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A Fractional Order Investigation of Smoking Model Using Caputo-Fabrizio Differential Operator

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Abstract: Smoking is a social trend that is prevalent around the world, particularly in places of learning and at some significant events. The World Health Organization defines smoking as the most important preventable cause of disease and the third major cause of death in humans. In order to analyze this matter, this study typically emphasizes analyzing the dynamics of the fractional order quitting smoking model via the Caputo-Fabrizio differential operator. For the numerical solution of the considered model, the Laplace transform with the Adomian decomposition method (LADM) and Homotopy perturbation method (HPM) is applied, and the comparison of both the achieved numerical solutions is presented. Moreover, numerical simulation for the suggested scheme has been presented in various fractional orders with the aid of Matlab and the numerical results are supported by illustrative graphics. The simulation reveals the aptness of the considered model.

Keywords: smoking model; Caputo-Fabrizio fractional derivative; Laplace transform; Adomian decomposition method; Homotopy perturbation method (HPM); numerical simulations

MSC: 26A33; 34K37

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Citation: Anjam, Y.N.; Shafqat, R.; Sarris, I.E.; ur Rahman, M.; Touseef, S.; Arshad, M. A Fractional Order Investigation of Smoking Model Using Caputo-Fabrizio Differential Operator. *Fractal Fract.* **2022**, *6*, 623. https://doi.org/10.3390/fractalfract6110623

Academic Editors: Carlo Cattani and Sourav Sen

Received: 7 September 2022
Accepted: 16 October 2022
Published: 26 October 2022

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1. Introduction

Mathematical biology is a wide-ranging field with several applications. In this field, researchers are focusing on the portrayal of different types of diseases with controls in the form of mathematical models. In 1909, Brownlee [1] took the initiative for the growth of the field of mathematical biology and the emphasis was on the theory of fortuitous. Furthermore, in 1912, he presented the basic laws for epidemic spreading [2]. In 1927, Kermark and McKendrik [3] discussed the minutiae of the epidemic study. Later, many researchers discoursed different models of various other diseases; see [4–7]. On the other hand, one of the social habits that are spreading all over the world speedily as an infectious disease is smoking. Smoking is the process by which people inhale the smoke of tobacco containing particles and gas or simply [8,9]. Smoking is the practice in which smoke is taken into the mouth and then released utilizing pipes or cigars. Columbus was the first one who introduced smoking in Europe in the sixteenth-century [10], but before and after this date, many other exotic species were offered, with great contrary effects on ecosystems and effect on human habits [11,12]. Nicot spread tobacco as a cash crop in England; he was the first who used it as a business, and that is why the word nicotine derived from his name. Smoking can root different kinds of diseases together with mouth cancer, lung cancer, larynan, heart diseases, vascular diseases, respiratory diseases, and many other diseases that are injurious to human health [13–15].

Nowadays, smoking is one of the foremost health problems in the world. According to the world health organization (WHO) report on the smoking, epidemic [16], smoking kills numerous people in their most active life. More than 5 million deaths in the world are caused due to the effect of smoking on different organs of the human body, which may increase to up to 8 million people per year by 2030 [17,18]. The chance of heart attack is 70% more in smokers equated to the persons who are not smoking. Smokers have a 10% higher incidence rate of lung cancer than that nonsmokers. The life of smokers is 10 to 13 years shorter than that of non-smokers. Mathematician tries to control smoking for securing the life expectancy of an individual. To give the best illustration of cigarette smoking phenomena, mathematicians tried to make different effective smoking models. The different smoking models were proposed by several authors. For the first time in the year of 1997, Castillo-Garsow [19] outlined a mathematical model for smoking and divide the total population into three different classes (potential smokers, chain smokers and permanently quit smokers). In 2007, Ham [20] documented the different stages and procedures of smoking among students through a survey in different vocational-technical schools in Korea. In 2008, Sharomi and Gumel [14] improved their model to present a new temporarily quit smokers class. Zaman [21] extended the model by presenting a new occasional smokers class and offered a dynamical interaction in an integer order. Several others presented the smoking models in integer and fractional order. Erturk [22] studied a giving up smoking model associated with the Caputo fractional derivative. Zeb [23] examined a fractional giving up smoking. Alkhudhari [24] analyzed the global dynamics of mathematical equations describing smoking. Khalid [25] described the fractional mathematical model of giving up smoking and many others.

