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A FOUR-VALUED MODAL LOGIC ARISING FROM MONTEIRO'S LAST ALGEBRAS

by

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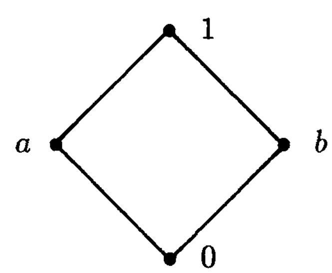
Abstract. We study the class of abstract logics projectively generated by the class of tetravalent modal algebras. They are modal logics and they turn out to be four-valued in the sense that they can be characterized using the natural logic on the four-element tetravalent modal algebra which generates this variety together with some special class of homomorphisms. We also characterize them by their abstract properties and prove a completeness theorem.

INTRODUCTION

In this paper we are going to deal with algebras $\mathfrak{A} = \langle A, \wedge, \neg, I, 0\rangle$ of type (2,1,1,0), and we will follow some notational conventions: We always put $1 = \neg 0$, $a \vee b = \neg(\neg a \wedge \neg b)$ for all $a, b \in A$, and $\mathfrak{A}^- = \langle A, \wedge, \neg, 0\rangle$ denotes the I-less reduct of $\mathfrak{A}$. We call $\mathfrak{M}_{4m}$ the four-element algebra such that $\mathfrak{M}_{4m}^- = \mathfrak{M}_4$ is the four-element De Morgan algebra (whose Hasse diagram is shown below) and whose unary operator I is defined in the following way:



$$I1 = 1$$
$$Ia = Ib = I0 = 0$$

This algebra has two subalgebras, namely the two-element and three-element chains with the inherited unary operator I; we call them $\mathfrak{B}_{2m}$ and $\mathfrak{M}_{3m}$ respectively, and we also put $\mathfrak{B}_{2m}^- = \mathfrak{B}_2$ (the two-element Boolean algebra) and $\mathfrak{M}_{3m}^- = \mathfrak{M}_3$ (the three-element Kleene algebra). Note that, aside from the presentation of the algebraic structure, $\mathfrak{M}_{3m}$ is the three-valued Łukasiewicz algebra.

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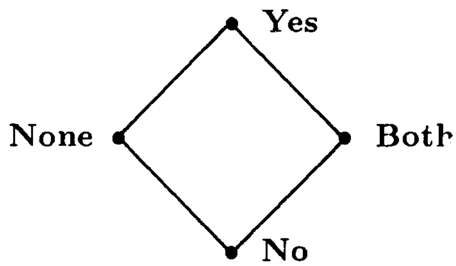
The algebra $\mathfrak{M}_{4m}$ generates the variety of the so-called **Tetravalent Modal Algebras (TMAs)**. These algebras were considered by Antonio Monteiro motivated by Luiz Monteiro's independence proof [**Mon**] of the axiomatization of three-valued Łukasiewicz algebras. During his last stay in Lisbon between 1977 and 1979 he suggested the study of TMAs to Isabel Loureiro, who began to work under his advice and finally (two years after Monteiro's death) wrote her Ph.D. Dissertation [**L3**] and a number of papers on such topic. It seems (see [**P**, page xxxvi]) that she was one of Monteiro's last disciples and that TMAs were his last original "creation".

So far, TMAs have received little attention, and still only from the algebraic point of view. We believe, however, that they have a genuine interest from the point of view of logic, both the multiple-valued branch and the abstract branch, and we will try in this paper to present some facts and results to support this claim. Our goal is to use $\mathfrak{M}_{4m}$ as a (generalized) matrix to generate logics by the usual semantic methods and to clarify the status of TMAs among these logics. This will give us the means to characterize these logics in an abstract form and as a corollary to find a completeness theorem for them, by using an ad-hoc sequent calculus.

The four-element De Morgan algebra $\mathfrak{M}_4$ has been used by several authors to define a semantical entailment relation. In [**Ma**], for instance, we find a very elegant presentation, although no interpretation is given to the four values. It was in [**Be**] where for the first time such an "epistemic" interpretation was given, by reading the two intermediate values respectively as "neither true nor false" (the well-known "undetermined" value of classical three-valued logic) and "both true and false" (the new value, sometimes called "overdetermined"). This seminal idea has lead to several developments, either in the study of question-answering data-bases and distributed logic programs dealing with information which might contain conflicts or gaps, either in some extensions of Kripke's theory of truth, where the truth predicate is partial. The behaviour of $\mathfrak{M}_4$ has been generalized to the nice concept of bilattice. See [**Fi1**], [**Fi2**] and [**Vi**] for more details and references.

The basic idea can be roughly explained as follows: If we admit that the information a computer can access to answer the questions he is presented concerning some topic might contain gaps (impossibility to handle the question, for instance due to the structure of the data base itself) and conflicts (for instance if the information is drawn from several independent sources) then we have to admit that the computer's answer to a question of the form "is ... true ?" should include at least the four possibilities: **Yes**, **No**, **Both** (conflict) and **None** (gap). These four values can be ordered according to their "degree of truth", and the resulting ordered set is a lattice; together with the natural negation it becomes a model of $\mathfrak{M}_{4m}$:



In this context we can consider adding a modal operator I of an epistemic character corresponding to questions of the type "can the computer confirm that ... is true ?". It is clear that the answer is **Yes** just in the case where the answer to the first question is also **Yes**, while it is **No** in all other cases; this modal operator is thus truth-functional and gives us a model of the four-element TMA $\mathfrak{M}_{4m}$.

We can extend the interpretation of the ordering relation of $\mathfrak{M}_{4m}$ in terms of "degrees (or modes) of truth" to find an appropriate way of definig a semantic entailment relation by taking valuations on $\mathfrak{M}_{4m}$. The basic principle should be that the value of the conclusion should have a greater (or equal) degree of truth than that of the premisses, taken together (that is, of their conjunction). Looking at the Hasse diagram for $\mathfrak{M}_{4m}$ and considering the two filters of the lattice: $F_1 = \{1, a\}$ and $F_2 = \{1, b\}$ we see we can implement the principle by asking that whenever the value of the (conjunction of the) premisses falls in F_i then the value of the conclusion should fall in F_i as well, for $i = 1, 2$. The readers familiar with the methods of algebraic logic will recognize that this proposal is just to take the two logical matrices $\langle \mathfrak{M}_{4m}, F_1 \rangle$ and $\langle \mathfrak{M}_{4m}, F_2 \rangle$ together, that is, to use the generalized matrix $\langle \mathfrak{M}_{4m}, \{F_1, F_2\} \rangle$, and to projectively generate a logic from it by the family of all the homomorphisms (valuations).

