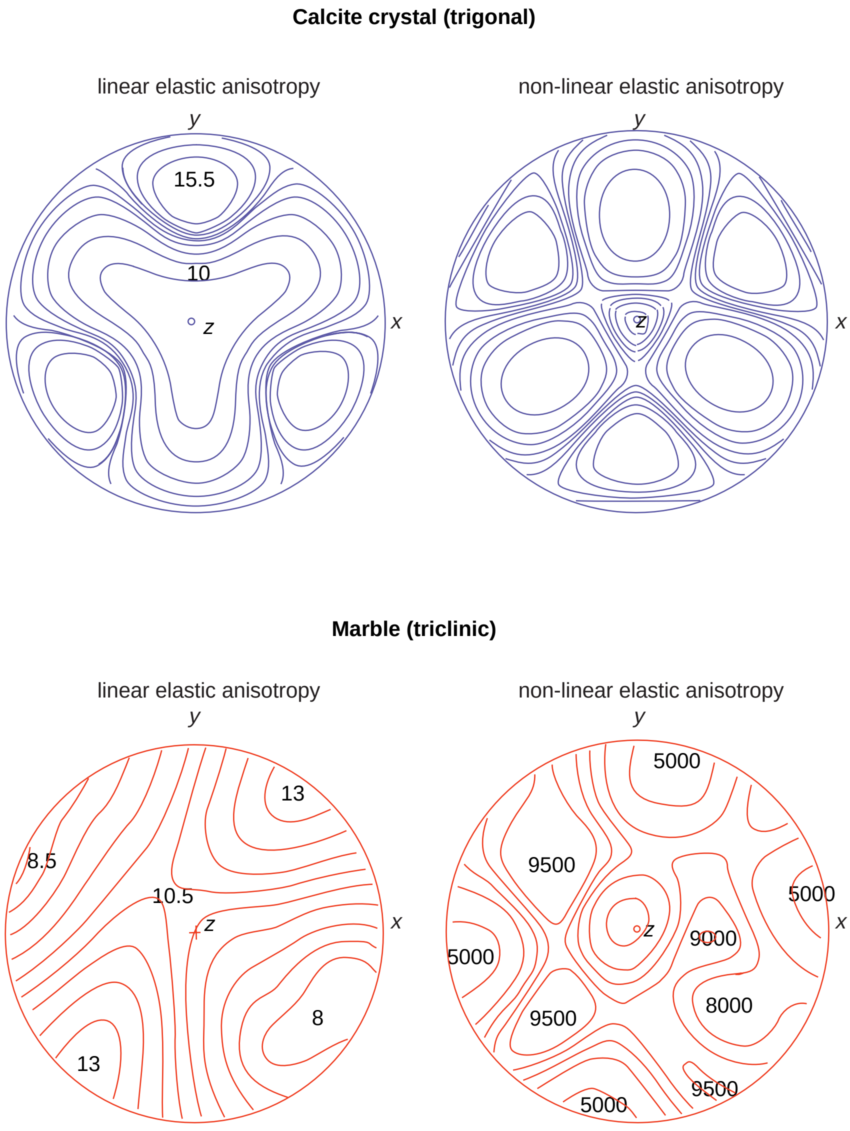


Figure 3

Characteristic surfaces of the rank four stiffness tensors (left) and the rank six stiffness tensors (right) of a calcite single crystal (top) and a marble sample. After Rasolofosaon and Yin (1996).

Calcite $CaCO_3$ trigonal, $\rho = 2710$ kg m^{-3}, $\varepsilon_{ref} = 0.162$

$$C_{pq} = \begin{pmatrix} 144 & 53.9 & 51.1 & -20.5 & 0 & 0 \\ * & (144) & (51.1) & (20.5) & 0 & 0 \\ * & * & 84 & 0 & 0 & 0 \\ * & * & * & 33.5 & 0 & 0 \\ * & * & * & * & (33.5) & (-20.5) \\ * & * & * & * & * & (45.05) \end{pmatrix} \text{ GPa}$$

