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A fractional Malthusian growth model with variable order using an optimization approach

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Abstract The objective is to study the population's growth with a fractional differential equation. The order of the fractional derivative is a function depending on time and the goal is to determine the fractional order function that better fits the given data. The model is than tested to describe the world population growth and of some countries. All the numerical experiments were done in MATLAB, using the routines lsqcurvefit, fminunc and spline.

Keywords Fractional calculus, fractional differential equation, least squares

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1. Introduction

The Malthusian growth model was proposed in 1798 by the English economist Thomas Malthus in his book *An Essay on the Principle of Population*. The theory states that population number has exponential growth based on a constant rate, applied to ideal circumstances or to a short period of time, when an individual lives in a region with no constraints on food and with no natural enemies. In this case, if $N(t)$ represents the size of the population at an instant t, the dynamic differential equation

$$N'(t) = P \cdot N(t)$$

models the growth of the population. The constant P, called the Malthusian parameter, is given by the difference between the fertility and the mortality rates, assuming that these rates are constant in time. If N_0 is the initial level of the population, the function

$$N(t) = N_0 \exp(Pt)$$

gives the exact number of individuals at a given time t. In reality, the growth of a population has some limitations due to environments restrictions, like food availability, competition with other species, competition for territory, etc. To model more realistic population growth, the Belgian mathematician Pierre Verhulst in 1838 proposed another formula, that illustrates how a population may increase exponentially until it reaches the carrying capacity of its environment. This model is known as the Logistic equation and it is given by the differential equation

$$N'(t) = P \cdot N(t) \left(1 - \frac{N(t)}{K}\right),$$

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where P is the Malthusian parameter (the rate of population growth) and K is the carrying capacity, that is, the maximum sustainable population. The solution to this problem is given by the curve

$$N(t) = \frac{K}{1 + (K/N_0 - 1)\exp(-Pt)}.$$

Although there are several models to describe the dynamics of the population growth, the Malthusian model has the advantage that it is given by a linear differential equation. Later, when we model the same problem but using a fractional differential equation, we know the analytic expression of the solution for the problem.

To test the different models that we propose here, we will see how close they are to real data, by fitting the solution with dependence on some parameters with the observations. One of the most used methods is the least squares technique. Suppose that the data consists in m points, say $(t_0, x_0), \ldots, (t_m, x_m)$, and we intend to fit these values in a theoretical model $t \mapsto x(t)$, where the form x is known but it depends on some unknown parameters $\beta_1, \ldots, \beta_k$. If we consider in each step the error $d_i := x_i - x(t_i)$, for $i = 0, \ldots, m$, then the total error is given by

$$E := \sum_{i=0}^{m} (d_i)^2.$$

The goal is to find the values of the parameters $\beta_1, \ldots, \beta_k$ for which E attains a minimum value.

For the readers convenience, we present here a short introduction on the fractional calculus theory. Following [10, 16], fractional derivatives are a generalization of ordinary derivatives, by considering an arbitrary real positive order. This subject has gain great importance during the past decades, since we can describe the dynamics of certain phenomena better when considering the problem modeled by a fractional differential equation. For example, we find applications in physics [7, 8], chemistry [11, 28], computer science [9, 15], engineering [18, 23, 24, 26], viscoelastic [2, 13], systems theory [3, 12] etc.

Two of the most important concepts in this theory are due to Riemann, and are the following. If x is a real valued integrable function in $[a, b]$ and α a positive real, the Riemann-Liouville fractional integral of x of order α is given by the integral

$$I^{\alpha}_{a+} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} x(\tau)\, d\tau.$$

When α is an integer, this expression is an n-tuple integral of form

$$\int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \ldots \int_a^{\tau_{n-1}} x(\tau_n)\, d\tau_n.$$

For the fractional derivative, let $n \in \mathbb{N}$ be such that $\alpha \in (n - 1, n)$. The Riemann–Liouville fractional derivative of x of order α is defined as

$$D^{\alpha}_{a+} x(t) := \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t - \tau)^{n - \alpha - 1} x(\tau)\, d\tau.$$