We will generalize this construction by taking arbitrary families of homomorphisms and by generating logics on any algebra of the type, obtaining a class of logics called **Tetravalent Modal Logics (TMLs)** because they are both modal (the operator I has an S5-type modal behaviour, see Proposition 3) and four-valued (by their own definition). We will see that certain logics canonically attached to TMAs have a distinguished position in this class of logics. This will allow us to obtain the abstract characterization of the original semantic operator and from it we will obtain the completeness result. For technical reasons the material in the paper follows a slightly different order: We begin by introducing the TMAs and some of their elementary properties, and then we present the class of TMLs and prove the main theorems about the relations linking TMAs and TMLs. After that we show the relation between TMLs and the generalized matrix on $\mathfrak{M}_{4m}$, while introducing the semantic operator, which receives the announced abstract characterization. Finally, after a change in the type of the algebras, we are able to define the syntactic operator and prove completeness.

NOTATION AND TERMINOLOGY

An **abstract logic** (briefly: a **logic**) is a pair $\mathbb{L} = \langle \mathfrak{A}, \mathbf{C} \rangle$ or $\mathbb{L} = \langle \mathfrak{A}, \mathcal{C} \rangle$, where $\mathfrak{A}$ is an algebra, $\mathbf{C}$ is a closure operator over A, the carrier of $\mathfrak{A}$, and $\mathcal{C}$ is a closure system on A. Given two closure operators $\mathbf{C}_1$ and $\mathbf{C}_2$ over the same set A, we say that $\mathbf{C}_1$ is **weaker** or **smaller** than $\mathbf{C}_2$ (in symbols $\mathbf{C}_1 \leqslant \mathbf{C}_2$) if and only if $\forall X \subseteq A$, $\mathbf{C}_1(X) \subseteq \mathbf{C}_2(X)$; it is equivalent to say that $\mathcal{C}_2 \subseteq \mathcal{C}_1$, that is, that $\mathcal{C}_1$ is **finer** than $\mathcal{C}_2$. We also say that $\mathbb{L}_1$ is weaker or smaller than $\mathbb{L}_2$.

Given any two logics $\mathbb{L}_1 = \langle \mathfrak{A}_1, \mathbf{C}_1 \rangle$ and $\mathbb{L}_2 = \langle \mathfrak{A}_2, \mathbf{C}_2 \rangle$, we say that $\mathbb{L}_1$ is **projectively generated** from $\mathbb{L}_2$ by a set $\mathcal{H} \subseteq Hom(\mathfrak{A}_1, \mathfrak{A}_2)$ if and only if $\mathcal{C}_1$ has the set $\{ h^{-1}(T) : T \in \mathcal{C}_2,\ h \in \mathcal{H} \}$ as a basis (where $\mathcal{C}_2$ can be substituted by any of its bases); it is equivalent to say that $\forall X \subseteq A_1$, $\mathbf{C}_1(X) = \{a \in A_1 : h(a) \in \mathbf{C}_2\big(h(X)\big)\ \forall h \in \mathcal{H}\}$. In the case where $\mathcal{H} = \{h\}$ reduces to only one epimorphism h we say that h is a **bilogical morphism** between $\mathbb{L}_1$ and $\mathbb{L}_2$. This is a central concept, as we see from the following properties. If h is a bilogical morphism between $\mathbb{L}_1$ and $\mathbb{L}_2$ then $\mathcal{C}_1 \cong \mathcal{C}_2$ as complete lattices, and we have both $\mathbf{C}_1 = h^{-1} \circ \mathbf{C}_2 \circ h$ and $\mathbf{C}_2 = h \circ \mathbf{C}_1 \circ h^{-1}$. Given any logic $\mathbb{L}$, to every $\theta \in Con(\mathfrak{A})$ one can associate the **quotient logic** of $\mathbb{L}$ by θ by taking $\mathcal{C}/\theta = \{T \subseteq A/\theta : \pi^{-1}(T) \in \mathcal{C}\}$, where $\pi : \mathfrak{A} \to \mathfrak{A}/\theta$ is the canonical projection, and putting $\mathbb{L}/\theta = \langle \mathfrak{A}/\theta, \mathcal{C}/\theta \rangle$. On the other hand, we naturally have the equivalence relation $\theta(\mathbf{C}) = \{ (a,b) \in A \times A : \mathbf{C}(a) = \mathbf{C}(b) \}$, and we say that an equivalence relation θ over A is an **$\mathbb{L}$-congruence** or a **logical congruence** whenever $\theta \in Con(\mathfrak{A})$ and $\theta \subseteq \theta(\mathbf{C})$; in this case the canonical projection π becomes a bilogical morphism between $\mathbb{L}$ and $\mathbb{L}/\theta$. Conversely, if h is a bilogical morphism between $\mathbb{L}_1$ and $\mathbb{L}_2$ then the relation $\theta_h = \{ (a,b) \in A_1 \times A_1 : h(a) = h(b) \}$ is an $\mathbb{L}_1$-congruence and by factorization we obtain a **logical isomorphism** between $\mathbb{L}_1/\theta_h$ and $\mathbb{L}_2$, that is, an algebraic isomorphism which is also an isomorphism between the closure systems; in other words, an "identification" of the logics. We thus see that in some sense bilogical morphisms play the role of epimorphisms in universal algebra.

In any lattice we usually denote by $\mathcal{F}$ the set of all its **filters**, and by $\mathcal{P}$ the set of all its **prime filters**. Our $\mathfrak{M}_{4m}$ is a lattice, with $\mathcal{P} = \{F_1, F_2\}$ and $\mathcal{F} = \{\{1\}, F_1, F_2, M_{4m}\}$. The abstract logic determined by all the filters is denoted by $\mathbb{L}_{4m} = \langle \mathfrak{M}_{4m}, \mathcal{F} \rangle$; remark that since $\mathfrak{M}_{4m}$ is a distributive lattice, $\mathcal{P}$ is a basis of $\mathcal{F}$. On any algebra $\mathfrak{A}$ of the same type we can consider the logic projectively generated from $\mathbb{L}_{4m}$ by the set $Hom(\mathfrak{A}, \mathfrak{M}_{4m})$; we denote it by $\mathbb{L}_{4m}(\mathfrak{A}) = \langle \mathfrak{A}, C_{4m}(\mathfrak{A}) \rangle$, or, if no confusion is likely to arise, by $\langle \mathfrak{A}, C_{4m} \rangle$, and remark that this C_{4m} has the set $\{h^{-1}(F_1), h^{-1}(F_2) \,:\, h \in Hom(\mathfrak{A}, \mathfrak{M}_{4m})\}$ as a basis.

Tetravalent modal algebras

We are going to recall their definiton and some properties, together with some new ones we will need later. Our definition will be slightly different from (although trivially equivalent to) that of Loureiro, to conform to the type of the algebras used here.

