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A FORMULA OF SIMONS' TYPE AND HYPERSURFACES WITH CONSTANT MEAN CURVATURE

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In a recent work [8] J. Simons has established a formula for the Laplacian of the second fundamental form of a submanifold in a Riemannian manifold and has obtained an important application in the case of a minimal hypersurface in the sphere, for which the formula takes a rather simple form. The application is made by means of the Laplacian of the function f on the hypersurface, which is defined to be the square of the length of the second fundamental form.

In the present paper, by a more direct route than Simons' we first obtain the same type of formula (see (16)) in the case of a hypersurface M immersed with constant mean curvature in a space $\tilde{M}$ of constant sectional curvature, and then derive a new formula (see (18)) for the function f which involves the sectional curvature of M. Based on this new formula our main results are the determination of hypersurfaces M of non-negative sectional curvature immersed in the Euclidean space R^{n+1} or the sphere S^{n+1} with constant mean curvature under the additional assumption that the function f is constant. This additional assumption is automatically satisfied if M is compact. We state the general results in a global form assuming completeness of M, but they are essentially of local nature.

1. Formula of Simons' type

Let $\tilde{M}$ be an $(n+1)$-dimensional space form, i.e., a Riemannian manifold of constant sectional curvature, say, c. Let $\phi\colon M \to \tilde{M}$ be an isometric immersion of an n-dimensional Riemannian manifold M into $\tilde{M}$. For simplicity, we say that M is a hypersurface immersed in $\tilde{M}$ and, for all local formulas and computations, we may consider ϕ as an imbedding and thus identify $x \in M$ with $\phi(x) \in M$. The tangent space $T_x(M)$ is identified with a subspace of the tangent space $T_x(\tilde{M})$, and the normal space $T_x^{\perp}$ is the subspace of $T_x(\tilde{M})$ consisting of all $X \in T_x(\tilde{M})$ which are orthogonal to $T_x(M)$ with respect to the Riemannian metric g. For the basic notations and formulas concerning differential geometry of submanifolds, we follow Chapter VII of Kobayashi-Nomizu [4].

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For an arbitrary point $x_0 \in M$, we may choose a field of unit normal vectors ξ defined in a neighborhood U. The second fundamental form h and the corresponding symmetric operator A are defined and related to covariant differentiations $\tilde{\nabla}$ and ∇ in $\tilde{M}$ and M, respectively, by the following formulas:

$$(1) \qquad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) ,$$

$$(2) \qquad \tilde{\nabla}_X \xi = -AX ,$$

where X and Y are vector fields tangent to M. The Gauss equation is:

$$(3) \qquad R(X, Y) = cX \wedge Y + AX \wedge AY , \qquad X, Y \in T_x(M) ,$$

where $X \wedge Y$ denotes the skew-symmetric endomorphism of $T_x(M)$ defined by $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$.

The Codazzi equation is expressed by

$$(4) \qquad (\nabla_X A)(Y) = (\nabla_Y A)(X) .$$

Since ξ is defined locally up to a sign, so is A, and A^2 is thus defined globally on M. We consider the function $f = \text{trace } A^2$ which is globally defined on M and wish to compute its Laplacian Δf. This is given as the trace of the symmetric bilinear form

$$(5) \qquad H_f(X, Y) = X(Yf) - (\nabla_X Y)f ;$$

in fact, H_f coincides with the usual Hessian of f at a critical point of f. If $\{e_1, \cdots, e_n\}$ is an arbitrary orthonormal basis in $T_x(M)$, then

$$(6) \qquad (\Delta f)(x) = \sum_{i=1}^{n} H_f(e_i, e_i) .$$

In order to compute Δf, we need to compute the "restricted" Laplacian of the tensor field A, which we now explain. Let T be an arbitrary tensor field of type (r, s) on M. Then the second covariant differential $\nabla^2 T$ is a tensor field of type $(r, s+2)$ which is given by

$$(7) \qquad (\nabla^2 T)(\,;Y;X) = \nabla_X(\nabla_Y T) - \nabla_{\nabla_X Y} T ,$$

