# A fractional Borel-Pompeiu type formula and a related fractional $\psi$-Fueter operator with respect to a vector-valued function 

José Oscar González-Cervantes ${ }^{(1)}$ and Juan Bory-Reyes ${ }^{(2)^{*}}$<br>${ }^{(1)}$ Departamento de Matemáticas, ESFM-Instituto Politécnico Nacional. 07338, Ciudad México, México<br>Email: jogc200678@gmail.com<br>${ }^{(2)}$ SEPI, ESIME-Zacatenco-Instituto Politécnico Nacional. 07338, Ciudad México, México<br>Email: juanboryreyes@yahoo.com


#### Abstract

In this paper we combine the fractional $\psi$-hyperholomorphic function theory with the fractional calculus with respect to another function. As a main result, a fractional Borel-Pompeiu type formula related to a fractional $\psi$-Fueter operator with respect to a vector-valued function, is proved.


Keywords. Quaternionic analysis; Borel-Pompeiu formula; Fractional Calculus; Fractional Fueter operator with respect to a vector-valued function.
AMS Subject Classification (2020): 30G30; 30G35; 35R11.

## 1 Introduction

The extension of real integer-order derivatives to fractional derivatives is an old topic introduced by Leibnitz in 1695. The date September 30, 1695 is regarded as the exact birth-date of the fractional calculus. For historical review of the theory we refer the reader to [1, 2].

The interest in fractional integrals and derivatives (Fractional Calculus) has been growing continuously during the last few years because of numerous applications in recent studies in engineering science.

[^0]There exists a vast literature on different definitions of fractional derivatives, see for instance [3-8. In [9] the authors introduce a unified two-parametric fractional derivative, from which, all the interesting derivatives can be obtained. The study prevent the ambiguous use of the concept of fractional derivative. For a review of definitions of fractional order derivatives and integrals that appear in mathematics, physics, and engineering we refer the reader to [10].

The theory of fractional derivatives and integrals with respect to another function was introduced by Erdélyi in [11,12, and it was extensively studied by Osler in [13/15. This operator theory generalizes several classical fractional derivatives, including Riemann-Liouville and Hadamard derivatives among others, see [16-18 as well as [19, Section 2.5] and [20, Section 18.2].

The skew field of real quaternions, denoted by $\mathbb{H}$, in combination with modern analytic methods, give rise to the development of the so-called Quaternionic Analysis. The classical works here are [21, 23]. Nowadays, it relies heavily on results on functions defined on domains in $\mathbb{R}^{4}$ with values in $\mathbb{H}$, associated to a generalized Cauchy-Riemann operator (the so-called $\psi$-Fueter operator), by using a general orthonormal basis in $\mathbb{R}^{4}$ (to be named structural set) $\psi$ of $\mathbb{H}^{4}$. The last goes back at least as far as [24]. The theory is centered around the concept of $\psi$-hyperholomorphic functions (i.e., null solutions to the $\psi$-Fueter operator). For direct constructions along classical lines we refer the reader to [25-28] and the references given there.

Combining a fractional $\psi$-hyperholomorphic function theory with the fractional calculus with respect to another function, we prove as the main result, a fractional Borel-Pompeiu type formula related to a fractional $\psi$-Fueter operator with respect to a vector-valued function to be introduced here for the first time as far as the authors know. As particular case of this study, the main results of [29], are recovered.

The structure of the paper is as follows. After this brief introduction, in the preliminary section we have compiled some basic facts of the quaternionic analysis associated to a structural set $\psi$, such as Stokes and the Borel-Pompieu formulas related to the $\psi$-Fueter operator, as well as a brief summary of the notions of fractional Riemann-Liouville integral and derivative with respect to another function. Section 3 discusses both Stokes and Borel-Pompieu type formulas related to a fractional $\psi$-Fueter operator with respect to a vector-valued function.

## 2 Preliminaries

### 2.1 Riemann-Liouville fractional integral and derivatives with respect to another function

Some definitions and standard facts on the fractional calculus with respect to another function are reviewed below.

Let $-\infty<a<b<\infty, \alpha>0$ and $g \in C^{1}([a, b], \mathbb{R})$ such that $g^{\prime}(x) \neq 0$ for all $x \in[a, b]$ a non-negative monotonously increasing and continuous function.

The (left) Riemann-Liouville fractional integral of order $\alpha$ with respect to $g$ is defined for any $f \in L^{1}([a, b], \mathbb{R})$ by

$$
\left({ }_{a} I_{g}^{\alpha} f\right)(x)=\frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(y) g^{\prime}(y)}{(g(x)-g(y))^{1-\alpha}} d y
$$

where $a<x<b$, see [7,30]
Let $A C^{1}([a, b], \mathbb{R})$ the set of real functions $f$ which are continuously differentiable on $[a, b]$ and $f^{\prime}$ is absolutely continuous on $[a, b]$.

The (left) Riemann-Liouville fractional derivative of order $\alpha$ with respect to $g$ for any $f \in A C^{1}([a, b], \mathbb{R})$ is given by

$$
\begin{aligned}
\left({ }_{a} D_{g}^{\alpha} f\right)(x) & =\left(\frac{1}{g^{\prime}(y)} \frac{d}{d y}\right)\left({ }_{a} I_{g}^{1-\alpha} f\right)(y) \\
& =\frac{\left(\frac{1}{g^{\prime}(y)} \frac{d}{\Gamma(1-\alpha)}\right)}{\int_{a}} \frac{f(y) g^{\prime}(y)}{(g(x)-g(y))^{\alpha}} d y
\end{aligned}
$$

The following semigroup property is valid

$$
\begin{equation*}
{ }_{a} D_{g}^{\alpha} \circ{ }_{a} I_{g}^{\alpha}=I, \tag{1}
\end{equation*}
$$

where $I$ is the identity operator, see 30. Note that the fractional RiemannLiouville integral and derivative with respect to another function are well defined linear operators for any $f \in A C^{1}([a, b], \mathbb{R})$.

Let us remark that, if $g(x)=x$ (resp. $\mathrm{g}(\mathrm{x})=\ln \mathrm{x}$ ), then the RiemannLiouville fractional integral and derivative with respect to $g$ reduces to the standard Riemann-Liouville (resp. the Hadamard), see [3, 7, 19].

### 2.2 Rudiments of quaternionic analysis

The skew field of real quaternions $\mathbb{H}$ is formed by $x=x_{0}+x_{1} \mathbf{i}+x_{2} \mathbf{j}+x_{3} \mathbf{k}$, $x_{k} \in \mathbb{R}, k=0,1,2,3$, where the basic elements satisfy $\mathbf{i}^{2}=\mathbf{j}^{2}=\mathbf{k}^{2}=-1, \mathbf{i} \mathbf{j}=$ $-\mathbf{j} \mathbf{i}=\mathbf{k} ; \mathbf{j} \mathbf{k}=-\mathbf{j} \mathbf{k}=\mathbf{i}$ and $\mathbf{k} \mathbf{i}=-\mathbf{k} \mathbf{i}=\mathbf{j}$. For $x \in \mathbb{H}$ we define the mapping of quaternionic conjugation: $x \rightarrow \bar{x}:=x_{0}-x_{1} \mathbf{i}-x_{2} \mathbf{j}-x_{3} \mathbf{k}$. Easily we see that $x \bar{x}=\bar{x} x=x_{0}^{2}+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}$ and $\overline{q x}=\bar{x} \bar{q}$ for $q, x \in \mathbb{H}$.