1. Definition. *We say that an algebra $\mathfrak{A} = \langle A, \wedge, \neg, I, 0 \rangle$ of type (2,1,1,0) is a* **tetravalent modal algebra (TMA)** *if and only if $\mathfrak{A}^- = \langle A, \wedge, \neg, 0 \rangle$ is a De Morgan algebra (that is, a De Morgan lattice with minimum 0) and the operator I satisfies $\forall a \in A$ the two following properties:*

(1) $Ia \wedge \neg a = 0$; *and*
(2) $\neg Ia \wedge a = \neg a \wedge a$.

Thus when dealing with TMAs we have all properties of De Morgan algebras; see for instance [**Ba-Dw**].

2. Proposition. *$\forall a, b \in A$, the folowing hold:*

(1) $\neg Ia \vee a = 1$;
(2) $Ia \vee \neg a = a \vee \neg a$;
(3) $Ia \leq a$;
(4) $I(a \wedge b) = Ia \wedge Ib$;
(5) $I1 = 1$;
(6) $I0 = 0$;
(7) $I(a \vee Ib) = Ia \vee Ib$;

(8) $I\neg Ia = \neg Ia$;

(9) $I^2 a = Ia$;

(10) $a \wedge I\neg a = 0$; *and*

(11) *TMAs form a variety.*

PROOF: Properties (1) to (9) are proved in [**L3**] (on the other hand, they are easy). To prove (10), from (1) we get $\neg a \vee \neg I \neg a = 1 \;\; \forall a \in A$, and the De Morgan rules for negation give $a \wedge I\neg a = 0$. Finally (11) comes directly from Definition 1. ∎

We say that an element $a \in A$ is **open** iff $a = Ia$, and denote by $B = \{a \in A : a = Ia\}$ the set of all open elements, which by 2.9 is equal to $I(A)$. We also say that $X \subseteq A$ is **open** iff $I(X) \subseteq X$. Then the next proposition is the algebraic expression of "being a modal operator of S5 type"; this kind of modal operators can also be characterized using abstract logics, with the methods of [**FV2**], which will be applied to TMAs in [**FR**].

3. PROPOSITION. *The operator I is a monadic interior operator, and moreover $\mathfrak{B} = \langle B, \wedge, \neg, 0\rangle$ is a subalgebra of $\mathfrak{A}^-$ which is a Boolean algebra.*

PROOF: From 2.3, 2.4, 2.5 and 2.9 we see that I is an interior operator, and 2.7 or 2.8 tell us that it is a monadic one (see [**H**]). From 2.4, 2.6 and 2.8 we conclude that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}^-$ (we might also say "of $\mathfrak{A}$" as well, because in $\mathfrak{B}$ the operator I is trivial). And finally 2.1 proves that it is a De Morgan algebra whose negation is a complement, that is, $\mathfrak{B}$ is a Boolean algebra. ∎

It can be proved, see [**L7**], that the subdirectly irreducible TMAs are precisely $\mathfrak{B}_{2m}$, $\mathfrak{M}_{3m}$ and $\mathfrak{M}_{4m}$; it follows that the variety of TMAs is generated by $\mathfrak{M}_{4m}$. This algebra is quasi-primal, and thus (see [**We**]) the variety is congruence-regular and enjoys the congruence-extension property. However, the congruences of a TMA are better understood in terms of filters. This topic will be dealt with in [**FR**]; here we are interested in filters and open filters because of their logical role.

4. DEFINITION. *We say that an $F \subseteq A$ is an* **open filter** *iff it is a filter of the lattice and $a \in F$ implies $Ia \in F$, $\forall a \in A$. We denote by $\mathcal{F}^+$ the set of all open filters of $\mathfrak{A}$, that is, $\mathcal{F}^+ = \{F \in \mathcal{F} : I(F) \subseteq F\}$. The closure operators associated with $\mathcal{F}$ and $\mathcal{F}^+$ are denoted by $\mathbf{F}$ and $\mathbf{F}^+$ respectively.*

Since TMAs are distributive lattices, the closure system of all filters has as a basis the set of all prime filters. It turns out that in TMAs there is also a close relationship between open filters and prime filters; to express this relationship one needs to consider a transformation defined on the set of all prime filters, which sometimes (see Loureiro's and Monteiro's papers) is called the **Birula-Rasiowa transformation** :

5. DEFINITION. *Let A be any set having a unary operation $\neg$. Then we denote by Φ the function defined for all subsets of A in the following way:*

$$\text{For any } X \subseteq A\,, \qquad \Phi(X) = \{x \in A : \neg x \notin X\} \subseteq A.$$

The definition given here is suitable for very general situations, but in the case of De Morgan algebras it is equivalent to the original one (which comes from the representation theory of these structures, see [**Bi-Ra**]) as shows the following result:

6. PROPOSITION. *If $\mathfrak{A}$ is a De Morgan algebra (and, a fortiori, if it is a TMA), then $\Phi(X) = A \setminus \neg X = A \setminus \{\neg x : x \in X\}$ $\forall X \subseteq A$; and if $F \in \mathcal{P}$ then also $\Phi(F) \in \mathcal{P}$ and $\Phi^2(F) = F$.* ∎

7. PROPOSITION. *In any TMA the following properties hold:*

(1) *If $\mathcal{P}' \subseteq \mathcal{P}$ is a family of prime filters closed under Φ, then $\bigcap \mathcal{P}' \in \mathcal{F}^+$;*
(2) *If $F \in \mathcal{F}^+$ then $\mathcal{P}' = \{P \in \mathcal{P} \,:\, F \subseteq P\}$ is a family of prime filters closed under Φ and such that $\bigcap \mathcal{P}' = F$*

PROOF: (1) We already know that $\bigcap \mathcal{P}' \in \mathcal{F}$. Let $a \in \bigcap \mathcal{P}'$ and suppose $Ia \notin \bigcap \mathcal{P}'$, that is, there is an $F \in \mathcal{P}'$ such that $Ia \notin F$. This implies $\neg Ia \in \Phi(F)$, but $\Phi(F) \in \mathcal{P}'$, so $a \in \Phi(F)$ and by 1.2 $a \wedge \neg a = a \wedge \neg Ia \in \Phi(F)$, and therefore $\neg a \in \Phi(F)$, but this means that $a \notin F$ which is a contradiction.

(2) We know that $\bigcap \mathcal{P}' = F$ because $\mathcal{P}$ is a basis of $\mathcal{F}$. If $x \in F \subseteq P$ with $x \notin \Phi(P)$, then $x = \neg y$ for some $y \in P$ and since $x \in F \in \mathcal{F}^+$, we have $Ix = I\neg y \in F$ and thus $y \wedge I\neg y = 0 \in F \subseteq P$, an absurdity. ∎

8. COROLLARY. *In any TMA the following hold $\forall P \in \mathcal{P}$ and $\forall a \in A$:*

(1) $P \cap \Phi(P) \in \mathcal{F}^+$;
(2) $a \in P \cap \Phi(P)$ *iff* $Ia \in P \cap \Phi(P)$; *and*
(3) $Ia \in P$ *ssi* $Ia \in \Phi(P)$.