We can utilize mathematical modeling to prevent the spread of tobacco smoking because it has been used as a significant tool for pandemic grasp in recent decades. The susceptible-exposed-infected-recovered (SEIR) is a general model. Therefore, the model under consideration for this study is described as follows:

$$\begin{cases} \dfrac{dX(t)}{d(t)} = \lambda - \beta_1 X(t)H(t) + \alpha\, Y(t) - \mu\, X(t), \\ \dfrac{dH(t)}{dt} = \beta_1 X(t)H(t) - \beta_2 H(t)J(t) - (\lambda_1 + \mu)H(t), \\ \dfrac{dJ(t)}{dt} = \beta_2 H(t)J(t) - (\omega + \lambda_2 + \mu)J(t)), \\ \dfrac{dY(t)}{dt} = \omega\, J(t) - (\gamma + \mu + \alpha)Y(t), \\ \dfrac{dZ(t)}{dt} = \gamma\, Y(t) - \mu\, Z(t). \end{cases} \tag{1}$$

We divided the total population into five classes, where $X(t)$ is susceptible smokers, $H(t)$ is the snuffing (ingestion) class, $J(t)$ is irregular smokers, $Y(t)$ is regular smokers, and $Z(t)$ is quit smokers, respectively, at a time t. The whole description of the parameters used in the model (1) is given in the Table 1 mbelow:

Table 1. Description of the model parameters.

Notation	Description of the Parameters
λ	The frequency of recruitment (birth or migration)
β_1	The rate of the vulnerable population transitions to the snuffing class
β_2	The Rate of snuffing becomes an irregular smokers
ω	Rate of irregular smokers turning to a regular smoker
γ	Departing rate
μ	Rate of natural death
α	Rate of recovery
λ_1	Rate of snuffing class deaths because of smoking
λ_2	Rate of death due to smoking

In the last few decades, several biological models were studied in detail with the thought of classical derivatives, including [26–33]. Since in recent years fractional calculus has fascinated great attention from researchers and various features of the said subject are under consideration for research. This is because of the reality that fractional derivative is an important tool to explain the dynamical behavior of various physical systems [34,35]. The specialty of these differential operators is their non-local physiognomies which do not exist in the integer order differential operators [36,37]. As a fact, fractional order models are more accurate and practical than the classical integer order models [38–40]. Fractional order derivative produces a better degree of freedom in these models. Arbitrary order derivatives are powerful tools for the discretion of the dynamical behavior of various bio-materials and systems [41–43]. This is why there are different fractional derivatives; Riemann and Liouville, and Caputo operators are the most conservative examples in traditional practice. Caputo [44] presented a fractional derivative that permits the conventional initial and boundary conditions allied with the real-world problem. Although the aforesaid derivatives show more accuracy in describing real phenomena compared to the integer-order derivatives, their kernel functions yield singularities which result in a multitude of computational deficiencies. Moreover, to examine fractional mathematical models outside of the traditional Caputo derivative notion, several methodologies, such as iterative and numerical methods, have been applied [45].

In a recent effort, a new derivative that is non-singular and comprises an exponential law kernel has latterly been brought into operation by Caputo and Fabrizio [46], and aptly titled after them as Caputo-Fabrizio (CF) derivative. Losada and Nieto [47] presented the fractional integral corresponding to the newly offered fractional derivative and analyzed some related fractional differential equations. The classical fractional derivatives, specifically the Caputo and Riemann derivatives, have their constraint as their kernel is singular which causes complications in fractional order derivatives. Since the kernel is employed to define the memory effect of the physical system, it is obvious that as a result of this weakness, both derivatives cannot exactly describe the full effect of the memory. Generally, it is not simple for non-linear fractional differential equations to obtain their general solutions. Therefore, the new type of non-singular derivatives of fractional order has been found more suitable for studying thermal problems instead of ordinary derivatives. Moreover, it is evident from the research which has been carried out pursuant to these advancements in recent years that these fractional derivatives provide scientists a good chance to describe diverse problems. To overwhelm this condition, various methods are outlined in the existing literature to determine the approximate solutions for the nonlinear systems [48]. Runge Kutta methods were widely employed in handling mathematical analysis [49]. The Homotopy perturbation method (HPM), Laplace Adomian decomposition method (LADM), and method of difference have been widely used to deal with both linear and non-linear fractional order differential equations [50–54]. Therefore, inspired by the applicability of these non-singular derivatives, we aim to further analyze the smoking model in view of the fractional concept and to explicate this problem in a better and more effective manner.