where X and Y are vector fields on M. At each point $x \in M$, we take an orthonormal basis $\{e_1, \cdots, e_n\}$ in $T_x(M)$ and set

$$(8) \qquad (\Delta' T)(x) = \sum_{i=1}^{n} (\nabla^2 T)(\,;e_i;e_i) .$$

This is independent of the choice of an orthonormal basis and the tensor field $\Delta' T$ of type (r, s) so defined is called the *restricted Laplacian* of T. When T is a function f, $\nabla^2 T$ coincides with H_f in (5) and $\Delta' T$ is nothing but Δf. The expression for $\Delta' T$ in conventional tensor notation is

$$(\Delta' T)^{i_1 \cdots i_r}_{j_1 \cdots j_s} = \sum_{p,q=1}^{n} g^{pq} T^{i_1 \cdots i_r}_{j_1 \cdots j_s; p; q} .$$

If T is a differential form ω of degree r, $\Delta' T$ does *not* coincide with the Laplacian $\Delta\omega$ as defined in the theory of harmonic integrals; indeed, $\Delta'\omega$ is part of $\Delta\omega$. This accounts for the name of "restricted Laplacian" which we are proposing. (In Simons [8], $\Delta' T$ is called simply the Laplacian; for results on the restricted Laplacian, see, for example, Lichnerowicz [5; pp. 1-4].)

Going back to the function $f = \operatorname{trace} A^2$ on the hypersurface M, we have

$$Yf = Y(\operatorname{trace} A^2) = \operatorname{trace}(\nabla_Y A^2) ,$$

since taking the trace is a contraction on tensor fields of type (1, 1), which commutes with covariant differentiation (cf. Kobayashi-Nomizu [3, p. 123]). Since

$$\begin{aligned} \operatorname{trace} \nabla_Y A^2 &= \operatorname{trace}(\nabla_Y A)A + \operatorname{trace} A(\nabla_Y A) \\ &= 2 \operatorname{trace}(\nabla_Y A)A , \end{aligned}$$

we have

$$Yf = 2 \operatorname{trace}(\nabla_Y A)A .$$

Thus we have

$$XYf = 2 \operatorname{trace}(\nabla_X(\nabla_Y A))A + 2 \operatorname{trace}(\nabla_Y A)(\nabla_X A)$$

as well as

$$(\nabla_X Y)f = 2 \operatorname{trace}(\nabla_{\nabla_X Y} A)A .$$

Hence

$$\frac{1}{2} f = \sum_{i=1}^{n} \{\operatorname{trace}(\nabla^2 A)(\, ; e_i; e_i)A + \operatorname{trace}(\nabla_{e_i} A)^2\} ,$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis in $T_x(M)$. Thus

$$\frac{1}{2} \Delta f = \operatorname{trace}(\Delta' A)A + \sum_{i=1}^{n} \operatorname{trace}(\nabla_{e_i} A)^2 .$$

By extending the metric g to the tensor space in the standard fashion, we may write

$$\frac{1}{2} \Delta f = g(\Delta' A, A) + g(\nabla A, \nabla A) . \tag{9}$$

We shall now compute $\Delta' A$. For this purpose, let us write $K(X, Y)$ for $(\nabla^2 A)(\,;Y;X)$ so that

$$K(X, Y) = \nabla_X(\nabla_Y A) - \nabla_{\nabla_X Y} A\ .$$

Using the identities $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ and $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, where the curvature transformation $R(X, Y)$ and the other terms are regarded as derivations of the algebra of tensor fields, we obtain

$$K(X, Y) = K(Y, X) + [R(X, Y), A]\ . \tag{10}$$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis in $T_x(M)$, and extend them to vector fields $E_1, \cdots, E_n$ in a neighborhood of x such that $\nabla E_i = 0$ at x. Let X be a vector field such that $\nabla X = 0$ at x. (Such vector fields can be easily obtained by using parallel displacement along each geodesic with origin x.) In (10) take E_i and X instead of X and Y, respectively, and apply each endomorphism to E_i. Since

