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A Fourth Derivative Test for Exponential Sums

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Abstract. We give an upper bound for the exponential sum $\sum_{m=1}^{M} \mathrm{e}^{2i\pi f(m)}$ in terms of M and λ, where λ is a small positive number which denotes the size of the fourth derivative of the real valued function f. The classical van der Corput's exponent 1/14 is improved into 1/13 by reducing the problem to a mean square value theorem for triple exponential sums.

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1. Introduction and Statement of the Result

The aim of this paper is to find upper bounds for the exponential sum

$$S_M = \sum_{m=1}^{M} \mathrm{e}(f(m)), \tag{1.1}$$

where we have set $\mathrm{e}(x)$ for $\mathrm{e}^{2i\pi x}$ and where M is a large integer and $f\colon [1, M] \to \mathbb{R}$ is a four times continuously differentiable function which satisfies van der Corput's condition

$$\lambda \leqslant f^{(4)}(x) \ll \lambda, \quad \text{for } 1 \leqslant x \leqslant M \tag{1.2}$$

where λ is a small positive number, and where the Vinogradov's symbol $u \ll v$ means that there exists an absolute positive constant C such that $|u| \leqslant Cv$. Under the condition (1.2), van der Corput has obtained the following classical bound (cf. [3], Theorem 2.8)

$$S_M \ll M\lambda^{1/14}, \quad \text{provided that } M \gg \lambda^{-4/7}. \tag{1.3}$$

The proof consists in applying twice Weyl and van der Corput's A-process (cf. [3], Lemma 2.5), and then van der Corput's inequality (cf. [3], Theorem 2.2). Slight improvements on (1.3) have been obtained later, but only under stronger hypothesis (see, e.g., [3] or [4]). It is interesting to notice that the bound (1.3) can be improved without any new hypothesis, as a consequence of a strong result of Bombieri and Iwaniec [2] on the mean value of eighth powers of simple cubic exponential sums. The deduction has been made in [7] with the bound

$$S_M \ll_\varepsilon M^{1+\varepsilon}\lambda^{3/40}, \text{ provided that } M \gg \lambda^{-3/5}. \tag{1.4}$$

Here and in the sequel, the symbol $\ll_\varepsilon$ means that the inequality holds for each $\varepsilon > 0$ and that the implied constant depends at most on ε and on the previous implied constants. Our result can be stated as follows:

THEOREM 1. *If the condition (1.2) is satisfied, then the two equivalent properties*

$$S_M \ll_\varepsilon M^{1+\varepsilon}\lambda^{1/13}, \text{ provided that } M \gg \lambda^{-8/13}, \tag{1.5}$$

and

$$S_M \ll_\varepsilon M^{\varepsilon}(M\lambda^{1/13} + \lambda^{-7/13}) \tag{1.6}$$

hold true.

We conclude this section with some remarks and comments while Sections 2, 3, and 4 are entirely devoted to the proof of Theorem 1.

Exponent pairs. Most of problems in analytic number theory where exponential sums occur, involve phase functions which satisfy much more than (1.2). Namely, these functions $f\colon [M, 2M] \to \mathbb{R}$ satisfy conditions (3.3.3) of [3] (we shall call them 'semi-monomial functions'). Bounds for exponential sums $S_M = \sum_{m=M+1}^{2M} \mathrm{e}(f(m))$ are then obtained in terms of exponent pairs (see §3.3 of [3]). For semi-monomial functions, the bound

$$S_M \ll_\varepsilon M^{1+\varepsilon}\lambda^{\theta} \tag{1.7}$$

corresponds to the property

$$(\theta + \varepsilon, 1 - 3\theta + \varepsilon) \text{ is an exponent pair for each } \varepsilon > 0. \tag{1.8}$$

Thus, our Theorem 1 implies that (1.8) holds for $\vartheta = 1/13$ (to see this, we only have to complete the proof in the case $M \ll \lambda^{-8/13}$ by means of the classical exponent pair $(2/18, 13/18) = ABA^2B(0, 1)$). But this value of ϑ is not the best known. Indeed, the refinement by Huxley and Kolesnik [5] (see also [4], §19.3) of Huxley's deep method for exponential sums with a large second derivative ([4], §17.4), yields a better value of ϑ. Namely, one can take out from table 19.2 of [4] the following result:

THEOREM (Huxley and Kolesnik). *The property (1.8) holds for*

$$\vartheta = \frac{516247}{6629696} = \frac{1}{12.84\ldots}.$$

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The interest of our Theorem 1 consists, on the one hand, in the simplicity of its proof and, on the other hand, in the wider range of its applications, particularly to short exponential sums.

Van der Corput's exponent. The exponent 1/14 in (1.3) can be sharpened to 1/13, at least with some restrictions on the relative size of M and λ. The question of knowing how much van der Corput's exponent 1/14 can be improved and under which conditions, arises naturally.

We have heuristic proofs of the two following assertions that we state as conjectures:

CONJECTURE 1. *Under the hypothesis (1.2), we have*

$$S_M \ll_\varepsilon M^{1+\varepsilon}\lambda^{3/38}, \text{ provided that } M \gg \lambda^{-13/19}. \tag{1.9}$$

CONJECTURE 2. *Under the hypothesis (1.2), we have*

$$S_M \ll_\varepsilon M^{1+\varepsilon}\lambda^{1/12}, \text{ provided that } M \gg \lambda^{-1}. \tag{1.10}$$

This last conjecture, if true, is far from implying that the pair $(1/12+\varepsilon, 9/12+\varepsilon)$ is an exponent pair for each $\varepsilon > 0$. The restriction $M \gg \lambda^{-1}$ in (1.10) is quite constraining and we think that, perhaps, it cannot be weakened. Furthermore, if we restrict conjecture 2 to semi-monomial phase functions, then Huxley's results already imply (1.10) (cf. [4], §17.4).