In the present study, to obtain a better understanding of the qualitative analysis as well as the numerical iterative analysis of the proposed model, the revised form of the model (1) in the sense of Caputo-Fabrizio fractional derivative (CFFD) is considered which is described as follows:

$$\begin{cases} {}^{CF}D_t^p(X)(t) = \lambda - \beta_1 X(t)H(t) + \alpha\, Y(t) - \mu\, X(t), \\ {}^{CF}D_t^p(H)(t) = \beta_1 X(t)H(t) - \beta_2 H(t)J(t) - (\lambda_1 + \mu)H(t), \\ {}^{CF}D_t^p J(t) = \beta_2 H(t)J(t) - (\omega + \lambda_2 + \mu)J(t), \\ {}^{CF}D_t^p(Y)(t) = \omega\, J(t) - (\gamma + \mu + \alpha)Y(t), \\ {}^{CF}D_t^p(Z)(t) = \gamma\, Y(t) - \mu\, Z(t), \end{cases} \tag{2}$$

subject to the initial conditions:

$$X(0) = m_1, H(0) = m_2, J(0) = m_3, Y(0) = m_4, Z(0) = m_5.$$

The goal of this work is to investigate the dynamical behavior of the smoking mathematical model with the help of the Caputo-Fabrizio differential operator and makes use of the utilities of fractional calculus. The Laplace transform with the Adomian decomposition method and the Homotopy perturbation method are employed to achieve the approximate solution of the considered model. As well as the comparison of the obtained results of both techniques is presented which shows the identical character. Moreover, the numerical solutions are facilitated with the aid of MATLAB and the graphical representation of each compartment is shown in arbitrary fractional order.

The distribution of this paper is ordered as follows: We start with some fundamental results and definitions in Section 2. In Section 3, the analytical solution of the suggested model utilizing the Laplace Adomian decomposition approach and the Homotopy perturbation method is estimated. The numerical simulations to validate and backing the analytical findings of the aforementioned sections are performed. In Section 4, the comparison of both numerical solutions achieved from the aforementioned schemes is shown graphically and the obtained numerical simulation results are briefly discussed. Finally, the concluding remarks of this study are given in Section 5.

2. Analytical Preliminaries

This section is devoted to certain essential terminologies and fractional calculus results in line with [55,56].

Definition 1. *Let $f \in H^1[0,\alpha]$, $\alpha > 0$, $p \in (0,1)$, then the Caputo-Fabrizio fractional derivative of the function $f(t)$ is described as follows:*

$$^{CF}D_t^p[f(t)] = \frac{M(p)}{1-p}\int_0^t f'(x)exp[-p\frac{t-x}{1-p}]\,dx,$$

where $M(p)$ represents the normalization function and $M(0) = M(1) = 1$.

Definition 2. *The Caputo-Fabrizio fractional integral with $p \in (0,1)$ for a function f is describes as follows:*

$$^{CF}I_t^p[f(t)] = \frac{1-p}{N(p)} + \frac{p}{N(p)}\int_0^t f(x)\,dx, t \geq 0.$$

Definition 3. *The general formula for the Laplace Transform of Caputo-Fabrizio fractional derivative is described as follows:*

$$\mathcal{L}[^{CF}D_t^p[f(t)] = \frac{s\mathcal{L}[f(t)] - f(0)}{s + p(1-s)}, s \geq 0.$$

Definition 4. *The homotopy perturbation method (HPM) is a semi-analytical technique for the solution of linear and nonlinear ordinary as well as partial differential equations and systems. A system of paired linear and nonlinear differential equations may also be estimated via this approach. One clear advantage of using the HPM compared to decomposition would be that it can handle nonlinear problems without the requirement for Adomian's polynomial. Moreover, "He" was the Chinese mathematician who first proposed the idea of HPM. The authors in [57] applied the HPM to nonlinear oscillator problem. It could be used in an equation having linear and nonlinear parts, and one can form a homotopy $\nu(s,p) : D \times [0,1] \to R$:*

$$F(\nu, p) = (1-p)[L(\nu) - L(u_0)] + p[L(\nu) + N(\nu) - f(s)] = 0, \tag{3}$$

where L is used for the linear part, N for the nonlinear part, $s \in D$, D is a topological space and $p \in [0,1]$ is the embedding parameter. Further, u_0 is an initial approximation that satisfies the boundary conditions.