$$C_{pqr} = \left(\begin{array}{l} \left. \begin{matrix} -579 & -147 & -193 & 218 & 0 & 0 \\ * & (243) & -41 & 10 & 0 & 0 \\ * & * & -239 & 82 & 0 & 0 \\ * & * & * & -69 & 0 & 0 \\ * & * & * & * & -139 & (124) \\ * & * & * & * & * & (-180) \end{matrix} \right) C_{1qr} \\ \\ \left. \begin{matrix} * & * & * & * & * & * \\ * & -675 & (-193) & (-238) & 0 & 0 \\ * & * & (-239) & (-82) & 0 & 0 \\ * & * & * & (-139) & 0 & 0 \\ * & * & * & * & (-69) & (104) \\ * & * & * & * & * & (-84) \end{matrix} \right) C_{2qr} \\ \\ \left. \begin{matrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & -498 & 0 & 0 & 0 \\ * & * & * & -195 & 0 & 0 \\ * & * & * & * & (-195) & (82) \\ * & * & * & * & * & (-76) \end{matrix} \right) C_{3qr} \\ \\ \left. \begin{matrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & 33 & 0 & 0 \\ * & * & * & * & (-33) & (35) \\ * & * & * & * & * & (10) \end{matrix} \right) C_{4qr} \\ \\ \left. \begin{matrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{matrix} \right) C_{5qr} \\ \\ \left. \begin{matrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & 0 \end{matrix} \right) C_{6qr} \end{array} \right) \text{ GPa} \qquad (19)$$

Marble, triclinic, $\rho = 2850$ kg m^{-3}, $\varepsilon_{ref} = 0.000761$

$$C_{pq} = \begin{pmatrix} 29.28 & 6.63 & 6.94 & 0.25 & -1.36 & 3.25 \\ * & 34.20 & 7.01 & 1.40 & -0.69 & 5.13 \\ * & * & 33.70 & 0.95 & -1.76 & 0.99 \\ * & * & * & 14.31 & 1.19 & -0.28 \\ * & * & * & * & 14.84 & -0.53 \\ * & * & * & * & * & 16.91 \end{pmatrix} \text{ GPa}$$

$$C_{pqr} = \begin{pmatrix} -50039 & 48343 & 46376 & 8807 & -5886 & 13119 \\ * & -66477 & -16762 & -2843 & 3820 & -21423 \\ * & * & -66981 & -4190 & 8784 & -8361 \\ * & * & * & -13340 & -4897 & 1279 \\ * & * & * & * & -14285 & -3697 \\ * & * & * & * & * & -12902 \\ & & & & & \\ * & * & * & * & * & * \\ * & -52811 & -54265 & 8160 & -13186 & 18266 \\ * & * & -57764 & -6111 & 4331 & -10300 \\ * & * & * & -10802 & -4226 & 3535 \\ * & * & * & * & -13077 & -2222 \\ * & * & * & * & * & -8884 \\ & & & & & \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & -48555 & 7090 & -8755 & 28286 \\ * & * & * & -9620 & -5347 & 2280 \\ * & * & * & * & -10808 & -25087 \\ * & * & * & * & * & -13384 \\ & & & & & \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & 2437 & -2264 & 6786 \\ * & * & * & * & 1949 & 1437 \\ * & * & * & * & * & 3868 \\ & & & & & \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & -3424 & 7938 \\ * & * & * & * & * & 4838 \\ & & & & & \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & 9182 \end{pmatrix} \textbf{\textit{GPa}} \qquad (20)$$

For the calcite single crystal the standard reference strain is $\varepsilon_{ref} = 0.1262$. This means that one has to expect significant changes in the stiffness tensor only for unrealistically large strains (~1/8). The reference strain for marble is only 0.000761. Strains of this order of magnitude can occur in experiments, thus it is likely that the non-linearity has to be taken into account.

To check this estimates, numerical tests were run with the data for the two materials. In these tests, a test strain parallel and antiparallel to one of the (unstrained) eigenstrains was increased in steps of $10^{-0.25}$ from a level well below the reference strain until one of the break-off criteria was satisfied: either the increase was larger than five orders of magnitude, or one of the six eigenstiffnesses had changed by a factor of $\sqrt{2}$.