$$\begin{aligned} K(E_i, X)E_i &= (\nabla_{E_i}(\nabla_X A))E_i - (\nabla_{\nabla_{E_i} X} A)E_i && \text{(the second term is 0 at } x\text{)} \\ &= \nabla_{E_i}((\nabla_X A)E_i) - (\nabla_X A)(\nabla_{E_i} E_i) && \text{(the second term is 0 at } x\text{)} \\ &= \nabla_{E_i}((\nabla_{E_i} A)X) && \text{(by virtue of Codazzi's equation)} \\ &= (\nabla_{E_i}(\nabla_{E_i} A))X + (\nabla_{E_i} A)(\nabla_{E_i} X) && \text{(the second term is 0 at } x\text{)} \\ &= K(E_i, E_i)X\ , \end{aligned}$$

we get at x

$$K(E_i, E_i)X = K(X, E_i)E_i + [R(E_i, X), A]E_i\ . \tag{11}$$

By a similar computation we get at x

$$K(X, E_i)E_i = \nabla_X((\nabla_{E_i} A)E_i)\ . \tag{12}$$

We now assume that M has constant mean curvature, that is, trace A = constant. Under this assumption we prove

$$\sum_{i=1}^{n} (\nabla_{E_i} A)E_i = 0\ . \tag{13}$$

Indeed, since $\nabla_{E_i} A$ is a symmetric operator together with A, we get, by using Codazzi's equation,

$$\begin{aligned} g\left(\sum_{i=1}^{n} (\nabla_{E_i} A)E_i, Z\right) &= \sum_{i=1}^{n} g(E_i, (\nabla_{E_i} A)Z) \\ &= \sum_{i=1}^{n} g(E_i, (\nabla_Z A)E_i) \\ &= \text{trace}\,(\nabla_Z A) = Z\cdot(\text{trace}\,A) = 0\ . \end{aligned}$$

Since this is valid for an arbitrary vector Z, we conclude (13). Substituting (13) in (12) we obtain

$$\sum_{i=1}^{n} K(X, E_i)E_i = 0 . \tag{14}$$

From (11) and (14) we get

$$(\Delta' A)(X) = \sum_{i=1}^{n} [R(E_i, X), A]E_i . \tag{15}$$

The right-hand side can be computed as follows. By the Gauss equation, we have

$$R(E_i, X) = c(E_i \wedge X) + AE_i \wedge AX .$$

Thus

$$\sum_{i=1}^{n} R(E_i, X)AE_i = \sum_{i=1}^{n} c\{g(AE_i, X)E_i - g(E_i, AE_i)X\} + \sum_{i=1}^{n} \{g(AE_i, AX)AE_i - g(AE_i, AE_i)AX\} .$$

Here

$$\sum_{i=1}^{n} g(E_i, AE_i) = \text{trace } A ,$$

$$\sum_{i=1}^{n} g(AE_i, AE_i) = \sum_{i=1}^{n} g(A^2E_i, E_i) = \text{trace } A^2 ,$$

$$\sum_{i=1}^{n} g(AE_i, X)E_i = \sum_{i=1}^{n} g(E_i, AX)E_i = AX ,$$

and

$$\sum_{i=1}^{n} g(AE_i, AX)AE_i = A \sum_{i=1}^{n} g(E_i, A^2X)E_i = A(A^2X) = A^3X .$$

Hence

$$\sum_{i=1}^{n} R(E_i, X)AE_i = cAX - c(\text{trace } A)X + A^3X - (\text{trace } A^2)AX .$$

Similarly, we get

$$\sum_{i=1}^{n} AR(E_i, X)E_i = cAX - cnAX + A^3X - (\text{trace } A)A^2X .$$

From these two equations we obtain

$$\sum_{i=1}^{n} [R(E_i, X), A]E_i = ncAX - (\text{trace } A^2)AX - c(\text{trace } A)X + (\text{trace } A)A^2X ,$$

that is, (15) gives

$$\Delta' A = ncA - (\text{trace } A^2)A - c(\text{trace } A)I + (\text{trace } A)A^2 , \tag{16}$$

where I is the identity transformation. From (9), we obtain

$$\frac{1}{2}\Delta f = cn(\text{trace } A^2) - (\text{trace } A^2)^2 - c(\text{trace } A)^2 + (\text{trace } A)(\text{trace } A^3) + g(\nabla A, \nabla A) . \tag{17}$$