Definition 5 ([58]). *The He's fractional derivative is defined as:*

$$D_t^p = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-p-1}[f_0(s) - f(s)]ds. \tag{4}$$

3. General Solution for the Model (2)

In this section, we discuss the general techniques to construct the solution for the considered model (2) along with the initial conditions.

3.1. General Solution for Model (2) with (LADM)

Now, taking the Laplace transform of the considered (2), and the normalization value $M(p) = 1$, one may get

$$\begin{cases} \mathcal{L}[{}^{CF}D_t^p(X)(t)] = \mathcal{L}[\lambda - \beta_1 X(t)H(t) + \alpha\, Y(t) - \mu\, X(t)], \\ \mathcal{L}[{}^{CF}D_t^p(H)(t)] = \mathcal{L}[\beta_1 X(t)H(t) - \beta_2 H(t)J(t) - (\lambda_1 + \mu)H(t)], \\ \mathcal{L}[{}^{CF}D_t^p(J)(t)] = \mathcal{L}[\beta_2 H(t)J(t) - (\omega + \lambda_2 + \mu)J(t)], \\ \mathcal{L}[{}^{CF}D_t^p(Y)(t)] = \mathcal{L}[\omega\, J(t) - (\gamma + \mu + \alpha)Y(t)], \\ \mathcal{L}[{}^{CF}D_t^p(Z)(t)] = \mathcal{L}[\gamma\, Y(t) - \mu\, Z(t)]. \end{cases} \tag{5}$$

Using the initial conditions yields

$$\begin{cases} \mathcal{L}[(X)(t)] = \frac{X(0)}{s} + \frac{s + p(1-s)}{s}\mathcal{L}[\lambda - \beta_1 X(t)H(t) + \alpha\, Y(t) - \mu\, X(t)], \\ \mathcal{L}[(H)(t)] = \frac{H(0)}{s} + \frac{s + p(1-s)}{s}\mathcal{L}[\beta_1 X(t)H(t) - \beta_2 H(t)J(t) - (\lambda_1 + \mu)H(t)], \\ \mathcal{L}[(J)(t)] = \frac{J(0)}{s} + \frac{s + p(1-s)}{s}\mathcal{L}[\beta_2 H(t)J(t) - (\omega + \lambda_2 + \mu)J(t)], \\ \mathcal{L}[(Y)(t)] = \frac{Y(0)}{s} - \frac{s + p(1-s)}{s}\mathcal{L}[\omega\, J(t) - (\gamma + \mu + \alpha)Y(t)], \\ \mathcal{L}[(Z)(t)] = \frac{Z(0)}{s} - \frac{s + p(1-s)}{s}\mathcal{L}[\gamma\, Y(t) - \mu\, Z(t)]. \end{cases} \tag{6}$$

Assuming the solution for the compartments $X(t), H(t), J(t), Y(t)$, and $Z(t)$ in an infinite series is given below:

$$\begin{cases} X(t) = \sum_{k=0}^{\infty} X_k(t), \;\; H(t) = \sum_{k=0}^{\infty} H_k(t), \\ J_k(t) = \sum_{k=0}^{\infty} J_k(t), Y(t) = \sum_{k=0}^{\infty} Y_k(t), \\ Z(t) = \sum_{k=0}^{\infty} Z_k(t), \end{cases} \tag{7}$$

where $X(t)H(t) = \sum_{k=0}^{\infty} S_k(t)$, $H(t)J(t) = \sum_{k=0}^{\infty} W(t)$ are the non-linear terms of Adomian polynomial wherein:

$$\begin{cases} S_k(t) = \frac{1}{m!}\frac{d^m}{d\lambda^m}[\sum_{k=0}^{m} \lambda^k X_k(t) \sum_{k=0}^{m} \lambda^k H_k(t)]|_{\lambda=0}, \\ W_k(t) = \frac{1}{m!}\frac{d^m}{d\lambda^m}[\sum_{k=0}^{m} \lambda^k H_k(t) \sum_{k=0}^{m} \lambda^k J_k(t)]|_{\lambda=0}. \end{cases} \tag{8}$$