PROOF: (1) Just apply 7.1 to the family $\{P, \Phi(P)\}$, which is closed under Φ.

(2) If $Ia \in P \cap \Phi(P)$ then, $a \in P \cap \Phi(P)$ because $Ia \leq a$. The reverse implication is given by (1).

(3) If $Ia \in P$ and $Ia \notin \Phi(P)$, then $\neg Ia \in P$ and thus $Ia \wedge \neg Ia = \neg a \wedge Ia = 0 \in P$, a contradiction. ∎

According to Proposition 7 there is a correspondence between families of prime filters closed under Φ and open filters. Different such families, though, can have the same meet, and thus we remark that such correspondence is not one-to-one. It can be proved, however, that it is so if we restric ourselves to families of the form shown in 7.2; in terms of the closure systems generated by those families of prime filters, this is to restrict ourselves to algebraic closure systems. See [**FR**] for more details and for similar results in more general situations.

TETRAVALENT MODAL LOGICS

We want to define these logics as the generalization of the logics determined on TMAs by the closure system of all filters. To this end we use some of the properties just proved, and Theorems 14 and 15 will confirm us that the choice made for the definition was right. As for the name given to them,it was so chosen to remark their relationship with TMAs, a term with a published tradition, although it seems somehow odd to use the latin term "tetravalent" in the place of "four-valued"; this last one would perhaps be linguistically better, but at the same time its meaning is too wide and ambiguous, and thus uninformative.

9. DEFINITION. *An abstract logic* $\mathbb{L} = \langle \mathfrak{A}, \mathcal{C} \rangle$ *is a* **quasi tetravalent modal logic (QTML)** *if and only if there is a basis* $\mathcal{E}$ *of* $\mathcal{C}$ *such that every* $P \in \mathcal{E}$ *satisfies*

(1) P *is an* $\wedge$**-filter** *(that is,* $\forall a, b \in A$, $a \wedge b \in P \iff a, b \in P)$;

(2) $\Phi(P) \in \mathcal{E}$ *and* $\Phi^2(P) = P$;

(3) $0 \notin P$;

(4) $Ia \in P$ *iff* $Ia \in \Phi(P)$ $\forall a \in A$; *and*

(5) $a \in P \cap \Phi(P)$ *iff* $Ia \in P \cap \Phi(P)$ $\forall a \in A$

We say that $\mathbb{L}$ *is a* **tetravalent modal logic (TML)** *iff it is a QTML and moreover* $\mathcal{C}$ *is algebraic (or finitary).*

10. EXAMPLES. (1) The **trivial** case of TML is obtained when $\mathcal{E} = \emptyset$; in such a case there is only one closed set in $\mathcal{C}$, namely A, and the five conditions of Definition 9 are trivially satisfied.

(2) As a result of Proposition 8, for any TMA $\mathfrak{A}$ we have one TML $\mathbb{L} = \langle \mathfrak{A}, \mathcal{F} \rangle$, where as usual $\mathcal{F}$ is the closure system of all its filters. As we have already announced, these kind of TMLs will be the "prototypes" of TMLs, in the precise sense stated in Theorem 15. Previously in Theorem 14 we will also see that the properties shown in Proposition 7 characterize the QTMLs, although they are better characterized in Theorem 17.

The logics satisfying 9.1 and 9.2 are called **De Morgan logics** in [**FV1**] and [**FV3**]. We are going to use some properties proved there for these logics, concerning mainly the non-modal part of our structures. For instance, we immediately have:

11. PROPOSITION. *Let $\mathbb{L}$ be a QTML. Then the following hold:*

(1) *$\mathbb{L}$ satisfies the* **Property of Conjunction**: $\mathbf{C}(a \wedge b) = \mathbf{C}(a, b)$ $\forall a, b \in A$;

(2) *Every $P \in \mathcal{E}$ is* $\vee$-**prime**: $a \vee b \in P \iff a \in P$ or $b \in P$ $\forall a, b \in A$;

(3) *$\mathbb{L}$ satisfies the* **Property of Disjunction**: $\mathbf{C}(X, a \vee b) = \mathbf{C}(X, a) \cap \mathbf{C}(X, b)$ $\forall a, b \in A, \forall X \subseteq A$. ∎

12. PROPOSITION. *Each one of the conditions 9.4 and 9.5 is preserved under bilogical morphisms, that is, if $\mathbb{L}_1$ and $\mathbb{L}_2$ are two abstract logics and there is a bilogical morphism between them, then one of them satisfies condition 9.4 (resp. 9.5) if and only if the other one does.*

PROOF: Put $\mathbb{L}_1 = \langle \mathfrak{A}_1, \mathcal{C}_1 \rangle$ and $\mathbb{L}_2 = \langle \mathfrak{A}_2, \mathcal{C}_2 \rangle$, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be bases of $\mathcal{C}_1$ and $\mathcal{C}_2$ resp. The existence of a bilogical morphism $h : \mathbb{L}_1 \to \mathbb{L}_2$ means that $\{h^{-1}(P) : P \in \mathcal{E}_2\}$ is a basis of $\mathcal{C}_1$ and conversely that $\{h(P) : P \in \mathcal{E}_1\}$ is a basis of $\mathcal{C}_2$. Also recall that a bilogical morphism is onto.

Suppose that $\mathbb{L}_1$ satisfies 9.4 and let $P \in \mathcal{E}_1$. If $Ia \in h(P)$, where $a \in A_2$, then $Ia = h(a')$ for some $a' \in P$. Let $b' \in A_1$ such that $h(b') = a$, then $h(Ib') = h(a')$ and therefore $Ib' \in P$ (because h is a bilogical morphism and $P \in \mathcal{E}_1$). But the assumption implies that $Ib' \in \Phi(P)$ and so $Ia = h(Ib') \in h(\Phi P) = \Phi(h(P))$. Conversely, if $Ia \in \Phi(h(P))$, we obtain that $Ia \in h(P)$, that is, $\mathbb{L}_2$ satisfies 9.4.

Suppose that $\mathbb{L}_1$ satisfies 9.5 and take an $a \in h(P) \cap \Phi h(P) = h(P \cap \Phi(P) \subseteq A_2$; then there is an $a' \in P \cap \Phi(P) \subseteq A_1$ with $h(a') = a$, but from $Ia' \in P \cap \Phi(P)$ and $h(Ia') = Ih(a') = Ia$ we obtain that $Ia \in h(P \cap \Phi(P)) = h(P) \cap \Phi h(P)$. Conversely, if $Ia \in h(P) \cap \Phi h(P) = h(P \cap \Phi(P)) \subseteq A_2$ then there is an $a' \in P \cap \Phi(P) \subseteq A_1$ with $h(a') = Ia$. Therefore there is a $b' \in A_1$ with $h(b') = a$, and thus $h(Ib') = Ia = h(a')$, which implies that Ib' and a' belong to the same closed sets, therefore $Ib' \in P \cap \Phi(P)$ and the assumption gives $b' \in P \cap \Phi(P)$ which also gives $a \in h(P \cap \Phi(P)) = h(P) \cap \Phi h(P)$. Thus we proved that $\mathbb{L}_2$ also satisfies 9.5.