The quaternionic scalar product of $q, x \in \mathbb{H}$ is given by

$$
\langle q, x\rangle:=\frac{1}{2}(\bar{q} x+\bar{x} q)=\frac{1}{2}(q \bar{x}+x \bar{q}) .
$$

A set of quaternions $\psi=\left\{\psi_{0}, \psi_{1}, \psi_{2}, \psi_{3}\right\}$ is called structural set if $\left\langle\psi_{k}, \psi_{s}\right\rangle=$ $\delta_{k, s}$, for $k, s=0,1,2,3$ and any quaternion $x$ can be rewritten as $x_{\psi}:=\sum_{k=0}^{3} x_{k} \psi_{k}$,
where $x_{k} \in \mathbb{R}$ for all $k$. Given $q, x \in \mathbb{H}$ we follow the notation used in [27] to write

$$
\langle q, x\rangle_{\psi}=\sum_{k=0}^{3} q_{k} x_{k}
$$

where $q_{k}, x_{k} \in \mathbb{R}$ for all $k$.
Given an structural set $\psi$, we will use the mapping

$$
\begin{equation*}
\sum_{k=0}^{3} x_{k} \psi_{k} \rightarrow\left(x_{0}, x_{1}, x_{2}, x_{3}\right) \tag{2}
\end{equation*}
$$

in essential way.
We have to say something about the set of complex quaternions, which are given by

$$
\mathbb{H}(\mathbb{C})=\left\{q=q_{1}+\mathrm{i} q_{2} \mid q_{1}, q_{2} \in \mathbb{H}\right\},
$$

where $i$ is the imaginary unit of $\mathbb{C}$. The main difference to the real quaternions is that not all non-zero elements are invertible. There are so-called zero-divisors.

Let us recall that $\mathbb{H}$ is embedded in $\mathbb{H}(\mathbb{C})$ as follows:

$$
\mathbb{H}=\left\{q=q_{1}+\mathrm{i} q_{2} \in \mathbb{H}(\mathbb{C}) \mid q_{1}, q_{2} \in \mathbb{H} \text { and } q_{2}=0\right\} .
$$

The elements of $\mathbb{H}$ are written in terms of the structural set $\psi$ hence those of $\mathbb{H}(\mathbb{C})$ can be written as $q=\sum_{k=0}^{3} \psi_{k} q_{k}$, where $q_{k} \in \mathbb{C}$.

Functions $\mathfrak{f}$ defined in a bounded domain $\Omega \subset \mathbb{H} \cong \mathbb{R}^{4}$ with value in $\mathbb{H}$ are considered. They may be written as: $\mathfrak{f}=\sum_{k=0}^{3} f_{k} \psi_{k}$, where $f_{k}, k=0,1,2,3$, are $\mathbb{R}$-valued functions in $\Omega$. Properties as continuity, differentiability, integrability and so on, which as ascribed to $\mathfrak{f}$ have to be posed by all components $f_{k}$. We will follow standard notation, for example $C^{1}(\Omega, \mathbb{H})$ denotes the set of continuously differentiable $\mathbb{H}$-valued functions defined in $\Omega$.

The left- and the right- $\psi$-Fueter operators are defined by ${ }^{\psi} \mathcal{D}[\mathfrak{f}]:=\sum_{k=0}^{3} \psi_{k} \partial_{k} \mathfrak{f}$ and ${ }^{\psi} \mathcal{D}_{r}[\mathfrak{f}]:=\sum_{k=0}^{3} \partial_{k} \mathfrak{f} \psi_{k}$, for all $\mathfrak{f}, \mathfrak{f} \in C^{1}(\Omega, \mathbb{H})$, respectively, where $\partial_{k} \mathfrak{f}=\frac{\partial \mathfrak{f}}{\partial x_{k}}$ for all $k$, see [27, 28].

Particularly, if $\partial \Omega$ is a 3-dimensional smooth surface then the Borel-Pompieu formula shows that

$$
\begin{align*}
& \int_{\partial \Omega}\left(K_{\psi}(y-x) \sigma_{y}^{\psi} \mathfrak{f}(y)+\mathbf{f}(y) \sigma_{y}^{\psi} K_{\psi}(y-x)\right) \\
& -\int_{\Omega}\left(K_{\psi}(y-x)^{\psi} \mathcal{D}[\mathfrak{f}](y)+{ }^{\psi} \mathcal{D}_{r}[\mathbf{f}](y) K_{\psi}(y-x)\right) d y \\
= & \begin{cases}\mathfrak{f}(x)+\mathbf{f}(x), & x \in \Omega, \\
0, & x \in \mathbb{H} \backslash \bar{\Omega} .\end{cases} \tag{3}
\end{align*}
$$

Differential and integral versions of Stokes' formulas for the $\psi$-hyperholomorphic functions theory are given by

$$
\begin{equation*}
\int_{\partial \Omega} \mathrm{f} \sigma_{x}^{\psi} \mathfrak{f}=\int_{\Omega}\left(\mathrm{f}^{\psi} \mathcal{D}[\mathfrak{f}]+{ }^{\psi} \mathcal{D}_{r}[\mathrm{f}] \mathfrak{f}\right) d x \tag{4}
\end{equation*}
$$

for all $\mathfrak{f}, \mathrm{f} \in C^{1}(\bar{\Omega}, \mathbb{H})$, see $23,27,28$. Here, $d$ stands for the exterior differentiation operator, $d x$ denotes the differential form of the 4-dimensional volume in $\mathbb{R}^{4}$ and

$$
\sigma_{x}^{\psi}:=-\operatorname{sgn} \psi\left(\sum_{k=0}^{3}(-1)^{k} \psi_{k} d \hat{x}_{k}\right)
$$

is the quaternionic differential form of the 3 -dimensional volume in $\mathbb{R}^{4}$ according to $\psi$, where $d \hat{x}_{k}=d x_{0} \wedge d x_{1} \wedge d x_{2} \wedge d x_{3}$ omitting factor $d x_{k}$. In addition, $\operatorname{sgn} \psi$ is 1 , or -1 , if $\psi$ and $\psi_{s t d}:=\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ have the same orientation, or not, respectively. Note that, $\left|\sigma_{x}^{\psi}\right|=d S_{3}$ is the differential form of the 3 -dimensional volume in $\mathbb{R}^{4}$ and write $\sigma_{x}=\sigma_{x}^{\psi_{s t d}}$. Let us recall that the $\psi$-hyperholomorphic Cauchy Kernel is given by

$$
K_{\psi}(y-x)=\frac{1}{2 \pi^{2}} \frac{\overline{y_{\psi}-x_{\psi}}}{\left|y_{\psi}-x_{\psi}\right|^{4}}
$$

and the integral operator

$$
\psi^{\psi} \mathcal{T}[\mathfrak{f}](x)=\int_{\Omega} K_{\psi}(y-x) \mathfrak{f}(y) d y
$$

defined for all $\mathfrak{f} \in L_{2}(\Omega, \mathbb{H}) \cup C(\Omega, \mathbb{H})$ satisfies

$$
\begin{equation*}
{ }^{\psi} \mathcal{D} \circ{ }^{\psi} \mathcal{T}[\mathfrak{f}]=\mathfrak{f}, \quad \forall \mathfrak{f} \in L_{2}(\Omega, \mathbb{H}) \cup C(\Omega, \mathbb{H}) \tag{5}
\end{equation*}
$$

This can be found in $[25-28]$.

## 3 Main results

For simplicity of notation, we write $\vec{\alpha}$ and $\vec{\beta}$ instead of the vectors ( $\alpha_{0}, \alpha_{1}, \alpha_{2}, \alpha_{3}$ ) and $\left(\beta_{0}, \beta_{1}, \beta_{2}, \beta_{3}\right)$ both in $(0,1)^{4}$.

