A FORMULA FOR AN INTEGRAL OCCURRING IN THE THEORY OF LINEAR SERVOMECHANISMS AND CONTROL-SYSTEMS*

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Introduction. Let t denote the time, $p = d/dt$ the differential operator with respect to time and

$$f_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \; ; \qquad a_0 \neq 0, n \geq 1 \tag{1}$$

a polynomial with real coefficients. If all zeros of $f_n(x)$ have negative real parts, every solution $y(t)$ of

$$f_n(p)y = 0 \tag{2}$$

and all derivatives $p^k y$ tend to zero with increasing t. Moreover the integral

$$Y = \int_0^\infty y^2(t)\, dt \tag{3}$$

exists. The purpose of this paper is to develop a formula for Y in terms of squared linear forms of the initial values

$$p^k y(0) = q_k \; ; \qquad k = 0, 1, \cdots, n - 1. \tag{4}$$

No further quantities but the coefficients a_i of (1) shall appear in this formula.

Such a formula may be useful for the design of linear servomechanisms and control-systems, governed by the equation

$$f_n(p)y = z(t). \tag{2'}$$

where $z(t)$ may be considered as an arbitrary disturbance function. For instance, let $z(t) \equiv 1$ for $t < 0$. At $t = 0$, $z(t)$ may step down to $z(t) \equiv 0$ for $t \geq 0$. The response $y(t)$ then is a solution of (2), and the integral Y measures, how fast the systems lines up with the stepping of z. The knowledge of Y makes it possible to choose the coefficients a_i of (1) under given conditions in order to minimize Y**. Two examples of such a minimization will be given in Sec. 4.

The development of this formula will also yield a new approach to the well known Hurwitz criterion of stability and to reductions of "stable" operator polynomials in p to such of a lower degree, including the reduction of Schur [1].

1. Auxiliary theorems and algorithms of reduction. Notation. Let J be the imaginary axis of the complex plane, J' the set of all points ωi, of J with $\omega > 0$, J'' the set of all points ωi of J with $\omega < 0$, and Re x the real, Im x the imaginary part of x.

Definitions. Let $f(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$ a polynomial with real or complex coefficients. We call m the proper degree of $f(x)$, if $b_0 \neq 0$. We define now

1) $F(x) = b_0 x^m + b_m$ as the "simplification" of $f(x)$, if $f(x)$ has the proper degree m,

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**Minimization of Y has already been investigated by P. Hazebroek and B. L. van der Waerden [2] who also gave a formula expressing Y as a symmetric function of the zeros of (1) for special systems (4).

2) $g(x) = b_m + b_{m-2}x^2 + \cdots$ as the even and $h(x) = f(x) - g(x)$ as the odd component of $f(x)$,
3) $f(x)$ as a "Hurwitz-polynomial", if all zeros of $f(x)$ are in the left-hand half-plane $\operatorname{Re} x < 0$ (the case $f(x) \equiv$ const. $\neq 0$ to be included),
4) $f(x)$ as definite (semidefinite) on a given set M of points of the complex plane, if a suitable constant $c \neq 0$ can be found, so that $cf(x) > 0$ (≥ 0) on M (for instance, x^m is definite on J'), c to be normalized by $|c| = 1$.

Lemma 1. Let $p(x)$ and $q(x)$ be two polynomials. The linear combination $r(x, t) = tp(x) + (1 - t)q(x)$ shall have proper degree m for all values $0 \leq t \leq 1$. We further assume $r(x, t) \neq 0$ on J for all these values of t. Then $p(x)$ and $q(x)$ have the same number of zeros for $\operatorname{Re} x > 0$ and for $\operatorname{Re} x < 0$.

Proof. No zero of $r(x, t)$ can pass J or can go to infinity, when t is running from 0 to 1. Hence the number of zeros for $\operatorname{Re} x > 0$ remains constant. The same holds for $\operatorname{Re} x < 0$.

