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A Formalism for the Investigation of Algebraically Special Metrics. I

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Abstract. A new formalism is proposed for the investigation of algebraically special metrics. Among its advantages are that the essential calculations are co-ordinate free and the equations are gauge invariant. The derived equations are simple in form hence easy to work with and the approach is rich in possibilities not explored by previous techniques.

This paper contains no new results but is an introduction to the technique and a demonstration of its use. New results will be presented in further papers.

Introduction

In the past ten years, several techniques for the investigation of Einstein's equations have appeared, notably those associated with the names of Robinson [1], Bondi *et al.* [2], Newman and Penrose [3], and Debever *et al.* [4]. Most of them share the following properties.

1) They are methods of reducing the Einstein equations to what might be called a minimal set of differential equations (MDE) where, with the exception of asymptotic properties, the analysis ends unless the MDE is soluble.

2) They work very well when applied to algebraically special metrics, but with the exception of the above mentioned asymptotic properties are more or less impotent in the face of an algebraically general metric.

3) They represent a formidable notational barrier to the uninitiate.

In this paper, yet another method is proposed. Since it is designed solely for algebraically special metrics it suffers a priori from defect number two. Similarly defect number three may be said to apply, although adherents of the Newman-Penrose (NP) formalism should find the notation comprehensible on first reading. The advantages claimed for the method are that not only is it an efficient method to arrive at the MDE (possibly the most efficient), but also that its usefulness does not end when these prove insoluble.

The system is an algebra of objects of good spin and boost weights and differential operators which are modified versions of those of

Geroch, Held, and Penrose (GHP) [5]. These operators are so chosen that certain integration and differentiation processes may be performed without recourse to a co-ordinate system. As a result, all calculations are done in a co-ordinate free manner and the end result is a set of relations between the variables which will include the effect of the differential operators on these variables. Further calculations may then be possible whether or not these relations are expressed in terms of co-ordinates. This will be demonstrated in a further paper where an analysis of algebraically special metrics possessing Killing vectors will be carried out. Despite the fact that the analysis necessarily involves working with metrics containing unsolved M.D.E.'s, a co-ordinate free expression for a general Killing vector will be obtained. Then, without the introduction of a co-ordinate system, several theorems concerning their properties will be proved.

The efficiency stems partly from the gauge invariance of the formalism (which simplifies the calculations), partly from the extreme simplicity of the resulting relations, and partly from the greatly reduced number of variables required. A co-ordinate system is introduced only if an explicit statement of the metric or of the M.D.E. is required, and then one has the additional advantage that since the solution is in a sense known, the co-ordinate system may be built on the solution rather than the converse.

In this system, unlike the other methods of analysis, the effect of a derivative operator on an element of the algebra is not obtained by a mechanical operation, and it may or may not be known. The object of the calculations is to build up a table of these relations, and a complete knowledge of the effect of the differential operators on the basic elements of the algebra is in general equivalent to a solution. Elements missing from the table lead to non-linearity in the M.D.E. In the case of Type D metrics, it is possible to build up a complete table. More complicated metrics such as those of the Robinson-Trautman type do not yield a complete table and so result in non-linear M.D.E.'s.

The section that follows, as well as describing the technique, contains a brief review of those aspects of the GHP paper which are applicable, so that this paper is essentially self-contained.

In further sections the method is applied to some type *D* metrics and a single metrical form for the six metrics referred to as Case II by Kinnersley [6] is derived. New results obtained through the use of this method will be presented in later papers[1].

[1] A geometrical description of this formalism in the language of fibre bundles is given in the note (by Ehlers) following this paper.

Section 2. The Formalism

The formalisms of GHP and of this paper deal solely with objects which have good spin and boost weights, a concept defined as follows. Consider a standard null tetrad l_a, n_a, m_a where

$$l_a n^a = -m_a \bar{m}^a = 1 \tag{2.1}$$

and all other scalar products vanish. A quantity η is said to have spin and boost weights s and t respectively if, under the tetrad transformations

$$l^a \to \lambda l^a, \quad n^a \to \lambda^{-1} n^a \tag{2.2a}$$

and

$$m^a \to e^{i\theta} m^a, \tag{2.2b}$$

η transforms as

$$\eta \to \lambda^t e^{is\theta} \eta. \tag{2.3}$$

(Note that η may be a tensor of any rank.)

Such quantities are referred to as being of type (p, q), with

$$(p, q) = (t+s, t-s). \tag{2.4}$$

Of the original 12 NP spin coefficients, eight are found to be of good weight. They are $\varrho, \sigma, \kappa, \tau$ and $\varrho'(-\mu), \sigma'(-\lambda), \kappa'(-\nu), \tau'(-\pi)$ where the operation of $'$ is equivalent to the symmetry transformation

$$l^a \leftrightarrow n^a, \quad m^a \leftrightarrow \bar{m}^a. \tag{2.5}$$

(Cf. Ref. [5] for details of this and other symmetry operators available within the formalism.) The remaining four NP spin coefficients are not quantities of good weight and so do not appear per se. Instead in GHP they are combined with the directional derivatives of the tetrad, also not objects of good weight, to form the operators Þ (Thorn), Þ′, ð (Edth), and ð′ which are of good weight.