### 3.1 Fractional $\psi$-Fueter operator of order $\vec{\alpha}$

Definition 1. Let $a=\sum_{k=0}^{3} \psi_{k} a_{k}, b=\sum_{k=0}^{3} \psi_{k} b_{k} \in \mathbb{H}$ such that $a_{k}<b_{k}$ for all $k$. Write

$$
\begin{aligned}
J_{a}^{b} & :=\left\{\sum_{k=0}^{3} \psi_{k} x_{k} \in \mathbb{H} \mid a_{k}<x_{k}<b_{k}, \quad k=0,1,2,3\right\} \\
& =\left(a_{0}, b_{0}\right) \times\left(a_{1}, b_{1}\right) \times\left(a_{2}, b_{2}\right) \times\left(a_{3}, b_{3}\right)
\end{aligned}
$$

and define $m\left(J_{a}^{b}\right):=\left(b_{0}-a_{0}\right)\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)\left(b_{3}-a_{3}\right)$.

Set $\mathfrak{f}=\sum_{i=0}^{3} \psi_{i} f_{i} \in A C^{1}\left(J_{a}^{b}, \mathbb{H}\right)$; i.e., the real components $f_{i}, i=0,1,2,3$ of $\mathfrak{f}$, belongs to $A C^{1}\left(\left(a_{i}, b_{i}\right), \mathbb{R}\right)$.

The mapping $x_{j} \mapsto f_{i}\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right)$ belongs to $A C^{1}\left(\left(a_{i}, b_{i}\right), \mathbb{R}\right)$ for each $q \in J_{a}^{b}$ and all $i, j=0,1,2,3$.

Now, given $q, x \in J_{a}^{b}$ and $i, j=0, \ldots, 3$, the (left) fractional Riemann-Liouville integral of order $\alpha_{j}$ with respect to a monotonously increasing functions $g_{j} \in$ $C^{1}\left(\left[a_{j}, b_{j}\right], \mathbb{R}\right)$ for the mapping $x_{j} \mapsto f_{i}\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right)$ is defined by

$$
\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{\alpha_{j}} f_{i}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right)=\frac{1}{\Gamma\left(\alpha_{j}\right)} \int_{a_{j}}^{x_{j}} \frac{f_{i}\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) g_{j}^{\prime}\left(y_{j}\right)}{\left(g_{j}\left(x_{j}\right)-g_{j}\left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j}
$$

By the above, as $\mathfrak{f}=\sum_{i=0}^{3} \psi_{i} f_{i}$ it follows that

$$
\begin{aligned}
\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right): & =\frac{1}{\Gamma\left(\alpha_{j}\right)} \int_{a_{j}}^{x_{j}} \frac{\mathfrak{f}\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) g_{j}^{\prime}\left(y_{j}\right)}{\left(g_{j}\left(x_{j}\right)-g_{j}\left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j} \\
& =\sum_{i=0}^{3} \psi_{i}\left(\mathbf{I}_{a_{j}, g_{j}+}^{\alpha_{j}} f_{i}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right)
\end{aligned}
$$

for every $\mathfrak{f} \in A C^{1}\left(J_{a}^{b}, \mathbb{H}\right)$ and $q, x \in J_{a}^{b}$.
What is more, the (left) fractional Riemann-Liouville derivative of order $\alpha_{j}$ with respect to a monotonously increasing function $g_{k} \in C^{1}[a, b]$ for all $k=0,1,2,3$ with $g_{k}^{\prime} \neq 0$ for $k=0,1,2,3$ for the mapping $x_{j} \mapsto \mathfrak{f}\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right)$ is given as

$$
\begin{aligned}
D_{a_{j}^{+}, g_{j}}^{\alpha_{j}} \mathfrak{f}\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right) & =\left(\frac{1}{g_{j}^{\prime}\left(x_{j}\right)} \frac{\partial}{\partial x_{j}}\right) \sum_{i=0}^{3} \psi_{i}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{\alpha_{j}} f_{i}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right) \\
& =\frac{\left(\frac{1}{g_{j}^{\prime}\left(x_{j}\right)} \frac{\partial}{\partial x_{j}}\right)}{\Gamma\left(\alpha_{j}\right)} \int_{a_{j}}^{x_{j}} \frac{\mathfrak{f}\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) g_{j}^{\prime}\left(y_{j}\right)}{\left(g_{j}\left(x_{j}\right)-g_{j}\left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j}
\end{aligned}
$$

Remark 1. Note that $\mathbf{I}_{a_{j}^{+}, g_{j}}^{\alpha_{j}} \mathfrak{f}$ and $D_{a_{j}^{+}, g_{j}}^{\alpha_{j}} \mathfrak{f}$ are $\mathbb{H}(\mathbb{C})$-valued functions for every $j$. In a similar way we can introduce the (right) fractional Riemann-Liouville integral and derivative, to be denoted by $\left(\mathbf{I}_{b_{j}^{-}, g_{j}}^{\alpha_{j}} \mathfrak{f}\right)$ and $D_{b_{j}^{-}, g_{j}}^{\alpha_{j}} \mathfrak{f}$ respectively, but we will not develop this point here.

Definition 2. Let $\mathfrak{f}, \mathrm{f} \in A C^{1}\left(J_{a}^{b}, \mathbb{H}\right)$ and let the vector-valued function $\mathbf{g}:=$ $\left(g_{0}, g_{1}, g_{2}, g_{3}\right)$ with monotonously increasing components $g_{k} \in C^{1}\left[a_{k}, b_{k}\right]$ with $g_{k}^{\prime} \neq$ 0 for $k=0,1,2,3$. The (left) fractional $\psi$-Fueter operator of order $\vec{\alpha}$ with respect
to $\mathbf{g}$ is defined by

$$
\begin{aligned}
\psi_{\mathfrak{D}_{a, g}}^{\vec{\alpha}}[\mathfrak{f}](q, x) & :=\sum_{j=0}^{3} \psi_{j}\left(D_{a_{j}^{+}, g_{j}}^{\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right) \\
& =\sum_{j=0}^{3} \psi_{j} \frac{\left(\frac{1}{g_{j}^{\prime}\left(x_{j}\right)} \frac{\partial}{\partial x_{j}}\right)}{\Gamma\left(\alpha_{j}\right)} \int_{a_{j}}^{x_{j}} \frac{\mathfrak{f}\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) g_{j}^{\prime}\left(y_{j}\right)}{\left(g_{j}\left(x_{j}\right)-g_{j}\left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j} .
\end{aligned}
$$

Particularly, $\left.{ }^{\psi} \mathfrak{D}_{a, \mathrm{~g}}^{\vec{\alpha}}[\mathrm{f}](q, x)\right|_{x=q}={ }^{\psi} \mathfrak{D}_{a, \mathrm{~g}}^{\vec{\alpha}}[f](q)$; i.e., it is ${ }^{\psi} \mathfrak{D}_{a, \mathrm{~g}}^{\vec{\alpha}}[\mathrm{f}]$ at point $q$.
Observe that $q$ is considered a fixed point since the integration and derivation variables are the real components of $x$ and ${ }^{\psi} \mathfrak{D}_{a, \mathrm{~g}}^{\vec{\alpha}}[\mathfrak{f}](q, \cdot)$ is a $\mathbb{H}(\mathbb{C})$-valued function.
Remark 2. Taking into account the non-commutativity of the $\mathbb{H}$-multiplication it is natural to introduce the right hand side analogue of $\psi \mathfrak{D}_{a, \mathrm{~g}}^{\vec{\alpha}}[\mathrm{f}]$ :

$$
\psi \mathfrak{D}_{r, a, \mathbf{g}}^{\vec{\alpha}}[\mathrm{f}](q, x):=\sum_{j=0}^{3} \frac{\left(\frac{1}{g_{j}^{\prime}\left(x_{j}\right)} \frac{\partial}{\partial x_{j}}\right)}{\Gamma\left(\alpha_{j}\right)} \int_{a_{j}}^{x_{j}} \frac{\mathrm{f}\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) g_{j}^{\prime}\left(y_{j}\right)}{\left(g_{j}\left(x_{j}\right)-g_{j}\left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j} \psi_{j},
$$

for $q, x \in J_{a}^{b}$.
A key observation is that if $g_{k}(x)=x$ for all $x \in\left[a_{k}, b_{k}\right]$ and $k=0,1,2,3$ then $\psi \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}$ becomes at the quaternionic fractional operator presented in [29]. Therefore, for $0<a_{k}<b_{k}$ with $k=01,2,3$ considering $\ln (x)=\left(\ln x_{0}, \ln x_{1}, \ln x_{2}, \ln x_{3}\right)$ for all $x_{k} \in\left[a_{k}, b_{k}\right]$ and $k=0,1,2,3$ we obtain the fractional $\psi$-Fueter operator of order $\vec{\alpha}$ with respect to $\ln$ associated to the Hadamard fractional derivative.

