A Formula for the Colored Jones Polynomial of 2-Bridge Knots

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Abstract. We derive a formula for the colored Jones polynomial of 2-bridge knots. For a twist knot, a more explicit formula is given and it leads to a relation between the degree of the colored Jones polynomial and the crossing number.

1. Introduction

For a positive integer n, the n-colored Jones polynomial is a Laurent polynomial invariant in q of a knot colored with n-dimensional irreducible representation of $U_q(sl_2)$. We denote by $J_K(n)$ the n-colored Jones polynomial of a knot K, which is normalized so that the colored Jones polynomial of the trivial knot is 1. Then, $J_K(2)$ coincides with the original Jones polynomial of a knot K defined by V. F. R. Jones [3]. K. Habiro [2] and T. T. Q. Le gave explicit formulas for the n-colored Jones polynomials of the trefoil knot and the figure eight knot. Moreover, a formula for the colored Jones polynomials of twist knots was given by G. Masbaum [7], using skein theory.

In this paper, we give a formula for the colored Jones polynomial of the 2-bridge knot $S(p, t)$, using a formula obtained by R. Lawrence and O. Ron in [5]. As an application of our formula, we obtain a relation between the degree of the colored Jones polynomial and the crossing number for the 2-bridge knots $S(4p + 1, 2p + 1)$ and $S(4p - 1, 2p - 1)$.

Let p and t be coprime odd integers with $p > t > 1$. We denote by $S(p, t)$ the 2-bridge knot of type (p, t) using Schubert notation. We explain how to draw a diagram of the 2-bridge knot $S(p, t)$(see [1], [4]). Drawn two disjoint vertical segments $w_1 = AB$ and $w_2 = CD$, divide each segments into p segments, and number from 0 to $2p - 1$ in both sides of the dividing points of each segment as pictured in Figure 1. The points B and D are numbered 0, and the points A and C numbered p. Next draw two disjoint simple curves v_1 from B to C and v_2 from D to A, which underpass the segments w_1 and w_2. Along the curve v_1, one starts

from 0, i.e. B of the segments w_1, and meets w_2 at t, and next meets w_1 at $2t$, and then meets w_2 at $3t$. This is to be repeated until one reaches $p \ (= C)$ of w_2. Similarly, along the curve v_2, one starts from $0 (= D)$ of w_2, and meets w_1 at t, and next meets w_2 at $2t$, and then meets w_1 at $3t$. This is to be repeated until one reaches $p \ (= A)$ of w_1. Here we take *it* mod $2p$. This regular projection is called Schubert's normal form. For example, Schubert's normal form for the $S(5,3)$ (the figure eight knot) is as in Figure 1.
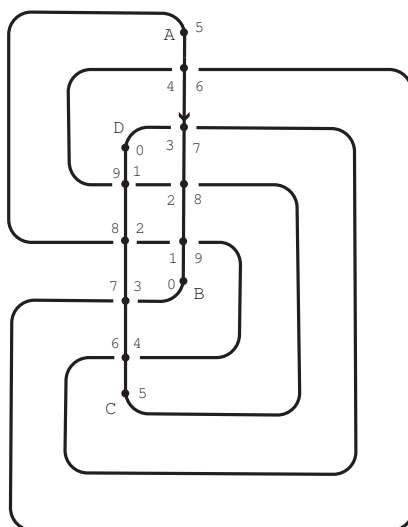
S(5,3)

Figure 1.

We will prepare with some notations. Let us put $p' = \dfrac{p-1}{2}$. For $1 \le j \le p'$, define integers $r(j)$ so that $r(j) \equiv (2j-1)t \bmod 2p$ and $-p < r(j) < p$. We put $\sigma_j := \dfrac{r(j)}{|r(j)|}$, $r'(j) := \dfrac{|r(j)|+1}{2}$ and $i_{r'(j)} = j$. Note that $\sigma_j = (-1)^{[(2j-1)t/p]}$, where $[x]$ denotes the greatest integer less than or equal to x. For an integer i, sign(i) is the sign of i. We put $\mathbf{m} = (m_1, \cdots, m_{p'})$, $m_0 = 0$. We will also define for $\sigma = \pm 1$

$$(q^\sigma)_k = (q^\sigma - 1)((q^\sigma)^2 - 1) \cdots ((q^\sigma)^k - 1), \quad \left\{ \begin{array}{c} n \\ m \end{array} \right\}_\sigma = \frac{(q^\sigma)_n}{(q^\sigma)_m (q^\sigma)_{n-m}}.$$

We obtain

Theorem 1. *A formula for the n-colored Jones polynomial of the 2-bridge knot*

$S(p,t)$ *is given by*

(1) $$J_{S(p,t)}(n) = \sum_{0 \le m_{j-1} \le m_j \le n-1} q^{a(\mathbf{m})n + b_1(\mathbf{m}) + b_2(\mathbf{m})} X(\mathbf{m}),$$

where

$$a(\mathbf{m}) = -\frac{1}{2}\sum_{j=1}^{p'}\left(\sum_{k=r'(j)}^{p'}(\sigma_{i_k} + \sigma_{i_{p'+1-k}})\right)(m_j - m_{j-1}) - \frac{1}{2}\sum_{j=1}^{p'-1}(\sigma_{j+1} + \sigma_{p'+1-j})m_j$$

$$-\frac{1}{2}(\sigma_{p'}+1)m_{p'} - \sum_{j=1}^{p'}\sigma_j,$$

$$b_1(\mathbf{m}) = -a(\mathbf{m})$$

$$+\sum_{1 \le k \le \frac{p-t}{2}}\frac{1-\sigma_{i_k}}{2}m_{i_k-1} - \sum_{\frac{p-t}{2}+1 \le k \le p'}m_{i_k-1} + \sum_{\frac{p-t}{2}+1 \le k \le p'}\frac{1+\sigma_{i_k}}{2}m_{i_k}$$

$$-(1+\sigma_{p'})m_{p'} + \frac{1}{2}\sum_{j=1}^{p'-1}(\sigma_{j+1} - \sigma_j)m_j$$

$$-\frac{1}{2}\sum_{1 \le k \le p'-1}\sum_{k+1 \le k' \le p'}\frac{1+\text{sign}(i_k - i_{k'})}{2}(\sigma_{i_k} - \sigma_{i_{k'}})(m_{i_k} - m_{i_k-1})(m_{i_{k'}} - m_{i_{k'}-1})$$

$$+\sum_{j=1}^{p'}\sigma_j\left(\sum_{k=1}^{r'(j)}(m_{i_k} - m_{i_k-1})\right)m_{j-1},$$

$$b_2(\mathbf{m}) = \begin{cases} \sum_{\frac{p-t}{2}+1 \le k \le \frac{t-1}{2}}\dfrac{1+\sigma_{i_k}}{2}m_{i_k-1} & \text{if} \quad p < 2t, \\ -\sum_{\frac{t+1}{2}+1 \le k \le \frac{p-t}{2}}\dfrac{1+\sigma_{i_k}}{2}m_{i_k-1} & \text{if} \quad p > 2t, \end{cases}$$

$$X(\mathbf{m}) = \frac{(\bar{q}^{\sigma_{p'}})_{n-1}(\bar{q}^{\sigma_{p'}})_{m_{p'}}}{(\bar{q}^{\sigma_{p'}})_{n-m_{p'}-1}}\left(\prod_{j=1}^{p'}(q^{-\sigma_j})_{m_j-m_{j-1}}\right)^{-1}(-1)^{m_{p'}}.$$

The plan of the present paper is as follows. In section 2, we review Lawrence and Ron's result and give a proof for Theorem 1. In section 3, we get a more explicit formula for the colored Jones polynomial of twist knots and estimate the maximal and minimal degrees.

2. Proof of Theorem 1

Firstly, we review a result in [5]. Let L be a knot in S^3. We consider a 1-tangle L' whose closure is L. We assume that L' is presented as a tangle diagram D in

generic position with respect to the height function with the end points at the top
and at the bottom. We orient D from the top to the bottom. Let c denote the
number of the crossings in a diagram D. There are eight crossing types given by
the orientations on the two strands and the sign of the crossing, two cap types,
and two cup types by the orientations on the strand in a tangle diagram. The two
crossoing types (1) and (2), the cap type (i), and the cup type (ii) as in Figure 2
are called leftpointing. We construct an *enhanced Gauss diagram* G for D, which is
a circle containing the combinatorial data which encodes D. We number crossings
by $j = 1, \cdots, c$ and denote by σ_j the sign of the j-th crossing. We mark the base
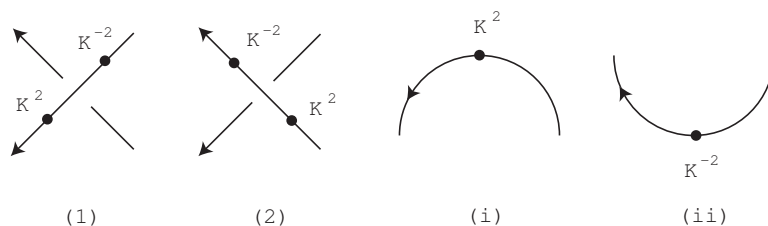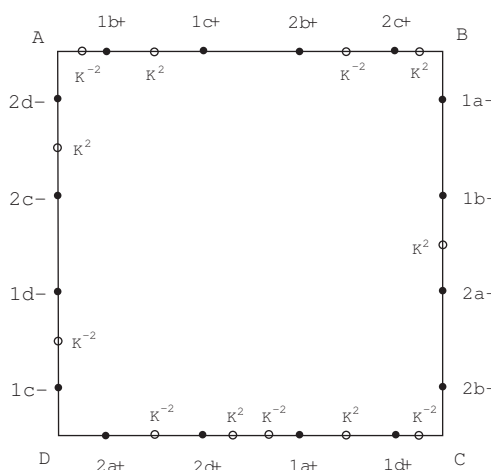

Figure 2.

