

A FORMULA FOR THE BETTI NUMBERS OF COMPACT LOCALLY SYMMETRIC RIEMANNIAN MANIFOLDS

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1. Let X be a simply connected symmetric Riemannian manifold and let G be a connected Lie group acting transitively and almost effectively on X as a group of isometries. We denote by K the isotropy group of G at a point o of X . If G is compact, it is a well-known theorem of Cartan-Hodge that a differential p -form is harmonic if and only if it is G -invariant. It follows from this theorem that the p -th Betti number of X is equal to the multiplicity with which the trivial representation enters in the linear isotropic representation of K in the vector space of p -covectors at the point o .

Let us suppose now that G is a connected semi-simple Lie group with finite center all of whose simple components are non-compact. Let Γ be a discrete subgroup of G such that the quotient $\Gamma \backslash G$ is compact. We denote by $h^p(X, \Gamma)$ the vector space of all harmonic p -forms on X which are invariant by Γ . We know that the dimension of the space $h^p(X, \Gamma)$ is finite. The results obtained in the previous papers [4] shows that in several cases the dimension of $h^p(X, \Gamma)$ is also equal to the multiplicity with which the trivial representation enters in the linear isotropic representation of K in the space of p -covectors at the point o , if the number $p/\dim X$ is small.

The purpose of the present paper is to prove a formula which relates the dimension of the space $h^p(X, \Gamma)$ with the decomposition of the unitary representation of G in the Hilbert space $L^2(\Gamma \backslash G)$ (see §2). This formula corresponds in a sense to the theorem of Cartan-Hodge and, in fact, if G is compact and Γ reduces to the identity, our formula is equivalent to Cartan-Hodge Theorem.

We shall also see as an example that, if X is the 3-dimensional hyperbolic space and if G is $SL(2, \mathbf{C})$ or the proper Lorentz group, the dimension of $h^1(X, \Gamma)$ is equal to the multiplicity in $L^2(\Gamma \backslash G)$ of the irreducible unitary representation $U_{2,0}$ of the principal series (see §5).

2. We retain the notations introduced in §1 so that G will denote a connected semi-simple Lie group with finite center all of whose simple components are non-compact. The group K is then a maximal compact subgroup of G . Let \mathfrak{g} be the Lie algebra of left-invariant vector fields on G , and \mathfrak{k} the subalgebra of \mathfrak{g} corresponding to K . We denote by $\varphi(X, Y)$ ($X, Y \in \mathfrak{g}$) the Killing form of the semi-simple Lie algebra \mathfrak{g} and by \mathfrak{m} the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to φ . We

know that

$$\begin{aligned} \mathfrak{g} &= \mathfrak{m} + \mathfrak{k}, & \mathfrak{m} \cap \mathfrak{k} &= (0), \\ [\mathfrak{m}, \mathfrak{m}] &= \mathfrak{k}, & [\mathfrak{k}, \mathfrak{m}] &= \mathfrak{m}. \end{aligned}$$

Moreover, $\varphi(X, X)$ is positive if $X \in \mathfrak{m}$, $X \neq 0$, and negative if $X \in \mathfrak{k}$, $X \neq 0$. Let $\{X_i\}_{i=1, \dots, r}$ and $\{X_a\}_{a=r+1, \dots, n}$ be bases of \mathfrak{m} and \mathfrak{k} respectively such that

$$\begin{aligned} \varphi(X_i, X_j) &= \delta_{ij} & (1 \leq i, j \leq r), \\ \varphi(X_a, X_b) &= -\delta_{ab} & (r+1 \leq a, b \leq n). \end{aligned}$$

In the following we shall make the convention that the indices i, j, \dots will range from 1 to r , while the indices a, b, \dots from $r+1$ to n .

A vector field $X \in \mathfrak{g}$ is left invariant by \mathbf{G} and hence by Γ so that X is projectable onto $\Gamma \backslash \mathbf{G}$. In the following we consider the elements X of \mathfrak{g} as vector fields on $\Gamma \backslash \mathbf{G}$. We denote by \mathbf{C} the differential operator on $\Gamma \backslash \mathbf{G}$ defined by

$$\mathbf{C} = \sum_{i=1}^r X_i^2 - \sum_{a=r+1}^n X_a^2.$$

The operator \mathbf{C} is called the Casimir operator of \mathbf{G} . We may consider \mathbf{C} as an element of the universal enveloping algebra $E(\mathfrak{g})$ of \mathfrak{g} . It is known that \mathbf{C} is in the center of $E(\mathfrak{g})$.