In particular, if M is minimal in $\bar{M}$, that is, trace $A = 0$, then

$$\Delta' A = ncA - (\text{trace } A^2)A , \tag{16'}$$

$$\frac{1}{2}\Delta f = cnf - f^2 + g(\nabla A, \nabla A) , \tag{17'}$$

In the case where M is the unit sphere S^{n+1} (so that $c = 1$), (16′) and (17′) are found in Simons [8].

We shall now transform (17) into a form which is convenient for our applications. We first prove

Lemma. *Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then, for any constant c,*

$$nc \operatorname{tr} A^2 - (\operatorname{tr} A^2)^2 - c(\operatorname{tr} A)^2 + (\operatorname{tr} A)(\operatorname{tr} A^3) = \sum_{i<j} (\lambda_i - \lambda_j)^2(c + \lambda_i\lambda_j) .$$

Proof. Since the equality is trivial for $n = 1$, assume that it is valid for the degree $n - 1$. Then the left-hand side is equal to

$$\begin{aligned} &nc\left(\sum_{i=1}^{n-1} \lambda_i^2 + \lambda_n^2\right) - \left(\sum_{i=1}^{n-1} \lambda_i^2 + \lambda_n^2\right)^2 \\ &\quad - c\left(\sum_{i=1}^{n-1} \lambda_i + \lambda_n\right)^2 + \left(\sum_{i=1}^{n-1} \lambda_i + \lambda_n\right)\left(\sum_{i=1}^{n-1} \lambda_i^3 + \lambda_n^3\right) \\ &= \left\{c(n-1)\left(\sum_{i=1}^{n-1} \lambda_i^2\right) - \left(\sum_{i=1}^{n-1} \lambda_i^2\right)^2 - c\left(\sum_{i=1}^{n-1} \lambda_i\right)^2 + \left(\sum_{i=1}^{n-1} \lambda_i\right)\left(\sum_{i=1}^{n-1} \lambda_i^3\right)\right\} \\ &\quad + \left\{c\left(\sum_{i=1}^{n-1} \lambda_i^2\right) - 2c\left(\sum_{i=1}^{n-1} \lambda_i\right)\lambda_n + c(n-1)\lambda_n^2\right\} \\ &\quad + \sum_{i=1}^{n-1} (\lambda_i^3\lambda_n - 2\lambda_i^2\lambda_n^2 + \lambda_i\lambda_n^3) . \end{aligned}$$

On the above right side the first term is, by inductive assumption, equal to

$$\sum_{1\le i<j<n} (\lambda_i - \lambda_j)^2(c + \lambda_i\lambda_j) ,$$

the second is equal to

$$\sum_{i<n} c(\lambda_i - \lambda_n)^2 ,$$

and the third is equal to

$$\sum_{i<n} \lambda_i\lambda_n(\lambda_i - \lambda_n)^2 .$$

Therefore the whole sum is equal to

$$\begin{aligned}\sum_{1\le i<j<n} (\lambda_i - \lambda_j)^2(c + \lambda_i\lambda_j) + \sum_{i<n} (\lambda_i - \lambda_n)^2(c + \lambda_i\lambda_n)\\ = \sum_{i<j} (\lambda_i - \lambda_j)^2(c + \lambda_i\lambda_j) ,\end{aligned}$$

which completes the proof of the lemma.

Now for each point x of the hypersurface M, let $\{e_1, \cdots, e_n\}$ be an orthonormal basis in $T_x(M)$ such that $Ae_i = \lambda_i e_i$, $1 \le i \le n$. By the Gauss equation (3) we see that the sectional curvature K_{ij} for the 2-plane spanned by e_i and e_j, $i \neq j$, is equal to $c + \lambda_i\lambda_j$. Thus (17) can be written as follows:

$$\frac{1}{2}\Delta f = \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} + g(\nabla A, \nabla A) . \tag{18}$$

2. Main results

Let M be a connected hypersurface immersed *with constant mean curvature* in a space form $\bar{M}$ of dimension $n + 1$ with constant curvature, say, c. We establish the following lemmas.