Figure 3.

point on a circle, which corresponds to the end at the top of D. As going around the
circle clockwise from the base point, with tracing the tangle strand according to its
orientation, we mark $2c$ points indexed by $i = 1, \cdots, 2c$, which correspond to points
passing crossings on the tangle strand. So, exactly two points of these $2c$ points
except the base point correspond to one crossing, and one comes from the underpass

and the other from the overpass. Join these two points on the circle by an oriented chord from overpass to underpass, and assign the sign of the corresponding crossing. Each i-th point on the circle is assigned with a pair $(j(i), \varepsilon_i)$, simply denoted by $j(i)\varepsilon_i$, where $j(i)$ is the number of the passing crossing, and ε_i is 1 for an overpass and -1 for an underpass. Moreover, the circle is marked with blobs $K^{\pm 2}$ placed on the leftpointing crossings and cups and caps as in Figure 2. The result is an enhanced Gauss diagram. For example, the enhanced Gauss diagram for Figure 1 is shown in Figure 3, where oriented chords are not drawn. We denote by $i_1, \cdots, i_b$ the position of the blobs $K^{\pm 2}$, where the a-th blob is placed between the i_a-th and $(i_a + 1)$-th positions on the circle, with $K^{2\delta_a}$.

We put

$$q_j = \frac{1}{2}\sum_{i>j+}\sigma_i - \frac{1}{2}\sum_{i>j-}\sigma_i - \frac{1}{2}\sum_{j=1}^{c}\sigma_j,$$

$$d(\mathbf{l}) = \sum_{j=1}^{c}q_j l_j - \frac{1}{2}\sum_{a}\delta_a - \frac{1}{2}\sigma_j,$$

$$C^0(\mathbf{l}) = -d(\mathbf{l}) - \sum_{\substack{a \\ k\leq i_a}}\delta_a \varepsilon_k l_k - \frac{1}{2}\sum_{j}\sigma_j l_j - \frac{1}{2}\sum_{\substack{i \\ k<i}}\sigma_i \varepsilon_i l_i \varepsilon_k l_k$$
$$+ \sum_{j}\sigma_j \left(\sum_{i<j-}\varepsilon_i l_i\right)\left(\sum_{k<j+}\varepsilon_k l_k\right).$$

Theorem 2([5]). *A formula for the n-colored Jones polynomial of a 0-framed knot L_0 is given by*

(2) $J_{L_0}(n)$

$$= \sum_{\substack{l_j=0 \\ j=0,\ldots,c}}^{\infty} q^{C^n(\mathbf{l})}\prod_{j=1}^{c}\left\{\begin{matrix} s(j)+l_j \\ l_j \end{matrix}\right\}_{\sigma_j}(q^{\sigma_j(n-s(j)-1)}-1)\cdots(q^{\sigma_j(n-s(j)-l_j)}-1),$$

where $s(j) = -\sum_{i<j-}\varepsilon_i l_{j(i)}$, and $C^n(\mathbf{l}) = d(\mathbf{l})n + C^0(\mathbf{l})$. Here $i < j-$ means $i < i_0$ for i_0 such that $j = j(i_0)$ and $\varepsilon_{i_0} = -1$.

We apply this formula to the 2-bridge knot $S(p, t)$.

From Schubert's normal form, we get a 1-tangle diagram as in Figure 4 and 5. The corresponding enhanced Gauss diagram without blobs and chords can be expressed in Figure 7. The number of the crossings equals $2(p-1) = 4p'$. We name by ja (resp. jd) the crossing of the label $(2j-1)t \mod 2p$ (resp. $2jt \mod 2p$) on CD for $1 \leq j \leq p'$ and name by jc (resp. jb) the crossing of the label $(2j-1)t \mod 2p$ (resp. $2jt \mod 2p$) on AB for $1 \leq j \leq p'$. For $1 \leq k \leq p'$, there exists a unique $1 \leq j \leq p'$ such that $2k-1 \equiv |r(j)| \mod 2p$, and then $i_k a = ja$ and $i_k c = jc$. For $1 \leq j \leq p'$, we define integers $\tilde{r}(j)$ so that $\tilde{r}(j) \equiv 2jt \mod 2p$ and

$-p < \tilde{r}(j) < p$. For $1 \le k \le p'$, there exists a unique $1 \le j \le p'$ such that $2k \equiv |\tilde{r}(j)| \mod 2p$, and then $i_k{'}b = jb$ and $i_k{'}d = jd$. We have the following lemma.

Lemma 3. (1) $\sigma_{jc} = \sigma_{ja} = (-1)^{[(2j-1)t/p]}$, and $\sigma_{jd} = \sigma_{jb} = (-1)^{[2jt/p]}$.
(2) $\sigma_{ja} = \sigma_{(p'+1-j)b}$ and $\sigma_{i_k{'}b} = \sigma_{i_{p'+1-k}a}$.

Proof. (1) It is clear from the definition.
(2) Since t is odd, $2(p'+1-j)t = (p+1-2j)t \equiv pt+(1-t)p-(2j-1)t = p-(2j-1)t$ mod $2p$. If $2jt \equiv \pm 2k \mod 2p$ for $1 \le k \le p'$, $2(p'+1-k)-1 = p-2k \equiv p \mp 2jt \equiv p \mp 2jt + (t-1)p \equiv (p \mp 2j)t \mod 2p$. So the claim holds. $\qquad\square$
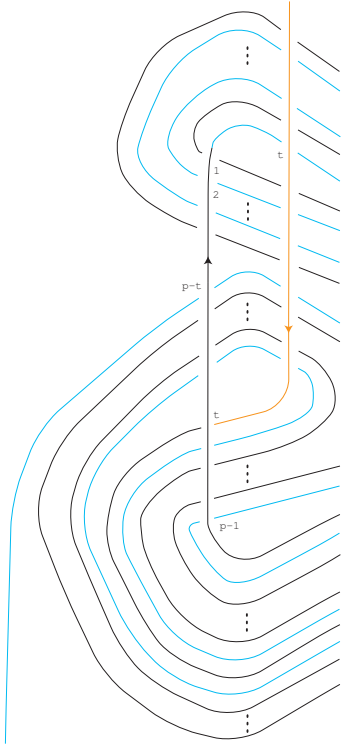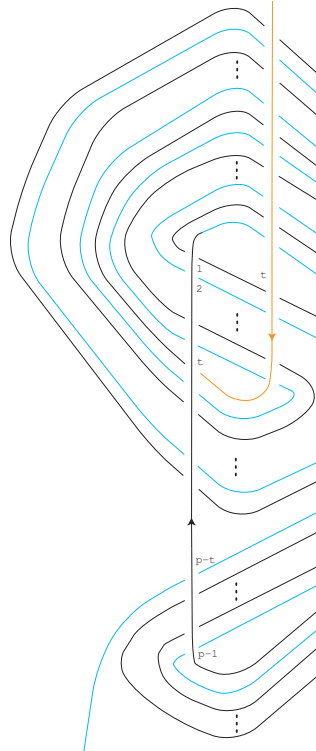
Figure 4. $p < 2t$ Figure 5. $p > 2t$

Let us compute $s(ja)$, $s(jb)$, $s(jc)$, $s(jd)$ for $j = 1, \cdots, p'$. Since the term with $s(j) < 0$ does not contribute to the sum in the formula (2), $s(1a) = -\sum_{j=1}^{p'}(l_{jb}+l_{jc})$ implies that $l_{jb}, l_{jc} = 0$ for $j = 1, \cdots, p'$ and $s(1a) = 0$. It also follows that $s(jb) = s((j+1)a) = \sum_{k=1}^{j} l_{ka}$ for $1 \le j \le p'-1$ and $s(p'b) = \sum_{k=1}^{p'} l_{ka}$. We also have $s(1c) = -\sum_{j=1}^{p'} l_{jd}$ and so it follows that $l_{jd} = 0$ for $1 \le j \le p'$ and $s(1c) = 0$. Thus only parameters l_{ja} appear in the formula (2). We may denote l_{ja} by simply
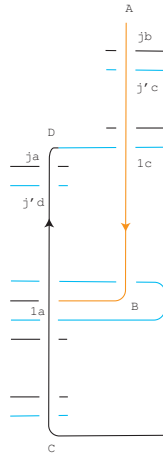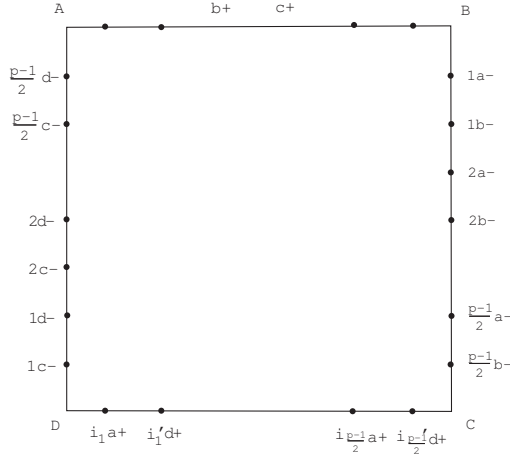
Figure 6.   Figure 7.

l_j for $1 \le j \le p'$. Moreover, we compute terms in $C^0(\mathbf{1})$. If $j'a+ < ja+$, then $\pm l_{j'a}$ appear in the sum $\sum_{i<ja+} \varepsilon_i l_i$. So, we have

$$(3) \qquad \sum_{i<ja+} \varepsilon_i l_i = - \sum_{j'a+\ge ja+} l_{j'} = - \sum_{r'(j')\le r'(j)} l_{j'} = - \sum_{k=1}^{r'(j)} l_{i_k}.$$

Similarly, noting that $l_{jd} = 0$ and that jd corresponds to $2jt$ on the segment CD, we have

$$(4) \qquad \sum_{i<jd+} \varepsilon_i l_i = - \sum_{r'(j')\le r'(j)} l_{j'} = - \sum_{k=1}^{r'(j)} l_{i_k}.$$

An easy observation gives

$$(5) \qquad \sum_{i<jb+} \varepsilon_i l_i = \sum_{i<jc+} \varepsilon_i l_i = 0.$$

So, from (3), (4), (5) and $s(jc) = s(jd) = 0$,

$$(6) \qquad \sum_j \sigma_j \Big(\sum_{i<j+} \varepsilon_i l_i\Big)\Big(\sum_{k<j-} \varepsilon_k l_k\Big) = \sum_{j=1}^{p'} \sigma_{ja}\Big(-\sum_{k=1}^{r'(j)} l_{i_k}\Big)(-s(ja)).$$

(Recall that $i_{r'(j)} = j$). Now we compute q_j. Since $ja+ > ja-$ for any ja, using

Lemma 3, we get

$$
\begin{aligned}
q_{ja} &= \frac{1}{2}\Big(\sum_{i>ja+}\sigma_i - \sum_{i>ja-}\sigma_i - \sigma_{ja}\Big) \\
&= -\frac{1}{2}\sum_{ja+\geq i\geq ja-}\sigma_i \\
&= -\frac{1}{2}\Big(\sum_{k=r'(j)}^{p'}(\sigma_{i_k a}+\sigma_{i'_k b}) + \sum_{k=j}^{p'}(\sigma_{ka}+\sigma_{kb})\Big) \\
&= -\frac{1}{2}\Big(\sum_{k=r'(j)}^{p'}(\sigma_{i_k a}+\sigma_{i_{p'+1-k}a}) + \sum_{k=j}^{p'}(\sigma_{ka}+\sigma_{(p'+1-k)a})\Big).
\end{aligned}
$$

We will write $\sigma_{ja}=\sigma_j$.