Now let \mathbf{T} be a unitary representation of \mathbf{G} in a Hilbert space \mathbf{H} . A vector $\varphi \in \mathbf{H}$ is called a *regular* vector if the function $s \rightarrow T(s)\varphi$ is of class \mathbf{C}^∞ . We denote by \mathbf{W} the subspace of all regular vectors of \mathbf{H} . It is known that \mathbf{W} is dense in \mathbf{H} . Let $X \in \mathfrak{g}$ and let $\exp tX$ be the 1-parameter subgroup of \mathbf{G} corresponding to X . For $\varphi \in \mathbf{W}$, put $\mathbf{T}(X)\varphi = \left[\frac{d}{dt} \mathbf{T}(\exp tX)\varphi \right]_{t=0}$. Then $i\mathbf{T}(X)$ is a self-adjoint operator with domain \mathbf{W} . We define the self-adjoint operator \mathbf{C}_T of \mathbf{H} with domain \mathbf{W} by putting

$$\mathbf{C}_T = \sum_{i=1}^r \mathbf{T}(X_i)^2 - \sum_{a=r+1}^n \mathbf{T}(X_a)^2,$$

and call it the Casimir operator of the unitary representation \mathbf{T} of \mathbf{G} . If \mathbf{T} is an irreducible unitary representation, there exists a real number λ_T such that $\mathbf{C}_T\varphi = \lambda_T\varphi$ for all $\varphi \in \mathbf{W}$.

In the following we shall denote by D_0 the set of irreducible unitary representations \mathbf{T} of \mathbf{G} such that $\lambda_T = 0$.

We denote by U the unitary representation of \mathbf{G} in the Hilbert space $L^2(\Gamma \backslash \mathbf{G})$. The vector space $\mathbf{C}^\infty(\Gamma \backslash \mathbf{G})$ of all complex valued \mathbf{C}^∞ -functions on $\Gamma \backslash \mathbf{G}$ is a subspace of the space of regular vectors of $L_2(\Gamma \backslash \mathbf{G})$, and we have $Cf = -C_U f$ for all $f \in \mathbf{C}^\infty(\Gamma \backslash \mathbf{G})$. The representation U decomposes into sum of a countable number of irreducible unitary representations in which each irreducible representation enters with a finite multiplicity [1]. We denote by $N(T)$ the multiplicity in U of an irreducible unitary representation \mathbf{T} of \mathbf{G} .

Now let \mathbf{T} be an irreducible unitary representation of \mathbf{G} , and T_K the restriction of \mathbf{T} onto K . It is well-known (see [2]) that the representation T_K of K decomposes into sum of a countable number of irreducible representations in which each irreducible representation enters with a finite multiplicity. We shall denote by $M(T_K; \tau)$ the multiplicity in T_K of an irreducible representation τ of K .

Let now \mathfrak{m}^C be the complexification of \mathfrak{m} . We denote by ad^p the representation of K in the vector space $\bigwedge^p \mathfrak{m}^C$ induced by the adjoint action of K in \mathfrak{m} . Let

$$(2.1) \quad ad^p = \tau_1^p + \cdots + \tau_{s_p}^p$$

be the decomposition of ad^p into a sum of irreducible representations.

Theorem. *Let \mathbf{G} be a connected semi-simple Lie group with finite center, K a maximal compact subgroup of \mathbf{G} , and Γ a discrete subgroup of \mathbf{G} with compact quotient space $\Gamma \backslash G$. Assume that Γ acts freely on the symmetric space $X = G/K$, and let $h^p(X, \Gamma)$ be the vector space of all harmonic p -forms on X invariant by Γ . Let \mathbf{T} be an irreducible unitary representation of \mathbf{G} , and T_K the restriction of \mathbf{T} on K . Let $N(\mathbf{T})$ denote the multiplicity of \mathbf{T} in the unitary representation U of \mathbf{G} in the Hilbert space $L^2(\Gamma \backslash G)$, and $M(T_K; \tau_i^p)$ the multiplicity of the irreducible representation τ_i^p of K in T_K . Then*

$$\dim h^p(X, \Gamma) = \sum_{T \in D_0} N(T) \left(\sum_{i=1}^{s_p} M(T_K; \tau_i^p) \right),$$

where D_0 denotes the set of all irreducible unitary representations of \mathbf{G} with vanishing Casimir operator.