Lemma 1. *If M is compact and has non-negative sectional curvature (for all 2-planes), then at every point of M we have*

$$\nabla A = 0 \quad \textit{and} \quad (\lambda_i - \lambda_j)^2 K_{ij} = 0 \qquad \textit{for all } i, j .$$

In particular, the eigenvalues of A are constant (where the field of unit normals ξ is defined).

Proof. By assumption, $K_{ij} \ge 0$. From the formula (18) we have $\Delta f \ge 0$. Since M is compact, we conclude that f is constant and $\Delta f = 0$ (see, for instance, Yano [10, p. 215] or Kobayashi-Nomizu [4, Note 14]). Thus we get $\nabla A = 0$ and $(\lambda_i - \lambda_j)K_{ij} = 0$ for all i, j.

Lemma 2. *If M has non-negative sectional curvature, and $f =$ trace A^2 is constant on M, then we have the same conclusions as Lemma 1.*

Proof. This is obvious from the formula (18) itself.

Lemma 3. *Under the assumptions of Lemma 1 or Lemma 2, either M is totally umbilical or A has exactly two distinct constants as eigenvalues at every point.*

Proof. As we already know, the eigenvalues of A remain constant (in its domain of definition). Thus the set of umbilics is an open set in M. Since it is obviously a closed set, either M is totally umbilical or M has no umbilic. In the second case, we show that A has at most (hence exactly) two eigenvalues at any point x. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A at x. We may assume that $\lambda_1 > 0$ for the following reason. If $\lambda_1 \leq 0$, then $\lambda_n \leq 0$. Since $\lambda_n = 0$ implies $\lambda_1 = \cdots = \lambda_n = 0$ contrary to our premise, we must have $\lambda_n < 0$. We may then change the field of unit normals ξ around x into $-\xi$ thus changing A into $-A$, whose largest eigenvalue $-\lambda_n$ is positive. Having assumed that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ with $\lambda_1 > 0$, we have $K_{12} \geq K_{13} \geq \cdots \geq K_{1n}$ and these are all non-negative by assumption. Assume that p is the largest integer such that $K_{1p} > 0$ and $K_{1p+1} = 0$ (set $p = n$ if $K_{1n} > 0$, although we see in a moment that this does not arise). From the second conclusion of Lemma 1 or 2, we get

$$(\lambda_1 - \lambda_i)^2 K_{1i} = 0 \qquad \text{for all } 1 \leq i \leq p ,$$

which imply that

$$\lambda_1 = \cdots = \lambda_p = \lambda , \quad \text{say.}$$

Here $p \neq n$, since x is not an umbilic. In addition we have

$$K_{1p+1} = \cdots = K_{1n} = 0 ,$$

that is,

$$c + \lambda_1\lambda_{p+1} = \cdots = c + \lambda_1\lambda_n = 0 ,$$

which imply that

$$\lambda_{p+1} = \cdots = \lambda_n = -c/\lambda .$$

This proves our assertion that A has at most two distinct eigenvalues.

With these preparations we shall now prove our main results.

Theorem 1. *Let M be a complete Riemannian manifold of dimension n with non-negative sectional curvature, and $\phi\colon M \to R^{n+1}$ an isometric immersion with constant mean curvature into a Euclidean space R^{n+1}. If $f =$ trace A^2 is constant on M, then $\phi(M)$ is of the form $S^p \times R^{n-p}$, $0 \leq p \leq n$, where R^{n-p} is an $(n-p)$-dimensional subspace of R^{n+p}, and S^p is a sphere in the Euclidean subspace perpendicular to R^{n-p}. Except for the case $p = 1$, ϕ is an imbedding.*

Poorf. We first assume that M is simply connected. By Lemma 3 we know that either M is totally umbilical or A has exactly two distinct constant eigenvalues λ, μ, where $\lambda \neq 0$ has multiplicity p, $1 \leq p \leq n-1$, and μ is actually 0 (since $c = 0$ in the proof of Lemma 3). In the first case, it follows that $\phi(M)$ is actually a Euclidean hyperplane R^n or a sphere S^n, depending on whether A is 0 or not. Since M and $\phi(M)$ are simply connected, we conclude that ϕ is an imbedding (cf. Theorem 4.6, p. 176 of Kobayashi-Nomizu [3]).