We consider the case where $p < 2t$. Let us calculate $\sum_a \delta_a$. From Figure 4, a blob of type (i) appears when the sign of the crossing corresponding to k on the segment CD with $1 \leq k \leq p-t-1$ is -1 and when the sign of the crossing corresponding to k on the segment CD with $p-t+1 \leq k \leq t-1$ is $+1$. Note that the blobs in the later case appear between the segment AB and CD (see Figrue 8 and 9). A blob of type (ii) appears when the sign of the crossing corresponding to k on the segment CD with $p-t+1 \leq k \leq p-1$ is -1. Moreover one blob of the type (i) appears before the curve v_2 reaches the crossing $p'd$ and one blob of type (ii) appears when the curve v_1 meets the point C.
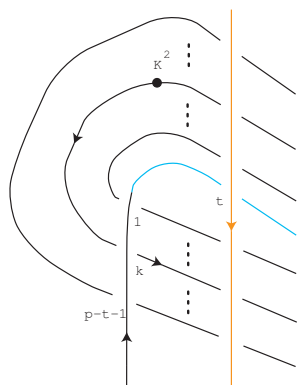

Figure 8.
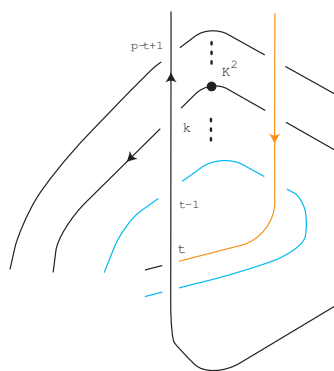

Figure 9.

Thus

$$
\begin{aligned}
\sum \delta_a &= \sum_{1 \le k \le \frac{p-t}{2}} \frac{1 - \sigma_{i_k}}{2} - \sum_{\frac{p-t}{2}+1 \le k \le p'} \frac{1 - \sigma_{i_k}}{2} + \sum_{\frac{p-t}{2}+1 \le k \le \frac{t-1}{2}} \frac{1 + \sigma_{i_k}}{2} \\
&\quad + \sum_{1 \le k \le \frac{p-t}{2}-1} \frac{1 - \sigma_{i_{p'+1-k}}}{2} - \sum_{\frac{p-t}{2}+1 \le k \le p'} \frac{1 - \sigma_{i_{p'+1-k}}}{2} + \sum_{\frac{p-t}{2}+1 \le k \le \frac{t-1}{2}} \frac{1 + \sigma_{i_{p'+1-k}}}{2} \\
&= \sum_{1 \le k \le \frac{p-t}{2}} \frac{1 - \sigma_{i_k}}{2} + \sum_{\frac{p-t}{2}+1 \le k \le \frac{t-1}{2}} \sigma_{i_k} - \sum_{\frac{t+1}{2}+1 \le k \le p'} \frac{1 - \sigma_{i_k}}{2} \\
&\quad + \sum_{1 \le k \le \frac{p-t}{2}-1} \frac{1 - \sigma_{i_{p'+1-k}}}{2} + \sum_{\frac{p-t}{2}+1 \le k \le \frac{t-1}{2}} \sigma_{i_{p'+1-k}} - \sum_{\frac{t+1}{2} \le k \le p'} \frac{1 - \sigma_{i_{p'+1-k}}}{2} \\
&= 2 \sum_{\frac{p-t}{2}+1 \le k \le \frac{t-1}{2}} \sigma_{i_k}.
\end{aligned}
$$

Here we used Lemma 3.

It is easy to see that $\sum \sigma_j = 4 \sum_{j=1}^{p'} \sigma_{ja}$. Therefore, we obtain

$$
\begin{aligned}
d(\mathbf{l}) &= -\frac{1}{2} \sum_{j=1}^{p'} \Big\{ \sum_{k=r'(j)}^{p'} (\sigma_{i_k} + \sigma_{i_{p'+1-k}}) + \sum_{k=j}^{p'} (\sigma_k + \sigma_{p'+1-k}) \Big\} l_j \\
&\quad - \sum_{\frac{p-t}{2}+1 \le k \le \frac{t-1}{2}} \sigma_{i_k} - 2 \sum_{j=1}^{p'} \sigma_j \\
&= -\frac{1}{2} \sum_{j=1}^{p'} \Big\{ \sum_{k=r'(j)}^{p'} (\sigma_{i_k} + \sigma_{i_{p'+1-k}}) + \sum_{k=j}^{p'} (\sigma_k + \sigma_{p'+1-k}) \Big\} l_j - \sum_{j=1}^{p'} \sigma_j.
\end{aligned}
$$

To show the second equality above, we need the following lemma.

Lemma 4. (1) If $p < 2t$, then $\displaystyle \sum_{\frac{p-t}{2}+1 \le k \le \frac{t-1}{2}} \sigma_{i_k} = -\sum_{j=1}^{p'} \sigma_j.$

(2) If $p > 2t$, then $\displaystyle \sum_{\frac{t+1}{2} \le k \le \frac{p-t}{2}} \sigma_{i_k} = \sum_{j=1}^{p'} \sigma_j.$

Proof. In the case where $p < 2t$, we focus the signs of the crossings labeled by even on the segment w_1. Note that the sum of their signs equals $\sum_{j=1}^{p'} \sigma_j$. On the other hand, from Figure 4, it follows that the sum of the signs of the crossings labeled even j so that $2t - p + 1 \le j \le p - 1$ equals 0 and that the sum of the sign of the crossing labeled even j so that $2 \le j \le 2t - p - 1$ equals $-\sum_{\frac{p-t}{2}+1 \le k \le \frac{t-1}{2}} \sigma_{i_k}$. So

(1) holds. Similarly, in the case where $p > 2t$, we examine the sum of the sign of the crossing labeled even on the segment w_1 in Figure 5. Note that the sum of their signs equals $\sum_{j=1}^{p'} \sigma_j$. From the diagram, it follows that the sum of the sign of the crossing labeled even j so that $2 \leq j \leq 2t$ equals 0 and that the sum of the sign of the crossing labeled even j so that $2t + 2 \leq j \leq p - 1$ equals $\sum_{\frac{t+1}{2} \leq k \leq \frac{p-t}{2}} \sigma_{i_k}$. So we get the equality (2). $\qquad\square$

Next, we calculate the term $\sum_a \delta_a \sum_{k < i_a} \varepsilon_k l_k$, which comes from the crossing of type (2) on CD and blobs of both types on the curve v_1. All blobs of type (i) give

$$- \sum_{1 \leq k \leq \frac{p-t}{2}} \frac{1 - \sigma_{i_k}}{2} \sum_{1 \leq s \leq i_k - 1} l_s - \sum_{\frac{p-t}{2} + 1 \leq k \leq \frac{t-1}{2}} \frac{1 + \sigma_{i_k}}{2} \sum_{1 \leq s \leq i_k - 1} l_s,$$

and all blobs of type (ii) except the special blob of type (ii) near C give

$$\sum_{\frac{p-t}{2} + 1 \leq k \leq p'} \frac{1 - \sigma_{i_k}}{2} \sum_{1 \leq s \leq i_k - 1} l_s,$$

and all crossings of type (2) give

$$\sum_{\frac{p-t}{2} + 1 \leq k \leq p'} \frac{1 + \sigma_{i_k}}{2} (-l_{i_k}),$$

and the special blob gives $\sum_{j=1}^{p'} l_j$.

Moreover, we compute the term $\sum_{\substack{i \\ k < i}} \sigma_i \varepsilon_i l_i \varepsilon_k l_k$. When $i < 1a-$, $i > i_1 a+$, $i = jb-$, we have $\sigma_i \varepsilon_i l_i = 0$. If $i = 1a-$, then $\sum_{\substack{i \\ k < i}} \sigma_i \varepsilon_i l_i \varepsilon_k l_k = 0$. If $i = ja-$ for $2 \leq j \leq p'$,

$$(7) \qquad \sigma_i \varepsilon_i l_i \sum_{k < i} \varepsilon_k l_k = \sigma_j (-1) l_j \left(-\sum_{k=1}^{j-1} l_k \right),$$

and if $i = i_{k'} a+$ for $1 \leq k' \leq p'$,

$$(8) \qquad \sigma_i \varepsilon_i l_i \sum_{k < i} \varepsilon_k l_k = \sigma_{i_{k'}} l_{i_{k'}} \left(-\sum_{k=1}^{k'} l_{i_k} \right).$$