We remark that, in opposite to ordinary derivatives, these fractional operators are nonlocal and contain memory and thus may be suitable when the current moment is most influenced by the past process. More recently, a similar concept was given, due to Michele Caputo [4]. If $x \in C^n[a, b]$ and given $\alpha \in (n - 1, n)$, the Caputo fractional derivative of x of order α is defined as

$${}^C D^{\alpha}_{a+} x(t) := \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n - \alpha - 1} \frac{d^n x}{d\tau^n}(\tau) d\tau.$$

Two important results are the relations between the fractional integral and the Caputo fractional derivative, to know

$$I^{\alpha}_{a+}\, {}^C D^{\alpha}_{a+} x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!} (t - a)^k,$$

and

$$^C D^{\alpha}_{a+} I^{\alpha}_{a+} x(t) = x(t).$$

A natural generalization of the previous concepts is to consider the order of the fractional derivative α to be a function $\alpha(\cdot)$, and so it may vary trough time. As it is known, fractional derivatives are nonlocal operators and they contain memory, and so it is reasonable to consider that the order of the derivative is not constant, and depends on time also. This subject is very recent, and was suggested by the first time by Samko and Ross [17]. If $\alpha : [a, b] \to (n-1, n)$ is the fractional variable order, fractional integrals and fractional derivatives are defined as

$$I^{\alpha(t)}_{a+} x(t) := \frac{1}{\Gamma(\alpha(t))} \int_a^t (t-\tau)^{\alpha(t)-1} x(\tau)\, d\tau,$$

and

$$D^{\alpha(t)}_{a+} x(t) := \frac{1}{\Gamma(n-\alpha(t))} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha(t)-1} x(\tau)\, d\tau,$$

respectively. In a similar way, a Caputo fractional derivative of variable order can be defined as (see e.g. [20])

$$^C D^{\alpha(t)}_{a+} x(t) := \frac{1}{\Gamma(n-\alpha(t))} \int_a^t (t-\tau)^{n-\alpha(t)-1} \frac{d^n x}{d\tau^n}(\tau) d\tau.$$

In particular, when $\alpha(\cdot)$ takes values in the open interval $(0, 1)$, this expression is written as

$$^C D^{\alpha(t)}_{a+} x(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_a^t (t-\tau)^{\alpha(t)} x'(\tau) d\tau.$$

Theorem 1
If $x(t) = (t-a)^{\gamma}$ with $\gamma > n-1$, we have that when $\alpha([a, b]) \subseteq (n-1, n)$,

$$^C D^{\alpha(t)}_{a+} x(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+1)} (t-a)^{\gamma-\alpha(t)}.$$

Proof
First, observe that

$$x^{(n)}(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-n)} (t-a)^{\gamma-n}.$$

Using the definition, we obtain

$$\begin{aligned} ^C D^{\alpha(t)}_{a+} x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-n)\Gamma(n-\alpha(t))} \int_a^t (t-\tau)^{n-\alpha(t)-1} (\tau-a)^{\gamma-n} d\tau \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-n)\Gamma(n-\alpha(t))} (t-a)^{n-\alpha(t)-1} \int_a^t \left(1 - \frac{\tau-a}{t-a}\right)^{n-\alpha(t)-1} (\tau-a)^{\gamma-n} d\tau. \end{aligned}$$

With the change of variables $s = (\tau-a)/(t-a)$, and recalling the Beta function

$$B(x, y) := \int_0^1 s^{x-1} (1-s)^{y-1}\, ds, \quad x, y > 0,$$

we deduce the following:

$$\begin{aligned} ^C D^{\alpha(t)}_{a+} x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-n)\Gamma(n-\alpha(t))} (t-a)^{\gamma-\alpha(t)} B(n-\alpha(t), \gamma-n+1) \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-n)\Gamma(n-\alpha(t))} (t-a)^{\gamma-\alpha(t)} \frac{\Gamma(n-\alpha(t))\Gamma(\gamma-n+1)}{\Gamma(\gamma-\alpha(t)+1)} \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+1)} (t-a)^{\gamma-\alpha(t)}. \end{aligned}$$

This completes the proof. □

We remark that, when $\gamma = k \in \mathbb{N}$ with $k \leq n-1$, from the definition of fractional derivative, it results that

$$^{C}D_{a+}^{\alpha(t)}(t-a)^k = 0.$$

In opposite to constant fractional order, this variable fractional order operator has no inverse operation like the Riemann–Liouville integral with respect to the Caputo derivative. For this reason, dealing and solving fractional differential equations for variable order is an extremely difficult problem, and often numerical methods are used [5, 25, 27].