In the outlook adopted here, the information contained in the four complex scalars $\alpha, \beta, \gamma, \varepsilon$ will appear in two non-gauge invariant vectors $\tilde{\alpha}_a$ (real) and $\tilde{\beta}_a$ (imaginary) where, [cf. (3.3) for definition of Ω^0]

$$\tilde{\alpha}_a = n^b \nabla_a l_b - \left\{ \tau\bar{\tau}\left(\frac{1}{\varrho} + \frac{1}{\bar{\varrho}}\right) + \frac{1}{2}\left(\frac{\Psi_2}{\varrho} + \frac{\bar{\Psi}_2}{\bar{\varrho}}\right)\right\} l_a + \frac{\varrho}{\bar{\varrho}}\bar{\tau} m_a + \frac{\bar{\varrho}}{\varrho}\tau \bar{m}_a \tag{2.6a}$$

$$\tilde{\beta}_a = \bar{m}^b \nabla_a m_b + \left\{ \Omega^0 \tau\bar{\tau} + \frac{1}{2}\left(\frac{\Psi_2}{\varrho} - \frac{\bar{\Psi}_2}{\bar{\varrho}}\right)\right\} l_a - \frac{\varrho}{\bar{\varrho}}\bar{\tau} m_a + \frac{\bar{\varrho}}{\varrho}\tau \bar{m}_a. \tag{2.6b}$$

Consider the differential operator

$$\tilde{\theta}_a = \nabla_a - t\tilde{\alpha}_a + s\tilde{\beta}_a. \tag{2.7}$$

If η is of type (p, q), it is easily shown that $\tilde{\theta}_a \eta$ is also of type (p, q). Therefore $\tilde{\theta}_a$ is of type $(0, 0)$. The operators Þ, $\tilde{Þ}'$, $\tilde{\eth}$, $\tilde{\eth}'$ are defined by

$$\tilde{\theta}_a = n_a Þ + l_a(\tilde{Þ}' + \bar{\tau}\tilde{\eth} + \tau\tilde{\eth}') - \bar{\varrho}\bar{m}_a\tilde{\eth} - \varrho m_a \tilde{\eth}' . \tag{2.8}$$

Since $l_a, n_a, m_a, \bar{m}_a$ are of types $(1, 1)$, $(-1, -1)$, $(1, -1)$, and $(-1, 1)$ respectively, it follows that the types of the operators are $Þ:(1, 1)$, $\tilde{Þ}':(-1, -1)$, $\tilde{\eth}:(0, -2)$, $\tilde{\eth}':(-2, 0)$. Objects of good weight behave as

$$\overline{(p, q)} = (q, p) . \tag{2.9}$$

where the bar represents complex conjugation.

Equation (2.8) forms the connecting link between the tetrad system used throughout the solution of the Einstein equations, and the expression of that solution in a normal co-ordinate system. In addition, together with the choice of gauge, it generates the vectors $\tilde{\alpha}_a$ and $\tilde{\beta}_a$ gratis — as will be seen in Section 5.

The Bianchi identities and those field equations of the NP formalism which have good weight are easily rewritten in terms of Þ, $\tilde{Þ}'$, $\tilde{\eth}$, and $\tilde{\eth}'$. The information contained in the remaining equations appears in the commutator relations of the operators.

Section 3. The Empty Space Algebraically Special Case

Let the vector l^a of the tetrad be chosen to lie in the direction of a degenerate principal null vector. Then by the Goldberg-Sachs theorem $\sigma = \kappa = 0$, and for empty space, Eq. (2.22) of Ref. [3] reduces to

$$Þ\varrho = \varrho^2 . \tag{3.1}$$

Henceforth we assume $\varrho \neq 0$ and put

$$\Omega^\circ = 1/\bar{\varrho} - 1/\varrho . \tag{3.2}$$

Then since Þ is both real and a derivation,

$$Þ\frac{1}{\varrho} = -1 , \tag{3.1a}$$

and

$$Þ\Omega^\circ = 0 . \tag{3.3}$$

A degree sign, $^\circ$, will be used to mark any quantity annihilated by the operator Þ, i.e. $Þ\eta^0 = 0$.

The equations in which the operator Þ occurs together with (3.1) enable most variables, and hence most equations of the problem to be expressed as polynomials in ϱ and $\bar{\varrho}$ with degree marked coefficients.

Since ϱ will in general be complex, one is not entitled to equate coefficients of like powers of ϱ and $\bar{\varrho}$. However, given a polynomial equation with all coefficients degree marked, relations between these coefficients may be extracted. The procedure is illustrated. Consider the equation

$$A^\circ + B^\circ \varrho + C^\circ \bar{\varrho} + D^\circ \varrho\bar{\varrho} + E^\circ \varrho^2 = 0 . \qquad (3.4)$$

Dividing (3.4) by $\varrho^2\bar{\varrho}$ gives

$$A^\circ (\varrho^2\bar{\varrho})^{-1} + B^\circ(\varrho\bar{\varrho})^{-1} + C^\circ(\varrho^2)^{-1} + D^\circ(\varrho)^{-1} + F(\bar{\varrho})^{-1} = 0 . \qquad (3.4a)$$

Operating on (3.4a) with Þ^3 yields

$$A^\circ = 0 . \qquad (3.5a)$$

Substituting this result into (3.4a) and operating with Þ^2,

$$B^\circ + C^\circ = 0 . \qquad (3.5b)$$

Substituting (3.5a) and (3.5b) into (3.4)

$$B^\circ(\varrho - \bar{\varrho}) + D^\circ \varrho\bar{\varrho} + E^\circ \varrho^2 = 0 , \qquad (3.6)$$

and using (3.2) in the form

$$\varrho - \bar{\varrho} = \varrho\bar{\varrho}\Omega^\circ , \qquad (3.2a)$$

(3.4) becomes

$$(\Omega^\circ B^\circ + D^\circ)\, \varrho\bar{\varrho} + E^\circ \varrho^2 = 0 . \qquad (3.7)$$

Dividing by $\varrho^2\bar{\varrho}$ and operating with Þ gives

$$\Omega^\circ B^\circ + D^\circ + E^\circ = 0 . \qquad (3.5c)$$

Finally substituting this into (3.7) and using (3.2a) once more,

$$E^\circ = 0 . \qquad (3.5d)$$

Therefore the results of Eq. (3.4) are

$$A^\circ = B^\circ + C^\circ = \Omega^\circ B^\circ + D^\circ = E^\circ = 0 . \qquad (3.5)$$

Assuming then that one is faced with a polynomial equation, all of whose coefficients are degree marked, the algorithm then is:

1. Equate the coefficients of the lowest power of $\varrho, \bar{\varrho}$ to 0.
2. Substitute the result back into the equation using (3.2a) if applicable. This will raise the lowest power of the equation.
3. Repeat the process till the equation is fully analysed.