We remark that $i_{k'+1} a \pm, \cdots, i_{p'} a \pm$ appear in the sum in the left hand side of (8). We can see that the term $\sigma_{i_k} l_{i_k} (-l_{i_m})$ with $k < m$ and $j' = i_k > i_m = j$ in (8) is

cancelled with the term $\sigma_{j'}(-1)l_{j'}(-l_j)$ in (7). Thus, we get

$$
\begin{aligned}
\sum_{\substack{i\\k<i}} \sigma_i \varepsilon_i l_i \varepsilon_k l_k &= -\sum_{j=1}^{p'} \sigma_j l_j^2 + \sum_{\substack{j>j'\\ja+>j'a+}} (\sigma_{ja} - \sigma_{j'a})l_{ja}l_{j'a} \\
&= -\sum_{j=1}^{p'} \sigma_j l_j^2 + \sum_{1\le k\le p'-1}\sum_{k+1\le k'\le p'} \frac{1+\text{sign}(i_k-i_{k'})}{2}(\sigma_{i_k}-\sigma_{i_{k'}})l_{i_k}l_{i_{k'}}.
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
C^0(\mathbf{l}) &= -d(\mathbf{l}) + \sum_{1\le k\le \frac{p-t}{2}} \frac{1-\sigma_{i_k}}{2} \sum_{1\le s\le i_k-1} l_s + \sum_{\frac{p-t}{2}+1\le k\le \frac{t-1}{2}} \frac{1+\sigma_{i_k}}{2} \sum_{1\le s\le i_k-1} l_s \\
&\quad - \sum_{\frac{p-t}{2}+1\le k\le p'} \frac{1-\sigma_{i_k}}{2} \sum_{1\le s\le i_k-1} l_s + \sum_{\frac{p-t}{2}+1\le k\le p'} \frac{1+\sigma_{i_k}}{2} l_{i_k} - \sum_{j=1}^{p'} l_j - \frac{1}{2}\sum_{j=1}^{p'} \sigma_j l_j \\
&\quad - \frac{1}{2}\left\{ -\sum_{j=1}^{p'} \sigma_j l_j^2 + \sum_{1\le k\le p'-1}\sum_{k+1\le k'\le p'} \frac{1+\text{sign}(i_k-i_{k'})}{2}(\sigma_{i_k}-\sigma_{i_{k'}})l_{i_k}l_{i_{k'}}\right\} \\
&\quad + \sum_{1\le k\le p'} \sigma_j \left(-\sum_{k=1}^{j-1} l_{i_k}\right)\left(-\sum_{k=1}^{r'(j)} l_{i_k}\right) \\
&= -d(\mathbf{l}) + \sum_{1\le k\le \frac{p-t}{2}} \frac{1-\sigma_{i_k}}{2} \sum_{1\le s\le i_k-1} l_s + \sum_{\frac{p-t}{2}+1\le k\le \frac{t-1}{2}} \frac{1+\sigma_{i_k}}{2} \sum_{1\le s\le i_k-1} l_s \\
&\quad + \sum_{\frac{p-t}{2}+1\le k\le p'} \frac{1+\sigma_{i_k}}{2} l_{i_k} - \sum_{\frac{p-t}{2}+1\le k\le p'} \frac{1-\sigma_{i_k}}{2} \sum_{1\le s\le i_k-1} l_s - \sum_{j=1}^{p'} l_j - \frac{1}{2}\sum_{j=1}^{p'} \sigma_j l_j \\
&\quad + \frac{1}{2}\sum_{j=1}^{p'} \sigma_j l_j^2 - \frac{1}{2}\left\{ \sum_{1\le k\le p'-1}\sum_{k+1\le k'\le p'} \frac{1+\text{sign}(i_k-i_{k'})}{2}(\sigma_{i_k}-\sigma_{i_{k'}})l_{i_k}l_{i_{k'}}\right\} \\
&\quad + \sum_{1\le k\le p'} \sigma_j \left(\sum_{k=1}^{j-1} l_{i_k}\right)\left(\sum_{k=1}^{r'(j)} l_{i_k}\right).
\end{aligned}
$$

Similarly, we compute $d(\mathbf{l})$ and $C^0(\mathbf{l})$ in the case where $p > 2t$. From Figure 5, one can see that a blob of type (i) appears when the sign of the crossing corresponding to k on the segment CD with $1 \le k \le p-t-1$ is -1. A blob of type (ii) appears when the sign of the crossing corresponding to k on the segment CD with $p-t+1 \le k \le p-1$ is -1 and when the sign of the crossing corresponding to k on the segment CD with $t+1 \le k \le p-t-1$ is $+1$. Moreover one blob of the

type (ii) appears before the curve v_1 arrives at the crossing $1a$ and one blob of the same type appears when the curve v_1 meets the point C. Therefore, by Lemma 3, it follows that

$$
\begin{aligned}
\sum \delta_a \;=\;& \sum_{1 \le k \le \frac{p-t}{2}} \frac{1 - \sigma_{i_k}}{2} - \sum_{\frac{p-t}{2}+1 \le k \le p'} \frac{1 - \sigma_{i_k}}{2} - \sum_{\frac{t+1}{2}+1 \le k \le \frac{p-t}{2}} \frac{1 + \sigma_{i_k}}{2} \\
&+ \sum_{1 \le k \le \frac{p-t}{2}-1} \frac{1 - \sigma_{i_{p'+1-k}}}{2} - \sum_{\frac{p-t}{2}+1 \le k \le p'} \frac{1 - \sigma_{i_{p'+1-k}}}{2} \\
&- \sum_{\frac{t+1}{2} \le k \le \frac{p-t}{2}-1} \frac{1 + \sigma_{i_{p'+1-k}}}{2} - 2 \\
=\;& \sum_{1 \le k \le \frac{t+1}{2}} \frac{1 - \sigma_{i_k}}{2} - \sum_{\frac{t+1}{2}+1 \le k \le \frac{p-t}{2}} \sigma_{i_k} - \sum_{\frac{p-t}{2}+1 \le k \le p'} \frac{1 - \sigma_{i_k}}{2} \\
&+ \sum_{1 \le k \le \frac{t-1}{2}} \frac{1 - \sigma_{i_{p'+1-k}}}{2} - \sum_{\frac{t+1}{2} \le k \le \frac{p-t}{2}-1} \sigma_{i_{p'+1-k}} - \sum_{\frac{p-t}{2}+1 \le k \le p'} \frac{1 - \sigma_{i_{p'+1-k}}}{2} - 2 \\
=\;& -2 \sum_{\frac{t+1}{2} \le k \le \frac{p-t}{2}} \sigma_{i_k}.
\end{aligned}
$$

Here we used $\sigma_{i_{\frac{t+1}{2}}} = 1$ in the last equality. Accordingly, from Lemma 4, we obtain

$$
\begin{aligned}
d(\mathbf{1}) \;=\;& -\frac{1}{2} \sum_{j=1}^{p'} \Big\{ \sum_{k=r'(j)}^{p'} (\sigma_{i_k} + \sigma_{i_{p'-k}}) + \sum_{k=j}^{p'} (\sigma_k + \sigma_{p'+1-k}) \Big\} l_j \\
&+ \sum_{\frac{t+1}{2}+1 \le k \le \frac{p-t}{2}} \sigma_{i_k} - 2 \sum_{j=1}^{p'} \sigma_j \\
=\;& -\frac{1}{2} \sum_{j=1}^{p'} \Big\{ \sum_{k=r'(j)}^{p'} (\sigma_{i_k} + \sigma_{i_{p'+1-k}}) + \sum_{k=j}^{p'} (\sigma_k + \sigma_{p'+1-k}) \Big\} l_j - \sum_{j=1}^{p'} \sigma_j.
\end{aligned}
$$

Next, we calculate the term $\sum_a \delta_a \sum_{k \le i_a} \varepsilon_k l_k$. Non-trivial terms correspond to the crossing of type (2) and blobs of both type on the curve v_1. All blobs of type (i) give

$$
- \sum_{1 \le k \le \frac{p-t}{2}} \frac{1 - \sigma_{i_k}}{2} \sum_{1 \le s \le i_k - 1} l_s,
$$

and all blobs of type (ii) except the special blob near C give

$$
\sum_{\frac{p-t}{2}+1 \le k \le p'} \frac{1 - \sigma_{i_k}}{2} \sum_{1 \le s \le i_k - 1} l_s + \sum_{\frac{t+1}{2}+1 \le k \le \frac{p-t}{2}} \frac{1 + \sigma_{i_k}}{2} \sum_{1 \le s \le i_k - 1} l_s,
$$

and all crossings of type (2) give

$$\sum_{\frac{p-t}{2}+1\le k\le p'}\frac{1+\sigma_{i_k}}{2}(-l_{i_k}),$$

and the special blob gives $\sum_{j=1}^{p'}l_j$.

From Figure 7, we can see

$$\sum_{\substack{i\\k<i_a}}\sigma_i\varepsilon_il_i\varepsilon_kl_k$$

$$=\quad -\sum_{j=1}^{p'}\sigma_jl_j{}^2+\sum_{\substack{j>j'\\ja+>j'a+}}(\sigma_{ja}-\sigma_{j'a})l_{ja}l_{j'a}$$

$$=\quad -\sum_{j=1}^{p'}\sigma_jl_j{}^2+\sum_{1\le k\le p'-1}\sum_{k+1\le k'\le p'}\frac{1+\text{sign}(i_k-i_{k'})}{2}(\sigma_{i_k}-\sigma_{i_{k'}})l_{i_k}l_{i_{k'}},$$

and so it is obtained that

$$
\begin{aligned}
C^0(\mathbf{1})\quad =\quad & -d(\mathbf{1})\\
& +\sum_{1\le k\le\frac{p-t}{2}}\frac{1-\sigma_{i_k}}{2}\sum_{1\le s\le i_k-1}l_s-\sum_{\frac{p-t}{2}+1\le k\le p'}\frac{1-\sigma_{i_k}}{2}\sum_{1\le s\le i_k-1}l_s\\
& -\sum_{\frac{t+1}{2}+1\le k\le\frac{p-t}{2}}\frac{1+\sigma_{i_k}}{2}\sum_{1\le s\le i_k-1}l_s-\sum_{\frac{p-t}{2}+1\le k\le p'}\frac{1+\sigma_{i_k}}{2}(-l_{i_k})\\
& -\sum_{j=1}^{p'}l_j-\frac{1}{2}\sum_{j=1}^{p'}\sigma_jl_j\\
& -\frac{1}{2}\{-\sum_{j=1}^{p'}\sigma_jl_j{}^2+\sum_{1\le k\le p'}\sum_{k+1\le k'\le p'}\frac{1+\text{sign}(i_k-i_{k'})}{2}(\sigma_{i_k}-\sigma_{i_{k'}})l_{i_k}l_{i_{k'}}\}\\
& +\sum_{1\le k\le p'}\sigma_j\left(-\sum_{k=1}^{j-1}l_k\right)\left(-\sum_{k=1}^{r'(j)}l_{i_k}\right)\\
=\quad & -d(\mathbf{1})+\sum_{1\le k\le\frac{p-t}{2}}\frac{1-\sigma_{i_k}}{2}\sum_{1\le s\le i_k-1}l_s+\sum_{\frac{p-t}{2}+1\le k\le p'}\frac{1+\sigma_{i_k}}{2}l_{i_k}\\
& -\sum_{\frac{p-t}{2}+1\le k\le p'}\frac{1-\sigma_{i_k}}{2}\sum_{1\le s\le i_k-1}l_s-\sum_{\frac{t+1}{2}+1\le k\le\frac{p-t}{2}}\frac{1+\sigma_{i_k}}{2}\sum_{1\le s\le i_k-1}l_s
\end{aligned}
$$

$$- \sum_{j=1}^{p'} l_j - \frac{1}{2} \sum_{j=1}^{p'} \sigma_j l_j + \frac{1}{2} \sum_{j=1}^{p'} \sigma_j l_j{}^2$$

$$- \frac{1}{2} \left\{ \sum_{1 \leq k \leq p'} \sum_{k+1 \leq k' \leq p'} \frac{1 + \operatorname{sign}(i_k - i_{k'})}{2} (\sigma_{i_k} - \sigma_{i_{k'}}) l_{i_k} l_{i_{k'}} \right\}$$

$$+ \sum_{1 \leq k \leq p'} \sigma_j (\sum_{k=1}^{j-1} l_k)(\sum_{k=1}^{r'(j)} l_{i_k}).$$