For each run, all six eigenstiffnesses and the direction of all six eigenstrains were monitored. The results are shown in the following figures.

calcite, epsilon > 0, eigval

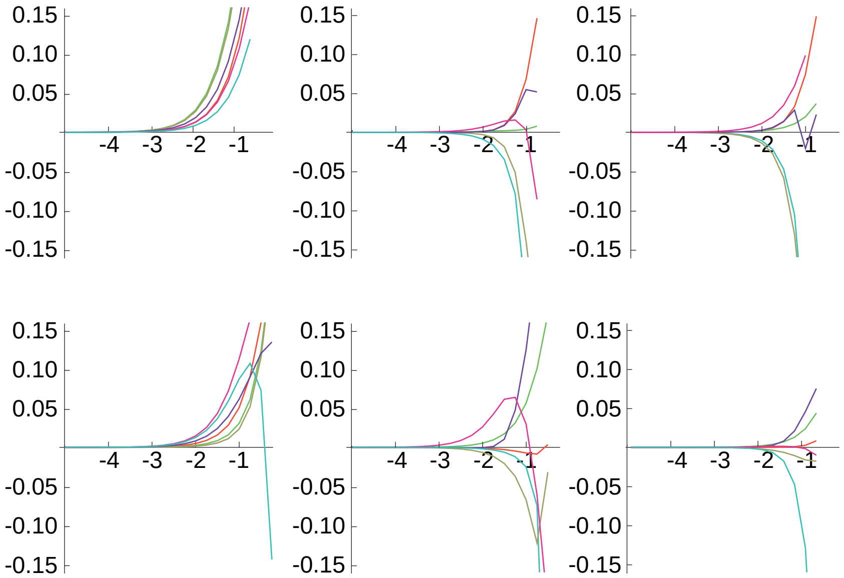


Figure 4a

Relative change of the system of eigenvalues of a calcite single crystal under strains "parallel" to the six eigenstrains. Each panel describes the effect of one eigenstrain, sorted according to the size of the corresponding eigenvalues. Abscissae: logarithm of the relative strain. Ordinates: logarithm of the relative change of the eigenvalues.

calcite, epsilon < 0, eigval

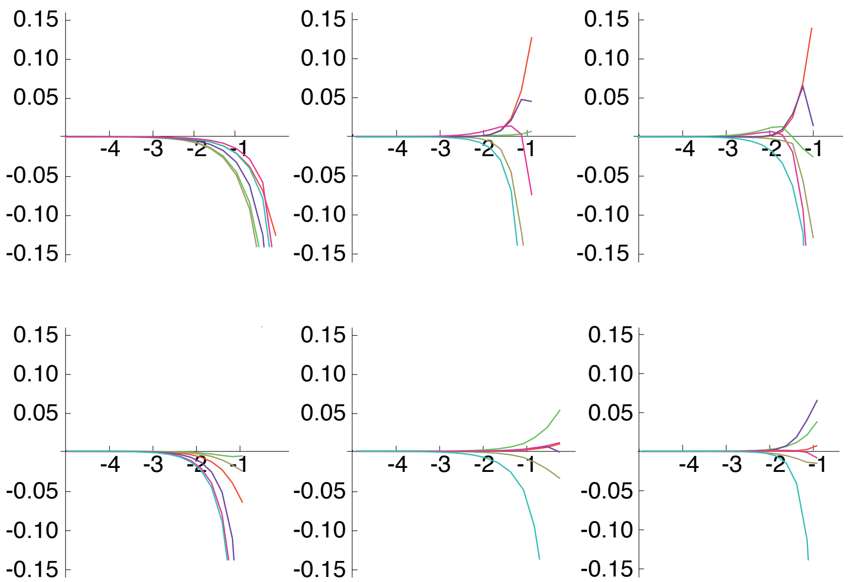


Figure 4b

As in Figure 4a for strains antiparallel to the six eigenstrains.