Using Equations (7) and (8) into (6), one can get

$$\begin{cases}\mathcal{L}[\sum_{k=0}^{\infty} X_k(t)] = \dfrac{X(0)}{s} + \dfrac{s+p(1-s)}{s}\mathcal{L}[\lambda - \beta_1 \sum_{k=0}^{\infty} S_k(t) + \alpha \sum_{k=0}^{\infty} Y_k(t) - \mu \sum_{k=0}^{\infty} X_k(t)],\\ \mathcal{L}[\sum_{k=0}^{\infty} H_k(t)] = \dfrac{H(0)}{s} + \dfrac{s+p(1-s)}{s}\mathcal{L}[\beta_1 \sum_{k=0}^{\infty} S_k(t) - \beta_2 \sum_{k=0}^{\infty} W(t) - (\lambda_1+\mu) \sum_{k=0}^{\infty} H_k(t)],\\ \mathcal{L}[\sum_{k=0}^{\infty} J_k(t)] = \dfrac{J(0)}{s} + \dfrac{s+p(1-s)}{s}\mathcal{L}[\beta_2 \sum_{k=0}^{\infty} W(t) - (\omega+\lambda_2+\mu) \sum_{k=0}^{\infty} J_k(t)],\\ \mathcal{L}[\sum_{k=0}^{\infty} Y_k(t)] = \dfrac{Y(0)}{s} + \dfrac{s+p(1-s)}{s}\mathcal{L}[\omega \sum_{k=0}^{\infty} J_k(t) - (\gamma+\mu+\alpha) \sum_{k=0}^{\infty} Y_k(t)],\\ \mathcal{L}[\sum_{k=0}^{\infty} Z_k(t)] = \dfrac{Z(0)}{s} + \dfrac{s+p(1-s)}{s}\mathcal{L}[\gamma \sum_{k=0}^{\infty} Y_k(t) - \mu \sum_{k=0}^{\infty} Z_k(t)].\end{cases} \tag{9}$$

Now, comparing like terms on both sides, one can get

$$\begin{cases}\mathcal{L}[X_0(t)] = \frac{m_1}{s}, \mathcal{L}[H_0(t)] = \frac{m_2}{s}, \mathcal{L}[J_0(t)] = \frac{m_3}{s}, \mathcal{L}[Y_0(t)] = \frac{m_4}{s}, \mathcal{L}[Z_0(t)] = \frac{m_5}{s},\\ \mathcal{L}[X_1(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\lambda - \beta_1 S_0(t) + \alpha Y_0(t) - \mu X_0(t)],\\ \mathcal{L}[H_1(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\beta_1 S_0(t) - \beta_2 W_0(t) - (\lambda_1+\mu)H_0(t)],\\ \mathcal{L}[J_1(t)] = [\beta_2 W_0(t) - (\omega+\lambda_2+\mu)J_0(t)],\\ \mathcal{L}[Y_1(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\omega J_0(t) - (\gamma+\mu+\alpha)Y_0(t)],\\ \mathcal{L}[Z_1(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\gamma Y_0(t) - \mu Z_0(t)],\\ \mathcal{L}[X_2(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\lambda - \beta_1 S_1(t) + \alpha Y_1(t) - \mu X_1(t)],\\ \mathcal{L}[H_2(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\beta_1 S_1(t) - \beta_2 W_1(t) - (\lambda_1+\mu)H_1(t)],\\ \mathcal{L}[J_2(t)] = [\beta_2 W_1(t) - (\omega+\lambda_2+\mu)J_1(t)],\\ \mathcal{L}[Y_2(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\omega J_1(t) - (\gamma+\mu+\alpha)Y_1(t)],\\ \mathcal{L}[Z_2(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\gamma Y_1(t) - \mu Z_1(t)],\\ \vdots\\ \mathcal{L}[X_{k+1}(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\lambda - \beta_1 S_k(t) + \alpha Y_k(t) - \mu X_k(t)],\\ \mathcal{L}[H_{k+1}(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\beta_1 S_k(t) - \beta_2 W_k(t) - (\lambda_1+\mu)H_k(t)],\\ \mathcal{L}[J_{k+1}(t)] = [\beta_2 W_k(t) - (\omega+\lambda_2+\mu)J_k(t)],\\ \mathcal{L}[Y_{k+1}(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\omega J_k(t) - (\gamma+\mu+\alpha)Y_k(t)],\\ \mathcal{L}[Z_{k+1}(t)] = \dfrac{s+p(1-s)}{s}\mathcal{L}[\gamma Y_k(t) - \mu Z_k(t)].\end{cases} \tag{10}$$