Definition 3. Given $\mathfrak{f}, \mathrm{f} \in A C^{1}\left(J_{a}^{b}, \mathbb{H}\right)$ define

$$
\begin{aligned}
& { }^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha}):=\sum_{j=0}^{3} \frac{1}{\Gamma\left(\alpha_{j}\right)} \int_{a_{j}}^{x_{j}} \frac{\mathfrak{f}\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) g_{j}^{\prime}\left(y_{j}\right)}{\left(g_{j}\left(x_{j}\right)-g_{j}\left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j} \\
& =\sum_{j=0}^{3}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right), \\
& \psi_{\mathfrak{C}_{a, \mathbf{g}}}^{\vec{\alpha}}[\mathfrak{f}](q, x):=\sum_{j=0}^{3}\left(g_{j}^{\prime}\left(x_{j}\right)-1\right) \psi_{j} \frac{1}{g_{j}^{\prime}\left(x_{j}\right)} \frac{\partial}{\partial x_{j}}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right) \\
& =\sum_{j=0}^{3}\left(g_{j}^{\prime}\left(x_{j}\right)-1\right) \psi_{j} D_{a_{j}^{+}, g_{j}}^{\alpha_{j}}[f]\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right), \\
& \psi_{\mathfrak{C}_{r, a, \mathbf{g}}}^{\vec{\alpha}}[\mathbf{f}](q, x):=\sum_{j=0}^{3}\left(g_{j}^{\prime}\left(x_{j}\right)-1\right) D_{a_{j}^{+}, g_{j}}^{\alpha_{j}}[\mathbf{f}]\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right) \psi_{j}, \\
& \psi_{\mathfrak{P}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathrm{f}](q, x):=\sum_{j=0}^{3} D_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}}[f]\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right), ~}^{\text {, }}
\end{aligned}
$$

Remark 4. If $0<a_{k}<b_{k}$ for $k=0,1,2,3$ the previous operators, associated to the Hadamard fractional derivative, are given by

$$
\begin{aligned}
\psi_{\mathfrak{D}_{a, \ln }^{\vec{\alpha}}[\mathfrak{f}](q, x)} & =\sum_{j=0}^{3} \psi_{j} \frac{x_{j}}{\Gamma\left(\alpha_{j}\right)} \frac{\partial}{\partial x_{j}} \int_{a_{j}}^{x_{j}} \frac{\mathfrak{f}\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right)}{y_{j}\left(\ln \left(x_{j}\right)-\ln \left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j}, \\
\psi_{\mathfrak{D}_{r, a, \ln }}^{\vec{\alpha}}[\mathfrak{f}](q, x) & =\sum_{j=0}^{3} \frac{x_{j}}{\Gamma\left(\alpha_{j}\right)} \frac{\partial}{\partial x_{j}} \int_{a_{j}}^{x_{j}} \frac{\mathfrak{f}\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right)}{y_{j}\left(\ln \left(x_{j}\right)-\ln \left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j} \psi_{j}, \\
\psi_{a, \ln }^{x}[\mathfrak{f}](q, x, \vec{\alpha}) & :=\sum_{j=0}^{3} \frac{1}{\Gamma\left(\alpha_{j}\right)} \int_{a_{j}}^{x_{j}} \frac{\mathfrak{f}\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right)}{y_{j}\left(\ln _{j}\left(x_{j}\right)-\ln n_{j}\left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j}, \\
\psi_{\mathcal{C}_{a, \ln } \vec{\alpha}}[\mathfrak{f}](q, x) & :=\sum_{j=0}^{3}\left(1-x_{j}\right) \psi_{j} \frac{\partial}{\partial x_{j}}\left(\mathbf{I}_{a_{j}^{+}, \ln _{j}}^{\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right) .
\end{aligned}
$$

for any $\mathfrak{f}, \mathfrak{f} \in A C^{1}\left(J_{a}^{b}, \mathbb{H}\right)$.
Proposition 5. Assume that $\mathfrak{f}, \mathfrak{f} \in A C^{1}\left(J_{a}^{b}, \mathbb{H}\right)$. Then we have
1.

$$
\begin{aligned}
&{ }^{\psi} \mathcal{D}_{x} \circ{ }^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[f](q, x, \vec{\alpha})={ }^{\psi} \mathfrak{C}_{a, \mathbf{g}}^{\vec{\alpha}}[f](q, x)+{ }^{\psi} \mathfrak{D}_{a, \mathbf{g}}^{\vec{a}}[f](q, x), \\
&{ }^{\psi} \mathcal{D}_{r, x} \circ{ }^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha})={ }^{\psi} \mathfrak{C}_{r, a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, x)+{ }^{\psi} \mathfrak{D}_{r, a, \mathbf{g}}^{\vec{a}}[\mathrm{f}](q, x),
\end{aligned}
$$

2. 

$$
\begin{aligned}
\psi_{\mathfrak{P}} \mathfrak{P}_{a, \mathbf{g}}^{\vec{\alpha}} \circ{ }^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[f](q, x, \vec{\alpha})= & \sum_{j=0}^{3} \mathfrak{f}\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right) \\
& +\sum_{\substack{j, k=0 \\
j \neq k}}^{3} \frac{\left(\mathbf{I}_{a_{k}^{+}, g_{k}}^{1-\alpha_{k}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right)}{\Gamma\left(1-\alpha_{j}\right)\left(g_{j}\left(x_{j}\right)-g_{j}(a)\right)^{1-\alpha_{j}}},
\end{aligned}
$$

3. 

$$
\begin{gathered}
\bar{\psi} \mathcal{D}_{x} \circ{ }^{\psi} \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, x)=\Delta_{\mathbb{R}^{2}} \circ{ }^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[f](q, x, \vec{\alpha})-{ }^{\bar{\psi}} \mathcal{D}_{x} \circ \psi \mathfrak{C}_{a, \mathbf{g}}^{\vec{\alpha}}[f](q, x), \\
\bar{\psi} \mathcal{D}_{r, x} \circ{ }^{\psi} \mathfrak{D}_{r, a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, x)=\Delta_{\mathbb{R}^{2}} \circ{ }^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha})-{ }^{\bar{\psi}} \mathcal{D}_{r, x} \circ \psi \mathfrak{C}_{r, a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, x),
\end{gathered}
$$

Proof. 1. and 3. Follow from direct computations.
2. From (1)

$$
\begin{aligned}
& \psi \mathfrak{P}_{a, \mathbf{g}}^{\vec{\alpha}} \circ{ }^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha})=\sum_{j=k=0}^{3} D_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}}\left(\mathbf{I}_{a_{k}^{+}, g_{k}}^{1-\alpha_{k}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right) \\
= & \sum_{j=0}^{3} \mathfrak{f}\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right)+\sum_{\substack{j, k=0 \\
j \neq k}}^{3} \frac{\left(\mathbf{I}_{a_{k}^{+}, g_{k}}^{1-\alpha_{k}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right)}{\Gamma\left(1-\alpha_{j}\right)\left(g_{j}\left(x_{j}\right)-g_{j}(a)\right)^{1-\alpha_{j}}},
\end{aligned}
$$

where the identity

$$
\begin{aligned}
D_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}}[1] & =\frac{1}{\Gamma\left(1-\alpha_{j}\right)} \frac{1}{g_{j}^{\prime}\left(x_{j}\right)} \frac{\partial}{\partial x_{j}} \int_{a_{j}}^{x_{j}} \frac{g_{j}^{\prime}\left(y_{j}\right)}{\left(g_{j}\left(x_{j}\right)-g_{j}\left(y_{j}\right)\right)^{1-\alpha_{j}}} d y_{j} \\
& =\frac{1}{\Gamma\left(1-\alpha_{j}\right)\left(g_{j}\left(x_{j}\right)-g_{j}(a)\right)^{1-\alpha_{j}}}
\end{aligned}
$$

was applied.