The following sections are devoted to proving this theorem.

3. Let η be a complex valued differential p -form in X invariant by Γ , and $\pi_0: G \rightarrow G/K = X$ the canonical projection of \mathbf{G} onto X . Put $\tilde{\eta} = \eta \circ \pi_0$. Then $\tilde{\eta}$ is a p -form on \mathbf{G} having the following properties:

$$\begin{aligned} \tilde{\eta} \circ L_\gamma &= \tilde{\eta} \quad (\gamma \in \Gamma), & \tilde{\eta} \circ R_k &= \tilde{\eta} \quad (k \in K), \\ i(Y)\tilde{\eta} &= 0 \quad (Y \in \mathfrak{k}), \end{aligned}$$

where L_g (resp. R_g) denotes the left (resp. right) translation of \mathbf{G} by $g \in G$, and $i(X)$ the operator of interior multiplication.

Now let $\omega^i (1 \leq i \leq r)$ be the left invariant 1-form on G such that $\omega^i(X_j) = \delta_j^i$. We denote by I an ordered set of p indices i_s such that $1 \leq i_1 < i_2 < \cdots < i_p \leq r$. Further put

$$\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}.$$

Then the p -form $\tilde{\eta}$ is written uniquely in the form

$$\tilde{\eta} = \sum_I \eta_I \omega^I,$$

where the coefficients η_I are functions on G . Now $\{\omega^I\}$ form a basis of $\overset{p}{\wedge} \mathfrak{m}^*C$, and we denote by ad^{*p} the representation of K in $\overset{p}{\wedge} \mathfrak{m}^*C$ which is contragredient to ad^p . Since the p -form ω^I is left-invariant, we have $\omega^I \circ R_k = ad^{*p}(k) \cdot \omega^I$ for all $k \in K$. Put

$$ad^{*p}(k) \cdot \omega^I = \sum_J \tau_J^I(k) \omega^J.$$

We then have $\tilde{\eta} \circ R_k = \sum_J \sum_I \tau_J^I(k) (\eta_I \circ R_k) \omega^J$ and, since $\tilde{\eta} \circ R_k = \tilde{\eta}$, we get

$$\eta_I(g \cdot k) = \sum_J \tau_I^J(k^{-1}) \eta_J(g) \quad (g \in G, k \in K).$$

It follows also from $\tilde{\eta} \circ L_\gamma = \tilde{\eta}$ and $\omega^I \circ L_\gamma = \omega^I$ that

$$\eta_I(\gamma \cdot g) = \eta_I(g) \quad (\gamma \in \Gamma).$$

Hence we may consider η_I as a function on $\Gamma \backslash G$ such that

$$\eta_I(x \cdot k) = \sum_J \tau_I^J(k^{-1}) \eta_J(x)$$

for $x \in \Gamma \backslash G$ and $k \in K$. We may also consider $\tilde{\eta}$ as a $\overset{p}{\wedge} \mathfrak{m}^*C$ -valued function on $\Gamma \backslash G$ defined by

$$\tilde{\eta}(x) = \sum_I \eta_I(x) \omega^I \quad (x \in \Gamma \backslash G).$$

We have then

$$(1) \quad \tilde{\eta}(x \cdot k) = ad^{*p}(k^{-1}) \tilde{\eta}(x).$$

Thus there corresponds to a differential p -form η on G/K invariant by Γ a $\overset{p}{\wedge} \mathfrak{m}^*C$ -valued function on $\Gamma \backslash G$ satisfying the condition (1), and conversely, to each of the functions satisfying (1) corresponds a Γ -invariant p -form and this correspondence is bijective. If the form η is of class C^∞ so is the corresponding function $\tilde{\eta}$; if η is measurable (with respect to the invariant measure on G/K), so is $\tilde{\eta}$ (with respect to the invariant measure on $\Gamma \backslash G$).

Now let Ω_p be the Hilbert space of all Γ -invariant measurable p -forms on G/K such that

$$\|\eta\|^2 = \int_F \langle \eta, \eta \rangle dv < +\infty,$$

where F denotes a compact fundamental domain for Γ , and \langle, \rangle the length of η with respect to the Riemannian metric of G/K . We can show that if η and θ are in Ω_p , and $\tilde{\eta}$ and $\tilde{\theta}$ are the corresponding $\overset{p}{\wedge} \mathfrak{m}^* \mathcal{C}$ -valued functions, then

$$(\theta, \eta) = M \sum_I \int_{\Gamma \backslash G} \theta_I \cdot \tilde{\eta}_I dx,$$

where M is a suitable constant independent of η, θ [5].