We compute the remaining term in the formula (2). Since $\sigma_1 = 1$, $s(1a) = 0$, and $s(ja) = \sum_{k=1}^{j-1} l_k (2 \leq j \leq p')$,

$$\prod_{j=1}^{c} \left\{ \begin{matrix} s(j) + l_j \\ l_j \end{matrix} \right\}_{\sigma_j} (q^{\sigma_j(n - s(j) - 1)} - 1) \cdots (q^{\sigma_j(n - s(j) - l_j)} - 1)$$

$$= \prod_{j=1}^{p'} \left\{ \begin{matrix} \sum_{k=1}^{j} l_k \\ l_j \end{matrix} \right\}_{\sigma_j} (q^{\sigma_j(n - \sum_{k=1}^{j-1} l_k - 1)} - 1) \cdots (q^{\sigma_j(n - \sum_{k=1}^{j-1} l_k - l_j)} - 1).$$

Putting $m_j := \sum_{k=1}^{j} l_k$ and $m_0 = 0$, the right hand side is changed to

$$(9) \qquad \prod_{j=1}^{p'} \left\{ \begin{matrix} m_j \\ m_j - m_{j-1} \end{matrix} \right\}_{\sigma_j} (q^{\sigma_j(n - m_{j-1} - 1)} - 1) \cdots (q^{\sigma_j(n - m_j)} - 1).$$

Noting that

$$(10) \qquad (q^{\sigma_j(n - m_{j-1} - 1)} - 1) \cdots (q^{\sigma_j(n - m_j)} - 1) = \frac{(q^{\sigma_j})_{n - m_{j-1} - 1}}{(q^{\sigma_j})_{n - m_j - 1}},$$

$$(11) \qquad \left\{ \begin{matrix} m_j \\ m_j - m_{j-1} \end{matrix} \right\}_{\sigma_j} = \frac{(q^{\sigma_j})_{m_j}}{(q^{\sigma_j})_{m_j - m_{j-1}} (q^{\sigma_j})_{m_{j-1}}},$$

$$(12) \qquad \frac{(q^{\sigma})_m}{(q^{\sigma'})_m} = (-1)^{\frac{\sigma - \sigma'}{2} m} q^{\frac{\sigma - \sigma'}{2} \cdot \frac{1}{2} m(m+1)},$$

(9) can be exchanged to

$$\frac{(q^{\sigma_{p'}})_{n-1} (q^{\sigma_{p'}})_{m_{p'}}}{(q^{\sigma_{p'}})_{n - m_{p'} - 1}} \left(\prod_{j=1}^{p'} (q^{\sigma_j})_{m_j - m_{j-1}} \right)^{-1}$$

$$\times \prod_{j=2}^{p'-1} (-1)^{\frac{\sigma_j - \sigma_{j+1}}{2} m_j} q^{\frac{\sigma_j - \sigma_{j+1}}{2} \frac{1}{2} m_j (m_j+1)}$$

$$\times \prod_{j=2}^{p'-1} (-1)^{\frac{\sigma_{j+1} - \sigma_j}{2}(n-m_j-1)} q^{\frac{\sigma_{j+1} - \sigma_j}{2} \frac{1}{2}(n-m_j)(n-m_j-1)}$$

$$= \frac{(q^{\sigma_{p'}})_{n-1}(q^{\sigma_{p'}})_{m_{p'}}}{(q^{\sigma_{p'}})_{n-m_{p'}-1}} \left(\prod_{j=1}^{p'} (q^{\sigma_j})_{m_j - m_{j-1}} \right)^{-1} q^{-n \sum_{j=1}^{p'-1} \frac{\sigma_{j+1} - \sigma_j}{2} m_j}.$$

Here we used $\sigma_2 = \sigma_{p'} = -1$ if $p < 2t$ and $\sigma_2 = \sigma_{p'} = 1$ if $p > 2t$. Recalling $m_j - m_{j-1} = l_j$, it is changed to

(13)
$$\frac{(q^{\sigma_{p'}})_{n-1}(q^{\sigma_{p'}})_{m_{p'}}}{(q^{\sigma_{p'}})_{n-m_{p'}-1}} (-1)^{\sum_{j=1}^{p'} l_j} q^{-\sum_{j=1}^{p'} \frac{1}{2} \sigma_j l_j (l_j+1)}$$

$$\times \left(\prod_{j=1}^{p'} (q^{-\sigma_j})_{m_j - m_{j-1}} \right)^{-1} q^{-n \sum_{j=1}^{p'-1} \frac{\sigma_{j+1} - \sigma_j}{2} m_j}.$$

Accordingly, we obtain

$$d(\mathbf{l}) - \sum_{j=1}^{p'-1} \frac{\sigma_{j+1} - \sigma_j}{2} m_j$$

$$= -\frac{1}{2} \sum_{j=1}^{p'} \{ \sum_{k=r'(j)}^{p'} (\sigma_{i_k} + \sigma_{i_{p'+1-k}}) + \sum_{k=j}^{p'} (\sigma_k + \sigma_{p'+1-k}) \}(m_j - m_{j-1})$$

$$- \sum_{j=1}^{p'} \sigma_j - \sum_{j=1}^{p'-1} \frac{\sigma_{j+1} - \sigma_j}{2} m_j$$

$$= -\frac{1}{2} \sum_{j=1}^{p'} \sum_{k=r'(j)}^{p'} (\sigma_{i_k} + \sigma_{i_{p'+1-k}})(m_j - m_{j-1}) - \frac{1}{2} \sum_{j=1}^{p'} \sum_{k=j}^{p'} (\sigma_k + \sigma_{p'+1-k})(m_j - m_{j-1})$$

$$- \sum_{j=1}^{p'} \sigma_j - \sum_{j=1}^{p'-1} \frac{\sigma_{j+1} - \sigma_j}{2} m_j$$

$$= -\frac{1}{2} \sum_{j=1}^{p'} \sum_{k=r'(j)}^{p'} (\sigma_{i_k} + \sigma_{i_{p'+1-k}})(m_j - m_{j-1}) - \frac{1}{2} \sum_{j=1}^{p'} (\sum_{k=j}^{p'} (\sigma_k + \sigma_{p'+1-k}) m_j$$

$$- \sum_{k=j+1}^{p'} (\sigma_k + \sigma_{p'+1-k}) m_j) - \sum_{j=1}^{p'} \sigma_j - \sum_{j=1}^{p'-1} \frac{\sigma_{j+1} - \sigma_j}{2} m_j$$

$$
= -\frac{1}{2}\sum_{j=1}^{p'}\sum_{k=r'(j)}^{p'}(\sigma_{i_k}+\sigma_{i_{p'+1-k}})(m_j-m_{j-1})-\frac{1}{2}\sum_{j=1}^{p'}(\sigma_j+\sigma_{p'+1-j})m_j
$$

$$
-\sum_{j=1}^{p'}\sigma_j-\frac{1}{2}\sum_{j=1}^{p'-1}(\sigma_{j+1}-\sigma_j)m_j
$$

$$
= -\frac{1}{2}\sum_{j=1}^{p'}\sum_{k=r'(j)}^{p'}(\sigma_{i_k}+\sigma_{i_{p'+1-k}})(m_j-m_{j-1})
$$

$$
-\frac{1}{2}\sum_{j=1}^{p'-1}(\sigma_{j+1}+\sigma_{p'+1-j})m_j-\frac{1}{2}(\sigma_{p'}+\sigma_1)m_{p'}-\sum_{j=1}^{p'}\sigma_j
$$

$$
= a(\mathbf{m}),
$$

and for the case where $p < 2t$,

$$
C^0(\mathbf{l})-\sum_{j=1}^{p'}\frac{1}{2}\sigma_j l_j(l_j+1)
$$

$$
= -a(\mathbf{m})-\frac{1}{2}\sum_{j=1}^{p'-1}(\sigma_{j+1}-\sigma_j)m_j
$$

$$
+\sum_{1\le k\le\frac{p-t}{2}}\frac{1-\sigma_{i_k}}{2}m_{i_k-1}+\sum_{\frac{p-t}{2}+1\le k\le\frac{t-1}{2}}\frac{1+\sigma_{i_k}}{2}m_{i_k-1}
$$

$$
+\sum_{\frac{p-t}{2}+1\le k\le p'}\frac{1+\sigma_{i_k}}{2}m_{i_k}-\sum_{\frac{p-t}{2}+1\le k\le p'}m_{i_k-1}-m_{p'}-\sum_{j=1}^{p'}\sigma_j(m_j-m_{j-1})
$$

$$
-\frac{1}{2}\left\{\sum_{1\le k\le p'}\sum_{k+1\le k'\le p'}\frac{1+\mathrm{sign}(i_k-i_{k'})}{2}(\sigma_{i_k}-\sigma_{i_{k'}})(m_{i_k}-m_{i_k-1})(m_{i_{k'}}-m_{i_{k'}-1})\right\}
$$

$$
+\sum_{1\le k\le p'}\sigma_j m_{j-1}\left(\sum_{k=1}^{r'(j)}(m_{i_k}-m_{i_k-1})\right)
$$

$$
= -a(\mathbf{m})-(1+\sigma_{p'})m_{p'}+\frac{1}{2}\sum_{j=1}^{p'-1}(\sigma_{j+1}-\sigma_j)m_j
$$

$$
+\sum_{1\le k\le\frac{p-t}{2}}\frac{1-\sigma_{i_k}}{2}m_{i_k-1}+\sum_{\frac{p-t}{2}+1\le k\le\frac{t-1}{2}}\frac{1+\sigma_{i_k}}{2}m_{i_k-1}
$$