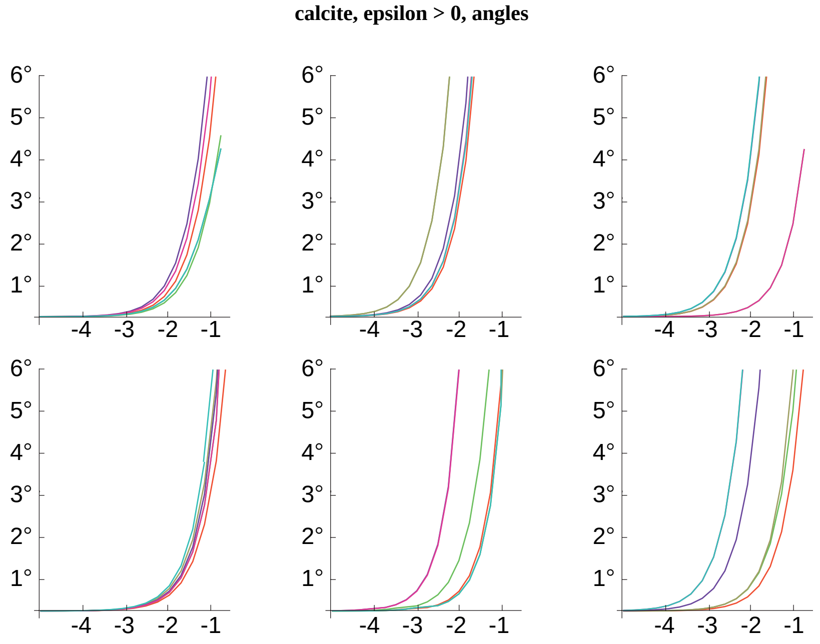


Figure 5a

Change of the direction of eigenstrains of a calcite single crystal under strains "parallel" to the six eigenstrains. Each panel describes the effect of one eigenstrain, sorted according to the size of the corresponding eigenvalues. Abscissae: logarithm of the strain. Ordinates: angle between the eigenstrains (defined as the inverse cosine of the scalar product of two unit vectors).

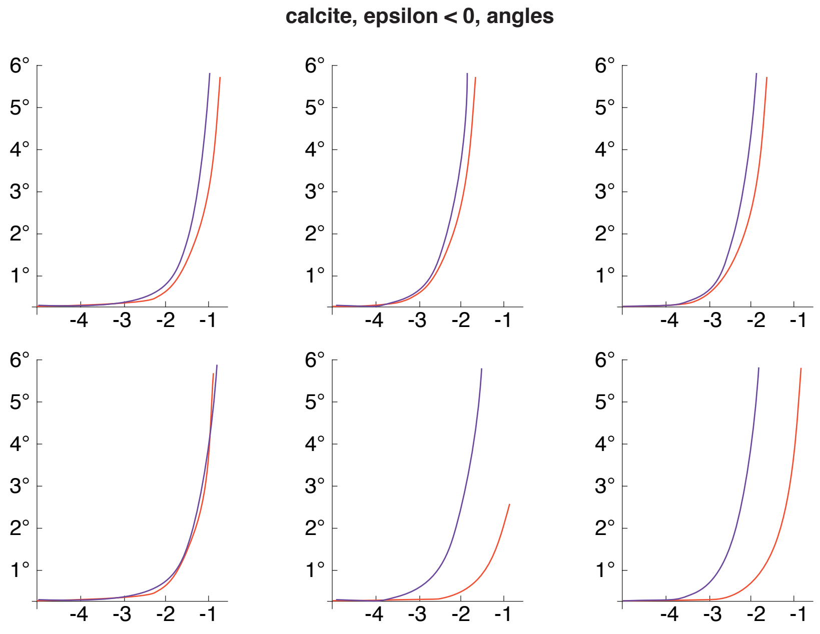


Figure 5b

As in Figure 5a for strains antiparallel to the six eigenstrains.