Very short exponential sums. In the opposite direction, we have the following improvement of (1.3) (cf. [7], Lemma 2.6):

$$S_M \ll M\lambda^{1/14}, \text{ provided that } M \gg \lambda^{-3/7}, \tag{1.11}$$

which concerns shorter exponential sums. It would be of interest, both in itself and for the applications, to find the infimum of positive real β such that the bound

$$S_M \ll M\lambda^{1/14}, \text{ provided that } M \gg \lambda^{-\beta}, \tag{1.12}$$

holds under the hypothesis (1.2). The example

$$f(m) = \frac{m^4}{100M^4},$$

in which we have $|S_M| \gg \lambda^{-1/4}$, shows that $\beta \geqslant 9/28$, so we have

$$9/28 \leqslant \beta \leqslant 3/7 \tag{1.13}$$

Outline of proof. At first, we apply van der Corput's A-process to the initial sum (1.1) and get a double sum in the variables h and m. Then we apply $A \times A-$process to the new double sum and get a quadruple sum in the variables r, q, h, m. At last,

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we shift the main variable m to produce a new variable n. This can be sketched in the following diagram:

$$\begin{aligned} S_M = \sum_m \mathrm{e}(f(m)) &\xrightarrow{A} \sum_h \sum_m \mathrm{e}(\Delta_h f(m)) \\ &\xrightarrow{A\times A} \sum_r \sum_q \sum_h \sum_m \mathrm{e}(\Delta_{h+r} f(m) - \Delta_h f(m+q)) \\ &\xrightarrow{\text{shift}} \sum_r \sum_m \left| \sum_h \sum_n \sum_q \mathrm{e}(\Delta_{h+r} f(m+n) - \Delta_h f(m+n+q)) \right|. \end{aligned} \tag{1.14}$$

where we have set $\Delta_h f(m)$ for $f(m+h) - f(m-h)$. By expanding the phase in the last exponential sum by means of Taylor's formula, we are in a position to apply Bombieri and Iwaniec's double large sieve [1]. Thus we have reduced the initial problem into that of counting the number of solutions of a (very particular) diophantine system, which is the purpose of our Theorem 2. The whole proof is self contained and elementary.

2. Preliminary Lemmas

We recall some basic lemmas.

2.1. WEYL AND VAN DER CORPUT A × A LEMMA

LEMMA 1. *Let M and H be positive integers and let $(a(m,h))_{(m,h)\in\mathbb{Z}^2}$ be complex numbers which are zero whenever (m,h) is outside the compact $[1, M] \times [1, H]$. We set*

$$S = \sum_{(m,h)\in\mathbb{Z}^2} a(m,h)$$

and we choose two integers Q and R such that $1 \leqslant Q \leqslant M$ and $1 \leqslant R \leqslant H$. We then have

$$S^2 \ll \frac{MH}{QR} \sum_{|q|<Q} \sum_{|r|<R} \left(1 - \frac{|q|}{Q}\right)\left(1 - \frac{|r|}{R}\right) \sum_{(m,h)\in\mathbb{Z}^2} a(m+q,h)\overline{a(m,h+r)} \tag{2.1}$$

For the proof, see [3], Lemma 6.1. □

2.2. PARTIAL SUMMATION FOR MULTIPLE SUMS

We give a general statement of partial summation for k-dimensional sums, where k is a positive integer. We need some notations.

Let $M_1, \ldots, M_k$ be positive integers and set:

$$\mathcal{P} = [1, M_1] \times \cdots \times [1, M_k] \subset \mathbb{R}^k. \tag{2.2}$$

Let I be any finite set and, for each fixed $i \in I$, let ϕ_i: $\mathcal{P} \to \mathbb{C}$ be a function which satisfies the following regularity condition.

For each integer r $(0 \leqslant r \leqslant k)$, for each $(j_1, \ldots, j_r)$ such that $1 \leqslant j_s \leqslant k$ $(1 \leqslant s \leqslant r)$ and $j_s \neq j_t$ for $s \neq t$, the rth order derivative $\frac{\partial^r \phi_i}{\partial x_{j_1} \ldots \partial x_{j_r}}$ exists and is continuous on $\mathcal{P}$ and satisfies the bound:

$$\left| \frac{\partial^r \phi_i}{\partial x_{j_1} \ldots \partial x_{j_r}}(\mathbf{x}) \right| \leqslant \frac{D}{M_{j_1} \ldots M_{j_r}}$$
$$\text{whenever } i \in I, \ \mathbf{x} \in \mathcal{P}, \ r \in \{0, \ldots, k\}, 1 \leqslant j_1 < \cdots < j_r \leqslant k \tag{2.3}$$

for some $D > 0$. We recall that the bound (2.3) in case $r = 0$ means that $|\varphi_i(\mathbf{x})| \leqslant D$ for each $i \in I$ and $\mathbf{x} \in \mathcal{P}$.

Let us now consider the k-dimensional sum

$$S_0 = \sum_{i \in I} \left| \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{N}^k} a_i(\mathbf{m}) \phi_i(\mathbf{m}) \right| \tag{2.4}$$

where $(a_i(\mathbf{m}))_{i \in I, \mathbf{m} \in \mathcal{P} \cap \mathbb{N}^k}$ is any given family of complex numbers. We can now state our lemma for k-dimensional partial summation.

LEMMA 2. *Let the above notations and hypothesis hold. We then have*

$$S_0 \leqslant 2^k D \max_{\mathcal{P}'} \sum_{i \in I} \left| \sum_{\mathbf{m} \in \mathcal{P}' \cap \mathbb{N}^k} a_i(\mathbf{m}) \right| \tag{2.5}$$

where the maximum has to be taken over all possible sets of the form $\mathcal{P}' = [1, M_1'] \times \cdots \times [1, M_k'] \subset \mathcal{P}$.