The properties of $\tilde{\eth}$, $\tilde{Þ}'$, $\tilde{\eth}'$ essential to this paper are

$$[Þ, \tilde{Þ}']\, \eta^\circ = [Þ, \tilde{\eth}]\, \eta^\circ = [Þ, \tilde{\eth}']\, \eta^\circ = 0 . \qquad (3.6)$$

(Additional properties will be demonstrated in a further paper.)

By virtue of these relations, the fundamental equations, when rewritten using these operators, give rise to polynomial equations which will admit of the analysis outlined above.

It is important to note that the symmetry between l^a and n^a has been broken, and unlike the operators of GHP,

$$(\tilde{\eth})' \neq \tilde{\eth}' \quad \text{and} \quad (\text{Þ})' \neq \tilde{\text{Þ}}' \,. \tag{3.7}$$

However there remains

$$\bar{\tilde{\eth}} = \tilde{\eth}' \quad \text{and} \quad \bar{\tilde{\text{Þ}}}' = \tilde{\text{Þ}}' \,. \tag{3.8}$$

It would appear that the equations should be more complex when expressed in terms of these new operators but in fact, as seen below, this is not necessarily the case, and when solving equations a considerable saving of labour is effected. The equations of GHP written in terms of the operators (2.8) are presented below.

Section 4

The Field equations, Bianchi Identities, and commutators adapted to an algebraically special free space metric.

Field Equations

$$\text{Þ}\varrho = \varrho^2 \tag{4.1a}$$

$$\text{Þ}\tau = \varrho(\tau - \bar{\tau}') \tag{4.1b}$$

$$\begin{aligned}\text{Þ}\kappa' - \tilde{\text{Þ}}'\tau' - \bar{\tau}\tilde{\eth}\tau' - \tau\tilde{\eth}'\tau' = {} & \tfrac{1}{2}\left(\frac{\Psi_2}{\varrho} - \frac{\bar{\Psi}_2}{\bar{\varrho}}\right)\tau' + \Omega^\circ\tau\bar{\tau}\tau' + \varrho'(\bar{\tau} - \tau') \\ & + \sigma'(\tau - \bar{\tau}') - \Psi_3\end{aligned} \tag{4.1c}$$

$$\text{Þ}\sigma' - \varrho\tilde{\eth}'\tau' = \varrho\sigma' - \tau'^2 + \varrho\bar{\tau}\tau'/\bar{\varrho} \tag{4.1e}$$

$$\text{Þ}\varrho' - \bar{\varrho}\tilde{\eth}\tau' = \varrho'\bar{\varrho} - (\bar{\tau}' + \varrho\tau/\bar{\varrho})\tau' - \Psi_2 \tag{4.1f}$$

$$\tilde{\eth}\tau = \Omega^\circ\tau^2 - \bar{\sigma}'\varrho/\bar{\varrho} \tag{4.2a}$$

$$\tilde{\eth}\varrho = \bar{\varrho}\tau/\varrho \tag{4.2b}$$

$$\begin{aligned}\varrho\tilde{\eth}'\varrho' - \bar{\varrho}\tilde{\eth}\sigma' = {} & -\bar{\varrho}\bar{\tau}\varrho'/\varrho - \varrho\tau\sigma'/\bar{\varrho} + (\varrho' - \bar{\varrho}')\,\tau' \\ & + (\bar{\varrho} - \varrho)\,\kappa' - \Psi_3\end{aligned} \tag{4.2c}$$

$$\begin{aligned}\tilde{\text{Þ}}'\varrho' + \bar{\tau}\tilde{\eth}\varrho' + \tau\tilde{\eth}'\varrho' - \bar{\varrho}\tilde{\eth}\kappa' = {} & -\left(\frac{1}{\varrho} + \frac{1}{\bar{\varrho}}\right)\bar{\tau}\tau\varrho' - \tfrac{1}{2}\left(\frac{\Psi_2}{\varrho} + \frac{\bar{\Psi}_2}{\bar{\varrho}}\right)\varrho' \\ & - \bar{\varrho}\Omega^\circ\tau\kappa' + \varrho'^2 + \sigma'\bar{\sigma}' - \bar{\kappa}'\tau'\end{aligned} \tag{4.3a}$$

$$\begin{aligned}\tilde{\text{Þ}}'\sigma' + \bar{\tau}\tilde{\eth}\sigma' + \tau\tilde{\eth}'\sigma' - \varrho\tilde{\eth}'\kappa' = {} & -\tfrac{1}{2}\left(3\frac{\Psi_2}{\varrho} - \frac{\bar{\Psi}_2}{\bar{\varrho}}\right)\sigma' + \left(\frac{1}{\varrho} - \frac{3}{\bar{\varrho}}\right)\tau\bar{\tau}\sigma' \\ & + \left(3\frac{\varrho}{\bar{\varrho}} - 1\right)\bar{\tau}\kappa' + (\varrho' + \bar{\varrho}')\,\sigma' - \tau'\kappa' + \Psi_4\end{aligned} \tag{4.3b}$$

$$\tilde{\text{Þ}}'\varrho + \bar{\tau}\tilde{\eth}\varrho + \tau\tilde{\eth}'\varrho - \varrho\tilde{\eth}'\tau = -\tfrac{1}{2}\left(\frac{\Psi_2}{\varrho} - \frac{\bar{\Psi}_2}{\bar{\varrho}}\right)\varrho + \varrho\bar{\varrho}' \,. \tag{4.3c}$$