The outline of this paper is the following. In Section 2 we present the Malthusian growth model. We start with the classical model, whose solution is given by an exponential function, and then we replace the ordinary derivative by a fractional derivative, and by doing so the solution is now given by the Mittag–Leffler function. Next, we substitute the fractional constant order α by a function $\alpha(t)$ and with it we model the problem by a variable fractional order dynamics. The goal is to determine the values of the parameters in each model that better fits with given data, and compare the error for each case. In the following Section 3 we proceed with a similar study, but considering this time four countries (China, India, USA and Indonesia) and we compare the classical model with the variable order fractional model. In the numerical experiments, some routines from `Matlab` are used.

2. World Population Growth

In [1], a fractional approach was considered to model the World Population Growth. Starting with the classical model

$$N'(t) = P \cdot N(t), \tag{1}$$

the ordinary derivative was replaced by the Caputo fractional derivative, and the dynamic was described by the fractional differential equation

$$^{C}D_{0+}^{\alpha}N(t) = P \cdot N(t), \quad t \geq 0,\, \alpha \in (0,1). \tag{2}$$

The solution to the fractional problem is given by the function

$$N(t) = N_0 E_\alpha(Pt^\alpha), \tag{3}$$

where $E_\alpha(\cdot)$ denotes the Mittag–Leffler function:

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \qquad z \in \mathbb{R}.$$

We remark that, when $\alpha = 1$, the Mittag–Leffler function is simply the exponential function, and so the fractional solution coincides with the classical one

$$N(t) = N_0 \exp(Pt). \tag{4}$$

Then, using the data available from the United Nations [21] from year 1910 until 2010, the best values for the parameters were found. To get a better accuracy for the model, the fractional order α was considered free, without any constraints. For the classical model (1), the values obtained were

$$P \approx 1.3501 \times 10^{-2} \quad \text{with error} \quad E_{classical} \approx 7.0795 \times 10^5.$$

When we considered the problem modeled by the fractional differential equation (2), the values were

$$\alpha \approx 1.3933, \quad P \approx 3.4399 \times 10^{-3} \quad \text{with error} \quad E_{fractional} \approx 2.0506 \times 10^5.$$

So, from these results, we see that the fractional approach is more efficient in modelling the problem than the ordinary one. These numerical tests were done in `Matlab` [14] using the routine `lsqcurvefit` that solves nonlinear data-fitting problems in least-squares sense. This routine is based on an iterative method with local convergence, *i.e.*, depending on the initial approximation to the parameters to estimate. In the computational tests the trust-region-reflective algorithm was selected. Theoretical details can be found in [19, 22].

The next step is to consider even a more general approach to this problem, by considering the fractional order to be a function depending on time $t \mapsto \alpha(t)$. Motivated by Eq. (3), and considering the Mittag–Leffler function with variable order

$$E_{\alpha(t)}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha(t)+1)}, \quad z \in \mathbb{R},$$

we propose the following theoretical model to study the world population problem:

$$N(t) = N_0 E_{\alpha(t)}(Pt^{\alpha(t)}). \tag{5}$$

Observe that, when $\alpha(\cdot)$ is constant, $\alpha(t) \equiv \alpha$, then expression (5) reduces to expression (3), which in turn when $\alpha \to 1$, we obtain the classical model (4). We test model (5) by the closeness to the observed data, from which we infer the values of the parameters. To start, we compare the fractional model with constant order with three new ones, with variable fractional orders. For example, we consider fractional derivatives with orders

$$\alpha_1(t) := at^2 + bt + c, \quad \alpha_2(t) := \cos(at+b), \quad \text{and} \quad \alpha_3(t) := \exp(at+b),$$

for some parameters $a, b, c \in \mathbb{R}$. For these cases, we determine the values of the parameters that better fit with the given data using the same routine `lsqcurvefit`. In Table 1 we summarize the obtained results, and the respective errors.