$$
+\sum_{\frac{p-t}{2}+1\le k\le p'}\frac{1+\sigma_{i_k}}{2}m_{i_k}-\sum_{\frac{p-t}{2}+1\le k\le p'}m_{i_k-1}
$$

$$-\frac{1}{2}\left\{\sum_{1\le k\le p'-1}\sum_{k+1\le k'\le p'}\frac{1+\mathrm{sign}(i_k-i_{k'})}{2}(\sigma_{i_k}-\sigma_{i_{k'}})(m_{i_k}-m_{i_k-1})(m_{i_{k'}}-m_{i_{k'}-1})\right\}$$

$$+\sum_{1\le k\le p'}\sigma_j m_{j-1}\sum_{k=1}^{r'(j)}(m_{i_k}-m_{i_k-1})$$

$$=\quad b_1(\mathbf{m})+b_2(\mathbf{m}),$$

and for the case where $p>2t$,

$$C^0(\mathbf{l})-\sum_{j=1}^{p'}\frac{1}{2}\sigma_j l_j(l_j+1)$$

$$=\quad -a(\mathbf{m})-\frac{1}{2}\sum_{j=1}^{p'-1}(\sigma_{j+1}-\sigma_j)m_j$$

$$+\sum_{1\le k\le\frac{p-t}{2}}\frac{1-\sigma_{i_k}}{2}m_{i_k-1}+\sum_{\frac{p-t}{2}+1\le k\le\frac{p-1}{2}}\frac{1+\sigma_{i_k}}{2}m_{i_k}-\sum_{\frac{p-t}{2}+1\le k\le\frac{p-1}{2}}m_{i_k-1}$$

$$-\sum_{\frac{t+1}{2}+1\le k\le\frac{p-t}{2}}\frac{1+\sigma_{i_k}}{2}m_{i_k-1}-m_{p'}-\sum_{j=1}^{p'}\sigma_j(m_j-m_{j-1})$$

$$-\frac{1}{2}\left\{\sum_{1\le k\le p'}\sum_{k+1\le k'\le p'}\frac{1+\mathrm{sign}(i_k-i_{k'})}{2}(\sigma_{i_k}-\sigma_{i_{k'}})(m_{i_k}-m_{i_k-1})(m_{i_{k'}}-m_{i_{k'}-1})\right\}$$

$$+\sum_{1\le k\le p'}\sigma_j m_{j-1}\left(\sum_{k=1}^{r'(j)}(m_{i_k}-m_{i_k-1})\right)$$

$$=\quad -a(\mathbf{m})-(1+\sigma_{p'})m_{p'}+\frac{1}{2}\sum_{j=1}^{p'-1}(\sigma_{j+1}-\sigma_j)m_j$$

$$+\sum_{1\le k\le\frac{p-t}{2}}\frac{1-\sigma_{i_k}}{2}m_{i_k-1}+\sum_{\frac{p-t}{2}+1\le k\le p'}\frac{1+\sigma_{i_k}}{2}m_{i_k}-\sum_{\frac{p-t}{2}+1\le k\le p'}m_{i_k-1}$$

$$-\sum_{\frac{t+1}{2}+1\le k\le\frac{p-t}{2}}\frac{1+\sigma_{i_k}}{2}m_{i_k-1}$$

$$-\frac{1}{2}\left\{\sum_{1\le k\le p'-1}\sum_{k+1\le k'\le p'}\frac{1+\mathrm{sign}(i_k-i_{k'})}{2}(\sigma_{i_k}-\sigma_{i_{k'}})(m_{i_k}-m_{i_k-1})(m_{i_{k'}}-m_{i_{k'}-1})\right\}$$

$$+\sum_{1\le k\le p'}\sigma_j m_{j-1}\left(\sum_{k=1}^{r'(j)}(m_{i_k}-m_{i_k-1})\right)=b_1(\mathbf{m})+b_2(\mathbf{m}).$$

Thus we obtain the formula (1).

For example, we give an explicit formula for the colored Jones polynomial of the knot 6_2, which is not a twist knot. The knot 6_2 has the Schubert's normal form $S(11, 7)$. We see $p' = 5$,

$$\sigma_1 = 1, \sigma_2 = -1, \sigma_3 = -1, \sigma_4 = 1, \sigma_5 = 1,$$
$$r'(1) = 4, r'(2) = 1, r'(3) = 5, r'(4) = 3, r'(5) = 2, i_1 = 2, i_2 = 5, i_3 = 4, i_4 = 1, i_5 = 3,$$
$$\sigma_{i_1} = -1, \sigma_{i_2} = -1, \sigma_{i_3} = 1, \sigma_{i_4} = 1, \sigma_{i_5} = -1,$$

and so

$$\sum_{k=r'(1)}^{5} (\sigma_{i_k} + \sigma_{i_{6-k}}) = -2, \ \sum_{k=r'(2)}^{5} (\sigma_{i_k} + \sigma_{i_{6-k}}) = -2, \ \sum_{k=r'(3)}^{5} (\sigma_{i_k} + \sigma_{i_{6-k}}) = -2,$$

$$\sum_{k=r'(4)}^{5} (\sigma_{i_k} + \sigma_{i_{6-k}}) = 0, \ \sum_{k=r'(5)}^{5} (\sigma_{i_k} + \sigma_{i_{6-k}}) = 0,$$

$$\sum_{j=1}^{4} (\sigma_{j+1} + \sigma_{6-j}) = -2m_1 - 2m_4.$$

Thus, we get $a(\mathbf{m}) = 1 + m_1 + m_3 + m_4$. Moreover, we calculate

$$\sum_{1 \le k \le 2} \frac{1 - \sigma_{i_k}}{2} m_{i_k - 1} = m_1 + m_4, \ \sum_{3 \le k \le 2} m_{i_k - 1} = m_3 + m_2, \ \sum_{3 \le k \le 5} \frac{1 + \sigma_{i_k}}{2} m_{i_k} = m_4 + m_1,$$

$$1 + \sigma_5 = 0, \quad \sum_{j=1}^{4} (\sigma_{j+1} - \sigma_j) m_j = -2m_1 + 2m_3 - 2m_4,$$

$$\sum_{1 \le k \le p'-1} \sum_{k+1 \le k' \le p'} \frac{1 + \mathrm{sign}(i_k - i_{k'})}{2} (\sigma_{i_k} - \sigma_{i_{k'}})(m_{i_k} - m_{i_k - 1})(m_{i_{k'}} - m_{i_{k'} - 1})$$

$$= \ -2(m_2 - m_1)m_1 - 2(m_5 - m_4)(m_4 - m_3) - 2(m_5 - m_4)m_1 + 2(m_4 - m_3)(m_3 - m_2),$$

$$\sum_{j=1}^{p'} \sigma_j \left(\sum_{k=1}^{r'(j)} (m_{i_k} - m_{i_k - 1}) \right) m_{j-1}$$

$$= \ -m_1(m_2 - m_1) - m_2 m_5) + m_3(m_2 - m_1 + m_5 - m_3) + (-m_4)(m_2 - m_1 + m_5 - m_4),$$

and so $b_1(\mathbf{m}) = -1 - m_2 - m_3 + m_1 m_5 - m_2 m_5 - m_1 m_3$. It is easy to compute $b_2(\mathbf{m}) = m_3$.

Finally, we obtain

$$
\begin{aligned}
&J_{S(11,7)}(n)\\
&=\sum_{\substack{0\leq m_{j-1}\leq m_j\leq n-1\\ 1\leq j\leq 5}} q^{n(1+m_1+m_3+m_4)-1-m_2+m_1m_5-m_2m_5-m_1m_3}\\
&\qquad\times\frac{(q)_{n-1}(q)_{m_5}}{(q)_{n-m_5-1}}\left((\bar{q})_{m_1}(q)_{m_2-m_1}(q)_{m_3-m_2}(\bar{q})_{m_4-m_3}(q)_{m_5-m_4}\right)^{-1}(-1)^{m_5}.
\end{aligned}
$$

3. Results for a twist knot

In this section, we will consider the twist knots $S(4p_1+1, 2p_1+1)$ and $S(4p_1-1, 2p_1-1)$.

From Theorem 1, we get explicit formulas for the colored Jones polynomials of these twist knots.

Corollary 5.

$$
\begin{aligned}
(14)\qquad &J_{S(4p_1+1,2p_1+1)}(n)\\
&=\sum_{0\leq m_{j-1}\leq m_j\leq n-1} q^{-n\sum_{i=1}^{2p_1-1}(-1)^i m_i+\sum_{i=1}^{p_1-1} m_{2i}(1+m_{2i+1})}\\
&\qquad\times\frac{(\bar{q})_{n-1}(\bar{q})_{m_{p'}}}{(\bar{q})_{n-m_{p'}-1}}(-1)^{m_{p'}}\left(\prod_{j=0}^{p_1-1}(\bar{q})_{m_{2j+1}-m_{2j}}\prod_{j=1}^{p_1}(q)_{m_{2j}-m_{2j-1}}\right)^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
(15)\qquad &J_{S(4p_1-1,2p_1-1)}(n)\\
&=\sum_{0\leq m_{j-1}\leq m_j\leq n-1} q^{n(-1+\sum_{i=1}^{2p_1-2}(-1)^i m_i-2m_{2p_1-1})+1-\sum_{i=1}^{p_1-1} m_{2i-1}(1+m_{2i})}\\
&\qquad\times\frac{(q)_{n-1}(q)_{m_{p'}}}{(q)_{n-m_{p'}-1}}(-1)^{m_{p'}}\left(\prod_{j=0}^{p_1-1}(\bar{q})_{m_{2j+1}-m_{2j}}\prod_{j=1}^{p_1-1}(q)_{m_{2j}-m_{2j-1}}\right)^{-1}.
\end{aligned}
$$

Proof. Firstly, let us consider the knot $S(4p_1+1, 2p_1+1)$ for $p_1\geq 1$. It is easy to see that

Lemma 6. (1) $\sigma_j=(-1)^{j-1}$

$$
(2)\ i_k=\begin{cases}2p_1+2-2k & \text{if } 1\leq k\leq p_1\\ 2(k-p_1)-1 & \text{if } p_1+1\leq k\leq 2p_1,\end{cases}\qquad
\sigma_{i_k}=\begin{cases}-1 & \text{if } 1\leq k\leq p_1\\ 1 & \text{if } p_1+1\leq k\leq 2p_1.\end{cases}
$$