Further, utilizing the inverse Laplace transform to equation (10), we have

$$\begin{cases} X_0 = m_1, H_0 = m_2, J_0 = m_3, Y_0 = m_4, Z_0 = m_5. \\ X_1(t) = [\lambda - \beta_1 m_1 m_2 + \alpha\, m_4 - \mu\, m_1][1 + p(t-1)], \\ H_1(t) = [\beta_1 m_1 m_2 - \beta_2 m_2 m_3 - (\lambda_1 + \mu) m_2][1 + p(t-1)], \\ J_1(t) = [\beta_2 m_2 m_3 - (\omega + \lambda_2 + \mu) m_3][1 + p(t-1)], \\ Y_1(t) = [\omega\, m_3 - (\gamma + \mu + \alpha) m_4][1 + p(t-1)], \\ Z_1(t) = [\gamma\, m_4 - \mu\, m_5][1 + p(t-1)]. \\ X_2(t) = [\lambda][1 + p(t-1)] + [-\beta_1(x_{11} m_2 + m_1 h_{11}) + (\alpha\, y_{11} - \mu\, x_{11})][p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ H_2(t) = [\beta_1(x_{11} m_2 + m_1 h_{11}) - \beta_2(m_2 J_{11} + m_3 h_{11}) + (\lambda_1 + \mu) h_{11}][p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ J_2(t) = [\beta_2(m_2 J_{11} + m_3 h_{11}) - (\omega + \lambda_2 + \mu) J_{11}][p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ Y_2(t) = [\omega\, J_{11} - (\gamma + \mu + \alpha) y_{11}][p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ Z_2(t) = [\gamma\, y_{11} - \mu\, z_{11}][p^2t^2 - 2p^2t + 2pt + (p-1)^2]. \end{cases} \quad (11)$$

Furthermore, the other terms can be obtained similarly, respectively, and the unknown values in the aforementioned equations are given underneath as:

$$\begin{cases} x_{11}(t) = [\lambda - \beta_1 m_1 m_2 + \alpha\, m_4 - \mu\, m_1][1 + p(t-1)], \\ h_{11}(t) = [\beta_1 m_1 m_2 - \beta_2 m_2 m_3 - (\lambda_1 + \mu) m_2][1 + p(t-1)], \\ j_{11}(t) = [\beta_2 m_2 m_3 - (\omega + \lambda_2 + \mu) m_3][1 + p(t-1)], \\ y_{11}(t) = [\omega\, m_3 - (\gamma + \mu + \alpha) m_4][1 + p(t-1)], \\ z_{11}(t) = [\gamma\, m_4 - \mu\, m_5][1 + p(t-1)]. \end{cases} \quad (12)$$

3.2. Numerical Results and Simulations

In this section, we accomplish the numerical simulations to help the analytical findings of our proposed model. Therefore, we apportioned some values for the parameters which are used in the proposed model (2) as:

$$\begin{cases} m_1 = 68,\ m_2 = 40,\ m_3 = 30,\ m_4 = 20,\ m_5 = 15, \\ \lambda = 0.1,\ \beta_1 = 0.003,\ \beta_2 = 0.002,\ \omega = 0.05, \\ \gamma = 0.05,\ \mu = 0.002,\ \alpha = 0.003,\ \lambda_1 = 0.003,\ \lambda_2 = 0.003. \end{cases}$$

Utilizing these parametric values, the following terms of the model (2) are obtained:

$$\begin{cases} X_0 = 68,\ H_0 = 40,\ J_0 = 30,\ Y_0 = 20,\ Z_0 = 15. \\ X_1 = (-8.13)[1 + p(t-1)],\ H_1 = (5.58)[1 + p(t-1)],\ J_1 = (2.13)[1 + p(t-1)], \\ Y_1 = (-0.98)[1 + (y-1)], Z_1 = (0.97)[1 + p(t-1)], \\ X_2 = (0.1)[1 + p(t-1)] - (0.1819)[p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ H_2 = (0.2267)[p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ J_2 = (0.486)[p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ Y_2 = (0.0624)[p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ Z_2 = (-0.0509)[p^2t^2 - 2p^2t + 2pt + (p-1)^2]. \end{cases} \quad (13)$$

Furthermore, the solutions to the first few terms are given as:

$$\begin{cases} X(t) = 68 - (8.03)[1 + p(t-1)] - (0.1819)[p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ H(t) = 40 + (5.58)[1 + p(t-1)] + (0.2267)[p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ J(t) = 30 + (2,13)[1 + p(t-1)] + (0.486)[p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ Y(t) = 20 - (-0.98)[1 + (y-1)] + (0.0624)[p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ Z(t) = 15 + (0.97)[1 + p(t-1)] - (-0.0509)[p^2t^2 - 2p^2t + 2pt + (p-1)^2]. \end{cases} \tag{14}$$

Note that from (14), one can obtain the approximate solution of the proposed model for various values of p.