Proposition 6. (Stokes type integral formula induced by $\psi_{\mathfrak{D}}^{a, \mathbf{g}} \overrightarrow{\vec{\alpha}}$ ) Suppose that $\mathbf{g}=\left(g_{0}, g_{1}, g_{2}, g_{3}\right)$ and $\mathbf{h}=\left(h_{0}, h_{1}, h_{2}, h_{3}\right)$ are two vector-valued functions with monotonously increasing components such that $g_{k}, h_{k} \in C^{1}\left[a_{k}, b_{k}\right]$ and $g_{k}^{\prime} \neq 0 \neq h_{k}^{\prime}$
for $k=0,1,2,3$. If $\mathfrak{f}, \mathfrak{f} \in A C^{1}\left(\overline{J_{a}^{b}}, \mathbb{H}\right)$ consider $q \in J_{a}^{b}$ such that the mappings $x \mapsto^{\psi} \mathcal{I}_{a}^{x}[\mathfrak{f}](q, x, \vec{\alpha})$ and $x \mapsto^{\psi} \mathcal{I}_{a}^{x}[\mathrm{f}](q, x, \vec{\beta})$ belong to $C^{1}\left(\overline{J_{a}^{b}}, \mathbb{H}(\mathbb{C})\right)$. Then

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}}{ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{x}[\mathrm{f}](q, x, \vec{\beta}) \sigma_{x}^{\psi \psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha}) \\
& =\int_{J_{a}^{b}}\left({ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{x}[\mathbf{f}](q, x, \vec{\beta}){ }^{\psi} \mathfrak{C}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, x)+{ }^{\psi} \mathcal{C}_{r, a, \mathbf{h}}^{\vec{\beta}}[\mathbf{f}](q, x)^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha})\right) d x \\
& +\int_{J_{a}^{b}}\left(\psi^{\psi} \mathcal{I}_{a, \mathbf{h}}^{x}[\mathbf{f}](q, x, \vec{\beta}){ }^{\psi} \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, x)+\psi^{\psi} \mathfrak{D}_{r, a, \mathbf{h}}^{\vec{\beta}}[\mathfrak{f}](q, x)^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha})\right) d x .
\end{aligned}
$$

Proof. Considering in (44) the functions ${ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{x}[\mathbf{f}](q, x, \vec{\beta})$ and ${ }^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha})$ and use Proposition 5

Corollary 7. (A version of the Cauchy theorem) Under the hypothesis of Proposition 6, if moreover

$$
\psi_{\mathfrak{D}_{a, \mathbf{g}}}^{\vec{\alpha}}[\mathfrak{f}](q, \cdot)=\psi_{\mathfrak{D}_{r, a, \mathbf{h}}}^{\vec{\beta}}[\mathbf{f}](q, \cdot)=0, \quad \text { on } \quad J_{a}^{b}
$$

Then

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}}{ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{x}[\mathbf{f}](q, x, \vec{\beta}) \sigma_{x}^{\psi \psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha}) \\
= & \left.\int_{J_{a}^{b}}\left({ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{x}[\mathbf{f}](q, x, \vec{\beta}) \psi_{\mathfrak{C}_{a, \mathbf{g}} \vec{\alpha}} \mathfrak{f}\right](q, x)+{ }^{\psi} \mathfrak{C}_{r, a, \mathbf{h}}^{\vec{\beta}}[\mathbf{f}](q, x)^{\psi} \mathcal{I}_{a, \mathbf{g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha})\right) d x .
\end{aligned}
$$

Remark 8. The Stokes type integral formula induced by $\psi_{\mathfrak{D}_{a, \ln }^{\vec{\alpha}}}$ associated to Hadamard fractional derivative holds. Set $0<a_{k}<b_{k}$ for $k=0,1,2,3$ and let $\mathfrak{f}, \mathfrak{f} \in A C^{1}\left(\overline{J_{a}^{b}}, \mathbb{H}\right)$. Consider $q \in J_{a}^{b}$ such that the mappings $x \mapsto{ }^{\psi} \mathcal{I}_{a}^{x}[\mathfrak{f}](q, x, \vec{\alpha})$ and $x \mapsto{ }^{\psi} \mathcal{I}_{a}^{x}[\mathrm{f}](q, x, \vec{\beta})$ belong to $C^{1}\left(\overline{J_{a}^{b}}, \mathbb{H}(\mathbb{C})\right)$. Then

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}}{ }^{\psi} \mathcal{I}_{a, \ln }^{x}[\mathrm{f}](q, x, \vec{\beta}) \sigma_{x}^{\psi \psi} \mathcal{I}_{a, \ln }^{x}[\mathfrak{f}](q, x, \vec{\alpha}) \\
& -\int_{J_{a}^{b}}\left({ }^{\psi} \mathcal{I}_{a, \ln }^{x}[\mathfrak{f}](q, x, \vec{\beta}){ }^{\psi} \mathfrak{C}_{a, \ln }^{\vec{\alpha}}[\mathfrak{f}](q, x)+{ }^{\psi} \mathfrak{C}_{r, a, \ln }^{\vec{\beta}}[\mathfrak{f}](q, x)^{\psi} \mathcal{I}_{a, \ln }^{x}[\mathfrak{f}](q, x, \vec{\alpha})\right) d x \\
= & \left.\int_{J_{a}^{b}}\left({ }^{\psi} \mathcal{I}_{a, \ln }^{x}[\mathfrak{f}](q, x, \vec{\beta})^{\psi} \mathfrak{D}_{a, \ln }^{\vec{\alpha}}[\mathfrak{f}](q, x)+{ }^{\psi}{ }_{\mathfrak{D}_{r, a, \ln }^{\vec{\beta}}}^{\vec{\beta}}\right](q, x)^{\psi} \mathcal{I}_{a, \ln }^{x}[\mathfrak{f}](q, x, \vec{\alpha})\right) d x
\end{aligned}
$$

and if

$$
\psi_{\mathfrak{D}_{a, \ln }^{\vec{\alpha}}[\mathfrak{f}](q, \cdot)={ }^{\psi} \mathfrak{D}_{r, a, \ln }^{\vec{\beta}}[\mathrm{f}](q, \cdot)=0, \quad \text { on } \quad J_{a}^{b}, ~}^{\text {, }}
$$

then

$$
\begin{aligned}
\int_{\partial J_{a}^{b}}{ }^{\psi} \mathcal{I}_{a, \ln }^{x}[\mathbf{f}](q, x, \vec{\beta}) \sigma_{x}^{\psi \psi} \mathcal{I}_{a, \ln }^{x}[\mathfrak{f}](q, x, \vec{\alpha})= & \int_{J_{a}^{b}}\left({ }^{\psi} \mathcal{I}_{a, \ln }^{x}[\mathbf{f}](q, x, \vec{\beta}) \psi \mathfrak{C}_{a, \ln }^{\vec{\alpha}}[\mathbf{f}](q, x)\right. \\
& \left.+{ }^{\psi} \mathfrak{C}_{r, a, \ln }^{\vec{\beta}}[\mathbf{f}](q, x)^{\psi} \mathcal{I}_{a, \ln }^{x}[\mathfrak{f}](q, x, \vec{\alpha})\right) d x
\end{aligned}
$$