Lemma 2. Let $f(x) = b_0x^m + \cdots + b_m$ be a polynomial with real coefficients b_k. The proper degree shall be $m \geq 1$; $f(x)$ and its simplification $F(x)$ shall not vanish on J. Then $f(x)$ and $F(x)$ have the same number of zeros for $\operatorname{Re} x > 0$ and for $\operatorname{Re} x < 0$, if at least one of the following conditions is satisfied:
a) the even component $g(x)$ of $f(x)$ is semidefinite on J';
b) m is odd, and the odd component $h(x)$ of $f(x)$ is semidefinite on J';
c) m is even, and the odd component $h(x)$ of $f(x)$ is definite on J'.

Proof. We set $r(x, t) = tf(x) + (1 - t)F(x) = b_0x^m + tb_1x^{m-1} + \cdots + tb_{m-1}x + b_m$. The proper degree of $r(x, t)$ is m for all values of t. We shall prove that $r(x, t) \neq 0$ on J for $0 \leq t \leq 1$. Application of the first lemma then completes the proof.

From the assumptions it follows that $r(x, 0) \neq 0$ on J and $r(x, 1) \neq 0$ on J; furthermore $r(0, t) = b_m \neq 0$. It is therefore sufficient to prove $r(x, t) \neq 0$ on J' or J'' for $0 < t < 1$. We denote by $G(x)$ the even and by $H(x)$ the odd component of $F(x)$. $G(x)$ is either the simplification of $g(x)$ or equal to $g(0) = b_m$; $H(x)$ is either the simplification of $h(x)$ or equal to $h(0) = 0$.

If any polynomial $s(x)$ is semidefinite on J', we have $cs(x) \geq 0$ on J' with a suitable constant c ($|c| = 1$). Considering extremely small and extremely great values of $|x|$, we find $cS(x) \geq 0$ on J' with the same constant c for the simplification $S(x)$ of $s(x)$. With this in mind, we distinguish the following three cases according to the conditions a, b, c of the lemma.

a) $g(x)$ is semidefinite on J'. This leads to $cg(x) \geq 0$ and to $cG(x) \geq 0$ on J'. We have either $G(x) = b_m$ or $G(x) = F(x)$, and in both cases $G(x) \neq 0$ on J. Therefore,

$$|r(x, t)| \geq |\operatorname{Re} r(x, t)| = c.tg(x) + c.(1 - t)G(x) \geq (1 - t)cG(x) > 0 \text{ on } J'.$$

b) m odd, $h(x)$ semidefinite on J'. We have $H(x) = b_0x^m \neq 0$ on J' and a suitable constant c, making $ch(x) \geq 0$ and $cH(x) \geq 0$ on J'. Hence for $0 < t < 1$ on J'

$$|r(x, t)| \geq |\operatorname{Im} r(x, t)| = c.th(x) + c.(1 - t)H(x) \geq c(1 - t)H(x) > 0.$$

c) m even, $h(x)$ definite on J'. We find $|r(x, t)| \geq t\,|h(x)| > 0$ on J' for $0 < t < 1$. Thus, $r(x, t) \neq 0$ on J.

Lemma 3. Let $p(x)$ and $q(x)$ be any two polynomials with real coefficients, p having proper degree m and q having proper degree $m' < m$. The polynomial $f(x) = p(x)q(-x)$ and its simplification $F(x)$ shall not vanish on J; $f(x)$ and $F(x)$ shall have the same number of zeros for $\operatorname{Re} x > 0$ and for $\operatorname{Re} x < 0$. From this it follows that

a) if $p(x)$ is a Hurwitz-polynomial, $q(x)$ is also one with $m' = m - 1$,
b) if $q(x)$ is a Hurwitz-polynomial, if furthermore $m = m' + 1$, and if all coefficients of $p(x)$ are positive, then $p(x)$ is also a Hurwitz-polynomial.