Suppose now that η is of class \mathcal{C}^∞ , and let Δ denote the laplacian operator for the p -forms. Then we have

$$(\Delta\theta)_I = C \cdot \theta_I,$$

where C denotes the Casimir operator [5]. Therefore we get

$$(\Delta\theta, \eta) = M \sum_I \int_{\Gamma \backslash G} C\theta_I \cdot \tilde{\eta}_I dx.$$

and θ is harmonic if and only if $C\theta_I = 0$ for all $I = (i_1, \dots, i_p)$.

The Killing form φ of \mathfrak{g} defines a positive definite hermitian inner product φ^* in $\overset{p}{\wedge} \mathfrak{m}^* \mathcal{C}$ invariant by the representation ad^{*p} of K for which $\{\omega^I\}$ is an orthonormal basis. We have then

$$(\theta, \eta) = M \int_{\Gamma \backslash G} \varphi^*(\tilde{\theta}(x), \tilde{\eta}(\tilde{x})) dx.$$

Let

$$\overset{p}{\wedge} \mathfrak{m}^* \mathcal{C} = F_1^* \oplus \dots \oplus F_{s_p}^*$$

be the decomposition of $\overset{p}{\wedge} \mathfrak{m}^* \mathcal{C}$ into the sum of mutually orthogonal irreducible K -invariant subspaces. We may assume that the irreducible representation of K in F_i^* is contragredient to τ_i^p (cf. (2.1)). Let P_i be the projection of $\overset{p}{\wedge} \mathfrak{m}^* \mathcal{C}$ onto F_i^* , and put

$$\tilde{\eta}_i(x) = P_i \tilde{\eta}(x) \quad (x \in \Gamma \backslash G).$$

Then $\tilde{\eta}_i$ is an F_i^* -valued function on $\Gamma \backslash G$ such that

$$\tilde{\eta}_i(xk) = \tau_i^{*p}(k^{-1})\tilde{\eta}_i(x) \quad (x \in \Gamma \backslash G, k \in K).$$

Let η_i be the Γ -invariant p -form corresponding to $\tilde{\eta}_i$. We then have $\eta = \sum_i \eta_i$, and η is harmonic if and only if each η_i is harmonic (cf. [5]).

We denote by $A_{p,i}$ the vector space of all F_i^* -valued C^∞ -functions f on $\Gamma \backslash G$ satisfying the conditions:

$$\begin{aligned} f(x \cdot k) &= \tau_i^{*p}(k^{-1})f(x) \quad (x \in \Gamma \backslash G, k \in K), \\ Cf &= 0. \end{aligned}$$

Then

$$(3.2) \quad \dim h^p(X, \Gamma) = \sum_{i=1}^{s_p} \dim A_{p,i}.$$

4. In this section we shall show that

$$(4.1) \quad \dim A_{p,i} = \sum_{T \in D_0} N(T) \cdot M(T_K; \tau_i^p).$$

Then the theorem follows from (3.2) and (4.1).

Let $\{\zeta^1, \dots, \zeta^m\}$ be an orthonormal basis of F_i^* , and $\{Z_1, \dots, Z_m\}$ the dual basis of the dual vector space F_i of F_i^* . We may consider F_i as an irreducible K -invariant subspace of $\overset{p}{\wedge} \mathfrak{m}^C$ such that

$$\overset{p}{\wedge} \mathfrak{m}^C = F_1 \oplus \dots \oplus F_{s_p},$$

and we may assume that the representation of K in F_i is τ_i^p . To simplify the notation we write τ instead of τ_i^p . Let

$$\tau^*(k)\zeta^\lambda = \sum_{\mu} a_{\mu}^\lambda(k)\zeta^\mu.$$

Then we have

$$\tau(k)z_\lambda = \sum_{\mu} a_{\lambda}^\mu(k^{-1})z_\mu.$$

Let now

$$L^2(\Gamma \backslash G) = \sum_{a=1}^{\infty} \oplus H_a$$

be the decomposition of the Hilbert space $L^2(\Gamma \backslash G)$ into the direct sum of irreducible \mathbf{G} -invariant closed subspaces, and U_a the irreducible unitary representation of \mathbf{G} in H_a induced by U . Further, let