Theorem 9. (Borel-Pompieu type formula induced by ${ }^{\psi} \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}$ and ${ }^{\psi} \mathfrak{D}_{r, a, \mathbf{h}}^{\vec{\beta}}$ ) Let $\mathbf{g}=\left(g_{0}, g_{1}, g_{2}, g_{3}\right)$ and $\mathbf{h}=\left(h_{0}, h_{1}, h_{2}, h_{3}\right)$ be two vector-valued functions with monotonously increasing components such that $g_{k}, h_{k} \in C^{1}\left[a_{k}, b_{k}\right]$ and $g_{k}^{\prime} \neq 0 \neq h_{k}^{\prime}$ for $k=0,1,2,3$. If $\mathfrak{f}, \mathfrak{f} \in A C^{1}\left(\overline{J_{a}^{b}}, \mathbb{H}\right)$, consider $q \in J_{a}^{b}$ such that the mappings $x \mapsto{ }^{\psi} \mathcal{I}_{a}^{x}[\mathfrak{f}](q, x, \vec{\alpha})$ and $x \mapsto{ }^{\psi} \mathcal{I}_{a}^{x}[\mathfrak{f}](q, x, \vec{\beta})$ belong to $C^{1}\left(\overline{J_{a}^{b}}, \mathbb{H}(\mathbb{C})\right)$. Then

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}}\left({ }^{\psi} \mathfrak{K}_{a, \mathbf{g}}^{\vec{\alpha}}(q, x, y) \sigma_{y}^{\psi \psi} \mathcal{I}_{a, \mathbf{g}}^{y}[\mathfrak{f}](q, y, \vec{\alpha})+{ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{y}[\mathfrak{f}](q, y, \vec{\beta}) \sigma_{y}^{\psi \psi} \mathfrak{K}_{a, \mathbf{h}}^{\vec{\beta}}(q, x, y)\right) \\
- & \int_{J_{a}^{b}}\left({ }^{\psi} \mathfrak{K}_{a, \mathbf{g}}^{\vec{\alpha}}(q, x, y)^{\psi} \mathfrak{C}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, y)+\psi_{\mathfrak{C}_{r, a, \mathbf{h}} \vec{\beta}}[\mathfrak{f}](q, y)^{\psi} \mathfrak{K}_{a, \mathbf{h}}^{\vec{\beta}}(q, x, y)\right) d y \\
- & \int_{J_{a}^{b}}\left({ }^{\psi} \mathfrak{K}_{a, \mathbf{g}}^{\vec{\alpha}}(q, x, y)^{\psi} \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, y)+{ }^{\psi} \mathfrak{D}_{r, a, \mathbf{h}}^{\vec{\beta}}[\mathbf{f}](q, y)^{\psi} \mathfrak{K}_{a, \mathbf{h}}^{\vec{\beta}}(q, x, y)\right) \\
= & \begin{cases}\sum_{j=0}^{3}(\mathfrak{f}+\mathbf{f})\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right)+M_{a, g}^{\vec{\alpha}}[\mathfrak{f}](q, x)+M_{a, h}^{\vec{\beta}}[\mathbf{f}](q, x), & x \in J_{a}^{b} \\
0, & x \in \mathbb{H} \backslash \overline{J_{a}^{b}}\end{cases}
\end{aligned}
$$

where

$$
\psi_{\mathfrak{K}_{a, \mathbf{g}}}^{\vec{\alpha}}(q, x, y)={ }^{\psi} \mathfrak{P}_{a, \mathbf{g}}^{\vec{\alpha}}\left[K_{\psi}(y-x)\right](q, x)
$$

and

$$
M_{a, \mathbf{g}}^{\vec{\alpha}}[\boldsymbol{f}](q, x)=\sum_{\substack{j, k=0 \\ j \neq k}}^{3} \frac{\left(\mathbf{I}_{a_{k}^{+}, g_{k}}^{1-\alpha_{k}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right)}{\Gamma\left(1-\alpha_{j}\right)\left(g_{j}\left(x_{j}\right)-g_{j}(a)\right)^{1-\alpha_{j}}} .
$$

Proof. Application of formula (3) with $\mathfrak{f}$ and f replaced by ${ }^{\psi} \mathcal{I}_{a, \mathrm{~g}}^{x}[\mathfrak{f}](q, x, \vec{\alpha})$ and ${ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{x}[\mathbf{f}](q, x, \vec{\beta})$ enables us to write

$$
\begin{align*}
& \int_{\partial J_{a}^{b}}\left(K_{\psi}(y-x) \sigma_{y}^{\psi \psi} \mathcal{I}_{a, \mathbf{g}}^{y}[\mathfrak{f}](q, y, \vec{\alpha})+{ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{y}[\mathfrak{f}](q, y, \vec{\beta}) \sigma_{y}^{\psi} K_{\psi}(y-x)\right) \\
& -\int_{J_{a}^{b}}\left(K_{\psi}(y-x)^{\psi} \mathcal{D}_{y}{ }^{\psi} \mathcal{I}_{a, \mathbf{g}}^{y}[\mathfrak{f}](q, y, \vec{\alpha})+{ }^{\psi} \mathcal{D}_{r, y}{ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{y}[\mathfrak{f}](q, y, \vec{\beta}) K_{\psi}(y-x)\right) d y \\
= & \begin{cases}\psi \mathcal{I}_{a, \mathbf{g}}^{x}[f](q, x, \vec{\alpha})+{ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{x}[\mathfrak{f}](q, x, \vec{\beta}), & x \in J_{a}^{b}, \\
0, & x \in \mathbb{H} \backslash \overline{J_{a}^{b}} .\end{cases} \tag{6}
\end{align*}
$$

From 1. in Proposition 5 we get

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}}\left(K_{\psi}(y-x) \sigma_{y}^{\psi \psi} \mathcal{I}_{a, \mathbf{g}}^{y}[\mathfrak{f}](q, y, \vec{\alpha})+{ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{y}[\mathfrak{f}](q, y, \vec{\beta}) \sigma_{y}^{\psi} K_{\psi}(y-x)\right) \\
& -\int_{J_{a}^{b}}\left(K_{\psi}(y-x)^{\psi} \mathcal{C}_{a, \mathbf{g}}^{\vec{\alpha}}[f](q, y)+{ }^{\psi} \mathcal{C}_{r, a, \mathbf{h}}^{\vec{\beta}}[\mathfrak{f}](q, y) K_{\psi}(y-x)\right) d y \\
& -\int_{J_{a}^{b}}\left(K_{\psi}(y-x)^{\psi} \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}[f](q, y)+{ }^{\psi} \mathfrak{D}_{r, a, \mathbf{h}}^{\vec{\beta}}[\mathfrak{f}](q, y) K_{\psi}(y-x)\right) d y \\
= & \begin{cases}\psi_{a} \mathcal{I}_{a, \mathbf{g}}^{x}[f](q, x, \vec{\alpha})+{ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{x}[\mathfrak{f}](q, x, \vec{\beta}), & x \in J_{a}^{b}, \\
0, & x \in \mathbb{H} \backslash \overrightarrow{J_{a}^{b}},\end{cases}
\end{aligned}
$$

Suppose $\mathrm{f}=0$, applying the operator ${ }^{\psi} \mathfrak{P}_{a, \mathbf{g}}^{\vec{\alpha}}$ on both sides using Fact 3. of Proposition 5, then Leibniz rule implies that