In the second case, we can define two distributions

$$T^1(x) = \{x \in T_x(M);\, AX = \lambda X\}\,,$$

and

$$T^0(x) = \{X \in T_x(M);\, AX = 0\}$$

of dimensions p and $n - p$, respectively. Knowing that λ is a constant, it is easy to see that both distributions are differentiable, involutive and totally geodesic on M. Thus M is the Riemannian direct product $M^1 \times M^0$, where M^1 and M^0 are the maximal integral manifolds of T^1 and T^0, respectively, through a certain point of M. From this point on, we may use the same arguments as those for Proposition 3 in Nomizu [6] to conclude that $\phi(M)$ is of the form $S^p \times R^{n-p}$. If $p \geq 2$, then $\phi(M)$ is simply connected and we conclude that ϕ is an imbedding. (If $p = 1$, then M may be $R \times R^{n-1}$ which is immersed onto $S^1 \times R^{n-1}$ in R^{n+1}.)

In the general case, let $\hat{M}$ be the universal covering manifold on M with the projection $\pi\colon \hat{M} \to M$. With respect to the naturally induced metric, $\hat{M}$ and $\hat{\phi} = \phi \circ \pi$ satisfy the same assumptions as those for M and ϕ. Thus $\hat{\phi}(\hat{M}) = \phi(M)$ is of the form $S^p \times R^{n-p}$. If $p \neq 1$, then $\hat{\phi}$ is an imbedding and so is ϕ.

Corollary 1. *If M is, in particular, minimal in Theorem* 1, *then $\phi(M)$ is a hyperplane and ϕ is an imbedding.*

Remark 1. Without completeness of M the corresponding local versions of Theorem 1 and Corollary 1 are valid.

Remark 2. Theorem 1 may be thought of as a partial extension of a result of Klotz and Osserman [2].

Corollary 2. *Let M be a connected compact Riemannian manifold of dimension n with non-negative sectional curvature. If $\phi\colon M \to R^{n+1}$ is an isometric immersion with constant mean curvature, then $\phi(M)$ is a hypersphere and ϕ is an imbedding.*

Proof. By Lemma 1, we know that f is a constant. Since $\phi(M)$ is compact, we must have $p = n$ in the conclusion of Theorem 1.

Remark. Corollary 2 is slightly stronger than the classical theorem of Süss [9], where M is assumed to be a convex hypersurface.

Before we prove our results for hypersurfaces in the unit sphere S^{n+1} (i.e. the standard model for a space form of dimension $n + 1$ with constant sectional curvature 1), we explain a few examples. In R^{n+2} with usual inner product, $S^{n+1} = \{x \in R^{n+2}; (x, x) = 1\}$.

For any unit vector a and for any r, $0 \leq r < 1$, let

$$\sum^n = \{x \in S^{n+1}; (x, a) = r\} .$$

When $r = 0$, $\sum^n$ is a *great sphere in* S^{n+1}. When $r > 0$, we call $\sum^n$ a *small sphere in* S^{n+1}. By elementary computation we find that the second fundamental form of $\sum^n$ as a hypersurface of S^{n+1} is given by

$$A = \frac{r}{\sqrt{1 - r^2}} I \qquad \text{(up to a sign)} ,$$

where I is the identity transformation. The mean curvature is constant and so is the function $f = \operatorname{trace} A^2$. It is known that a totally umbilical hypersurface in S^{n+1} is locally (globally if it is complete) $\sum^n$; in particular, it is a great sphere if it is totally geodesic.