$$H_a = \sum_{b=1}^{\infty} \oplus H_{a,b}$$

be the decomposition of H_a into the direct sum of irreducible K -invariant closed subspaces. We take an index a such that $U_a \in D_0$, and suppose that the representations of K in $H_{a,1}, \dots, H_{a,b_i}$ ($b_i = M((U_a)_K; \tau_i^p)$) are equivalent to $\tau (= \tau_i^p)$. We fix an index b such that $1 \leq b \leq b_i$, and take a basis $\{f_\lambda\}_{\lambda=1, \dots, m}$ of $H_{a,b}$ such that

$$(4.2) \quad U_a(k)f_\lambda = \sum_{\mu} a_\lambda^\mu(k^{-1})f_\mu.$$

If $\{g_\lambda\}_{\lambda=1, \dots, m}$ is another basis of $H_{a,b}$ which satisfies (4.2), then there exists a complex number α such that $g_\lambda = \alpha f_\lambda$ ($\lambda = 1, \dots, m$) by Schur's lemma.

We define an F_i^* -valued function f on $\Gamma \backslash G$ by putting

$$f(x) = \sum_{\lambda} f_\lambda(x)\zeta^\lambda.$$

Then we have

$$f(x \cdot k) = \tau^*(k^{-1})f(x).$$

Let η be the Γ -invariant p -form on G/K corresponding to the function f . We are going to show that η is harmonic. For this purpose we remark first that we have

$$(4.3) \quad (C \cdot h, \varphi) = 0$$

for all $h \in C^\infty(\Gamma \backslash G)$ and $\varphi \in H_a$. In fact, let W_a be the space of regular vectors of H_a , and let $\varphi \in W_a$. Since C is equal to the opposite of the Casimir operator C_U of the representation U , C_U is self-adjoint, and φ is in the domain of C_U , we get $(Ch, \varphi) = -(h, C_U\varphi)$. Now $C_U\varphi = C_{U_a}\varphi = 0$, and hence $(Ch, \varphi) = 0$. Since W_a is dense in H_a , we get $(Ch, \varphi) = 0$ for all $\varphi \in H_a$.

Now let θ be a Γ -invariant p -form of class C^∞ , and $\tilde{\theta}$ the corresponding $\overset{p}{\wedge} m^*C$ -valued function on $\Gamma \backslash G$. Take an orthonormal basis (ξ^1, \dots, ξ^N) of $\overset{p}{\wedge} m^*C$ such that $\xi^\lambda = \zeta^\lambda$ ($\lambda = 1, \dots, m$), and let $\tilde{\theta}(x) = \sum_{\lambda=1}^N \theta_\lambda(x)\xi^\lambda$. We have $\tilde{\eta}(x) = f(x) = \sum_{\lambda=1}^m f_\lambda(x)\xi^\lambda$, and

$$(\Delta\theta, \eta) = M \sum_{\lambda=1}^m (C\theta_\lambda, f_\lambda).$$

Since $f_\lambda \in H_a$, we get $(\Delta\theta, \eta) = 0$ by (4.3). Thus η is orthogonal to the p -forms $\Delta\theta$ and, as is well known, it follows from this that η is of class C^∞ and harmonic. Therefore the functions f_λ are of class C^∞ and satisfy the equation $Cf_\lambda = 0$. It follows then that the function f belongs to $A_{p,i}$. Thus we have shown that to each $H_{a,b}$ with $U_a \in D_0, 1 \leq b \leq M((U_a)_K; \tau_i^p)$, and to each basis $\{f_\lambda\}_{\lambda=1, \dots, m}$ of $H_{a,b}$ satisfying (4.2) there corresponds a function $f_{a,b} \in A_{p,i}$. Moreover, $f_{a,b}$ is independent

of the choice of such a basis $\{f_\lambda\}$ up to a scalar multiple, and these functions $f_{a,b}$ are linearly independent. Therefore we get

$$\dim A_{p,i} \geq \sum_{T \in D_0} N(T)M(T_K; \tau_i^p).$$

Let conversely $f \in A_{p,i}$. We show that f is a linear combination of the functions $f_{a,b}$. Put

$$f(x) = \sum_{\lambda} f_\lambda(x)\zeta^\lambda.$$

We have then

$$(4.4) \quad U(k)f_\lambda = \sum_{\mu} a_\lambda^\mu(k^{-1})f_\mu, \quad Cf_\lambda = 0.$$