Proof. The number of zeros of $F(x)$ in the half-plane Re $x < 0$ may be n, the number of zeros in Re $x > 0$ may be n'. All zeros of $F(x)$ form a regular polygon for $n + n' \geq 3$, and no zero can appear on J. Hence $|n - n'| \leq 1$. Should $p(x)$ be a Hurwitz polynomial, $f(x)$ and $F(x)$ have at least m zeros in Re $x < 0$ and not more than $m' \leq m - 1$ zeros in Re $x > 0$. Therefore $n = m$ and $n' = m' = m - 1$. The $m - 1$ zeros of $f(x)$ in Re $x > 0$ are those of $q(-x)$. This means, that $q(x)$ is a Hurwitz polynomial. Should the conditions of b) be satisfied, then at least $m - 1$ zeros of $F(x)$ and consequently of $f(x)$ appear in Re $x < 0$. Therefore $p(x)$ has $m - 1$ zeros in Re $x < 0$. Should the last zero of $p(x)$ be situated in Re $x > 0$, it must necessarily be real, i.e. positive. But no such zero can exist, since $p(x)$ is assumed to have positive coefficients. This completes the proof of the lemma.

Note. The condition, that all coefficients of $p(x)$ are positive is—apart from a constant factor—a necessary condition for $p(x)$ to be a Hurwitz polynomial. It is well known and it can easily be proved by splitting $p(x)$ into root factors. No coefficient can vanish without reducing the degree of the polynomial.

Algorithms can be based on Lemmas 2 and 3 in order to reduce a Hurwitz polynomial to such of a lower degree. It may be worthwhile to explain, how the well-known reduction of Schur (see [1]) can be obtained in this way.

Schur's algorithm of reduction. We consider the polynomial (1) with real coefficients, but we do not assume that it is a Hurwitz polynomial. We denote by g^+ the even and by h^+ the odd component of $f_n(x)$. With Schur we introduce

$$f_{n-1}(x) = (2a_1 - a_0x)[g^+(x) + h^+(x)] + (-1)^n a_0 x[g^+(x) - h^+(x)] \tag{5}$$

with lower degree than n. The odd component of the polynomial $f(x) = f_n(x)f_{n-1}(-x)$ is

$$h(x) = -2a_0xh^{+2}(x) \text{ for even } n, \qquad h(x) = 2a_0xg^{+2}(x) \text{ for odd } n. \tag{6}$$

This component is obviously semidefinite on J' and on J''. It can easily be seen, that $f_n(x) = 0$ on J in any point x leads to $f_{n-1}(x) = 0$ for the same point. *Vice versa*, $a_1f_n(x) = 0$ is a consequence of $f_{n-1}(x) = 0$ in any point x of J. We now assume that

$$a_1 \neq 0. \tag{7}$$

This is necessary and sufficient for f_{n-1} to have the proper degree $n - 1$. The polynomial $f(x)$ then has the proper degree $2n - 1$. If either f_n or f_{n-1} is a Hurwitz polynomial, f cannot vanish on J. Also $F(x)$, the simplification of f, cannot vanish on J. Hence Lemma 2 is applicable to f and F and the Lemma 3 to f_n and f_{n-1}. Thus, if f_n is a Hurwitz polynomial, f_{n-1} is also one; if f_{n-1} is a Hurwitz polynomial and if f_n has positive coefficients, then f_n is a Hurwitz polynomial too.

Another algorithm. Assume that $f_n(x)$ and $f_n(-x)$ do not have common zeros. Then two polynomials $r(x)$ and $t(x)$ with real coefficients and with no higher degree than $n - 1$ exist, satisfying

$$f_n(x)r(x) + f_n(-x)t(x) \equiv 2. \tag{8}$$

From this it follows that

$$f_n(x)f_{n-1}(-x) + f_n(-x)f_{n-1}(x) \equiv 2 \quad \text{with} \quad 2f_{n-1}(x) = r(-x) + t(x), \tag{9}$$

the degree of f_{n-1} being at most $n - 1$. Any other polynomial $p(x)$ of any degree, which satisfies (9) instead of f_{n-1} can be written as