Bianchi Identities

$$Þ\Psi_2 = 3\varrho\Psi_2 \tag{4.4a}$$

$$Þ\Psi_3 - \varrho\tilde{ð}'\Psi_2 = -3\tau'\Psi_2 + 2\varrho\Psi_3 \tag{4.4b}$$

$$Þ\Psi_4 - \varrho\tilde{ð}'\Psi_3 = 3\sigma'\Psi_2 + \left(2\frac{\varrho}{\bar{\varrho}}\bar{\tau} - 4\tau'\right)\Psi_3 + \varrho\Psi_4 \tag{4.4c}$$

$$\tilde{ð}\Psi_2 = 3\frac{\tau}{\bar{\varrho}}\Psi_2 \tag{4.5a}$$

$$\tilde{Þ}'\Psi_2 + \bar{\tau}\tilde{ð}\Psi_2 + \tau\tilde{ð}'\Psi_2 - \bar{\varrho}\tilde{ð}\Psi_3 = 3\varrho'\Psi_2 - 2\tau\Psi_3 \tag{4.6a}$$

$$\tilde{Þ}'\Psi_3 + \bar{\tau}\tilde{ð}\Psi_3 + \tau\tilde{ð}'\Psi_3 - \bar{\varrho}\tilde{ð}\Psi_4 = -3\kappa'\Psi_2 + \left(4\varrho' - 2\frac{\tau\bar{\tau}}{\bar{\varrho}} - \frac{\Psi_2}{\varrho}\right)\Psi_3 - \tau\Psi_4 . \tag{4.6b}$$

Commutators

$$[Þ, \tilde{Þ}'] = \left(\frac{\bar{\tau}\bar{\tau}'}{\bar{\varrho}} + \frac{\tau\tau'}{\varrho} - \frac{1}{2}\frac{\Psi_2}{\varrho} - \frac{1}{2}\frac{\bar{\Psi}_2}{\bar{\varrho}}\right)Þ \tag{4.7a}$$

$$[Þ, \tilde{ð}] = -\left(\frac{\tau}{\varrho} + \frac{\bar{\tau}'}{\bar{\varrho}}\right)Þ \tag{4.7b}$$

$$[\tilde{ð}, \tilde{ð}'] = \frac{1}{\varrho\bar{\varrho}}(\bar{\varrho}' - \varrho')\,Þ + \Omega^\circ\tilde{Þ}' + p\Sigma^\circ - q\bar{\Sigma}^\circ \tag{4.7c}$$

where

$$\Sigma^\circ = \left(\frac{\varrho'}{\bar{\varrho}} + \frac{\Psi_2}{2\varrho}\left(\frac{1}{\varrho} + \frac{1}{\bar{\varrho}}\right) + \tilde{ð}\left(\frac{\bar{\tau}}{\bar{\varrho}}\right)\right) \tag{4.8}$$

$$[\tilde{Þ}', \tilde{ð}'] = \left(\frac{\bar{\tau}}{\varrho\bar{\varrho}}(\varrho' - \bar{\varrho}') - \frac{\kappa'}{\varrho}\right)Þ + p\Gamma^\circ \tag{4.7d}$$

where

$$\Gamma^\circ = \left(\kappa' + \frac{\Psi_3}{\varrho} + \tilde{Þ}'\left(\frac{\bar{\tau}}{\bar{\varrho}}\right) - \frac{\bar{\tau}\Psi_2}{\varrho\bar{\varrho}} - \frac{\bar{\tau}\varrho'}{\bar{\varrho}} - \frac{1}{2}\tilde{ð}'\left(\frac{\Psi_2}{\varrho}\right) - \frac{\sigma'\tau}{\varrho}\right). \tag{4.9}$$

The two remaining commutators are obtained by complex conjugation of (4.7b) and (4.7d).

Section 5. A Class of Type D Solutions

In this section the Formalism is used to set up the equations associated with a class of Type D metrics. Since the metric is of Type D, both l^a and n^a may be chosen to lie in the direction of degenerate principal null vectors, so that

$$\kappa = \sigma = \kappa' = \sigma' = \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 . \tag{5.1}$$

The metric is then further restricted by requiring that

$$\tau = \varrho\bar{\varrho}\,\tau^\circ\,, \tag{5.2}$$

a class of solutions called Case II by Kinnersley [6].

In order to do the analysis, expressions for $\tilde{\eth}\varrho$, $\tilde{\eth}'\varrho$, and $\tilde{\text{Þ}}'\varrho$ are required. Of these, the first and the third are readily available from the equations of Section 4. To obtain the second, the expression

$$[\text{Þ}, \tilde{\eth}']\,\Psi_2 \tag{5.3}$$

is calculated first by successive applications of the operators, then by use of the appropriate commutator. Equating the two results in

$$\text{Þ}\tau' - \varrho\tilde{\eth}'\varrho = -\frac{\varrho^2}{\bar{\varrho}}\bar{\tau}\,. \tag{5.4}$$

When considered in its natural order, this equation will yield the factor $\tilde{\eth}'\varrho$.

The integration procedure is straightforward. Using assumption (5.2), Eq. (4.1b) gives

$$\tau' = -\varrho^2\bar{\tau}^\circ\,, \tag{5.5}$$

and Eq. (4.2b)

$$\tilde{\eth}\varrho = \varrho^2\tau^\circ\,. \tag{5.6}$$

Then substituting (5.5) into (5.4) yields

$$\tilde{\eth}'\varrho = -\varrho^2\bar{\tau}^\circ\,. \tag{5.7}$$

(5.6) and the complex conjugate of (5.7) give the useful result

$$\tilde{\eth}\Omega^\circ = 2\tau^\circ\,. \tag{5.8}$$

From the Bianchi identity (4.4a)

$$\Psi = \varrho^3\Psi^\circ\,. \tag{5.9}$$

(Since there is no possibility of confusion, for the balance of the paper the Subscript 2 will be dropped from Ψ_2.)