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}} \psi_{\mathfrak{P}_{a, \mathbf{g}}^{\vec{a}}}^{\vec{\alpha}}\left[K_{\psi}(y-x)\right](q, x) \sigma_{y}^{\psi \psi} \mathcal{I}_{a, \mathbf{g}}^{y}[\mathfrak{f}](q, y, \vec{\alpha})-\int_{J_{a}^{b}} \psi_{\mathfrak{P}_{a, \mathbf{g}}}^{\vec{\alpha}}\left[K_{\psi}(y-x)\right](q, x)^{\psi} \mathfrak{C}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, y) d y \\
& -\int_{J_{a}^{b}} \psi \mathfrak{P}_{a, \mathbf{g}}^{\vec{\alpha}}\left[K_{\psi}(y-x)\right](q, x)^{\psi} \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, y) \\
= & \left\{\begin{array}{lll}
\sum_{j=0}^{3} \mathfrak{f}\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right)+\sum_{\substack{k=0 \\
j \neq k}}^{3} \frac{\left(\mathbf{I}_{a_{k}, g_{k}}^{1-\alpha_{k}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right)}{\Gamma\left(1-\alpha_{j}\right)\left(g_{j}\left(x_{j}\right)-g_{j}(a)\right)^{1-\alpha_{j}},} & x \in J_{a}^{b}, \\
0, & x \in \mathbb{H} \backslash \overline{J_{a}^{b}} .
\end{array}\right.
\end{aligned}
$$

Suppose now $\mathfrak{f}=0$ in (6) and compute similarly as before for $f$. After addition of the two readily inferred relations, the theorem follows.

Remark 10. To provide an explicit representation of $\psi_{\mathfrak{K}_{a, \mathbf{g}}^{\vec{\alpha}}}(y, x, \tau)$ one can use a decomposition of the hyperholomorphic Cauchy kernel in terms of Gegenbauer polynomials given in [31, page 93] such that

$$
\psi \mathfrak{K}_{a, \mathbf{g}}^{\vec{\alpha}}(q, x, y):=\frac{1}{2 \pi^{2}} \sum_{k=0}^{\infty} \frac{1}{|y|^{k+3}} \psi \mathfrak{P}_{a, g}^{\vec{\alpha}}\left[|x|^{k} A_{4, k}(x, y)\right](q, x)
$$

with

$$
2 A_{4, k}(x, y):=\left[(k+1) C_{k+1}^{1}(s)+(2-n) C_{k}^{2}(s) \omega_{y} \wedge \omega_{x}\right] \bar{\omega}_{x}
$$

where $C_{k+1}^{1}$ and $C_{k}^{2}$ are the Gegenbauer polynomials, $x=|x| \omega_{x}, y=|y| \omega_{y}$ and $s=\left(\omega_{x}, \omega_{y}\right)$.

Corollary 11. (Cauchy type formula induced by ${ }^{\psi} \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}$ and $\psi \mathfrak{D}_{r, a, \mathbf{h}}^{\vec{\beta}}$ ) Suppose that $\mathbf{g}=\left(g_{0}, g_{1}, g_{2}, g_{3}\right)$ and $\mathbf{h}=\left(h_{0}, h_{1}, h_{2}, h_{3}\right)$ are two vector-valued functions with monotonously increasing components such that $g_{k}, h_{k} \in C^{1}\left[a_{k}, b_{k}\right]$ and $g_{k}^{\prime} \neq 0 \neq h_{k}^{\prime}$
for $k=0,1,2,3$. If $\mathfrak{f}, \mathrm{f} \in A C^{1}\left(\overline{J_{a}^{b}}, \mathbb{H}\right)$, consider $q \in J_{a}^{b}$ such that the mappings $x \mapsto{ }^{\psi} \mathcal{I}_{a}^{x}[\mathfrak{f}](q, x, \vec{\alpha}), x \mapsto{ }^{\psi} \mathcal{I}_{a}^{x}[\mathfrak{f}](q, x, \vec{\beta})$ belong to $C^{1}\left(\overline{J_{a}^{b}}, \mathbb{H}(\mathbb{C})\right)$ and

$$
\psi \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, y)={ }^{*} \mathfrak{D}_{r, a, \mathbf{h}}^{\vec{\beta}}[\mathrm{f}](q, y)=0 .
$$

Then

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}}\left({ }^{\psi} \mathfrak{K}_{a, \mathbf{g}}^{\vec{\alpha}}(q, x, y) \sigma_{y}^{\psi \psi} \mathcal{I}_{a, \mathbf{g}}^{y}[\mathfrak{f}](q, y, \vec{\alpha})+{ }^{\psi} \mathcal{I}_{a, \mathbf{h}}^{y}[\mathfrak{f}](q, y, \vec{\beta}) \sigma_{y}^{\psi \psi} \mathfrak{K}_{a, \mathbf{h}}^{\vec{\beta}}(q, x, y)\right) \\
- & \int_{J_{a}^{b}}\left(\psi \mathfrak{K}_{a, \mathbf{g}}^{\vec{\alpha}}(q, x, y)^{\psi} \mathfrak{C}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, y)+{ }^{\psi} \mathfrak{C}_{r, a, \mathbf{h}}^{\vec{\beta}}[\mathfrak{f}](q, y)^{\psi} \mathfrak{K}_{a, \mathbf{h}}^{\vec{\beta}}(q, x, y)\right) d y \\
= & \begin{cases}\sum_{j=0}^{3}(\mathfrak{f}+\mathfrak{f})\left(q_{0}, \ldots, x_{k}, \ldots, q_{3}\right)+M_{a, \mathbf{g}}^{\vec{a}}[f](q, x)+M_{a, \mathbf{h}}^{\vec{\beta}}[\mathfrak{f}](q, x), & x \in J_{a}^{b}, \\
0, & x \in \mathbb{H} \backslash \overline{J_{a}^{b}},\end{cases}
\end{aligned}
$$

Remark 12. If $g_{k}$ and $h_{k}$ are the identities functions for $k=0,1,2,3$ then the previous Borel-Pompieu and Cauchy type formulas reduce to that appear in [29]. Simultaneously, if $g_{k}=h_{k}=\ln$ for $k=0,1,2,3$ they induce analogous formulas associated to the Hadamard fractional derivative.

Remark 13. As an particular case, under the hypothesis of Theorem 9 with $g_{j} \in C^{2}\left(a_{j}, b_{j}\right)$ for all $j$, and appealing to Theorem 3 for

$$
\left(g_{j}^{\prime}\left(y_{j}\right)\right)^{-1}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right)
$$

for $j=0,1,2,3$ yields

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}} K_{\psi}(y-x) \sigma_{y}^{\psi}\left(g_{j}^{\prime}\left(y_{j}\right)\right)^{-1}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) \\
& -\int_{J_{a}^{b}} K_{\psi}(y-x)^{\psi} \mathcal{D}_{y}\left[\left(g_{j}^{\prime}\left(y_{j}\right)\right)^{-1}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right)\right] d y \\
= & \begin{cases}\left(g_{j}^{\prime}\left(x_{j}\right)\right)^{-1}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right), & x \in J_{a}^{b}, \\
0, & x \in \mathbb{H} \backslash \overline{J_{a}^{b}} .\end{cases}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}} K_{\psi}(y-x) \sigma_{y}^{\psi}\left(g_{j}^{\prime}\left(y_{j}\right)\right)^{-1}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) \\
& \left.-\int_{J_{a}^{b}} K_{\psi}(y-x) \psi_{j} \frac{1}{g_{j}^{\prime}\left(y_{j}\right)} \frac{\partial}{\partial y_{j}}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right)\right] d y \\
& \left.+\int_{J_{a}^{b}} K_{\psi}(y-x) \psi_{j} \frac{g_{j}^{\prime \prime}\left(y_{j}\right)}{\left(g_{j}^{\prime}\left(y_{j}\right)\right)^{2}}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right)\right] d y \\
= & \begin{cases}\left(g_{j}^{\prime}\right)^{-1}\left(x_{j}\right)\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right), & x \in J_{a}^{b}, \\
0, & x \in \mathbb{H} \backslash \overline{J_{a}^{b}}\end{cases}
\end{aligned}
$$