Proof. The proof goes by recurrence on k. When $k = 1$, the result is nothing but the classical one-dimensional partial summation. We suppose that the result is true up to k-dimensional sums, and we want to prove that it is true for $(k+1)$-dimensional sums.

The $(k+1)$-dimensional sum S_0 may be written as

$$S_0 = \sum_{i \in I} \left| \sum_{\mathbf{m} \in \mathcal{P}_k} \sum_{n=1}^{N} a_i(\mathbf{m}, n) \phi_i(\mathbf{m}, n) \right|$$

In order to apply one dimensional partial summation to the sum in n, we set

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$A_i(\mathbf{m}, n) = \sum_{\nu=1}^{n} a_i(\mathbf{m}, \nu)$ and $\psi_{i,n}(\mathbf{m}) = \phi_i(\mathbf{m}, n) - \phi_i(\mathbf{m}, n+1)$. We have

$$S_0 \leqslant \sum_i \left| \sum_{\mathbf{m}\in\mathcal{P}_k} A_i(\mathbf{m}, N)\phi_i(\mathbf{m}, N) \right| + \sum_i \sum_{n=1}^{N-1} \left| \sum_{\mathbf{m}\in\mathcal{P}_k} A_i(\mathbf{m}, n)\psi_{i,n}(\mathbf{m}) \right|.$$

We apply the recurrence hypothesis to both terms

$$S_0 \leqslant 2^k D \max_{\mathcal{P}'_k} \sum_i \left| \sum_{\mathbf{m}\in\mathcal{P}'_k} \sum_{\nu=1}^{N} a_i(\mathbf{m}, \nu) \right| + 2^k \frac{D}{N} \max_{\mathcal{P}'_k} \sum_i \sum_{n=1}^{N-1} \left| \sum_{\mathbf{m}\in\mathcal{P}'_k} \sum_{\nu=1}^{n} a_i(\mathbf{m}, \nu) \right|,$$

and the desired result follows. □

2.3. THIRD DERIVATIVE TEST AND PARTIAL SUMMATION

The following lemma is not essential in the proof of Theorem 1, but it gives rise to simplifications.

LEMMA 3. Let *M be a positive integer, and let g and u*: $[1, M] \to \mathbb{R}$ *be two functions, respectively $\mathcal{C}^3$ and $\mathcal{C}^1$, such that*

$$\mu \leqslant |g'''(x)| \ll \mu \quad \text{and} \quad u'(x) \ll \mu^{1/2}, \quad \text{for } 1 \leqslant x \leqslant M, \tag{2.6}$$

where μ is a small positive number. We then have

$$\sum_{m=1}^{M} \mathrm{e}(g(m) + u(m)) \ll M\mu^{1/6} + \mu^{-1/3}. \tag{2.7}$$

Proof. If $u \equiv 0$, Lemma 3 is the third derivative test for exponential sums ([7], Corollary 4.2). If $M \ll \mu^{-1/2}$, we can eliminate without cost the term $u(m)$ by a (one-dimensional) partial summation, and (2.7) follows. Now, we suppose $M \gg \mu^{-1/2}$. Then we divide the initial sum into $\mathrm{O}(M\mu^{1/2})$ sums of length $\mu^{-1/2}$ and we apply the previous case to each short sum. □

2.4. DOUBLE LARGE SIEVE INEQUALITY

We consider the exponential sum

$$\widetilde{S} = \sum_{0<|r|<R} \sum_{m=1}^{M} \left| \sum_{0<|q|<Q} \sum_{h=H}^{2H-1} \sum_{n=1}^{N} b_r(q, h, n)\mathrm{e}(x_m P_1(r, q, h, n) + y_m P_2(r, q, h, n)) \right|, \tag{2.8}$$

with the following notations: R, M, Q, H, N are positive integers; $b_r(q, h, n)$ are complex numbers with modulus at most one; P_1 and P_2 are polynomials in four

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variables, P_1 with integer coefficients and P_2 with real coefficients; $(x_m)_{1 \leqslant m \leqslant M}$ and $(y_m)_{1 \leqslant m \leqslant M}$ are two families of real numbers.

We suppose that the next two inequalities hold

$$\max_{1 \leqslant m, m' \leqslant M} |y_m - y_{m'}| \leqslant \mu \tag{2.9}$$

$$\max_{r,q,h,n} |P_i(r, q, h, n)| \leqslant X_i \quad (i = 1, 2) \tag{2.10}$$

the latter maximum being taken over all quadruples (r, q, h, n) of integers such that

$$0 < |r| < R, \ 0 < |q| < Q, \ H \leqslant h < 2H, \ 1 \leqslant n \leqslant N.$$

We introduce the numbers $\mathcal{N}$ and $\mathcal{B}$ which correspond to spacing problems

$$\mathcal{N} = \max_{1 \leqslant Q_1 < Q} \# \left\{ \begin{array}{c} (r, q_1, q_2, h_1, h_2, n_1, n_2) \in \mathbb{Z}^7 \text{ which satisfy} \\ \text{conditions (2.11a)} \ldots \text{(2.11d)} \end{array} \right\} \tag{2.11}$$

$$0 < |r| < R, |Q_1| \leqslant |q_i| < \min(2Q_1, Q), H \leqslant h_i < 2H, 1 \leqslant n_i \leqslant N,$$
$$\text{for } i = 1, 2, \tag{2.11a}$$
$$q_1 q_2 > 0, \tag{2.11b}$$
$$P_1(r, h_1, q_1, n_1) = P_1(r, h_2, q_2, n_2), \tag{2.11c}$$
$$|P_2(r, h_1, q_1, n_1) - P_2(r, h_2, q_2, n_2)| \leqslant 1/\mu \tag{2.11d}$$

and

$$\mathcal{B} = \# \left\{ (m_1, m_2) \in \{1, \ldots, M\}^2 \left| \begin{array}{l} \|x_{m_1} - x_{m_2}\| \leqslant X_1^{-1} \\ \text{and} \\ |y_{m_1} - y_{m_2}| \leqslant X_2^{-1} \end{array} \right. \right\} \tag{2.12}$$

with the usual notation: $\|x\| = \min_{m \in \mathbb{Z}} |x - m|$.