Using these results Eq. (4.1f) may be rewritten as

$$\text{Þ}\frac{\varrho'}{\bar{\varrho}} = -2\varrho^3\tau^\circ\bar{\tau}^\circ - \varrho^2\tilde{\eth}\bar{\tau}^\circ - \frac{\varrho^3}{\bar{\varrho}}\Psi^\circ \tag{5.10}$$

and integrated to give

$$\varrho' = \bar{\varrho}\varrho'^\circ - 1/2\,\varrho^2\Psi^\circ - \varrho\bar{\varrho}(\tilde{\eth}\bar{\tau}^\circ + 1/2\,\Psi^\circ) - \varrho^2\bar{\varrho}\tau^\circ\bar{\tau}^\circ\,. \tag{5.11}$$

The "thorn" integrations, which correspond to the D integrations of the NP formalism, are now complete and all the variables have been expanded as polynomials in ϱ and $\bar{\varrho}$ with degree marked coefficients.

For the balance of this section only the equation numbers and new results will be indicated. Equation (4.2c) is worked through in detail in the appendix.

$$(4.2a) \to \tilde{\eth}\tau^\circ = 0 \qquad (5.12)$$

$$(4.4b) \to \tilde{\eth}'\Psi^\circ = 0 \qquad (5.13)$$

$$(4.6a) \to \tilde{\eth}\Psi^\circ = 0 \qquad (5.14)$$

$$(4.2c) \to \tilde{\eth}'\varrho'^\circ = 0 \qquad (5.15)$$

$$\tilde{\text{Þ}}'\bar{\tau}^\circ = 0 \qquad (5.16)$$

$$\tilde{\eth}\bar{\tau}^\circ = -1/2\Omega^\circ(\varrho'^\circ + \bar{\varrho}'^\circ) + 1/2(\bar{\Psi}^\circ - \Psi^\circ) \qquad (5.17)$$

$$\varrho'^\circ = \bar{\varrho}'^\circ \qquad (5.18)$$

$$(4.6a) \to \tilde{\text{Þ}}'\Psi^\circ = 0 \qquad (5.19)$$

$$(4.3a) \to \tilde{\text{Þ}}'\varrho'^\circ = 0 \qquad (5.20)$$

$$(4.3c) \to \tilde{\text{Þ}}'\varrho = \varrho^2\varrho'^\circ - 1/2\varrho^2(\varrho\Psi^\circ + \bar{\varrho}\bar{\Psi}^\circ) - \varrho^3\bar{\varrho}\tau^\circ\bar{\tau}^\circ). \qquad (5.21)$$

Finally, using this last equation and its complex conjugate,

$$\tilde{\text{Þ}}'\Omega^\circ = 0. \qquad (5.22)$$

This completes the integration. A complete table of the action of the derivative operators on the variables ϱ, Ω°, ϱ'°, Ψ°, τ°, has now been established. The remaining equations and the commutators merely confirm the above.

The next step is to obtain expressions for the vectors $\tilde{\alpha}_a$ and $\tilde{\beta}_a$. This is done through judicious choice of gauge. Noting that

$$\text{Þ}\varrho'^\circ = \tilde{\text{Þ}}'\varrho'^\circ = \tilde{\eth}\varrho'^\circ = \tilde{\eth}'\varrho'^\circ = 0$$

and that ϱ'° is of type $(-2, -2)$, (2.8) applied to ϱ'° becomes

$$(\nabla_a + 2\tilde{\alpha}_a)\,\varrho'^\circ = 0. \qquad (5.23)$$

Since under the transformation (2.2a),

$$\varrho'^\circ \to \lambda^{-2}\varrho'^\circ \qquad (5.24)$$

choosing $= \lambda = |\varrho'^\circ|^{\frac{1}{2}}$ sets

$$\varrho'^\circ = \pm 1. \qquad (5.25)$$

From (5.23) it follows that with this choice of gauge $\tilde{\alpha}_a = 0$.

It follows immediately that

$$\nabla_a\Psi^\circ = 0, \qquad (5.26)$$

so that

$$\Psi^\circ = \Psi_R + \mathrm{i}\,\Psi_I = \text{constant}. \qquad (5.27)$$

If $\varrho'^\circ=0$, $\tilde{\alpha}_a=0$ is obtained by setting either Ψ_R or Ψ_I, one of which must be non-vanishing, equal to a constant.

Proceeding along similar lines for Ω° and ϱ (in the case of ϱ it is convenient to work with $1/\varrho+1/\bar{\varrho}$ since the imaginary part of $1/\varrho$ appears as Ω°),

$$\nabla_a\Omega^\circ=2\varrho\bar{\tau}^\circ m_a-2\bar{\varrho}\tau^\circ\bar{m}_a \tag{5.28}$$

and

$$\nabla_a\left(\frac{1}{\varrho}+\frac{1}{\bar{\varrho}}\right)=-2Tl_a-2n_a \tag{5.29}$$

where

$$T=\varrho'^\circ-\tfrac{1}{2}\varrho\Psi^\circ-\tfrac{1}{2}\bar{\varrho}\bar{\Psi}^\circ-\varrho\bar{\varrho}\tau^{\circ 2}. \tag{5.30}$$