Now suppose that $\mathbb{L}_2$ satisfies 9.4 and take a $P \in \mathcal{E}_2$. $Ia \in h^{-1}(P)$ iff $h(Ia) = Ih(a) \in P$ iff $Ih(a) = h(Ia) \in \Phi(P)$ iff $Ia \in h^{-1}(\Phi(P))$, and thus $\mathbb{L}_1$ satisfies 9.4.

Finally suppose that $\mathbb{L}_2$ satisfies 9.5. Then $a \in h^{-1}(P) \cap \Phi(h^{-1}(P)) = h^{-1}(P \cap \Phi(P))$, iff $h(a) \in P \cap \Phi(P)$ iff $Ih(a) = h(Ia) \in P \cap \Phi(P)$, iff $Ia \in h^{-1}(P \cap \Phi(P)) = h^{-1}(P) \cap \Phi(h^{-1}(P))$. We have thus proved that $\mathbb{L}_1$ satisfies 9.5. ▮

13. COROLLARY. *Let $\mathbb{L}_1$ and $\mathbb{L}_2$ be two abstract logics such that there is a bilogical morphism between them. Then one of them is a QTML (resp. a TML) if and only if the other one is.*

PROOF: We must prove that each one of the conditions of Definition 9 is preserved under bilogical morphisms. Proposition 12 contains the proof for conditions 9.4 and 9.5; condition 9.3 is trivial; condition 9.2 is proved in Proposition 3 of [**FV1**] and condition 9.1 in Proposition 2.5 of [**FV2**]. Finally finitarity is dealt with for instance in a Corollary to Proposition 6 of [**V**]. ▮

Recall that to any abstract logic $\mathbb{L} = \langle \mathfrak{A}, \mathbf{C} \rangle$ we naturally associate the equivalence relation $\theta(\mathbf{C}) = \{ (a,b) \in A \times A : \mathbf{C}(a) = \mathbf{C}(b) \}$. If no confusion is likely to arise (which is the normal situation) we will denote this relation simply as θ, and if necessary will attach to it the sub- or superscripts corresponding to the logic. Also remark that if $\mathcal{E}$ is a basis of $\mathcal{C}$, then $(a,b) \in \theta(\mathbf{C})$ if and only if for any $P \in \mathcal{E}$, $a \in P$ iff $b \in P$.

14. THEOREM. *Let $\mathbb{L} = \langle \mathfrak{A}, \mathcal{C} \rangle$ be any abstract logic. Then the following conditions are equivalent:*

(1) *$\mathbb{L}$ is a QTML ;*
(2) *$\theta \in Con(\mathfrak{A})$, $\mathfrak{A}/\theta$ is a TMA, and $\mathcal{C}/\theta$ is a closure system on A/θ having a basis of prime filters closed under Φ ; and*
(3) *There is a bilogical morphism between $\mathbb{L}$ and a logic $\mathbb{L}' = \langle \mathfrak{A}', \mathcal{C}' \rangle$ where $\mathfrak{A}'$ is a TMA and $\mathcal{C}'$ is a closure system on A/θ having a basis of prime filters closed under Φ.*

PROOF: (1) $\Longrightarrow$ (2): In [**FV2**] it is proved that $\theta \in Con(\mathfrak{A}^-)$, that $\mathfrak{A}^-/\theta$ is a De Morgan lattice, and that $\mathcal{C}/\theta$ has a basis of prime filters closed under Φ; it remains only to prove that $\theta \in Con(\mathfrak{A})$ and that $\mathfrak{A}/\theta$ is a TMA. Let $(a,b) \in \theta$ and

supppose $Ia \in P$ for some $P \in \mathcal{E}$. Then by the assumption $Ia \in P \cap \Phi(P)$ and also $a \in P \cap \Phi(P)$; then from $(a,b) \in \theta$ we obtain $b \in P \cap \Phi(P)$ and also $Ib \in P \cap \Phi(P)$; a fortiori $Ib \in P$, therefore $(Ia, Ib) \in \theta$ and this means that $\theta \in Con(\mathfrak{A})$. From now on we denote by a_θ the equivalence class of $a \in A$ in the quotient A/θ. Condition 9.3 says that 0_θ is the infimum of the De Morgan lattice $\mathfrak{A}/\theta$, therefore it is a De Morgan algebra. To prove that it is a TMA we must show that it satisfies:
(1.1) $Ia_\theta \wedge \neg a_\theta = 0 \quad \forall a_\theta \in \mathfrak{A}/\theta$; and
(1.2) $a_\theta \wedge \neg a_\theta = a_\theta \wedge \neg Ia_\theta \quad \forall a_\theta \in \mathfrak{A}/\theta$
To prove (1.1) we have that $Ia_\theta \wedge \neg a_\theta \leq 0_\theta$ iff $\forall P \in \mathcal{E},\ \ Ia \wedge \neg a \notin P$. If $Ia \in P$ then $Ia \in P \cap \Phi(P)$ and also $a \in P \cap \Phi(P)$, but $a \in \Phi(P)$ iff $\neg a \notin P$, that is, $Ia \wedge \neg a \notin P$; if $Ia \notin P$ then $Ia \wedge \neg a \notin P$ either, because P is a filter.
To prove (1.2), note that $a_\theta \wedge \neg a_\theta \leq a_\theta \wedge \neg Ia_\theta$ iff $a \wedge \neg a \in P$ implies $a \wedge \neg Ia \in P$, $\forall P \in \mathcal{E}$. If $a \wedge \neg a \in P$ and $\neg Ia \notin P$, then $Ia \in \Phi(P)$ and also $Ia \in P \cap \Phi(P)$, that is, $\neg a \wedge Ia \in P$, a contradiction; thus we must have $\neg Ia \in P$ and thus $a \wedge \neg Ia \in P$. Conversely, $a_\theta \wedge \neg Ia_\theta \leq a_\theta \wedge \neg a_\theta$ iff $a \wedge \neg Ia \in P$ implies $a \wedge \neg a \in P$, $\forall P \in \mathcal{E}$. If $a \wedge \neg Ia \in P$ and $\neg a \notin P$, then $a \in \Phi(P)$ and also $a \in P \cap \Phi(P)$ and thus $Ia \in P \cap \Phi(P)$, a contradiction, because from $\neg Ia \in P$ we get $Ia \notin \Phi(P)$.

(2) $\Longrightarrow$ (3): Just take $\mathbb{L}' = \langle \mathfrak{A}/\theta, \mathcal{C}/\theta \rangle$, and the canonical projection is the requested bilogical morphism.