We can now state the double large sieve inequality in the particular form which will be needed later.

LEMMA 4. *With the above notations, we have*

$$\widetilde{S}^2 \ll R(1 + X_1)(1 + \mu X_2)\mathcal{B}(\log Q)^2. \tag{2.13}$$

Proof. We set

$$\widetilde{S}(r, Q_1) = \sum_{m=1}^{M} \left| \sum_{Q_1 < |q| < \max(2Q_1, Q)} \sum_{h=H}^{2H-1} \sum_{n=1}^{N} b_r(q, h, n) \right.$$
$$\left. \mathrm{e}(x_m P_1(r, q, h, n) + y_m P_2(r, q, h, n)) \right|,$$

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so that we have

$$\widetilde{S} \ll \max_{1 \leqslant Q_1 < Q} \left(\sum_{0<|r|<R} S(r, Q_1) \right) \log Q. \tag{2.14}$$

We apply Bombieri and Iwaniec's double large sieve [1] (cf. also [3], Lemma 7.5 or [4], Lemma 5.6.6) to each sum $\widetilde{S}(r, Q_1)$ and we get

$$\widetilde{S}(r, Q_1)^2 \ll (1 + X_1)(1 + \mu X_2)\mathcal{N}(r, Q_1)\mathcal{B}, \tag{2.15}$$

where $\mathcal{N}(r, Q_1)$ is the number of $(q_1, q_2, h_1, h_2, n_1, n_2) \in \mathbb{Z}^6$ which satisfy conditions (2.11.a), ..., (2.11.d).

Next we apply Cauchy's inequality and we substitute (2.15) into (2.14) to get (2.13). □

2.5. IWANIEC AND MOZZOCHI'S ARITHMETIC LEMMA

For a positive integer n, we set

$$\tau(n) = \sum_{d|n} 1 \quad \text{and} \quad \sigma(n) = \sum_{d|n} d. \tag{2.16}$$

The following lemma is contained in the proof of theorem 14.1 of [6]. A complete and independent proof may be found in Lemma 13.1.2 of [4] (see also [8]).

LEMMA 5. *Let a, b, c be three nonzero integers, with $\gcd(a, b, c) = 1$ and $c > 0$. Let $V \geqslant 1$, $\alpha < \beta$ be real numbers. We denote by $\mathcal{V}$ the number of triplets (u, v, w) with nonzero integers such that $\gcd(u, v, w) = 1$, $V \leqslant v \leqslant 2V$ and*

$$au + bv + cw = 0 \text{ and } \alpha \leqslant \frac{u}{v} \leqslant \beta. \tag{2.17}$$

We then have

$$\mathcal{V} \ll \tau(c) + (\beta - \alpha)V^2 \frac{\sigma(c)}{c^2} \tag{2.18}$$

3. The Diophantine Problem

3.1. STATEMENT OF THE RESULT

Let R, H, Q, N be real numbers $\geqslant 1$ with $R \leqslant H/2$ and let δ be a positive number. We denote by $\mathcal{N}(R, Q, H, N, \delta)$ the number of integer points $(r, q_1, q_2, h_1, h_2, n_1, n_2) \in \mathbb{Z}^7$ lying in the domain

$$0 < |r| < R,\ Q \leqslant |q_i| < 2Q,\ H \leqslant h_i < 2H,\ 1 \leqslant n_i \leqslant N \quad \text{for } i = 1, 2, \quad \text{and} \quad q_1 q_2 > 0, \tag{3.1}$$

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and satisfying the system

$$\begin{aligned}&rn_1 + h_1q_1 = rn_2 + h_2q_2,\\ &|rn_1^2 + 2h_1q_1n_1 + h_1q_1^2 - (rn_2^2 + 2h_2q_2n_2 + h_2q_2^2)| \leqslant \delta HQ^2.\end{aligned} \tag{3.2}$$

THEOREM 2. *The number of solutions of the diophantine system (3.2), lying in the domain (3.1) satisfies the bound*

$$\mathcal{N}(R, H, Q, N, \delta) \ll_\varepsilon (RNHQ)^{1+\varepsilon}(1 + \delta Q). \tag{3.3}$$

The rest of this section is devoted to the proof of Theorem 2.

3.2. REDUCTION OF THE PROBLEM

We reduce the diophantine system (3.2) into a simpler one by means of easy calculations. Let $\mathcal{J}_1(R, Q, H, \delta)$ be the number of integer points $(r, q_1, q_2, h_1, h_2, d) \in \mathbb{Z}^6$ lying in the domain

$$\begin{aligned}&0 < |r| < R,\ \ Q \leqslant |q_i| < 2Q,\ \ H \leqslant h_i < 2H,\\ &\quad \text{for } i = 1, 2,\ \ q_1q_2 > 0 \quad \text{and} \quad 0 < |d| \ll (1 + \delta)Q\end{aligned} \tag{3.4}$$

satisfying the system

$$\begin{aligned}&rd + h_1q_1 - h_2q_2 = 0\\ &|rd^2 + 2h_1q_1d + h_1q_1^2 - h_2q_2^2| \leqslant \delta HQ^2,\end{aligned} \tag{3.5}$$

just as the additional condition

$$\gcd(d, q_1, q_2) = 1, \quad \gcd(r, h_1, h_2) = 1. \tag{3.6}$$

LEMMA 6. *With the above notations, we have*

$$\begin{aligned}&\mathcal{N}(R, Q, H, N, \delta)\\ &\quad \ll_\varepsilon (RNHQ)^{1+\varepsilon}(1 + \delta Q) + N \sum_{1 \leqslant j \leqslant R} \sum_{1 \leqslant k \leqslant Q} \mathcal{J}(R/j, Q/k, H/j, \delta).\end{aligned} \tag{3.7}$$