and adding all terms we get the following:

$$
\begin{aligned}
& \int_{\partial J_{a}^{b}} K_{\psi}(y-x) \sigma_{y}^{\psi} \sum_{j=0}^{3}\left(g_{j}^{\prime}\left(y_{j}\right)\right)^{-1}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) \\
& \\
& -\int_{J_{a}^{b}} K_{\psi}(y-x)^{\psi} \mathfrak{D}_{a, \mathbf{g}}^{\vec{\alpha}}[\mathfrak{f}](q, y) d y \\
& \left.+\int_{J_{a}^{b}} K_{\psi}(y-x) \sum_{j=0}^{3} \psi_{j} \frac{g_{j}^{\prime \prime}\left(y_{j}\right)}{\left(g_{j}^{\prime}\left(y_{j}\right)\right)^{2}}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right)\right] d y \\
& = \begin{cases}\sum_{j=0}^{3}\left(g_{j}^{\prime}\left(x_{j}\right)\right)^{-1}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right), & x \in J_{a}^{b}, \\
0, & x \in \mathbb{H} \backslash \overline{J_{a}^{b} .} .\end{cases}
\end{aligned}
$$

If ${ }^{\psi} \mathfrak{D}_{a, \mathrm{~g}}^{\vec{\alpha}}[f](q, \cdot) \equiv 0$ on $J_{a}^{b}$ we have that

$$
\begin{aligned}
& \quad \int_{\partial J_{a}^{b}} K_{\psi}(y-x) \sigma_{y}^{\psi} \sum_{j=0}^{3}\left(g_{j}^{\prime}\left(y_{j}\right)\right)^{-1}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right) \\
& \\
& \left.=\int_{J_{a}^{b}} K_{\psi}(y-x) \sum_{j=0}^{3} \psi_{j} \frac{g_{j}^{\prime \prime}\left(y_{j}\right)}{\left(g_{j}^{\prime}\left(y_{j}\right)\right)^{2}}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{j}\right)\left(q_{0}, \ldots, y_{j}, \ldots, q_{3}\right)\right] d y \\
& \sum_{j=0}^{3}\left(g_{j}^{\prime}\left(x_{j}\right)\right)^{-1}\left(\mathbf{I}_{a_{j}^{+}, g_{j}}^{1-\alpha_{j}} \mathfrak{f}\right)\left(q_{0}, \ldots, x_{j}, \ldots, q_{3}\right), \\
& 0, \\
& 0 \in J_{a}^{b}, \\
& 0, \\
& x \in \mathbb{H} \backslash \overline{J_{a}^{b} .}
\end{aligned}
$$

## Acknowledgments

The authors wish to thank the Instituto Politécnico Nacional (grant numbers SIP20220017, SIP20221274) for partial support.

## References

[1] Miller, K. S., Ross, B. An Introduction to the Fractional Calculus and Fractional Differential Equations. A Wiley-Interscience Publication. John Wiley \& Sons, Inc., New York (1993).
[2] Ross B. A brief history and exposition of the fundamental theory of fractional calculus. In: Ross B. (eds) Fractional Calculus and Its Applications. Lecture Notes in Mathematics, vol 457. Springer, Berlin, Heidelberg (1975).
[3] Marichev, O. I., Kilbas, A. A., Samko, S. G., Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon (1993).
[4] Oldham, K. B., Spanier, J. The Fractional Calculus. Dover Publ. Inc. (2006).
[5] Ortigueira, M. D. Fractional calculus for scientists and engineers. Lecture Notes in Electrical Engineering, 84. Springer, Dordrecht (2011).
[6] Ortigueira, M. D., Machado, J. A. T. On fractional vectorial calculus. Bull. Pol. Acad. Sci. Tech.Sci. 66 389-402 (2018).
[7] Podlubny, I. Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA (1999).
[8] Gorenflo, R., Mainardi, F. Fractional calculus: integral and differential equations of fractional order. Fractals and fractional calculus in continuum mechanics (Udine, 1996), 223-276, CISM Courses and Lect., 378, Springer, Vienna (1997).
[9] Valério , D.; Ortigueira, M.D.; Lopes, A.M. How Many Fractional Derivatives Are There? Mathematics, 10 (5), 737 (2022). doi.org/10.3390/math10050737.
[10] de Oliveira, E. C., Tenreiro Machado, J. A. A review of definitions for fractional derivatives and integral. Math. Probl. Eng. 2014, Art. ID 238459, 6 pp.
[11] Erdélyi, A., An integral equation involving Legendre functions, SIAM J. Appl. Math., 12, 15-30 (1964).
[12] Erdélyi, A., Axially symmetric potentials and fractional integration, SIAM J. Appl. Math., 13, 216-228 (1965).
[13] Osler, T.J., The fractional derivative of a composite function, SIAM. J. Math. Anal., 1(2), 288-293 (1970).
[14] Osler, T.J., Leibniz rule for fractional derivatives generalized and an application to infinite series, SIAM. J. Appl. Math., 18(3), 658-674 (1970).
[15] Osler, T.J., Taylor's series generalized for fractional derivatives and applications, SIAM. J. Math. Anal., 2(1), 37-48 (1971).
[16] Benjemaa, M. Taylor's formula involving generalized fractional derivatives, Appl. Math. Comput., 335, 182-195 (2018).
[17] Katugampola, U.N., A New approach to generalized fractional derivatives, Bull. Math. Anal. Appl., 6(4), 1-15 (2014).
[18] Sneddon, I.N., The Use of Operators of Fractional Integration in Applied Mathematics, PWN - Polish Sci. Publishers, Warszawa-Poznan (1979).
[19] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam (2006).
[20] Samko, S.G., Kilbas, A.A., Marichev, O.I. Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach Sci. Publ. London, New York, 1993 Nauka, Minsk (1987).
[21] Fueter, R. Reguläre Funktionen einer Quaternionenvariablen. Lecture notes, Spring Semester, Math. Inst. Univ. Zürich (1940).
[22] Fueter, R. Functions of a Hyper Complex Variable. Lecture notes written and supplemented by E. Bareiss, Math. Inst. Univ. Zürich, Fall Semester (1949).
[23] Sudbery, A. Quaternionic analysis, Math. Proc. Phil. Soc. 85, 199-225 (1979).
[24] Nôno, K. On the quaternion linearization of Laplacian 1. Bull. Fukuoka Univ. Ed. III 35, 5-10 (1986).
[25] Mitelman, I., Shapiro, M. Differentiation of the Martinelli-Bochner integrals and the notion of the hyperderivability. Math. Nachr., 172, 211-238 (1995).
[26] Shapiro, M. Quaternionic analysis and some conventional theories. In: Alpay, D. (ed.) Operator Theory, 1423-1446. Springer, Basel (2015).
[27] Shapiro, M, Vasilevski, N. L. Quaternionic $\psi$-monogenic functions, singular operators and boundary value problems. I. $\psi$-Hyperholomorphy function theory. Compl. Var. Theory Appl. 27, 17-46 (1995).
[28] Shapiro, M., Vasilevski, N. L. Quaternionic $\psi$-hyperholomorphic functions, singular operators and boundary value problems II. Algebras of singular integral operators and Riemann type boundary value problems. Compl. Var. Theory Appl. 27, 67-96 (1995).
[29] González Cervantes J. O., Bory Reyes, J. A quaternionic fractional BorelPompieu type formula, Fractals 30 (1), 2250013 (15 pages) (2022).
[30] Abdeljawad, T., Jarad, F., Generalized Fractional derivatives and Laplace Transform. Discrete and Continuous Dynamical Systems Series S, 13 3, 709722 (2020).
[31] Gürlebeck, K., Sprössig, W. Quaternionic and Clifford calculus for physicists and engineers. John Wiley and Sons (1997).


[^0]:    * corresponding author