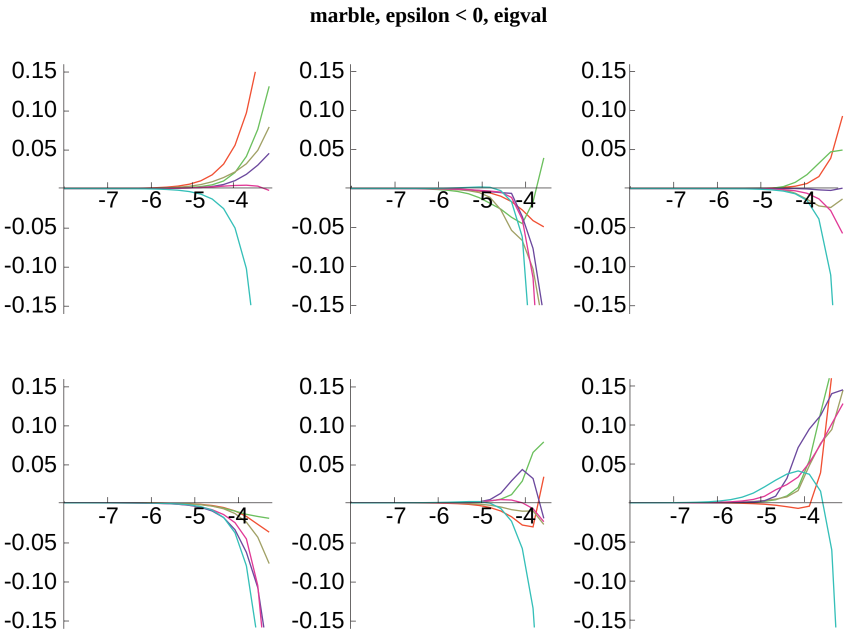


Figure 6a

Relative change of the system of eigenvalues of a triclinic marble sample under strains “parallel” to the six eigenstrains. Each panel describes the effect of one eigenstrain, sorted according to the size of the corresponding eigenvalues. Abscissae: logarithm of the strain. Ordinates: logarithm of the relative change of the eigenvalues.

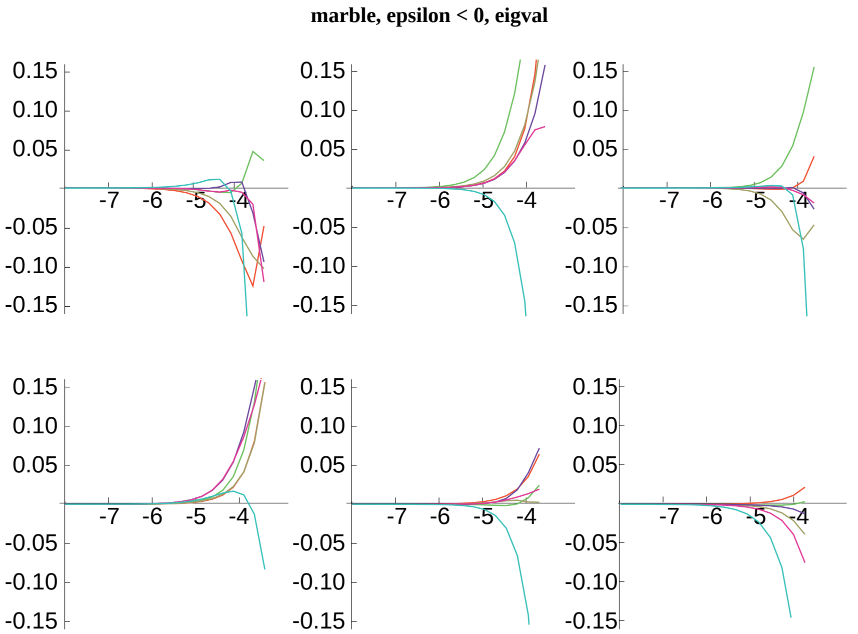


Figure 6b

As in Figure 6a for strains antiparallel to the six eigenstrains.