$\alpha(t)$	a	b	c	P	$E_{fractional}$
$\alpha_1(t)$	-4.4865×10^{-5}	7.5331×10^{-3}	0.8560	7.5849×10^{-3}	1.48133×10^4
$\alpha_2(t)$	-9.8731×10^{-3}	0.8660	–	1.3880×10^{-2}	1.7250×10^4
$\alpha_3(t)$	-1.2401×10^{-3}	0.9117	–	2.1398×10^{-4}	6.3825×10^4

Table 1. World population from 1910 until 2010.

From these values, we observe that we already obtain better results, even when compared to the fractional model with constant order. In these cases, we considered some particular fractional orders, but other orders could be considered in order to improve the method. We remark that we do not impose any restriction on $\alpha(t_i)$ in order to obtain a better accuracy for the procedure. Anyway, when $\alpha(t) \equiv \alpha$, that is, when the fractional order is constant, and as $\alpha \to 1^{\pm}$, the solution of the fractional differential equation converges to the solution of the differential equation obtained when $\alpha = 1$ (cf [1, Theorem 1]).

One more interesting question is to determine the function $t \mapsto \alpha(t)$ that better fits with the data. Since we are working with discrete data, we are interested in finding the optimal values for $\alpha(t_i) = \alpha_i$, in each instant of time t_i and also the P value. A nonlinear unconstrained optimization problem is formulated:

$$\min_{\alpha_i,\, P} \quad \sum_{i=0}^{m} (x_i - N_0 E_{\alpha_i}(Pt_i^{\alpha_i}))^2 \tag{6}$$

whose variables are $\alpha_i,\ i = 0 \ldots, m$, and P.

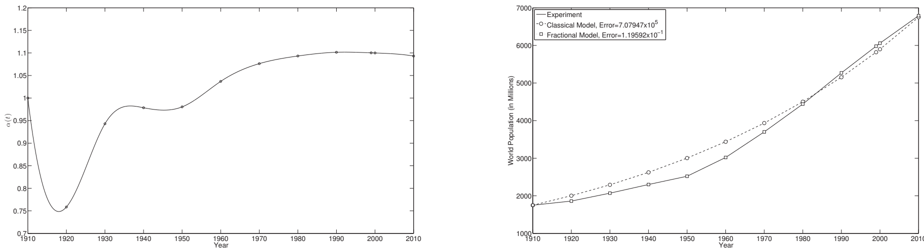
In our case study, the data (t_i, x_i) consists in 12 values, and we need to find the best values for $\alpha(t_i),\ i = 0, \ldots, 11$ and P. The results are then

$$\begin{array}{llll} \alpha(t_0) \approx 1.0000, & \alpha(t_1) \approx 0.7585, & \alpha(t_2) \approx 0.9430, & \alpha(t_3) \approx 0.9785, \\ \alpha(t_4) \approx 0.9805, & \alpha(t_5) \approx 1.0370, & \alpha(t_6) \approx 1.0763, & \alpha(t_7) \approx 1.0932, \\ \alpha(t_8) \approx 1.1015, & \alpha(t_9) \approx 1.1002, & \alpha(t_{10}) \approx 1.0999, & \alpha(t_{11}) \approx 1.0931, \end{array}$$

with $P \approx 9.6850 \times 10^{-3}$. The error that comes from this procedure is

$$E_{fractional} \approx 0.1196,$$

and in Figure 1 we have the plot of the results. On left side, we exhibit the graph of the obtained fractional order curve using the routine `spline`. On the right side we present a comparison between the data, the classical model and the variable fractional model. In these numerical experiments the `fminunc` routine from the `Matlab` Optimization toolbox is used. This routine attempts to find a minimum of a scalar function of several variables, starting at an initial estimate. The interior-point optimization algorithm was selected.



(a) Optimal fractional order

(b) Experimental data vs classical model vs fractional model

Figure 1. World Population Growth.