(3) $\Longrightarrow$ (1): By Propositions 6 and 8, $\mathbb{L}'$ is a QTML, and thus by Proposition 12 so is $\mathbb{L}$. ∎

The preceding result tells us that QTMLs can all be generated from TMAs by taking families of prime filters closed under Φ and then projectively generating logics by arbitrary epimorphisms from arbitrary algebras. Later on (Theorem 17) we will give another, more "constructive" method of finding all possible QTMLs, using the four-element algebra. First we give the parallel characterization of TMLs.

15. THEOREM. *Let $\mathbb{L} = \langle \mathfrak{A}, \mathcal{C} \rangle$ be any abstract logic. Then the following conditions are equivalent:*

(1) *$\mathbb{L}$ is a TML;*
(2) *$\theta \in Con(\mathfrak{A})$, $\mathfrak{A}/\theta$ is a TMA, and $\mathcal{C}/\theta$ is the set of all filters of $\mathfrak{A}/\theta$; and*
(3) *There is a bilogical morphism between $\mathbb{L}$ and a logic $\mathbb{L}' = \langle \mathfrak{A}', \mathcal{F}' \rangle$ where $\mathfrak{A}'$ is a TMA and $\mathcal{F}'$ is the set of all its filters.*

PROOF: It consists in adding finitarity to Theorem 14; but this is independent of the modal part of the structure, and the proof given in [**FV1**] suits also here. ∎

This Theorem offers a number of algebraic applications, among which we can quote the following results: It establishes a clear relationship between the categories of TMLs and TMAs. For every fixed $\mathfrak{A}$, the correspondence $\theta \mapsto \theta(\mathbf{C})$ establishes an isomorphism between the lattice of all closure operators $\mathbf{C}$ on $\mathfrak{A}$ such that $\langle\mathfrak{A},\mathbf{C}\rangle$ is a TML and the lattice of all congruences θ of $\mathfrak{A}$ such that the quotient $\mathfrak{A}/\theta$ is a TMA. An algebra $\mathfrak{A}$ is a TMA iff there is a closure operator $\mathbf{C}$ on A such that $\langle\mathfrak{A},\mathbf{C}\rangle$ is a TML and $\theta(\mathbf{C}) = \Delta$. An abstract logic $\mathbb{L} = \langle\mathfrak{A},\mathcal{C}\rangle$ is a simple TML (that is, a TML whose only logical congruence is Δ) iff $\mathfrak{A}$ is a TMA and $\mathcal{C}$ is the set of all filters of $\mathfrak{A}$. More details on such topics will appear in [**FR**].

FOUR-VALUED LOGICS

Now we want to characterize, given any algebra $\mathfrak{A}$, the logic $\mathbb{L}_{4m}(\mathfrak{A})$ semantically defined from $\mathbb{L}_{4m} = \langle\mathfrak{M}_{4m},\mathcal{F}\rangle$ by using the set of all homomorphisms. We will describe it as a distinguished member of the class of all TMLs over $\mathfrak{A}$. To this end we first need to consider logics defined on $\mathfrak{A}$ by subsets of the set of all homomorphisms. Although these logics have not as yet been given a clear logical significance, it is not difficult to imagine several reasons to consider restricting the class of valuations to a smaller class of "admissible" ones.

16. PROPOSITION. *Let $\mathbb{L} = \langle\mathfrak{A},\mathcal{C}\rangle$ be a QTML, and let $\mathcal{E}$ be the basis of $\mathcal{C}$ mentioned in Definition 9. Then for every $P \in \mathcal{E}$ there is an $h \in Hom(\mathfrak{A}, \mathfrak{M}_{4m})$ such that $P = h^{-1}(F_1)$ or $P = h^{-1}(F_2)$.*

PROOF: For each $P \in \mathcal{E}$ we consider the logic $\mathbb{L}_P = \langle\mathfrak{A},\mathcal{C}_P\rangle$ where $\mathcal{C}_P$ is the closure system generated by $\{P,\Phi(P)\}$. Clearly this is a TML, and thus $\mathfrak{A}/\theta_P$ is a TMA, and it can have two, three, or four elements, depending on the relative positions of P and $\Phi(P)$. Its structure as a De Morgan algebra in the three possible cases is determined in Proposition 4 of [**FV1**], but this also determines its structure as a TMA, because it follows from the main results of [**L4**] that there can be at most one structure of TMA on a given De Morgan algebra (the same paper gives necessary and sufficient conditions for a De Morgan algebra to be a TMA). The three cases are the following:

<u>Case 1</u>: $P = \Phi(P)$. Then there are just two equivalence classes, and $(\mathfrak{A}/\theta_P)^- = \mathfrak{B}_2$, so $\mathfrak{A}/\theta_P = \mathfrak{B}_{2m}$. If we denote by π_P the canonical projection and by i the embedding of $\mathfrak{B}_{2m}$ into $\mathfrak{M}_{4m}$, then $P = (i\circ\pi_P)^{-1}(F_1)$.

Case 2: $P \subsetneq \Phi(P)$, or $\Phi(P) \subsetneq P$. The quotient has three different elements and $(\mathfrak{A}/\theta_P)^- = \mathfrak{M}_3$, therefore $\mathfrak{A}/\theta_P = \mathfrak{M}_{3m}$; if we denote by j the embedding of $\mathfrak{M}_{3m}$ into $\mathfrak{M}_{4m}$, with for instance $j(a) = a$, then $P = (j \circ \pi_P)^{-1}(F_2)$, or $P = (j \circ \pi_P)^{-1}(F_1)$ respectively.

Case 3: When P and $\Phi(P)$ are not comparable, the quotient has four elements and $(\mathfrak{A}/\theta_P)^- = \mathfrak{M}_4$, and thus $\mathfrak{A}/\theta_P = \mathfrak{M}_{4m}$ with $P = \pi_P^{-1}(F_1)$. ∎

17. THEOREM. *An abstract logic $\mathbb{L}$ is a QTML if and only if $\mathbb{L}$ is projectively generated from $\mathbb{L}_{4m}$ by some set $\mathcal{H} \subseteq Hom(\mathfrak{A}, \mathfrak{M}_{4m})$.*

PROOF: ($\Longrightarrow$) Just take as $\mathcal{H}$ the family of homomorphisms found when applying Proposition 16 to all $P \in \mathcal{E}$, the basis of $\mathcal{C}$ mentioned in the definition of QTMLs.