$$p(x) = f_{n-1}(x) + s(x)f_n(x), \tag{10}$$

with a suitable odd polynomial $s(x) = -s(-x)$. Hence f_{n-1} is the only polynomial satisfying (9) with no higher degree than $n - 1$. From (9) it follows that

$$g(x) \equiv 1 \tag{11}$$

for the even component of the product $f(x) = f_n(x)f_{n-1}(-x)$; $g(x)$ is definite on J' and on J'', it is even definite on J. The product $f(x)$ cannot vanish on J. Also its simplification $F(x)$ cannot vanish on J, since the degree of $F(x)$ is odd and $F(0) = 1$. Therefore $f(x)$ and $F(x)$ have the same number of zeros for Re $x > 0$ and for Re $x < 0$ according to Lemma 2. Lemma 3 is applicable to f_n and f_{n-1}. Thus, if f_n is a Hurwitz polynomial, f_{n-1} is also one with proper degree $n - 1$. If f_{n-1} with proper degree $n - 1$ is a Hurwitz polynomial and if f_n has positive coefficients, f_n is a Hurwitz polynomial too, and this is a consequence of (9).

2. Details concerning the second algorithm of reduction. The second algorithm will be useful for the development of the formula announced. Some necessary details will therefore be developed. We assume $f_n(x) = a_0x^n + \cdots + a_n$ to be a Hurwitz polynomial of proper degree n with real coefficients. As already stated, the polynomial $f_{n-1}(x)$ defined by (9) is also a Hurwitz polynomial with real coefficients and with proper degree $n - 1$. The method leading from f_n to f_{n-1} can now be applied to f_{n-1} and so on. Thus we obtain a sequence of Hurwitz polynomials

$$f_n\,,\, f_{n-1}\,,\, f_{n-2}\,,\, \cdots\,,\, f_1\,,\, f_0\,, \tag{12}$$

with f_0 as a constant; f_k has the proper degree k and real coefficients; any two adjacent polynomials f_k, f_{k-1} satisfy

$$f_k(x)f_{k-1}(-x) + f_k(-x)f_{k-1}(x) \equiv 2. \tag{13}$$

It means no loss of generality to assume

$$f_n(0) = a_n = 1; \tag{14}$$

(13) and (14) then lead to

$$f_k(0) = 1 \qquad \text{for} \qquad k = 0, 1, \cdots, n = 1. \tag{15}$$

This in turn causes positive coefficients for all polynomials f_k (see Sec. 1, Note). We increase all subscripts in (13) by 1 and subtract the new equation from (13); hence $p(x)f_k(-x) + p(-x)f_k(x) \equiv 0$ with $p(x) = f_{k+1}(x) - f_{k-1}(x)$; $f_k(x)$ and $f_k(-x)$ have no common zeros. Therefore,

$$f_{k+1}(x) - f_{k-1}(x) = c_{k+1}\, x\cdot f_k(x) \qquad \text{for} \qquad k = 1, 2, \cdots, n - 1 \tag{16}$$

with a suitable constant

$$c_{k+1} > 0. \tag{16$'$}$$

In addition to (16), we write

$$f_1(x) = 1 + c_1x; \qquad c_1 > 0. \tag{16$''$}$$

Regarding the positive constants c_1, c_2, $\cdots$, c_n as given, we can solve the system (16) with regard to f_2, $\cdots$, f_n. We find:

$$f_k(x) = \begin{vmatrix} 1 + c_1x & -1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 1 & c_2x & -1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & c_3x & -1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 1 & c_kx \end{vmatrix}; \qquad k = 2, 3 \tag{17}$$

This is a representation of all Hurwitz polynomials of proper degree k with $f_k(0) = 1$. Vice versa all determinants (17) with coefficients $c_i > 0$ give Hurwitz polynomials. Another representation may be given by means of the determinants

$$\begin{vmatrix} c_1x & -1 & 0 & \cdot & \cdot & 0 & 0 \\ 1 & c_2x & -1 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & c_3x & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 & c_kx \end{vmatrix} = F(x; c_1, c_2, \cdots c_k). \tag{18}$$

We can write then

$$f_k(x) = F(x; c_1, c_2, \cdots, c_k) + F(x; c_2, c_3, \cdots, c_k), \tag{17'}$$

the right-hand-side showing the even and the odd component of f_k. The functions (18) have imaginary zeros in x or real zeros in ix, which can easily be recognized as the eigenvalues of a Hermitian matrix. The result about the zeros of the components of a Hurwitz polynomial with real coefficients is well known and has been found by E. J. Routh. So far this represents a minor application of (18).