Finally for τ°,

$$\begin{aligned}(\nabla_a+\tilde{\beta}_a)\tau^\circ=&-(\varrho\Omega^\circ\varrho'^\circ+i\varrho\Psi_I-\varrho^2\bar{\tau}^\circ\tau^\circ)m_a-\bar{\varrho}^2\tau^{\circ 2}\bar{m}_a\\ &\{\varrho\bar{\varrho}(\Omega^\circ\varrho'^\circ+i\Psi_I)-\tau^\circ\bar{\tau}^\circ(\varrho^2\bar{\varrho}-\varrho\bar{\varrho}^2)-\tfrac{1}{2}(\varrho^2\Psi^\circ-\bar{\varrho}^2\bar{\Psi}^\circ)\}\tau^\circ l_a.\end{aligned} \tag{5.31}$$

As τ° is of type $(-3,-1)$, under the transformation (2.2b)

$$\tau^\circ\rightarrow e^{-i\theta}\tau^\circ. \tag{5.32}$$

Choosing θ such that τ° is real allows (5.31) to be separated into its real and imaginary parts. The real part is

$$\nabla_a\tau^\circ=-\tfrac{1}{2}\varrho(\Omega^\circ\varrho'^\circ+i\Psi_I^\circ)m_a+\tfrac{1}{2}\bar{\varrho}(\Omega^\circ\varrho'^\circ+i\Psi_I)\bar{m}_a. \tag{5.33}$$

The vector $\tilde{\beta}_a$, if desired, may be obtained from the imaginary part of (5.31). The tetrad is now determined to within a two dimensional reflection.

Finally (5.28) and (5.33) are combined as

$$\tfrac{1}{2}(\varrho'^\circ\Omega^\circ+i\Psi_I)\nabla_a\Omega^\circ=-2\tau^\circ\nabla_a\tau^\circ \tag{5.34}$$

and integrate to give

$$(\Omega^\circ\varrho'^\circ+i\Psi_I)^2=\pm 4a^2-4\varrho'^\circ\tau^{\circ 2} \tag{5.35}$$

where $\pm 4a^2$ is a constant of integration.

The "co-ordinate free integration" part of the procedure is now complete. We have established a complete table of the action of the operators on the basic variables ϱ and τ°. It remains now to pick a co-ordinate system and in it describe a properly normalized tetrad such that Eqs. (5.28), (5.29) and the other derived relationships, all of which have been expressed in a co-ordinate free manner, are satisfied. We know from the work of Talbot [8] that for an empty space algebraically special metric formally the tetrad must exist. He has given a prescription for writing out a tetrad for the most general possible such metric in terms of several unknown functions together with equations (the M.D.E.'s) relating these functions (but no proof of the solubility of those equations). Therefore his tetrad, with suitable simplifications, will suffice to describe a less general metric and will naturally satisfy the derived co-ordinate free relations. However as was indicated in the introduction, one of the advantages of the technique under discussion is that we can base our co-ordinate system on a known solution. Therefore we will not use the Talbot co-ordinate system but rather build one on the results of this section.

For the construction of a co-ordinate system, the optimal situation arises if the co-ordinate free integration procedure yields a complete involutive table of the operators on three linearly independent complex functions, all of which have $p \neq \pm q$. In this case, one of these functions is set equal to a real (or imaginary) constant using the transformations (2.2). This gives an expression for the vectors $\tilde{\alpha}_a$ and $\tilde{\beta}_a$ in terms of the tetrad vectors and the other two functions [see statement following (5.33)]. The four co-ordinate functions are then chosen as the real and imaginary parts of the remaining two functions. Equation (2.8) (which is an identity!) is then applied to each of the co-ordinate functions. This yields the tetrad components directly [cf. Eqs. (6.11) and (6.12)].

Variations on this theme arise if one or more of the functions has $p = \pm q$. Then either additional functions are required or a method must be found to generate additional co-ordinate surfaces. In the case of the metric under consideration we have three basic functions ϱ, τ°, and ϱ'°. These are not optimal as ϱ'° has $p = q$ and the imaginary part of ϱ and τ° are not linearly independent. We have used ϱ'° to establish $\tilde{\alpha}_a = 0$, and the phase freedom of τ° to establish $\tilde{\beta}_a$. There remains only ϱ, which can supply but two co-ordinate surfaces. Therefore we must choose a further two co-ordinate surfaces and develop the attendant tetrad components. This is done in the next section.

Complications arise if the table of differential operations is not complete. In this case an unknown function may be substituted for the missing element(s) and the procedure carried on as before. The missing element(s) is then calculated directly from the final form of the tetrad, giving rise to a non-linear M.D.E. for the substituted function. Complications may also arise if the smallest involutive system of the operators involves more than three variables. The basic procedure is the same as for the missing element case and the results are again M.D.E.'s.

Section 6. The Development of a Tetrad

The co-ordinate system used is that of Ref. [7]. With this choice, l^a and l_a take the forms

$$l^a = (0, 0, 0, 1) \tag{6.1}$$

$$l_a = (x^3, 1, 0, 0) \tag{6.2}$$

where x^4 is an affine parameter and x^3 may be any real function which obeys [7]

$$l_{[a,b}x^3{}_{,c]}=0\,. \tag{6.3}$$

Let the affine parameter be chosen as follows. Set

$$x^4=r=-\tfrac{1}{2}\left[\frac{1}{\varrho}+\frac{1}{\bar{\varrho}}\right]. \tag{6.4}$$

Then from (5.33)

$$\nabla_a r=Tl_a+n_a\,. \tag{6.5}$$

Since $n^a\underset{\cdot}{\alpha}_a=0$, l^a is affinely paramatrized and transvecting (6.5) with l^a shows that r is indeed an affine parameter.