($\Longleftarrow$) By assumption $\mathcal{E} = \{h^{-1}(F_i) \ : \ i = 1,2 \ ; \ h \in \mathcal{H}\}$ is a basis of $\mathcal{C}$. Theorem 2 of [**FV1**] proves conditions 9.1 and 9.2. Condition 9.3 is trivial. To prove 9.4 consider that $Ia \in h^{-1}(F_i)$ iff $h(Ia) = Ih(a) \in F_i$ iff $Ih(a) = h(Ia) \in \Phi(F_i)$, and thus $Ia \in h^{-1}(\Phi(F_i))$. To prove 9.5 we have that $a \in h^{-1}(F_i) \cap \Phi h^{-1}(F_i)$ iff $a \in h^{-1}(F_i \cap \Phi(F_i))$ iff $h(a) \in F_i \cap \Phi(F_i)$ iff $Ih(a) = h(Ia) \in F_i \cap \Phi(F_i)$ iff $Ia \in h^{-1}(F_i) \cap \Phi(F_i)$. ∎

18. COROLLARY. *For any algebra $\mathfrak{A}$ of type (2,1,1,0), the logic $\mathbb{L}_{4m}(\mathfrak{A})$ is the finest QTML over $\mathfrak{A}$.*

PROOF: From Theorem 17 and the fact that by definition $\mathbb{L}_{4m}(\mathfrak{A})$ is projectively generated by the set of all homomorphisms. ∎

Now we want to prove that $\mathbb{L}_{4m}(\mathfrak{A})$ is also the finest TML over $\mathfrak{A}$. We need only prove that it is really a TML, that is, that it is finitary. If the algebra $\mathfrak{A}$ is the formula algebra then this is usually called the **compactness theorem**, and is a well known result of the theory of logical matrices (see [**Wo**] for instance). But for an arbitrary $\mathfrak{A}$ we need to complete the proof ourselves:

19. THEOREM. *For any algebra $\mathfrak{A}$, the logic $\mathbb{L}_{4m}(\mathfrak{A})$ is finitary, and as a consequence $\mathbb{L}_{4m}(\mathfrak{A})$ is the finest TML over $\mathfrak{A}$.*

PROOF: We know that $\mathbb{L}_{4m}(\mathfrak{A})$ is a QTML, therefore by Theorem 14 there is a bilogical morphism h between it and a logic $\mathbb{L}' = \langle \mathfrak{A}', \mathcal{C}' \rangle$ where $\mathfrak{A}'$ is a TMA and $\mathcal{C}'$ is a closure system of filters of $\mathfrak{A}$ having a basis of prime filters closed under Φ. If $\mathbb{L}_{4m}(\mathfrak{A})$ is not a TML then $\mathbb{L}'$ is not either, by Theorem 14, and thus it is strictly less fine than the logic of all filters $\mathbb{L}'' = \langle \mathfrak{A}', \mathcal{F}' \rangle$, because this one is really a TML. Now the same epimorphism h projectively generates on $\mathfrak{A}$ another TML $\mathbb{L}^* = \langle \mathfrak{A}, \mathcal{C}'' \rangle$ from $\mathbb{L}''$, which will be strictly finer than $\mathbb{L}_{4m}(\mathfrak{A})$; but this is against Corollary 18, since $\mathbb{L}^*$ is a fortiori a QTML. ∎

Completeness Theorems

Sometimes, results in the form of Theorem 19 are thought of as a kind of **Abstract Completeness Theorems**, for two main reasons: First, they characterize a semantic operator by means of its abstract properties. Second, because from these properties a syntactic operator can be defined on the algebra of formulas in such a way that the proof of the usual completeness theorem is rather straightforward. However, if we observe the conditions of Definition 9, we realize that all properties are expressed in terms of the closure system rather than in terms of the closure operator. To obtain an equivalent formulation allowing us to extract from it a syntactical definition we need to enrich our language by treating the connective $\vee$ as primitive; in this new language we will find an equivalent concept of TML which will serve our purposes.

20. Definition. *Let $\mathfrak{A}' = \langle A, \wedge, \vee, \neg, I, 0 \rangle$ be an algebra of type (2,2,1,1,0), and let $\mathbb{L}' = \langle \mathfrak{A}', \mathbf{C} \rangle$ be an abstract logic over $\mathfrak{A}'$. We say that $\mathbb{L}'$ is a* **(Q)TML** *when the following conditions are satisfied, for all $a, b \in A$ and for all $X \subseteq A$:*

(1) $\mathbf{C}(a \wedge b) = \mathbf{C}(a, b)$;
(2) $\mathbf{C}(X, a \vee b) = \mathbf{C}(X, a) \cap \mathbf{C}(X, b)$;
(3) $\mathbf{C}(a) = \mathbf{C}(\neg\neg a)$;
(4) $a \in \mathbf{C}(b) \implies \neg b \in \mathbf{C}(\neg a)$;
(5) $\mathbf{C}(0) = A$;
(6) $\mathbf{C}(a) \cap \mathbf{C}(\neg I a) = \mathbf{C}(\emptyset)$;
(7) $\mathbf{C}(a, \neg I a) = \mathbf{C}(a, \neg a)$; *and*
(8) $\mathbf{C}$ *is finitary (only for TMLs).*

21. Theorem. *Let $\mathfrak{A} = \langle A, \wedge, \neg, I, 0 \rangle$ be an algebra of type (2,1,1,0) and $\mathbb{L} = \langle \mathfrak{A}, \mathbf{C} \rangle$ be an abstract logic over $\mathfrak{A}$. Then $\mathbb{L}$ is a (Q)TML in the sense of Definition 9 if and only if there is a binary operation $\vee$ in A such that putting $\mathfrak{A}' = \langle A, \wedge, \vee, \neg, I, 0 \rangle$ and $\mathbb{L}' = \langle \mathfrak{A}', \mathbf{C} \rangle$, $\mathbb{L}'$ is a (Q)TML in the sense of Definition 20. Moreover, in such a case it holds that $\mathbf{C}(a \vee b) = \mathbf{C}\big(\neg(\neg a \wedge \neg b)\big)$.*

PROOF: If $\mathbb{L}$ is a (Q)TML in the sense of Definition 9, then Theorem 14 tells us that $\theta \in Con(\mathfrak{A})$ and that $\mathfrak{A}/\theta$ is a TMA. But 11.3 tells us that $\mathbb{L}$ satisfies 20.2 for $a \vee b = \neg(\neg a \wedge \neg b)$, and thus $\theta \in Con(\mathfrak{A}')$. It is well-known that 20.1 is equivalent to 9.1, and then conditions 20.3 to 20.7 can be re-written as equations or implications which do hold in $\mathfrak{A}'/\theta$, therefore they hold as stated in $\mathfrak{A}'$, because $\mathfrak{A}'/\theta$ is precisely the quotient of $\mathfrak{A}'$ by $\mathbf{C}$ (that is, the equality in $\mathfrak{A}'/\theta$ is exactly the $\mathbf{C}$-equivalence in $\mathfrak{A}'$). We conclude that $\mathbb{L}'$ is a (Q)TML in the sense of Definition 20. Note that, in view of 20.2, condition 20.6 is equivalent to $\mathbf{C}(a \vee \neg Ia) = \mathbf{C}(\emptyset)$.