Proof. We set $n_1 = n_2 + d$ and we insert this in (3.2). We observe that the terms containing n_2 cancel out each other and we obtain (3.5). On the other hand, the system

$$\begin{aligned}&rd + h_1q_1 - h_2q_2 = 0,\\ &|(h_1q_1 + h_2q_2)d + h_1q_1^2 - h_2q_2^2| \leqslant \delta HQ^2,\end{aligned} \tag{3.8}$$

is equivalent to (3.5). From it, since q_1 and q_2 have the same sign, we deduce that

$$d \ll (1 + \delta)Q. \tag{3.9}$$

Using only the first line of (3.2), we see that the number of solutions of (3.2) with $d = 0$ (i.e. $n_1 = n_2$) is $O_\varepsilon((RQHN)^{1+\varepsilon})$, so that we may suppose now $d \neq 0$. Let j and k be two positive integers and let $\mathcal{J}(j,k)$ be the number of integer points $(r, q_1, q_2, h_1, h_2, d) \in \mathbb{Z}^6$, lying in the domain (3.4), satisfying system (3.5) and the additional condition

$$\gcd(d, q_1, q_2) = k, \quad \gcd(r, h_1, h_2) = j.$$

The following bound is then obvious:

$$\mathcal{N}(R, Q, H, N, \delta) \ll_\varepsilon (RNHQ)^{1+\varepsilon} + N \sum_{1 \leqslant j \leqslant R} \sum_{1 \leqslant k < 2Q} \mathcal{J}(j,k) \tag{3.10}$$

and it may be transformed into

$$\mathcal{N}(R, Q, H, N, \delta) \ll_\varepsilon (RNHQ)^{1+\varepsilon}(1+\delta) + N \sum_{1 \leqslant j \leqslant R} \sum_{1 \leqslant k \leqslant Q} \mathcal{J}(j,k). \tag{3.11}$$

Indeed, if we assume that $k = \gcd(d, q_1, q_2)$ is greater than Q, then we have $q_1 = q_2 = k$ and there are $O(1+\delta)$ possibilities for d, so that the total number of solutions of (3.5) with $\gcd(d, q_1, q_2) > Q$, is $O(RQHN(1+\delta))$ (we have only to use the first line of 3.5). We have thus proved (3.11). But, for j and k fixed, with $1 \leqslant j \leqslant R$ and $1 \leqslant k \leqslant Q$, we may divide the first line of (3.5) by jk and the second line of (3.5) by jk^2. The real numbers $R/j, Q/k$ and H/j are $\geqslant 1$, with $R/j \leqslant \frac{1}{2} H/j$ and we have

$$\mathcal{J}(j,k) = \mathcal{J}(R/j, Q/k, H/j, \delta).$$

Thus (3.11) implies (3.7) and the proof of Lemma 6 is complete. □

From Lemma 6, we deduce that Theorem 2 is a consequence of the following lemma:

LEMMA 7. *Let R, Q, H, δ be positive real numbers with $R \geqslant 1$, $Q \geqslant 1$ and $H \geqslant 2R$. We have:*

$$\mathcal{J}(R, Q, H, \delta) \ll_\varepsilon (RQH)^{1+\varepsilon}(1 + Q\delta), \tag{3.12}$$

which remains to be proved.

3.3. PROOF OF LEMMA 7

(a) First we treat the case $\delta \geqslant 1$. The system (3.8) reduces then to

$$\begin{cases} h_2 q_2 = h_1 q_1 + rd \\ \quad d \ll \delta Q \end{cases}$$

from which we deduce (3.12) at once. From now on, we suppose $0 < \delta < 1$.

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(b) We fix the integers r, h_1 and h_2. In order to apply Lemma 5, we transform the system (3.8). We use the first line of (3.8) to express d and we substitute this expression into the second line; we divide the inequality so obtained by $q_1^2 h_2(h_2 - r)$ and we get

$$\left| \frac{q_2^2}{q_1^2} - \frac{h_1(h_1 - r)}{h_2(h_2 - r)} \right| \leqslant 2\delta \frac{|r|}{H},$$

since $h_2 - r \geqslant H/2$. Finally, the system (3.8) implies

$$\begin{cases} rd + h_1 q_1 - h_2 q_2 = 0 \\ \dfrac{q_2}{q_1} = \sqrt{\dfrac{h_1(h_1 - r)}{h_2(h_2 - r)}} + \mathrm{O}\left(\delta \dfrac{|r|}{H}\right) \end{cases} \tag{3.13}$$

By Lemma 5, the number of triplets (d, q_1, q_2) solutions of (3.13) is

$$\ll_\varepsilon H^\varepsilon \left(1 + \frac{\delta Q^2}{H}\right),$$

so that we have

$$\mathcal{J}(R, Q, H, \delta) \ll_\varepsilon H^\varepsilon (RH^2 + RHQ^2\delta), \tag{3.14}$$

and this proves (3.12) in the case $H \ll Q$.