Let P_a be the projection operator of $L^2(\Gamma \backslash G)$ such that $P_a\varphi = \varphi$ for $\varphi \in H_a$, and $P_a\varphi = 0$ for $\varphi \in H_b$, $b \neq a$. Then $f_\lambda = \sum_a P_a f_\lambda$. Let W (resp. W_a) be the space of regular vectors of $L^2(\Gamma \backslash G)$ (resp. H_a). Since f_λ is of class C^∞ , f_λ belongs to W , and moreover $P_a f_\lambda \in W_a$ for all a . We have $P_a C_U \varphi = C_{U_a} P_a \varphi$ for $\varphi \in W$, and hence we get $C_{U_a} P_a f_\lambda = 0$, because $C_U f_\lambda = -C f_\lambda = 0$. It follows that $P_a f_\lambda = 0$ for the index a such that $U_a \notin D_0$. Now suppose that $U_a \in D_0$ and $P_a f_\lambda \neq 0$ for an index λ . We see from (4.4) that

$$U_a(k)P_a f_\lambda = \sum_{\mu} a_\lambda^\mu(k^{-1})P_a f_\mu \quad (k \in K).$$

Let F be the linear subspace of H_a spanned by the elements $P_a f_\lambda (\lambda = 1, \dots, m)$. Then F is a K -invariant subspace of H_a , and there exists a K -module homomorphism of F_i onto F which maps Z_λ onto $P_a f_\lambda$. Since $F \neq (0)$ and F_i is an irreducible K -module, this homomorphism is an isomorphism. It follows then that $P_a f_\lambda$ are linearly independent, and F is contained in the direct sum $\sum_{b=1}^{b_i} H_{a,b} (b_i = M((U_a)_K; \tau_i^p))$. Let $\{f_{a,b;\lambda}\}_{\lambda=1, \dots, m}$ be a basis of $H_{a,b}$ satisfying (4.2), and put

$$P_a f_\lambda = \sum_b \sum_{\mu} \alpha_{b,\lambda}^\mu f_{a,b;\mu}.$$

We see easily that the matrix $(\alpha_{b,\lambda}^\mu)_{\lambda, \mu=1, \dots, m}$ commutes with the matrix $(a_\lambda^\mu(k))_{\lambda, \mu=1, \dots, m}$ for all $k \in K$, and hence $(\alpha_{b,\lambda}^\mu)$ is a scalar matrix. Therefore $P_a f_\lambda = \sum_b \alpha_b \cdot f_{a,b;\lambda}$ with $\alpha_b \in \mathcal{C}$, and hence $f = \sum_{\alpha} \sum_{\lambda} P_a f_\lambda \zeta^\lambda = \sum_{a,b} \alpha_b f_{a,b}$. Thus f is a linear combination of the functions $f_{a,b}$. We have thus completed the proof of (4.1) and the theorem is proved.

5. We consider now the special cases where G is the complex uni-modular group $SL(2, \mathbf{C})$ or the proper Lorentz group.

Let $G = SL(2, \mathbf{C})$. A maximal compact subgroup is the special unitary group $SU(2)$, and put $K = SU(2)$. Then G/K is the 3-dimensional hyperbolic space.

The irreducible unitary representations of the compact group K are given as follows:

There is a 1-1 correspondence between the set of equivalence classes of irreducible unitary representations of K and the set of non-negative integers and non-negative half-integers. The irreducible representation ρ_k corresponding to $\frac{k}{2}$ (k : non-negative integer) is realized in the vector space of covariant symmetric tensors of order k constructed over the 2-dimensional complex vector space on which K operators (see [6]).

Now let \mathfrak{m} be the vector space of 2×2 hermitian matrices of trace 0. We then have $\mathfrak{g} = \mathfrak{m} + \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] = \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}$, and the representation $ad_{\mathfrak{m}}$ of \mathfrak{k} in \mathfrak{m} is absolutely irreducible and equivalent to the representation ρ_2 .