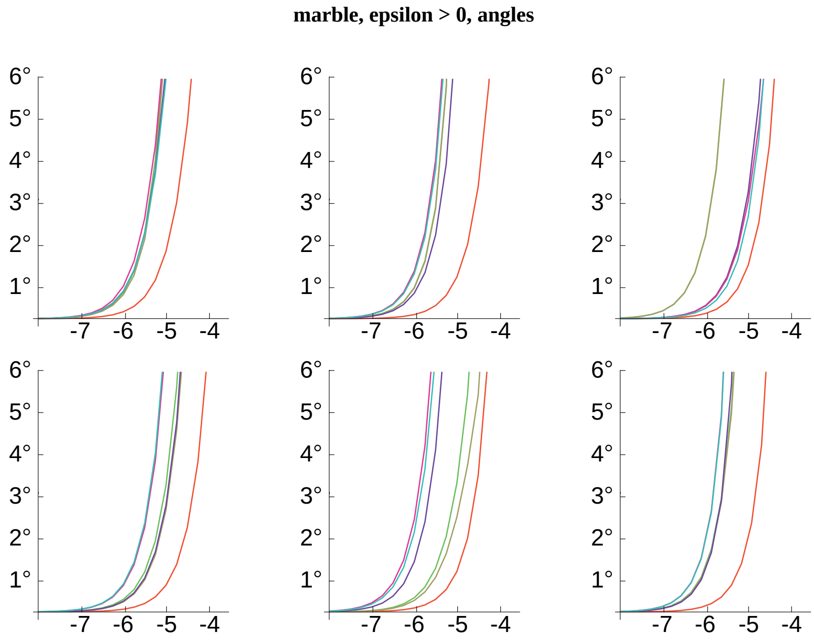


Figure 7a

Change of the direction of eigenstrains of a triclinic marble sample under strains "parallel" to the six eigenstrains. Each panel describes the effect of one eigenstrain, sorted according to the size of the corresponding eigenvalues. Abscissae: logarithm of the strain. Ordinates: angle between the eigenstrains (defined as the inverse cosine of the scalar product of two unit vectors).

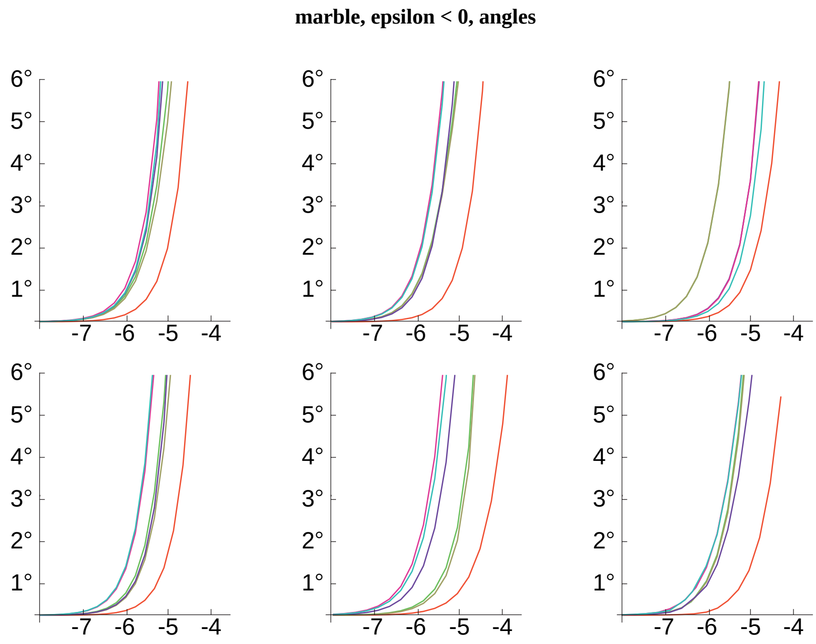


Figure 7b

As in Figure 7a for strains antiparallel to the six eigenstrains.

CONCLUSIONS

There are a few rules of thumb one can tentatively deduce from the two examples:

- At the standard reference strain one has to expect drastic changes of the eigensystem: at a third of the reference strain, eigenstiffnesses may change as much as 25% (corresponding to a decimal logarithm of ± 0.1). The effect on the direction of the eigenstrains is similar: a change of 5° can occur already for 3% of the standard reference strain.
- The 72 deviation curves for the direction of eigenstrains are similar in appearance, but can be separated by a factor of at least ten in strain. Thus an individual test calculation—or a simple estimate based on the reference strain—can be significantly in error. Note that this cautionary remark does not automatically apply to isotropioc media, where the orientation eigentrains are arbitrary.
- The sets of curves for strains "parallel" and "anti parallel" to an eigenstrain can differ significantly both for the magnitude of eigenvalues and for the direction of eigenstrains. Note that an eigenstrain has no a priori direction: after multiplication with -1 it is still an eigenstrain.

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Final manuscript received in July 1998