(c) It only remains to prove Lemma 7 in the following two cases

$$0 < \delta < 1, \ \ Q \ll H \text{ and } \delta \ll R/H \tag{3.15}$$

and

$$0 < \delta < 1, \ \ Q \ll H \text{ and } \delta \gg R/H \tag{3.16}$$

We could fix the integers d, q_1, q_2 with the aim of applying Lemma 5 to bound the number of triplets (r, h_1, h_2) which satisfy (3.8), as in the previous case. But this direct method does not yield (3.12) and some extra work is needed. First we want to prove that (3.8) implies the two systems

$$\begin{cases} dr + q_1 h_1 - q_2 h_2 = 0, \\ \dfrac{h_1}{h_2} = \dfrac{q_2(q_2 - d)}{q_1(q_1 + d)} + \mathrm{O}(\delta), \quad \text{if (3.15) holds}, \\ 2d + q_1 - q_2 \ll RQ/H \end{cases} \tag{3.17}$$

and

$$\begin{cases} dr + q_1 h_1 - q_2 h_2 = 0, \\ \dfrac{h_1}{h_2} = \dfrac{q_2}{q_1} + O(R/H), \quad \text{if (3.16) holds}, \\ 2d + q_1 - q_2 \ll Q\delta. \end{cases} \tag{3.18}$$

For this, we recall that q_1 and q_2 are of the same sign, so that we have either $|q_1 + d| \geqslant Q$ or $|q_2 - d| \geqslant Q$. For example, we assume that q_1, q_2 and d are of

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the same sign. From (3.8), we deduce that

$$\frac{h_1}{h_2} = \frac{q_2(q_2 - d)}{q_1(q_1 + d)} + \mathrm{O}(\delta). \tag{3.19}$$

From the first line of (3.8) and the bound $d \ll Q$, we deduce at once

$$\frac{h_1}{h_2} = \frac{q_2}{q_1} + \mathrm{O}(R/H). \tag{3.20}$$

At last, from (3.19) and (3.20), we deduce

$$2d + q_1 - q_2 \ll Q\delta + RQ/H, \tag{3.21}$$

so that the systems (3.17) and (3.18) follow from (3.8) as claimed above.

Now, we suppose that (3.15) holds. We suppose furthermore that $|d|$ has a fixed size D, that is $D \leqslant |d| < 2D$, with $D \ll Q$. We then fix the integers d, q_1 and q_2 with only $O(DQ + DQ^2R/H)$ possibilities, by (3.21). By Lemma 5, the number of triplets (r, h_1, h_2) which satisfy (3.17) is $O_\varepsilon(Q^\varepsilon(1 + H^2\delta/D))$, so that the total number of integer points $(d, q_1, q_2, r, h_1, h_2)$ lying in the domain (3.4) and satisfying (3.17) is

$$\ll_\varepsilon Q^\varepsilon \max_{1 \leqslant D \ll Q} \big(1 + H^2\delta/D\big)\big(DQ + DQ^2R/H\big),$$

$$\ll_\varepsilon Q^\varepsilon\big(Q^2 + \delta H^2 Q + Q^3 R/H + Q^2 HR\delta\big)$$

which proves (3.12) in this case. The proof of Lemma 7 in case (3.16) is completely similar and we have only to use (3.18) instead of (3.17). The proofs of Lemma 7 and of Theorem 2 are complete. □

4. Proof of Theorem 1

We are now going to prove Theorem 1. We may suppose that hypothesis (1.2) holds with λ small enough. We split up the proof into short steps.

STEP 0: THE SIZE OF M

For proving Theorem 1, we may suppose that

$$M \asymp \lambda^{-8/13} \tag{4.1}$$

(where the notation $u \asymp v$ means that we have both $u \ll v$ and $v \ll u$). Indeed, we set $M_0 = [\lambda^{-8/13}]$. If we have $M \geqslant M_0$, we divide the sum S_M into $\mathrm{O}(M\lambda^{8/13})$ shorter sums and the problem reduces to (4.1). Now we consider the case

$\lambda^{-7/13} \ll M < M_0$. We perform a C^4 continuation of f by setting

$$\widetilde{f}(M+t) = \sum_{j=0}^{4} f^{(j)}(M) \frac{t^j}{j!}$$

for $t > 0$. Then by Lemma 5.2.3 of [4], we have

$$S_M \ll \max_{0 \leqslant \vartheta \leqslant 1} \left| \sum_{m=1}^{M_0} \mathrm{e}(\widetilde{f}(m) + \vartheta m) \right| \log M_0$$

and the problem reduces again to (4.1).

STEP 1: A-PROCESS

We start with the sum $S_M = \sum_{m=1}^{M} \mathrm{e}(f(m))$ and we apply Weyl and van der Corput's A-process in the form that uses symmetrical differences (cf. [4], Lemma 5.6.2). We set $\Delta_h f(m) = f(m+h) - f(m-h)$ and choose a positive integer H such that

$$H \asymp \lambda^{-2/13} \tag{4.2}$$

We then have

$$S_M^2 \ll \frac{M^2}{H} + \frac{M}{H} \left| \sum_{h=1}^{H-1} \left(1 - \frac{h}{H}\right) \sum_{m=h+1}^{M-h} \mathrm{e}(\Delta_h f(m)) \right|. \tag{4.3}$$

Next, we remove the factor $(1 - h/H)$ by partial summation and we use the following remark: given any complex numbers $a_1, a_2, \ldots, a_H$, there exists a positive integer $H_1 \leqslant H/2$ such that

$$\sum_{h=1}^{H-1} a_h \ll \left(\max_{1 \leqslant h \leqslant H-1} |a_h| + \left| \sum_{h=H_1}^{2H_1 - 1} a_h \right| \right) \log H. \tag{4.4}$$

Taking $a_h = \sum_{m=h+1}^{M-h} \mathrm{e}(\Delta_h f(m))$, with $|a_h| \leqslant M$, we get

$$S_M^2 \ll \frac{M^2}{H} \log H + \frac{M}{H} |S(H_1)| \log H, \tag{4.5}$$

for some integer $H_1 \leqslant H/2$, where we have set

$$S(H_1) = \sum_{h=H_1}^{2H_1 - 1} \sum_{m=h+1}^{M-h} \mathrm{e}(\Delta_h f(m)). \tag{4.6}$$