Lemma 7. (1) $\sigma_{i_k} + \sigma_{i_{p'+1-k}} = 0.$ (2) $\sigma_{j+1} + \sigma_{p'+1-j} = (-2)^j.$

From the above lemmas, we obtain

$$(16) \qquad a(\mathbf{m}) = - \sum_{j=1}^{2p_1-1} (-1)^j m_j.$$

We compute $b_1(\mathbf{m})$ and $b_2(\mathbf{m})$. One calculate

$$(17) \qquad \sum_{1 \leq k \leq \frac{p-t}{2}} \frac{1 - \sigma_{i_k}}{2} m_{i_k-1} = \sum_{j=1}^{p_1} m_{2j-1}, \qquad \sum_{\frac{p-t}{2}+1 \leq k \leq p'} m_{i_k-1} = \sum_{j=1}^{p_1-1} m_{2j},$$

$$(18) \qquad \sum_{\frac{p-t}{2}+1 \leq k \leq p'} \frac{1 + \sigma_{i_k}}{2} m_{i_k}$$

$$= \sum_{p_1+1 \leq k \leq 2p_1} m_{i_k} = \sum_{p_1+1 \leq k \leq 2p_1} m_{2k-2p_1-1} = \sum_{j=1}^{p_1} m_{2j-1}$$

$$(19) \qquad (1 + \sigma_{p'}) m_{p'} = 0,$$

$$(20) \qquad \sum_{j=1}^{p'-1} (\sigma_{j+1} - \sigma_j) m_j = -2 \sum_{j=1}^{p_1} m_{2j-1} + 2 \sum_{j=1}^{p_1-1} m_{2j} = 2 \sum_{j=1}^{2p_1-1} (-1)^j m_j.$$

Moreover, it follows that $k < k'$ and $\sigma_{i_k} \neq \sigma_{i_{k'}}$ if and only if $1 \leq k \leq p_1$ and $p_1 + 1 \leq k' \leq 2p_1$. We note that $\mathrm{sign}(i_k - i_{k'}) = 1$ if and only if $i_k > i_{k'}$ and that $i_k = 2p_1 + 2 - 2k$ for $1 \leq k \leq p_1$ and $i_{k'} = 2k' - 2p_1 - 1$ for $p_1 + 1 \leq k' \leq 2p_1$. So, we obtain

$$\sum_{1 \leq k \leq p'} \sum_{k+1 \leq k' \leq p'} \frac{1 + \mathrm{sign}(i_k - i_{k'})}{2} (\sigma_{i_k} - \sigma_{i_{k'}})(m_{i_k} - m_{i_k-1})(m_{i_{k'}} - m_{i_{k'}-1})$$

$$= -2 \sum_{1 \leq k \leq p_1} \sum_{p_1+1 \leq k' \leq 2p_1+1-k} (m_{2p_1+2-2k} - m_{2p_1+1-2k})(m_{2k'-2p_1-1} - m_{2k'-2p_1-2})$$

$$= -2 \sum_{1 \leq j \leq p_1} \sum_{1 \leq j' \leq j} (m_{2j} - m_{2j-1})(m_{2j'-1} - m_{2j'-2}),$$

and

$$\sum_{j=1}^{p'} \sigma_j \Big(\sum_{k=1}^{r'(j)} (m_{i_k} - m_{i_k-1}) \Big) m_{j-1}$$

$$= \sum_{j=1}^{p_1} \sigma_{2j-1} \Big(\sum_{k=1}^{r'(2j-1)} (m_{i_k} - m_{i_k-1}) \Big) m_{2j-2} + \sum_{j=1}^{p_1} \sigma_{2j} \Big(\sum_{k=1}^{r'(2j)} (m_{i_k} - m_{i_k-1}) \Big) m_{2j-1}$$

$$= \sum_{j=1}^{p_1} \sigma_{2j-1} \Big(\sum_{k=1}^{p_1+j} (m_{i_k} - m_{i_k-1}) \Big) m_{2j-2} + \sum_{j=1}^{p_1} \sigma_{2j} \Big(\sum_{k=1}^{p_1-j+1} (m_{i_k} - m_{i_k-1}) \Big) m_{2j-1}$$

$$= \sum_{j=1}^{p_1} \Big(\sum_{k=1}^{p_1} (m_{2p_1+2-2k} - m_{2p_1+1-2k}) + \sum_{k=p_1+1}^{p_1+j} (m_{2k-2p_1-1} - m_{2k-2p_1-2}) \Big) m_{2j-2}$$

$$- \sum_{j=1}^{p_1} \Big(\sum_{k=1}^{p_1-j+1} (m_{2p_1+2-2k} - m_{2p_1+1-2k}) \Big) m_{2j-1}$$

$$= \sum_{j=1}^{p_1} \Big(\sum_{j'=1}^{p_1} (m_{2j'} - m_{2j'-1}) + \sum_{j'=1}^{j} (m_{2j'-1} - m_{2j'-2}) \Big) m_{2j-2}$$

$$- \sum_{j=1}^{p_1} \Big(\sum_{j'=j}^{p_1} (m_{2j'} - m_{2j'-1}) \Big) m_{2j-1}$$

$$= \sum_{j=1}^{p_1} \Big(m_{2j} + \sum_{j'=j+1}^{p_1} (m_{2j'} - m_{2j'-1}) \Big) m_{2j-2} - \sum_{j=1}^{p_1} \Big(\sum_{j'=j}^{p_1} (m_{2j'} - m_{2j'-1}) \Big) m_{2j-1}$$

$$= \sum_{j=1}^{p_1} \Big(\sum_{j'=j}^{p_1} (m_{2j'} - m_{2j'-1}) + m_{2j-1} \Big) m_{2j-2} - \sum_{j=1}^{p_1} \Big(\sum_{j'=j}^{p_1} (m_{2j'} - m_{2j'-1}) \Big) m_{2j-1}.$$

Summing up, we obtain

$$(21) \qquad\qquad b_1(\mathbf{m}) + b_2(\mathbf{m}) = \sum_{j=1}^{p_1-1} m_{2j}(1 + m_{2j+1}).$$

Next, we consider the knot $S(4p_1 - 1, 2p_1 - 1)$ for $p_1 \geq 1$. It is not hard to show the following two lemmas.

Lemma 8. *It follows that (1)* $\sigma_j = (-1)^{j-1}$, *and*

$$(2)\ i_k = \begin{cases} 2p_1 - 2k + 1 & \text{if } 1 \leq k \leq p_1, \\ 2(k - p_1) & \text{if } p_1 + 1 \leq k \leq 2p_1 - 1, \end{cases} \qquad \sigma_{i_k} = \begin{cases} 1 & \text{if } 1 \leq k \leq p_1, \\ -1 & \text{if } p_1 + 1 \leq k \leq 2p_1 - 1. \end{cases}$$

Lemma 9. *(1)* $\sigma_{i_k} + \sigma_{i_{p'+1-k}} = \begin{cases} 2 & \text{if } k = p_1, \\ 0 & \text{if } k \neq p_1. \end{cases}$ *(2)* $\sigma_{j+1} + \sigma_{p'+1-j} = 0.$

From these lemmas, it follows that

$$
(22) \qquad a(\mathbf{m}) = \sum_{i=1}^{2p_1-2} (-1)^i m_i - 2m_{2p_1-1} - 1,
$$

We compute

$$
(23) \qquad \sum_{1 \le k \le \frac{p-t}{2}} \frac{1-\sigma_{i_k}}{2} m_{i_k-1} = \sum_{1 \le k \le p_1} \frac{1-\sigma_{i_k}}{2} m_{i_k-1} = 0,
$$

$$
(24) \qquad \sum_{\frac{p-t}{2}+1 \le k \le p'} m_{i_k-1} = \sum_{p_1+1 \le k \le 2p_1-1} m_{i_k-1} = \sum_{p_1+1 \le k \le 2p_1-1} m_{2k-2p_1-1}
$$

$$
= \sum_{j=1}^{p_1-1} m_{2j-1},
$$

$$
(25) \qquad \sum_{\frac{p-t}{2}+1 \le k \le p'} \frac{1+\sigma_{i_k}}{2} m_{i_k} = \sum_{p_1+1 \le k \le 2p_1-1} \frac{1+\sigma_{i_k}}{2} m_{i_k} = 0,
$$

$$
(26) \qquad (1+\sigma_{p'}) m_{p'} = 2m_{2p_1-1},
$$

$$
(27) \qquad \sum_{j=1}^{p'-1} (\sigma_{j+1} - \sigma_j) m_j = \sum_{j=1}^{2p_1-2} (\sigma_{j+1} - \sigma_j) m_j = 2 \sum_{i=1}^{2p_1-2} (-1)^i m_i.
$$

Moreover, it follows that $k < k'$ and $\sigma_{i_k} \neq \sigma_{i_{k'}}$ if and only if $1 \le k \le p_1$ and $p_1 + 1 \le k' \le 2p_1 - 1$. We note that $\mathrm{sign}(i_k - i_{k'}) = 1$ if and only if $i_k > i_{k'}$ and that $i_k = 2p_1 - 2k + 1$ for $1 \le k \le p_1$ and $i_{k'} = 2k' - 2p_1$ for $p_1 + 1 \le k' \le 2p_1 - 1$.