3. Countries Population Growth

In this section we study the population growth model for several countries. We choose the World's four most populous countries in 2015: People's Republic of China, India, USA and Indonesia. For data source, we use the ones available from [6], and we study from the year 1910 until 2010, measured every 10 years, consisting in 11 values. In each case, we determine the best variable fractional order for each country, taking into account the 11 values. In Table 2 we present the growth rates P, as well the error, for each country.

Country	P – Classical Model	Error	P – Fractional Model	Error
China	1.1823×10^{-2}	5.0155×10^{4}	9.3059×10^{-3}	8.1067×10^{-4}
India	1.5321×10^{-2}	4.2133×10^{4}	9.1063×10^{-3}	3.8354×10^{-3}
USA	1.2363×10^{-2}	5.9101×10^{2}	1.3453×10^{-2}	8.8584×10^{-6}
Indonesia	1.7417×10^{-2}	8.2862×10^{2}	1.3200×10^{-2}	6.1555×10^{-5}

Table 2. Population by countries: classical vs fractional models.

In Figure 2 we present the plots of the fractional optimal order for each country, using again the routine `spline`, and in Figure 3 we compare for each country the data, with the classical and fractional models.

4. Conclusion

In this paper we studied the population growth problem, by modeling the dynamics by a fractional differential equation. Improving the results obtained in [1], where the fractional order is constant, here we generalize the previous work by considering the order a function depending on time. As we saw, the variable order model is much more efficient in modelling the World Population Growth, compared to the constant order one. This may suggest that, instead of considering constant order fractional differential equations, in some situations, we can improve the method by considering the order as a function depending on time. This is an important issue, with very few works done in this direction, and we hope to attract the attention of the community to this kind of problems. One

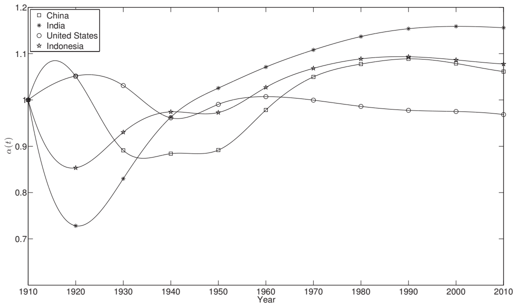
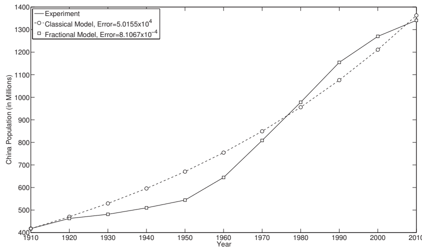
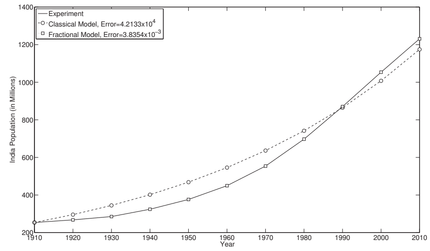


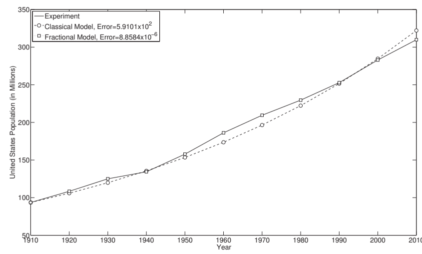
Figure 2. Optimal fractional order.



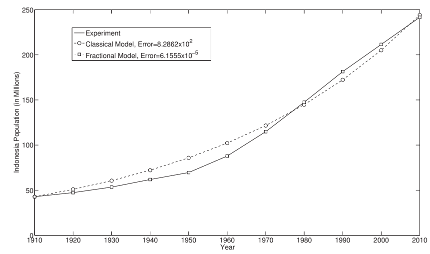
(a) China.



(b) India



(c) USA.



(d) Indonesia

Figure 3. Countries Population Growth.

interesting question, that we hope to answer in the future, is what is the relation between the dynamics of the population growth with the fractional order? Is there a connection between the monotonicity of $\alpha(t)$ and the rate of growth of the population? This will be our next problem to be treated.

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