We are now going to develop another formula for f_k where only the coefficients a_i of the given polynomial f_n appear. For this purpose we introduce the column-vector

$$\mathfrak{a}_{ij} = \begin{pmatrix} a_{i-2j+2} \\ a_{i-2j+4} \\ \cdot \\ \cdot \\ \cdot \\ a_i \end{pmatrix}, \qquad a_k = 0 \quad \text{for} \quad k > n \quad \text{and for} \quad k < 0;$$

with j components, the matrices

$$\mathfrak{H}_k = (\mathfrak{a}_{2k-1,k}\,,\ \mathfrak{a}_{2k-2,k}\,,\ \cdots\ \mathfrak{a}_{k,k}); \qquad k = 1, 2, \cdots, n,$$

the so-called Hurwitz determinants

$$D_0 = \operatorname{sign} a_0 = 1, \qquad D_k = \|\ \mathfrak{H}_k\ \| \qquad \text{for} \qquad k = 1, 2, \cdots, n, \tag{19}$$

and the column-vector

$$\beta_k = \begin{pmatrix} b_{0k} \\ \cdot \\ \cdot \\ b_{k-1,k} \end{pmatrix}$$

of the coefficients of the polynomial

$$f_k(-x) = b_{0k}x^k + b_{1k}x^{k-1} + \cdots + b_{k-1,k}x + 1.$$

We then consider the polynomials

$$f_n(x)f_k(-x) - (-1)^{n-k}f_n(-x)f_k(x) = w_{n-k-1}(x); \qquad k = 0, 1, \cdots, n \tag{20}$$

with the two significant special cases

$$w_{-1}(x) \equiv 0, \qquad w_0(x) \equiv 2. \tag{20'}$$

From (16) and (20) follows for $n - k \geq 1$

$$w_{n-k} = c_{k+1}xw_{n-k-1} + w_{n-k-2}\,, \tag{21}$$

and we derive from (16′), (20′) and (21) that $w_{n-k}(x)$ has the proper degree $n - k$. This means: the product $f_n(x)f_k(-x)$ does not contain the powers x^{n+k-1}, x^{n+k-3}, $\cdots$, x^{n-k+1}. This is expressed by

$$\mathfrak{H}_k\beta_k = -\mathfrak{a}_{k-1,k}\,. \tag{22}$$

There is only one polynomial f_{n-1} of degree $n - 1$ according to (9). Hence there is only one solution β_{n-1} of (22) for $k = n - 1$, and this leads to $D_{n-1} \neq 0$. Let $D_{k+1} \neq 0$; consequently the matrix $\mathfrak{H}_{k+1}$ is of rank $k + 1$, while the matrix $(\mathfrak{H}_k\,,\ \mathfrak{a}_{k-1,k})$ consisting of all rows but the last of $\mathfrak{H}_{k+1}$ is of rank k. This very matrix appears in (22), so only one solution of (22) for β_k exists, and therefore $D_k \neq 0$. Hence

$$D_k \neq 0; \qquad k = 1, 2, \cdots, n - 1. \tag{23}$$

All systems (22) have only one solution β_k, and this belongs to

$$f_k(x) = \frac{1}{D_k}\begin{vmatrix} x^k & -x^{k-1} & \cdot\cdot & (-1)^k \\ \mathfrak{a}_{2k-1,k} & \mathfrak{a}_{2k-2,k} & \cdot\cdot & \mathfrak{a}_{k-1,k} \end{vmatrix} = a_0\,\frac{D_{k-1}}{D_k}\,x^k + \cdots + 1. \tag{24}$$