By the third derivative test (Lemma 3), we see that, if $H_1 \ll \lambda^{-1/7}$, we have $S_M \ll M\lambda^{1/13} \log M$, and the theorem is proved. Thus it remains to prove that

$$S(H_1) \ll_\varepsilon M^{1+\varepsilon}\ , \ \text{for } \lambda^{-1/7} \ll H_1 \ll \lambda^{-2/13}. \tag{4.7}$$

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STEP 2: A $\times$ A-PROCESS

We choose two integers R and Q such that

$$R \asymp \lambda^{-1/13} \quad \text{and} \quad Q \asymp \lambda^{-3/13}. \tag{4.8}$$

We apply Lemma 1 to get

$$S(H_1)^2 \ll \frac{MH_1}{QR} \sum_{|r|<R} \sum_{|q|<Q} \left(1 - \frac{|r|}{R}\right)\left(1 - \frac{|q|}{Q}\right) \sum_{h \in J_1(r)} \sum_{m \in J_2(h,q)} \times \\ \times \, \mathrm{e}(\Delta_h f(m+q) - \Delta_{h+r} f(m)), \tag{4.9}$$

where $J_1(r)$ and $J_2(h,q)$ are intervals defined by

$$J_1(r) = [\max(H_1, H_1 - r), \min(2H_1 - 1, 2H_1 - 1 - r)],$$

and

$$J_2(h,q) = [\max(1+h, 1+h-q), \min(M-h, M-h-q)]$$

In the sum in (4.9), we want to remove all terms with $r = 0$ or $q = 0$ to get

$$S(H_1)^2 \ll M^2 + \frac{MH_1}{QR} \sum_{0<|r|<R} \sum_{0<|q|<Q} \left(1 - \frac{|r|}{R}\right)\left(1 - \frac{|q|}{Q}\right) \times \\ \times \sum_{h \in J_1(r)} \sum_{m \in J_2(h,q)} \mathrm{e}(\Delta_h f(m+q) - \Delta_{h+r} f(m)). \tag{4.10}$$

In order to prove (4.10), we first notice that the terms in the sum (4.9) corresponding to $r = q = 0$ have a contribution

$$\ll \frac{(MH_1)^2}{QR} \ll M^2.$$

The terms corresponding to $r = 0$ and $q \neq 0$ may be treated as exponential sums on the variable m, by van der Corput's inequality ([3], Theorem 2.2). Their contribution is $\ll M^2 \lambda^{1/13} \ll M^2$. The terms corresponding to $q = 0$ and $r \neq 0$ may be treated similarly, but with Lemma 3. In order to see that the hypotheses are satisfied, we make use of Taylor's formula to write the phase as:

$$\Delta_h f(m) - \Delta_{h+r} f(m) = g(m) + u(m),$$

with $g(m) = -2rf'(m)$ and

$$u(m) = \int_0^h \frac{(h-t)^2}{2!} (f'''(m+t) + f'''(m-t))\mathrm{d}t - \\ - \int_0^{h+r} \frac{(h+r-t)^2}{2!} (f'''(m+t) + f'''(m-t))\mathrm{d}t.$$

If we set $\mu = |r|\lambda$, so that $|g'''(m)| \asymp \mu$, we have $u'(m) \ll H^3\lambda \ll \mu^{1/2}$. An application of Lemma 3 shows that the contribution of these terms is $\ll M^2\lambda^{1/13}$. We have completed the proof of (4.10).

STEP 3: SHIFT

We want to apply the following obvious equality:

$$\sum_{1\leqslant m\leqslant M} a(m) = \frac{1}{N}\sum_{n=1}^{N}\sum_{1-n\leqslant m\leqslant M-n} a(m+n),$$

where N is any positive integer and where $(a(m))_{1\leqslant m\leqslant M}$ are any given complex numbers. Here we choose

$$N \asymp \lambda^{-3/13}. \tag{4.11}$$

If $g(m)$ is any real valued function, we have

$$\sum_{m\in J_2(h,q)} \mathrm{e}(g(m)) = \frac{1}{N}\sum_{n=1}^{N}\sum_{m\in J_3(h,q,n)} \mathrm{e}(g(m+n)),$$

where $J_2(h, q)$ is as in (4.10). Set now $J_0 = [H_1 + Q, M - 2H_1 - Q - N]$; we have $J_0 \subset J_3(h, q, n) \subset [1, M]$, so that

$$\sum_{m\in J_2(h,q)} \mathrm{e}(g(m)) = \frac{1}{N}\sum_{n=1}^{N}\sum_{m\in J_0} \mathrm{e}(g(m+n)) + O(H_1 + Q + N).$$

Inserting the above equality in (4.10), we finally deduce

$$S(H_1)^2 \ll M^2 + \frac{MH_1}{QRN}\sum_{0<|r|<R}\sum_{m\in J_0} \times \\ \times \left|\sum_{0<|q|<Q}\left(1-\frac{|q|}{Q}\right)\sum_{h=H_1}^{2H_1-1}\sum_{n=1}^{N} \mathrm{e}(\Delta_h f(m+n+q) - \Delta_{h+r} f(m+n))\right|. \tag{4.12}$$

STEP 4: TAYLOR'S FORMULA AND PARTIAL SUMMATION

We write

$$f(m+y) = f(m) + f'(m)y + f''(m)\frac{y^2}{2!} + f'''(m)\frac{y^3}{3!} + v_m(y),$$

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with

$$v_m(y) = \int_0^y \frac{(y-t)^3}{3!} f^{(4)}(m+t)t.$$

Then v_m is a $\mathcal{C}^4$ function which satisfies

$$v_m^{(j)}(y) \ll Q^{4-j}\lambda \quad \text{for } 0 \leqslant j \leqslant 4 \text{ and } y \ll Q. \tag{4.13}$$