So, we compute

$$
\sum_{1 \le k \le p'} \sum_{k+1 \le k' \le p'} \frac{1 + \mathrm{sign}(i_k - i_{k'})}{2} (\sigma_{i_k} - \sigma_{i_{k'}})(m_{i_k} - m_{i_k-1})(m_{i_{k'}} - m_{i_{k'}-1})
$$

$$
= 2 \sum_{1 \le k \le p_1} \sum_{p_1+1 \le k' \le 2p_1-k} (m_{2p_1-2k+1} - m_{2p_1-2k})(m_{2k'-2p_1} - m_{2k'-2p_1-1})
$$

$$
= 2 \sum_{0 \le j \le p_1-1} \sum_{1 \le j' \le j} (m_{2j+1} - m_{2j})(m_{2j'} - m_{2j'-1}),
$$

and

$$\sum_{j=1}^{p'} \sigma_j \left(\sum_{k=1}^{r'(j)} (m_{i_k} - m_{i_k-1}) \right) m_{j-1}$$

$$= \sum_{j=1}^{p_1} \sigma_{2j-1} \left(\sum_{k=1}^{r'(2j-1)} (m_{i_k} - m_{i_k-1}) \right) m_{2j-2} + \sum_{j=1}^{p_1-1} \sigma_{2j} \left(\sum_{k=1}^{r'(2j)} (m_{i_k} - m_{i_k-1}) \right) m_{2j-1}$$

$$= \sum_{j=1}^{p_1} \left(\sum_{k=1}^{p_1-j+1} (m_{i_k} - m_{i_k-1}) \right) m_{2j-2} - \sum_{j=1}^{p_1-1} \left(\sum_{k=1}^{p_1-j} (m_{i_k} - m_{i_k-1}) \right) m_{2j-1}$$

$$= \sum_{j=1}^{p_1} \left(\sum_{k=1}^{p_1-j+1} (m_{2p_1-2k+1} - m_{2p_1-2k}) \right) m_{2j-2}$$

$$- \sum_{j=1}^{p_1-1} \left(\sum_{k=1}^{p_1} (m_{2k-2p_1+1} - m_{2k-2p_1}) + \sum_{k=p_1+1}^{p_1+j} (m_{2p_1-2k} - m_{2p_1-2k-1}) \right) m_{2j-1}$$

$$= \sum_{j=1}^{p_1} \left(\sum_{j'=j}^{p_1} (m_{2j'-1} - m_{2j'-2}) m_{2j-2} - \sum_{j=1}^{p_1-1} \left(\sum_{j'=1}^{p_1-1} (m_{2j'+1} - m_{2j'}) + \sum_{j'=1}^{j} (m_{2j'} - m_{2j'-1}) \right) m_{2j-1}$$

$$= \sum_{j=1}^{p_1} \left(\sum_{j'=j}^{p_1} (m_{2j'-1} - m_{2j'-2}) \right) m_{2j-2} - \sum_{j=1}^{p_1-1} \left(\sum_{j'=j+1}^{p_1} (m_{2j'-1} - m_{2j'-2}) + m_{2j} \right) m_{2j-1}$$

$$= \sum_{0 \le j' \le p_1-1} \sum_{1 \le j \le j'} (m_{2j'+1} - m_{2j'})(m_{2j} - m_{2j-1}) - \sum_{j=1}^{p_1-1} m_{2j} m_{2j-1}.$$

Thus we obtain

(28)
$$b_1(\mathbf{m}) + b_2(\mathbf{m}) = 1 - \sum_{j=1}^{p_1-1} m_{2j-1}(1 + m_{2j}).$$

□

Using Corollary 5, we give explicit formulas for the trefoil knot $S(3,1)$ and the figure eight knot $S(5,3)$.
By the formula (15) with $p_1 = 1$ and $p' = 1$, we get

$$J_{S(3,1)}(n) = \sum_{0 \le m_1 \le n-1} \frac{(q)_{n-1}(q)_{m_1}(-1)^{m_1}}{(q)_{n-m_1-1}(\bar{q})_{m_1}} q^{n(-1-2m_1)+1}$$

$$= \sum_{0 \le m_1 \le n-1} (\bar{q}^{n-1} - 1) \cdots (\bar{q}^{n-m_1} - 1)(-1)^{m_1} q^{n(-1-m_1)+1}.$$

By the formula (14) with $p_1 = 1$ and $p' = 1$,

$$J_{S(5,3)}(n) = \sum_{0 \leq m_1 \leq m_2 \leq n-1} \frac{(\bar{q})_{n-1}(\bar{q})_{m_2}}{(\bar{q})_{n-m_2-1}(\bar{q})_{m_1}(q)_{m_2-m_1}} q^{nm_1}(-1)^{m_2},$$

and using the formula

$$\sum_{i=0}^{k} \frac{(q)_k}{(q)_{k-i}(\bar{q})_i} q^{-\frac{i}{2}(k+b+1)} = \prod_{j=1}^{k}(1 - q^{-\frac{1}{2}(k+b+1-2j)}),$$

we obtain a formula of K. Habiro [2] or T. T. Q. Le as follows,

$$J_{S(5,3)}(n) = \sum_{0 \leq m \leq n-1} \frac{(q)_{n-1}(q)_{n+m}}{(q)_n(q)_m(q)_{n-m-1}} q^{-nm}.$$

Finally, as an application of Corollary 5, we estimate the maximal and minimal degree of $J_K(n)$ for a twist knot K. We denote by $\deg_+(J_K(n))$ and $\deg_-(J_K(n))$ the maximal degree and the minimal degree in q of $J_K(n) \in \mathbb{Z}[q^{\pm 1}]$ respectively, and let $\mathrm{br}(J_K(n)) = \deg_+(J_K(n)) - \deg_-(J_K(n))$. From Corollary 5, we obtain

Corollary 10. *For* $K = S(4p_1 + 1, 2p_1 + 1)$,

(29) $$\deg_+(J_K(n)) = \frac{1}{2}n(n-1)(2p_1), \quad \deg_-(J_K(n)) = -n(n-1),$$

$$\mathrm{br}(J_K(n)) = \frac{1}{2}n(n-1)(2p_1 + 2),$$

and for $K = S(4p_1 - 1, 2p_1 - 1)$,

(30) $$\deg_+(J_K(n)) = -n + 1, \quad \deg_-(J_K(n)) = -n^2 + 1 - \frac{1}{2}n(n-1)(2p_1 - 1),$$

$$\mathrm{br}(J_K(n)) = \frac{1}{2}n(n-1)(2p_1 + 1).$$

Thus, it follows that for $K = S(4p_1 + 1, 2p_1 + 1)$ *or* $K = S(4p_1 - 1, 2p_1 - 1)$,

(31) $$\mathrm{br}(J_K(n)) = \frac{1}{2}n(n-1)C(K),$$

where $C(K)$ *is the crossing number of* K.

Proof. From Corollary 5, we obtain

$$J_{S(4p_1+1,2p_1+1)}(n) = \sum_{0 \leq m_{j-1} \leq m_j \leq n-1} q^{-n\sum_{i=1}^{2p_1}(-1)^i m_i + \frac{1}{2}\sum_{i=1}^{2p_1-1} m_i(1+m_i)}(-1)^{\sum_{i=1}^{2p_1} m_i}$$

$$\times (q^{n-1} - 1) \cdots (q^{n-m_{p'}} - 1) \prod_{j=2}^{2p_1} \frac{(q)_{m_j}}{(q)_{m_{j-1}}(q)_{m_j - m_{j-1}}}.$$

Notice that the maximal degree in q of $\dfrac{(q)_a}{(q)_b(q)_{a-b}}$ is $b(a-b)$ and the minimal degree is 1. Noting that

$$
\begin{aligned}
&-n(n-1)\\
\leq\quad & n(m_1 + \sum_{i=1}^{p_1-1}(m_{2i+1}-m_{2i}) - m_{2p_1}) + \frac{1}{2}\sum_{i=1}^{2p_1-1} m_i(1+m_i) + \sum_{j=2}^{2p_1} m_{j-1}(m_j - m_{j-1})\\
=\quad & -n\sum_{i=1}^{p_1-1} l_{2i} - \frac{1}{2}\sum_{i=1}^{2p_1-1}(m_i - m_{i+1})^2 + \frac{1}{2}\sum_{i=1}^{2p_1-1} m_{i+1}^2 + \frac{1}{2}\sum_{i=1}^{2p_1-1} m_i\\
\leq\quad & \frac{1}{2}n(n-1)(2p_1 - 1),
\end{aligned}
$$

and

$$
0 \leq \frac{1}{2}m_{2p_1}(2n - 1 - m_{2p_1}) \leq \frac{1}{2}n(n-1),
$$

it follows that the term with $m_i = n - 1$ contributes to $\deg_+(J_K(n))$ and the term with $m_i = 0$ $(i \neq p')$ and $m_{p'} = n - 1$ to $\deg_-(J_K(n))$. So, the formula (29) holds. Similarly, from Corollary 5,

$$
\begin{aligned}
J_{S(4p_1-1, 2p_1-1)}(n) \quad=\quad & \sum_{0 \leq m_{j-1} \leq m_j \leq n-1} q^{n(-1+\sum_{i=1}^{2p_1-1}(-1)^i m_i) + 1 - \frac{1}{2}\sum_{i=1}^{2p_1-2} m_i(1+m_i)}(-1)^{\sum_{i=1}^{2p_1-1} m_i}\\
& \times (\bar{q}^{n-1} - 1) \cdots (\bar{q}^{n-m_{p'}} - 1) \prod_{j=2}^{2p_1-1} \frac{(\bar{q})_{m_j}}{(\bar{q})_{m_{j-1}}(\bar{q})_{m_j - m_{j-1}}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&-n+1\\
\geq\quad & n(-1 - m_1 - \sum_{i=1}^{p_1-1}(m_{2i+1}-m_{2i})) + 1 - \frac{1}{2}\sum_{i=1}^{2p_1-2} m_i(1+m_{i+1}) - \sum_{j=2}^{2p_1-1} m_{j-1}(m_j - m_{j-1})\\
=\quad & n(-1 - \sum_{i=1}^{p_1} l_{2i-1} + 1) + \frac{1}{2}\sum_{i=1}^{2p_1-2}(m_i - m_{i-1})^2 - \frac{1}{2}\sum_{i=1}^{2p_1-2} m_{i+1}^2 - \frac{1}{2}\sum_{i=1}^{2p_1-2} m_i\\
\geq\quad & -n^2 + 1 - \frac{1}{2}n(n-1)(2p_1 - 2),
\end{aligned}
$$

and

$$
0 \geq -\frac{1}{2}(2n - 1 - m_{2p_1+1})m_{2p_1+1} \geq -\frac{1}{2}n(n-1),
$$

the term with $m_i = 0$ contributes to $\deg_+(J_K(n))$ and the term with $m_i = n - 1$ $(i \neq p')$ and $m_{p'} = n - 1$ to $\deg_-(J_K(n))$. Thus, we get the formula (30).

Moreover, since $S(4p_1 + 1, 2p_1 + 1)$ (resp. $S(4p_1 - 1, 2p_1 - 1)$) is given in the Conway notation by $C(2p_1, 2)$ (resp. $C(2p_1 - 1, 2)$), the equality (31) holds. $\square$

Remark 11. Recently, Le proved that the equality (31) holds for any non-trivial alternating knot (see Proposition 2.1 of [6]).

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