3.3. General Solution for Model (2) with (HPM)

Now, we will employ the Homotopy perturbation method (HPM) to construct the general solution of the Model (2) as:

$$\begin{cases} (1-q)[{}^{CF}D_t^p(X)(t) -{}^{CF} D_t^p(X_0)(t)] + q[{}^{CF}D_t^p(X)(t) - \lambda + \beta_1 X(t)H(t) - \alpha\, Y(t) + \mu\, X(t)] = 0, \\ (1-q)[{}^{CF}D_t^p(H)(t) -{}^{CF} D_t^p(H)_0(t)] + q[{}^{CF}D_t^p(H)(t) - \beta_1 X(t)H(t) + \beta_2 H(t)J(t) + (\lambda_1 - \mu)H(t)] = 0, \\ (1-q)[{}^{CF}D_t^p(J)(t) -{}^{CF} D_t^p(J)_0(t)] + q[{}^{CF}D_t^p(J)(t) - \beta_2 H(t)J(t) + (\omega + \lambda_2 - \mu)J(t)] = 0, \\ (1-q)[{}^{CF}D_t^p(Y)(t) -{}^{CF} D_t^p(Y_0)(t)] + [{}^{CF}D_t^p(Y)(t) - \omega\, J(t) + (\gamma + \mu + \alpha)Y(t)] = 0, \\ (1-q)[{}^{CF}D_t^p(Z)(t) -{}^{CF} D_t^p(Z_0)(t)] + q[{}^{CF}D_t^p(Z)(t) - \gamma\, Y(t) + \mu\, Z(t)] = 0. \end{cases} \tag{15}$$

Substituting $q = 0$ in (15) yields the following system of fractional differential equations:

$$\begin{cases} {}^{CF}D_t^p(X)(t) -{}^{CF} D_t^p(X_0)(t) = 0, \\ {}^{CF}D_t^p(H)(t) -{}^{CF} D_t^p(H)_0(t) = 0, \\ {}^{CF}D_t^p(J)(t) -{}^{CF} D_t^p(J_0)(t) = 0, \\ {}^{CF}D_t^p(Y)(t) -{}^{CF} D_t^p(Y_0)(t) = 0, \\ {}^{CF}D_t^p(Z)(t) -{}^{CF} D_t^p(Z_0)(t) = 0, \end{cases} \tag{16}$$

and the solution to the above equation is simple. Now, by setting $q = 1$ in (15) yields the similar model as (2). Now, assuming the solution in an infinite series form as:

$$\begin{cases} X(t) = \sum_{n=0}^{\infty} q^n X_n(t), \\ H(t) = \sum_{n=0}^{\infty} q^n H_n(t), \\ J(t) = \sum_{n=0}^{\infty} q^n J_n(t), \\ Y(t) = \sum_{n=0}^{\infty} q^n Y_n(t), \\ Z(t) = \sum_{n=0}^{\infty} q^n Z_n(t). \end{cases} \tag{17}$$

Furthermore, the original system can be attained by substituting $q = 1$ in (15). Now, by substituting (17) into (15) and comparing the terms with the power of q generates:

$$q^0 : \begin{cases} X_0(t) = X(0) = m_1, \\ H_0(t) = H(0) = m_2, \\ J_0(t) = J(0) = m_3, \\ Y_0(t) = Y(0) = m_4, \\ Z_0(t) = Z(0) = m_5. \end{cases} \tag{18}$$

$$q^1 : \begin{cases} X_1(t) = [\lambda - \beta_1 m_1 m_2 + \alpha\, m_4 - \mu\, m_1][1 + p(t-1)], \\ H_1(t) = [\beta_1 m_1 m_2 - \beta_2 m_2 m_3 - (\lambda_1 + \mu) m_2][1 + p(t-1)], \\ J_1(t) = [\beta_2 m_2 m_3 - (\omega + \lambda_2 + \mu) m_3][1 + p(t-1)], \\ Y_1(t) = [\omega\, m_3 - (\gamma + \mu + \alpha) m_4][1 + p(t-1)], \\ Z_1(t) = [\gamma\, m_4 - \mu\, m_5][1 + p(t-1)]. \end{cases} \tag{19}$$