The proof is clear. The coefficients of f_k are positive. We have $D_1 = a_1 > 0$, $a_0D_{k-1}D_k^{-1} > 0$, $D_n = a_nD_{n-1}$ and thus,

$$D_i > 0 \qquad \text{for} \qquad i = 1, 2, \cdots, n. \tag{25}$$

The coefficients c_i in (16) are the quotients of the highest-power-terms of f_i and f_{i-1}. Therefore

$$c_1 = a_0 a_1^{-1}, \qquad c_k = D_{k-1}^2 D_k^{-1} \cdot D_{k-2}^{-1} \qquad \text{for} \qquad k = 2, 3, \cdots, n. \tag{26}$$

The inequalities (25) form the well known Hurwitz criterion of stability.

3. The formula for Y. Let

$$P(u, v) = \sum_{i,k=0}^{m} a_{ik} u^i v^k \tag{27}$$

be a polynomial of two variables u and v. Let $y(t)$ and $z(t)$ be two functions with continuous derivatives $p^i y$, $p^k z$ up to the order i, $k = m + 1$. We then set

$$P^*(y, z) = \sum_{i,k=0}^{m} a_{ik} p^i y \cdot p^k z. \tag{28}$$

We introduce $Q(u, v) = (u + v)P(u, v)$. Obviously,

$$\int_a^b Q^*(y, z)\, dt = \int_a^b pP^*(y, z)\, dt = P^*(y, z) \Big|_a^b \tag{29}$$

We consider the special polynomials

$$Q_k(u, v) = f_k(u) f_{k-1}(v) + f_k(v) f_{k-1}(u) - 2; \qquad k = 1, 2, \cdots, n. \tag{30}$$

From (16) it follows that

$$Q_k(u, v) = (u + v) c_k f_{k-1}(u) f_{k-1}(v) + Q_{k-1}(u, v). \tag{31}$$

Therefore,

$$Q_n(u, v) = (u + v) \sum_{k=1}^{n} c_k f_{k-1}(u) f_{k-1}(v). \tag{32}$$

We apply (29) to (32) with y and z as solutions of (2), i.e., $f_n(p)y = 0$ and $f_n(p)z = 0$. Hence

$$2 \int_a^b y(t) z(t)\, dt = -\int_a^b Q_n^*(y, z)\, dt = \sum_{k=1}^{n} c_k f_{k-1}^*(y) f_{k-1}^*(z) \Big|_b^a \tag{33}$$

with $f_{k-1}^*(y) = f_{k-1}(p)y$ and $f_{k-1}^*(z) = f_{k-1}(p)z$. Setting $y = z$ and $a = 0$, $b = \infty$ we find the announced formula

$$2Y = 2 \int_0^\infty y^2(t)\, dt = \sum_{k=1}^{n} c_k (f_{k-1}^*(y)_0)^2. \tag{34}$$

We express c_k and f_{k-1} according to (26) and (24). We obtain

$$2Y = a_0 a_1^{-1} q_0^2 + \sum_{k=1}^{n-1} D_{k-1}^{-1} D_{k+1}^{-1} \begin{vmatrix} q_k & -q_{k-1} & \cdot\cdot & (-1)^k q_0 \\ \mathfrak{a}_{2k-1,k} & \mathfrak{a}_{2k-2,k} & \cdot\cdot & \mathfrak{a}_{k-1,k} \end{vmatrix}^2 \tag{35}$$

with the initial values q_k as explained by (4). In this formula, squared linear forms of the q_k appear together with the coefficients a_i of the given equation. Formula (35) has already a form which makes it independent of the restriction $a_n = 1$. It holds quite generally.