Another example is a product of spheres $S^p(r) \times S^q(s)$, where $p + q = n$ and $r^2 + s^2 = 1$. For such $p, q > 0$, consider R^{n+2} as $R^{p+1} \times R^{q+1}$ and let

$$S^p(r) = \{x \in R^{p+1}; (x, x) = r^2\} ,$$
$$S^q(s) = \{y \in R^{q+1}; (y, y) = s^2\} .$$

Then

$$S^p(r) \times S^q(s) = \{(x, y) \in R^{n+2}; x \in S^p(r), y \in S^q(s)\}$$

is a hypersurface of S^{n+1}. The second fundamental form A has eigenvalues s/r of multiplicity p and $-r/s$ of multiplicity q. Both the mean curvature and the function f are constants. $S^p(r) \times S^q(s)$ is minimal if and only if $r = \sqrt{p/n}$.

In particular, consider the case $n = 2$. For r, s such that $r^2 + s^2 = 1$, $S^1(r) \times S^1(s)$ in S^3 is called a *flat torus*. When $r = s = 1/\sqrt{2}$, it is a minimal surface in S^3.

We now prove

Theorem 2. *Let M be an n-dimensional complete Riemannian manifold with non-negative sectional curvature, and ϕ: $M \to S^{n+1}$ an isometric immersion with constant mean curvature. If $f = \operatorname{trace} A^2$ is constant on M, then either*

(1) *$\phi(M)$ is a great or small sphere in S^{n+1}, and ϕ is an imbedding;*

or

(2) *$\phi(M)$ is a product of spheres $S^p(r) \times S^q(s)$, and for $p \neq 1$, $n - 1$, ϕ is an imbedding.*

Proof. We may assume that M is simply connected. By Lemma 3 we know that either M is totally umbilical, in which case we get the conclusion (1), or A has two constants λ, μ such that $\lambda\mu = -1$ as the eigenvalues at all points. Let p, q be the multiplicities of λ, μ (so that $p + q = n$). It follows that M is the direct product $M_1 \times M_2$, where M_1 is a p-dimensional space of constant curvature $1 + \lambda^2$, and M_2 is a q-dimensional space of constant curvature $1 + \mu^2$. (We may prove this fact again by considering the distributions of eigenspaces for λ and μ; for the detail, see Ryan [7]). If $p \neq 1$, then $M_1 = S^p(r)$ where $r = 1/\sqrt{1 + \lambda^2}$. Similarly, if $q \neq 1$, then $M_2 = S^q(s)$ where $s = 1/\sqrt{1 + \mu^2}$. Of course, $r^2 + s^2 = 1$. If $p = 1$ or $q = 1$, we take R^1 instead of $S^1(r)$ or $S^1(s)$. At any rate, the type number for ϕ (i.e. the rank of A) is equal to n everywhere. Thus if $n \geq 3$, the classical rigidity theorem (cf., for example, Ryan [7]) shows that $\phi(M)$ is the product of spheres $S^p(r) \times S^q(s)$ in S^{n+1} and that ϕ is an imbedding unless $p = 1$ or $q = 1$. It remains to show that, for $n = 2$, $\phi(M)$ is a flat torus. But this can be done by an elementary argument. We have thus proved Theorem 2.

Corollary 1. *If M is, in particular, minimal in Theorem* 2, *then $\phi(M)$ is a great sphere or $S^p(\sqrt{p/n}) \times S^{n-p}(\sqrt{(n-p)/n})$.*

Remark. Without completeness of M, the corresponding local versions of Theorem 2 and Corollary 1 are valid.

Corollary 2. *Let M be a connected compact Riemannian manifold of dimension n with non-negative sectional curvature. If ϕ: $M \to S^{n+1}$ is an isometric immersion with constant mean curvature, then* (1) *or* (2) *of Theorem* 2 *holds.*

The following special case is worth mentioning.

Corollary 3. *Let M be a connected compact minimal hypersurface immersed in S^{n+1}. If M has positive sectional curvature, then M is imbedded as a great sphere.*

Remark. Corollary 3 is a generalization of a result of Almgren [1] which says that a compact minimal surface of genus 0 in S^3 is a great sphere.

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