Using the results of Section V it can be shown that

$$l_{[a,b}\Omega^\circ{}_{,c]}=0\,, \tag{6.7}$$

so that x^3 may be defined by

$$x^3=i\Omega^\circ\,. \tag{6.8}$$

With these choices ϱ has the simple form

$$\varrho=-\left[r+i\frac{x^3}{2}\right]^{-1}. \tag{6.9}$$

The remaining co-ordinate freedom is [7]

$$x^1\to x^1+f(x^3)\,. \tag{6.10}$$

Therefore

$$\delta_a^4=Tl_a+n_a \tag{6.11}$$

$$i\delta_a^3=2\varrho\,\tau^\circ m_a-2\bar{\varrho}\tau^\circ\bar{m}_a\,. \tag{6.12}$$

It is now a reasonably simple matter to construct the tetrad. The values of l^3, l^4, n^3, n^4, m^3, m^4, n_1, n_2, n_3, n_4 may be read directly from (6.11) and (6.12). Taking into account the orthogonality relations one obtains

$$n_a=(-Tx^3,\,-T,0,1) \tag{6.13}$$

$$n^a=(n^1,1-n^1x^3,0,T) \tag{6.14}$$

$$m_a=\left(\bar{\varrho}m_2\left[x^3-\frac{1}{n^2}\right],\bar{\varrho}m_2,i\frac{m^1m_2}{2n^1\tau^\circ},0\right) \tag{6.15}$$

$$m^a=(m^1,\,-x^3m^1,\,-2i\bar{\varrho}\tau^\circ,0) \tag{6.16}$$

plus the equation

$$\varrho m^1+\bar{\varrho}\bar{m}^1=\frac{n^1}{m_2}\,. \tag{6.17}$$

Transvecting the relation

$$l_{[a,b]} = \delta_a{}^1 \delta_b{}^3 - \delta_a{}^3 \delta_b{}^1 \tag{6.18}$$

with

$$m^a n^b \quad \text{and} \quad m^a \bar{m}^b$$

gives

$$n^1 = \varrho\bar{\varrho}\frac{x^3}{2} \tag{6.19}$$

and

$$m_2 = \tau^\circ, \tag{6.20}$$

and m^1 is reduced to the form

$$m^1 = \bar{\varrho}\left[\frac{x^3}{4\tau^\circ} + iv\right] \tag{6.21}$$

where v is an as yet undetermined function.

Since the co-ordinate system itself contains the freedom (6.10), it is not surprising that the tetrad components are not uniquely defined. This ambiguity is removed by applying the commutators to x^1.

$$[\text{Þ}, \tilde{\eth}]\, x^1 \to v = v^\circ \tag{6.22a}$$

$$[\tilde{\text{Þ}}', \tilde{\eth}]\, x^1 \to v = v^\circ(x^1, x^3) \tag{6.22b}$$

$$[\tilde{\eth}', \tilde{\eth}]\, x^1 \to v = -\frac{2\tau^\circ x^1}{x^3} + F(x^3)\,. \tag{6.22c}$$

The function $F(x^3)$ is then absorbed using the remaining co-ordinate freedom (6.10) and the final form of the tetrad is

$$l^a = (0, 0, 0, 1) \tag{6.23a}$$

$$n^a = \left(\varrho\bar{\varrho}\frac{x^3}{2}, 1 - \varrho\bar{\varrho}\frac{(x^3)^2}{2}, 0, T\right) \tag{6.23b}$$

$$m^a = \left(\bar{\varrho}\left[\frac{x^3}{4\tau^\circ} - i\frac{2\tau^\circ x^1}{x^3}\right], -\bar{\varrho}x^3\frac{x^3}{4\tau^\circ} - i\frac{2\tau^\circ x^1}{x^3}, -2i\bar{\varrho}\,\tau^\circ, 0\right) \tag{6.23c}$$

$$l_a = (x^3, 1, 0, 0) \tag{6.24a}$$

$$n_a = (-Tx^3, -T, 0, 1) \tag{6.24b}$$

$$m_a = \left(\bar{\varrho}\tau^\circ\left[x^3 - \frac{2}{\varrho\bar{\varrho}x^3}\right], \bar{\varrho}\tau^\circ, \frac{1}{\varrho}\frac{2\tau^\circ x^1}{(x^3)^2} + \frac{i}{4\tau^\circ}, 0\right) \tag{6.24c}$$

where

$$\tau^{\circ 2} = \frac{1}{4\varrho'^\circ}\left[\varrho'^\circ\frac{x^3}{2} + \Psi_1\right]^2 \pm a^2 \tag{6.25}$$

$$T = \varrho'^\circ + \varrho\bar{\varrho}\left[r\Psi_R - \frac{x^3}{2}\Psi_I - \tau^{\circ 2}\right] \tag{6.26}$$

$$\varrho'^\circ = \pm 1\,, \tag{6.27}$$

and a^2, Ψ_R, Ψ_I are arbitrary real constants.

If $\varrho'^{\circ} = 0$, $\Psi_I \neq 0$, the metrical form remains the same. Equation (5.38) simplifies to

$$\frac{i}{2} \nabla_a \Omega^{\circ} = 2\tau^{\circ} \nabla_a \tau^{\circ} \tag{6.28}$$

and τ° is given by

$$\tau^{\circ 2} = x^3 \pm a^2 . \tag{6.29}$$

Finally if $\varrho'^{\circ} = \Psi_I = 0$, Ψ_R is set equal to one by choice of gauge and Eq. (5.35) indicates that

$$\tau^{\circ} = \text{constant} . \tag{6.30}$$

The metric g_{ab} is defined by

$$g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)} . \tag{6.31}$$

The co-ordinate x^2 does not appear in the metric, therefore the the vector $\partial/\partial x^2$ is a Killing vector. From Ref. [6] we know that these metrics possess a second Killing vector. A derivation of an alternative co-ordinate system which displays the existence of the second Killing vector directly has been carried out by Stewart and Walker [9].

Conclusions

The technique of GHP has been adapted to the problem of algebraically special metrics. It was shown that when applied to a class of Type D metrics it resulted in equations of an exceptionally simple form. Through the use of the auxilliary vectors $\tilde{\alpha}_a$ and $\tilde{\beta}_a$ equations relating the tetrad system and all co-ordinate systems were constructed. This enabled a co-ordinate system to be chosen which took maximum advantage of the known solution. Having done this, the tetrad was then easily calculated.