In the special case $q_0 = 1$, $q_1 = q_2 = \cdots = q_{n-1} = 0$ we find

$$2Y = \sum_{k=1}^{n} c_k \,. \tag{36}$$

This sum can be easily computed from (16). Addition of all formulae (16) gives

$$f_n + f_{n-1} - 2 = -2 + f_0 + f_1 + \sum_{i=1}^{n-1} (f_{i+1} - f_{i-1}) = x \sum_{i=0}^{n-1} c_{i+1} f_i$$

or $\sum_{i=1}^{n} c_i =$ coefficient of x in $(f_n + f_{n-1})$.
Therefore

$$2Y = a_{n-1}a_n^{-1} + D_n^{-1} \cdot \mid \mathfrak{a}_{2n-3,n-1}\mathfrak{a}_{2n-4,n-1} \cdots \mathfrak{a}_{n,n-1}\mathfrak{a}_{n-2,n-1} \mid . \tag{36$'$}$$

This formula too is not restricted to $a_n = 1$.

4. Two applications. 1) We set $a_0 = a_n = 1$, which is no essential restriction. All other coefficients of f_n may be variable in order to minimize Y according to (36). This means, that the sum of all coefficients c_i is to be minimized under the restriction $c_1c_2 \cdots c_n = 1$. An elementary calculation gives Min $2Y = n$ for $c_1 = c_2 = \cdots = c_n = 1$ with

$$\begin{aligned} f_n(x) = x^n + \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} + \cdots \\ + \binom{n-1}{0}x^{n-1} + \binom{n-2}{1}x^{n-3} + \cdots . \end{aligned} \tag{37}$$

This formula can be proved by induction on n.

(2) There are servomechanisms with an arbitrary input $\theta_i(t)$ and with a servo controlled output $\theta_0(t)$. The control depends on

$$\epsilon(t) = \theta_0(t) - \theta_i(t) \tag{38}$$

and shall make $|\epsilon|$ as small as possible. According to the definitions given in [2], ϵ can be called the regulated variable and θ_0 the regulating flow. Let the servocontrol be of the proportional plus integral type, i.e.

$$a_0\ddot{\theta}_0 + a_1\dot{\theta}_0 = -a_2\epsilon - \int_0^t a_3 \, \epsilon \, dt \tag{39}$$

with constants $a_i > 0$ for $i = 0, 1, 2, 3$. Combination of (38) and (39) gives

$$a_0\dddot{\epsilon} + a_1\ddot{\epsilon} + a_2\dot{\epsilon} + a_3\epsilon = -a_0\dddot{\theta}_i - a_1\ddot{\theta}_i \,. \tag{40}$$

Due to the integral in (39), $\epsilon(t)$ tends to zero with increasing t if the right-hand-side vanishes identically and if

$$D_2 = a_1a_2 - a_0a_3 > 0. \tag{41}$$

Now we consider the case

$$\theta_i \equiv 0 \quad \text{for} \quad t < 0; \qquad \theta_i = Ct \quad \text{for} \quad t \geq 0. \tag{42}$$

Then $\epsilon(t)$ is a solution of the equation (40) made homogeneous. If the servomechanism is to start from rest at $t = 0$, the initial values are

$$\epsilon(0) = q_0 = q_0 = 0; \qquad \dot{\epsilon}(0) = q_1 = -C; \qquad \ddot{\epsilon}(0) = q_2 = 0. \tag{43}$$

Application of (35) leads to

$$2Y = C^2 \frac{a_1^2 + a_3 a_0^2}{a_3 D_2}. \tag{44}$$

It is obvious that Y becomes smaller with increasing a_2 . Therefore a_2 should be made as large as possible. For practical reasons (saturation and overcontrol of amplifiers or the like) an upper bound for a_2 is given. With this in mind we minimize $2Y$ for a given a_2 by variation of a_3 . Setting $b_i = a_i/a_0$ we find

$$\text{Min } 2Y = C^2 \frac{2b_1^2 + b_2 + 2b_1(b_1^2 + b_2)^{1/2}}{2b_1 b_2^2}, \tag{45}$$

$$b_3 = b_1^2\{(b_1^2 + b_2)^{1/2} - b_1\}. \tag{46}$$

This gives the best design with respect to the important case (42).

References

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