For the converse part we are going to prove directly each one of the conditions 9.1 to 9.5. In Proposition 1 of [**FV3**] it is proved that conditions 20.1 to 20.4 imply 9.1 and 9.2, where the basis $\mathcal{E}$ of $\mathcal{C}$ is taken to be the set of all $\vee$-prime $\wedge$-filters of $\mathfrak{A}$; it is also proved that $\mathbf{C}(a \vee b) = \mathbf{C}(\neg(\neg a \wedge \neg b))$, because they are equal in the quotient. Moreover 9.3 is equivalent to 20.5.

We now prove: (*) $\forall a \in A$, $a \in \mathbf{C}(Ia)$: By using 20.6, 20.2, 20.7 and 20.2 again we have $\mathbf{C}(\neg a) = \mathbf{C}(\neg a, a \vee \neg Ia) = \mathbf{C}(\neg a, a) \cap \mathbf{C}(\neg a, \neg Ia) = \mathbf{C}(a, \neg Ia) \cap \mathbf{C}(\neg a, \neg Ia) = \mathbf{C}(a \vee \neg a, \neg Ia)$. Therefore $\neg Ia \in \mathbf{C}(\neg a)$ and thus by 20.4 and 20.3 we obtain (*).

To prove 9.4 it is enough to prove that, for any $P \in \mathcal{E}$, $Ia \in P$ implies $Ia \in \Phi(P)$, because $\Phi^2(P) = P$. Suppose that $Ia \in P$ but $Ia \notin \Phi(P)$, that is, $\neg Ia \in P$; by (*) and 20.7 we would have $\neg a \in P$, that is, $a \notin \Phi(P)$. But $\Phi(P) \in \mathcal{E}$ and from 9.1 and 9.2 it follows that it is $\vee$-prime, and 20.6 implies that $a \vee \neg Ia \in \Phi(P)$; thus we must have $\neg Ia \in \Phi(P)$ which is the same as $Ia \notin P$, a contradiction.

Finally we prove 9.5: If $Ia \in P \cap \Phi(P)$ then (*) implies $a \in P \cap \Phi(P)$. To prove the converse, observe that 20.7 says that if $a \in P$, then $\neg a \in P$ iff $\neg Ia \in P$, that is, $a \in \Phi(P)$ iff $Ia \in \Phi(P)$; and using $\Phi^2(P) = P$ it also says that if $a \in \Phi(P)$, then $a \in P$ iff $Ia \in P$. Therefore we conclude that if $a \in P \cap \Phi(P)$ then $Ia \in P \cap \Phi(P)$. ∎

As a consequence of this result we will treat the two notions of (Q)TML as equivalent, and thus all the results proved until now for (Q)TMLs in the first setting can be used as if they were proved for the second one.

We are now ready to introduce a syntactic operator on the algebra of formulas $\mathfrak{F} = \langle Form, \wedge, \vee, \neg, \Box, \bot \rangle$, that is, the absolutely free algebra of the type (2,2,1,1,0). In this case the semantic operator $\mathbb{L}_{4m}(\mathfrak{F})$ is usually denoted by $\langle \mathfrak{F}, \models \rangle$. The syntactic operator will be defined by means of a **sequent calculus**. Remark that our sequents are expressions of the form $\Gamma \vdash \varphi$, where $\varphi \in Form$ and Γ is a finite (possibly empty) unordered set of formulas of $Form$.

22. DEFINITION. *$\mathbb{L}_S = \langle \mathfrak{F}, \vdash_S \rangle$ is the logic defined on $\mathfrak{F}$ by: $\Gamma \vdash_S \varphi$ if and only if there is a finite $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \vdash \varphi$ is derivable in the sequent calculus which has the following axiom and rules (for any $\Delta \subseteq Form$, and for any $\alpha, \beta, \gamma \in Form$) :*

(Structural axiom) $\Delta, \alpha \vdash \alpha$ *(Modal axiom)* $\vdash \alpha \vee \neg\Box\alpha$

(Weakening) $\dfrac{\Delta \vdash \alpha}{\Delta, \beta \vdash \alpha}$ *(Cut)* $\dfrac{\Delta \vdash \alpha \qquad \Delta, \alpha \vdash \beta}{\Delta \vdash \beta}$

$(\wedge \vdash)$ $\dfrac{\Delta, \alpha, \beta \vdash \gamma}{\Delta, \alpha \wedge \beta \vdash \gamma}$ $(\vdash \wedge)$ $\dfrac{\Delta \vdash \alpha \qquad \Delta \vdash \beta}{\Delta \vdash \alpha \wedge \beta}$

$(\vee \vdash)$ $\dfrac{\Delta, \alpha \vdash \gamma \qquad \Delta, \beta \vdash \gamma}{\Delta, \alpha \vee \beta \vdash \gamma}$ $(\vdash \vee)$ $\dfrac{\Delta \vdash \alpha}{\Delta \vdash \alpha \vee \beta}$ $\dfrac{\Delta \vdash \beta}{\Delta \vdash \alpha \vee \beta}$

$$(\neg) \quad \frac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}$$

$(\neg\neg \vdash)$ $\dfrac{\Delta, \alpha \vdash \beta}{\Delta, \neg\neg\alpha \vdash \beta}$ $(\vdash \neg\neg)$ $\dfrac{\Delta \vdash \alpha}{\Delta \vdash \neg\neg\alpha}$

$$(\bot) \quad \frac{\Delta \vdash \bot}{\Delta \vdash \alpha}$$

$(\Box \vdash)$ $\dfrac{\Delta, \alpha, \neg\alpha \vdash \beta}{\Delta, \alpha, \neg\Box\alpha \vdash \beta}$ $(\vdash \Box)$ $\dfrac{\Delta \vdash \alpha \wedge \neg\alpha}{\Delta \vdash \alpha \wedge \neg\Box\alpha}$

This definition has been chosen to make the proof of the following theorem rather straightforward:

23. THEOREM. *The logic $\mathbb{L}_S = \langle \mathfrak{F}, \vdash_S \rangle$ is the least TML over $\mathfrak{F}$.*

PROOF: Each of the conditions 20.1 to 20.8 can be proved for $\mathbb{L}_S$ from its definition; and given a proof of some sequent $\Gamma_0 \vdash \varphi$ in the sequent calculus, and any other TML $\langle \mathfrak{F}, \mathbf{C} \rangle$ over $\mathfrak{F}$, a routine inductive process tells us that the same "proof" can be "reproduced" for $\mathbf{C}$ to conclude that $\varphi \in \mathbf{C}(\Gamma_0)$, that is, $\vdash_S \leqslant \mathbf{C}$. ∎

24. COROLLARY. **(Completeness)** $\mathbb{L}_S = \mathbb{L}_{4m}(\mathfrak{F})$, *that is,* $\vdash_S \; = \; \models$.

PROOF: By Theorem 19 and Theorem 23. ∎

We have thus reached our last goal. In our way to it we have found some interesting results, and this was actually our main goal.

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