The metric derived reproduces the six case II solutions of Kinnersley. It was shown to have four arbitrary constants, one of which (ϱ'°) had been set equal to ± 1 by appropriate choice of gauge. The space is influenced by the change of sign of two of these, Ψ_R and Ψ_I, only to the extent apparent in the metric. However, the signs of ϱ'° and $\pm a^2$ extend deeper and determine the underlying topology as well as the obvious local change in the metric. This effect is reflected in the allowable range of the co-ordinate x^3 and is best demonstrated by performing the co-ordinate transformation

$$y^3 = 1/4\varrho'^{\circ} x^3 + 1/2\,\Psi_I \tag{7.1}$$

and substituting into Eq. (6.25). This then appears as

$$\tau^{\circ 2} = \varrho'^{\circ}(y^3)^2 \pm a^2\,, \tag{7.2}$$

with the three different topologies being determined by

$$\text{I.}\ \ \varrho'^{\circ} = +1,\ +a^2 \rightarrow -\infty < y^3 < +\infty \tag{7.3}$$

$$\text{II.}\ \ \varrho'^{\circ} = +1,\ -a^2 \rightarrow -\infty < y \leqq -a,\ a \leqq y < +\infty \tag{7.4}$$

$$\text{III.}\ \ \varrho'^{\circ} = -1,\ +a^2 \rightarrow -a \leqq y \leqq a\,. \tag{7.5}$$

A detailed discussion of the physics of this class of metric is found in Ref. [6].

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Appendix

In this appendix the integration of Eq. (4.2c) is demonstrated in detail.

The equation may be rewritten as

$$\tilde{\eth}'\varrho' = \varrho\bar{\tau}^{\circ}(\bar{\varrho}' - 2\varrho')\,. \tag{A-1}$$

Using (5.11), operating through with $\tilde{\eth}'$ and using the commutator $[\tilde{\eth}', \tilde{\eth}]$, where required yields

$$\begin{aligned}
&\bar{\varrho}\tilde{\eth}'\varrho'^{\circ} + \varrho\bar{\varrho}(\Omega^{\circ}\tilde{\text{Þ}}'\bar{\tau}^{\circ} - 3\varrho'^{\circ}\bar{\tau}^{\circ} + \bar{\varrho}'^{\circ}\bar{\tau}^{\circ}) + \bar{\varrho}^2\varrho'^{\circ}\bar{\tau}^{\circ}\\
&\quad + \varrho^3\bar{\tau}^{\circ}\Psi^{\circ} + \varrho^2\bar{\varrho}\bar{\tau}^{\circ}(\tilde{\eth}\bar{\tau}^{\circ} + 1/2\,\Psi^{\circ} - \tilde{\eth}'\tau^{\circ}) - \varrho\bar{\varrho}^2(\tilde{\eth}\bar{\tau}^{\circ} + 1/2\,\Psi^{\circ})\\
&\quad + 2\bar{\varrho}\,\varrho^3\tau^{\circ}\bar{\tau}^{\circ 2} - \varrho^2\bar{\varrho}^2\tau^{\circ}\bar{\tau}^{\circ 2}\\
&= \varrho^2(\bar{\tau}^{\circ}\bar{\varrho}'^{\circ}) - 2\varrho\,\bar{\varrho}\varrho'^{\circ}\bar{\tau}^{\circ} + \varrho^2\bar{\varrho}\bar{\tau}^{\circ}(-\tilde{\eth}'\tau^{\circ} - 1/2\,\bar{\Psi}^{\circ} + 2\tilde{\eth}\bar{\tau}^{\circ} + \Psi^{\circ})\\
&\quad + \varrho^3\bar{\tau}^{\circ}\Psi^{\circ} - 1/2\,\varrho\bar{\varrho}^2\bar{\tau}^{\circ}\bar{\Psi}^{\circ} + 2\varrho^3\bar{\varrho}\tau^{\circ}\bar{\tau}^{\circ 2} - \varrho^2\bar{\varrho}^2\tau^{\circ}\bar{\tau}^{\circ 2}\,.
\end{aligned} \tag{A-2}$$

The lowest powers of ϱ are equated to give

$$\tilde{\eth}'\varrho'^{\circ} = 0\,. \tag{A-3}$$

Proceeding to the power '2' the situation is more complicated. Equating the coefficients of ϱ^2, $\varrho\bar{\varrho}$, and $\bar{\varrho}^2$ yields

$$\tilde{\text{Þ}}'\bar{\tau}^{\circ} = 0 \tag{A-4}$$

and the terms

$$\varrho^2\bar{\varrho}\Omega^{\circ}\bar{\tau}^{\circ}\bar{\varrho}'^{\circ} + \varrho\bar{\varrho}^2\Omega^{\circ}\bar{\tau}^{\circ}\varrho'^{\circ} \tag{A-5}$$

to be added to the R.H.S. Including these terms in the '3' equation gives

$$\tilde{\eth}\bar{\tau}^{\circ} = -1/2\,\Omega^{\circ}(\varrho'^{\circ} + \bar{\varrho}'^{\circ}) + 1/2(\bar{\Psi}^{\circ} - \Psi^{\circ}) \tag{A-6}$$

and a 'carry' of

$$1/2\,\varrho^2\bar{\varrho}^2\bar{\tau}^\circ\Omega^{\circ 2}(\varrho'^\circ-\bar{\varrho}'^\circ)\,. \tag{A-7}$$

From the revised '4' equation it follows that

$$\varrho'^\circ=\bar{\varrho}'^\circ \tag{A-8}$$

The '5' equation is satisfied identically.

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