We introduce the function

$$u_{m,r}(q,h,n) = v_m(n+q+h) - v_m(n+q-h) - v_m(n+h+r) + v_m(n-h-r) \tag{4.14}$$

and the two polynomials

$$P_1(r,q,h,n) = qh - rn, \;\; P_2(r,q,h,n) = hq^2 + 2hqn - rn^2 - rh^2 - r^2h, \tag{4.15}$$

so that we have

$$\begin{aligned}&\Delta_h f(m+n+q) - \Delta_{h+r} f(m+n) - \\ &\quad = -2rf'(m) + 2f''(m)P_1(r,q,h,n) + \\ &\qquad + f'''(m)P_2(r,q,h,n) + \frac{r^3}{3}f'''(m) + u_{m,r}(q,h,n).\end{aligned} \tag{4.16}$$

We bring this formula into (4.12). Our aim now is to remove the term $u_{m,r}(q,h,n)$ from the triple exponential sum by partial summation. But for the function

$$(q,h,n) \to \mathrm{e}(u_{m,r}(q,h,n)),$$

the bound (2.3) holds with $D \ll 1$, for we have $Q^4\lambda \ll 1$. If we use coefficients in (2.5) instead of the sets $\mathcal{P}'$, we have finally obtained

$$S(H_1)^2 \ll M^2 + \frac{MH_1}{QRN}\widetilde{S}(H_1),$$

where we have set

$$\widetilde{S}(H_1) = \sum_{0<|r|<R} \sum_{m=1}^{M} \left| \sum_{0<|q|<Q} \sum_{h=H_1}^{2H_1-1} \sum_{n=1}^{N} b_r(q,h,n) \mathrm{e}(x_m P_1(r,q,h,n) + y_m P_2(r,q,h,n)) \right|, \tag{4.17}$$

and

$$x_m = 2f''(m), \;\; y_m = f'''(m), \text{ for } m = 1, \ldots, M, \tag{4.18}$$

and where $b_r(q,h,n)$ are complex numbers of modulus at most one. Taking (4.7) into

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account, we see that Theorem 1 will be proved if we obtain the bound

$$\widetilde{S}(H_1) \ll_\varepsilon M^{1+\varepsilon}\lambda^{-5/13}. \tag{4.19}$$

STEP 5: DOUBLE LARGE SIEVE

We want to apply Lemma 4 to the sum $\widetilde{S}(H_1)$. We set

$$\mu = M\lambda, \;\; X_1 = QH, \;\; X_2 = QHN. \tag{4.20}$$

We define $\mathcal{N}$ and $\mathcal{B}$ as in (2.11) and (2.12). The size of parameters M, H, H_1, R, Q and N (cf. (4.1), (4.2), (4.7), (4.8) and (4.11)) shows that the hypothesis of Lemma 4 are satisfied and that (2.13) implies

$$\widetilde{S}(H_1)^2 \ll RX_1\mu X_2\mathcal{N}\mathcal{B}(\log Q)^2. \tag{4.21}$$

It only remains to bound $\mathcal{B}$ and $\mathcal{N}$.

STEP 6: A BOUND FOR $\mathcal{B}$

In (2.12), we set $m_2 = m$ and $m_1 = m + k$. We then have

$$\mathcal{B} \ll \big\{(k, m) \mid 1 \leqslant m \leqslant m + k \leqslant M \text{ and such that the properties (4.22) and (4.23) are satisfied}\big\}$$

with

$$|\overline{\Delta}_k f'''(m)| \leqslant X_2^{-1}, \tag{4.22}$$

$$\|2\overline{\Delta}_k f''(m)\| \leqslant X_1^{-1}, \tag{4.23}$$

where we have set $\overline{\Delta}_k\varphi(x) = \varphi(x+k) - \varphi(x)$.

The inequality (4.22) yields a bound for k, say $0 \leqslant k \leqslant K$, with $K \asymp (\lambda X_2)^{-1}$, while the inequality (4.23) may be treated with respect to m, with fixed k, by the first derivative test for integer points close to a curve (cf. [4], Lemma 3.1.2). We then obtain

$$\mathcal{B} \ll M + \sum_{k=1}^{K}\left(\frac{M}{X_1} + Mk\lambda + \frac{1}{k\lambda X_1} + 1\right).$$

The final bound is

$$\mathcal{B} \ll M \log M. \tag{4.24}$$

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STEP 7: A BOUND FOR $\mathcal{N}$

We have to bound the number $\mathcal{N}$ of integral solutions of the system

$$h_1 q_1 - r n_1 = h_2 q_2 - r n_2,$$
$$(h_1 q_1^2 + 2h_1 q_1 n_1 - r n_1^2 - r h_1^2 - r^2 h_1) - (h_2 q_2^2 + 2h_2 q_2 n_2 - r n_2^2 - r h_2^2 - r^2 h_2) \ll \frac{1}{M\lambda}. \tag{4.25}$$

We failed in solving this problem in its right generality. If we were able to prove the expected bound (under some suitable restrictions), then we should obtain conjecture 1 by the same method (we should only need to change the size of the parameters).

Presently, we reduce the system (4.25) to the simpler one of Theorem 2 by transferring the terms rh_i^2 and $r^2 h_i$ in the error term. This is possible since we have imposed the condition $RH_1^2 \ll 1/M\lambda$.

Furthermore, the hypothesis $R \leqslant H_1/2$ of theorem 2 is satisfied when λ is small enough, because of (4.7) and (4.8). From Theorem 2, we deduce

$$\mathcal{N} \ll_\varepsilon M^\varepsilon \lambda^{-9/13}. \tag{4.26}$$

We take back (4.24) and (4.26) into (4.21). This gives (4.19) and the proof is complete. □

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