The irreducible unitary representation of $SL(2, \mathbf{C})$ are the following [6]:

1. Principal series $U_{m,\rho}$. These representations depend on two parameters m and ρ with $m \in \mathbf{Z}$ and $\rho \in \mathbf{R}$. $U_{m,\rho}$ is the representation in the Hilbert space $H = L^2(\mathbf{C})$, and the unitary operator $U_{m,\rho}(g)$ is defined by

$$(U_{m,\rho}(g)f)(z) = (bz + d)^m |bz + d|^{-m+i\rho-2} f\left(\frac{az + c}{bz + d}\right),$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C}).$$

The representations $U_{m,\rho}$ and $U_{n,\sigma}$ are equivalent if and only if $n = -m$ and $\sigma = -\rho$.

The Casimir operator $C_{m,\rho}$ of $U_{m,\rho}$ is:

$$C_{m,\rho} = \frac{1}{16} \left\{ \left(\frac{m}{2}\right)^2 - \left(\frac{\rho}{2}\right)^2 - 1 \right\} \cdot 1.$$

The irreducible representation ρ_k is contained in $U_{m,\rho}|K$ at most once, and ρ_k is actually contained in $U_{m,\rho}|K$ if and only if $\frac{m}{2}$ equals one of the numbers $\frac{k}{2}, \frac{k}{2} - 1, \frac{k}{2} - 2, \dots$.

2. Supplementary series U_σ ($0 < \sigma < 2$). The representation U_σ is realized in the Hilbert space H of complex-valued function on \mathcal{C} , the inner product (f_1, f_2) in H and the unitary operator $U_\sigma(g)$ are defined as follows:

$$(f_1, f_2) = \int_c \int_c |z_1 - z_2|^{-2+\sigma} f_1(z_1) \overline{f_2(z_2)} dz_1 dz_2,$$

$$(U_\sigma(g)f)(z) = |bz + d|^{-2-\sigma} f\left(\frac{az + c}{bz + d}\right),$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathcal{C}).$$

The Casimir operator C_σ of U_σ is :

$$C_\sigma = \frac{1}{16} \left\{ \left(\frac{\sigma}{2}\right)^2 - 1 \right\} \cdot 1 \quad (0 < \sigma < 2).$$

The representation $U_\sigma|K$ decomposes as follows:

$$U_\sigma|K = \sum_{k=0}^{\infty} \rho_{2k}.$$

Now the Casimir Operator C_σ does not vanish, and the Casimir Operator $C_{m,\rho}$ ($m \geq 0$) vanishes if and only if $\rho = \pm\sqrt{m^2 - 4}$. As ρ is real, we have $m \geq 2$. On the other hand, $U_{m,\rho}|K$ ($m \geq 0$) contains ρ_2 if and only if $m = 2$. Therefore there is one and only one irreducible unitary representation \mathbf{T} of $SL(2, \mathcal{C})$ with vanishing Casimir operator such that $\mathbf{T}|K$ contains ρ_2 , that is, $\mathbf{T} = U_{2,0}$. Moreover, the multiplicity of ρ_2 in $U_{2,0}|K$ is 1.

Let now \mathbf{G} be the proper Lorentz group. Then $G \cong SL(2, \mathcal{C})/\{\pm 1\}$ and $K \cong SU(2)/\{\pm 1\}$. The irreducible unitary representations of K are ρ_{2k} ($k = 0, 1, 2, \dots$), and the irreducible unitary representations \mathbf{T} of \mathbf{G} are those of $SL(2, \mathcal{C})$ satisfying the condition $\mathbf{T}(-1) = 1$, and therefore these representations are $U_{m,\rho}$ with even m and U_σ . Just as in the of $SL(2, \mathcal{C})$, the only irreducible unitary representation \mathbf{T} of \mathbf{G} with vanishing Casimir operator such that $\mathbf{T}|K$ contains ρ_2 is the representation $U_{2,0}$. The multiplicity of ρ_2 in $U_{2,0}|K$ is 1.

From our theorem we then have the following result:

Let \mathbf{G} be the complex unimodular group $SL(2, \mathcal{C})$ or the proper Lorentz group. Let Γ be a discrete subgroup of \mathbf{G} such that $\Gamma \backslash \mathbf{G}$ is compact. Assume that Γ acts freely on the 3-dimensional hyperbolic space G/K . Then the multiplicity of the irreducible unitary representation $U_{2,0}$ of \mathbf{G} in the unitary representation \mathbf{T} of \mathbf{G} in $L^2(\Gamma \backslash \mathbf{G})$ equals the rank of the finitely generated abelian group Γ/Γ' , Γ' being the commutator subgroup of Γ .

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