$$q^2 : \begin{cases} X_2(t) = [\lambda][1 + p(t-1)] - [\beta_1(x_{11} m_2 + m_1 I_{11}) + (\alpha\, y_{11} - \mu\, X_{11})][p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ H_2(t) = [\beta_1(x_{11} m_2 + m_1 I_{11} - \beta_2 m_2 J_{11} + m_3 I_{11}][p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ J_2(t) = [\beta_2 m_2 J_{11} + m_3 I_{11} - (\omega + \lambda_2 + \mu) J_{11}][p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ Y_2(t) = [\omega\, J_{11} - (\gamma + \mu + \alpha) y_{11}][p^2t^2 - 2p^2t + 2pt + (p-1)^2], \\ Z_2(t) = [\gamma\, y_{11} - \mu\, z_{11}][p^2t^2 - 2p^2t + 2pt + (p-1)^2]. \end{cases} \tag{20}$$

Similarly, higher-order power terms can be identified, and unknown terms have earlier been mentioned in the previous section. Thus, we were able to reach the same terms as the LADM technique. Moreover, both approaches are useful in solving fractional-order nonlinear differential equations.

4. Simulations Results and Discussion

We offer numerical simulation for the proposed model to allow for the variation of our semi-analytical findings in both ways. The simulation makes use of qualitative point analysis and takes into account the parameters from a biological feasibility aspect. The starting class sizes for each compartment, namely, Susceptible $X(t)$, Snuffing $H(t)$, Irregular $J(t)$, Regular $Y(t)$, and quitter $Z(t)$, are taken from the Section 3.2 together with the numerical values of the parameters. In this section graph of all five compartments is signified, and the explanation of the behavior of all the classes is discussed. Moreover, the results are shown in Figures 1–5.

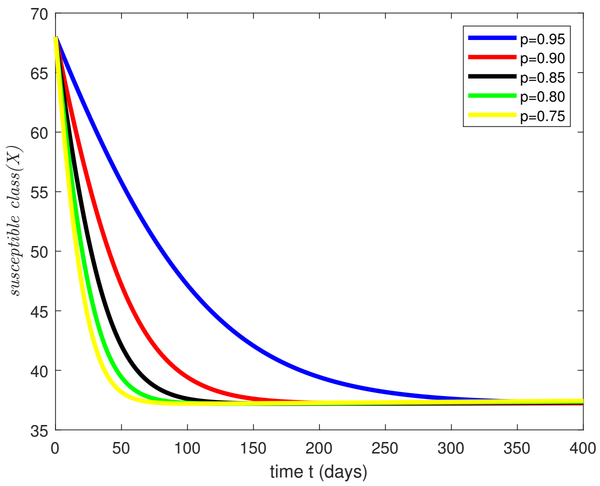


Figure 1. Adaptive nature of the approximated result for the susceptible class $X(t)$ class of the considered model at different arbitrary fractional orders.

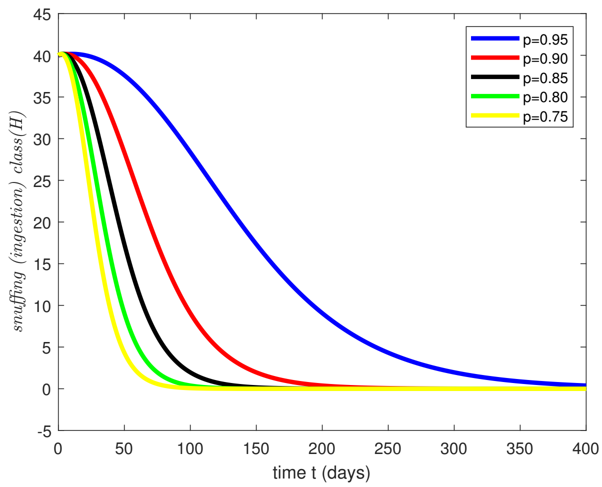


Figure 2. Adaptive nature of the approximated result for the snuffing class $H(t)$ class of the proposed model at different arbitrary fractional orders.

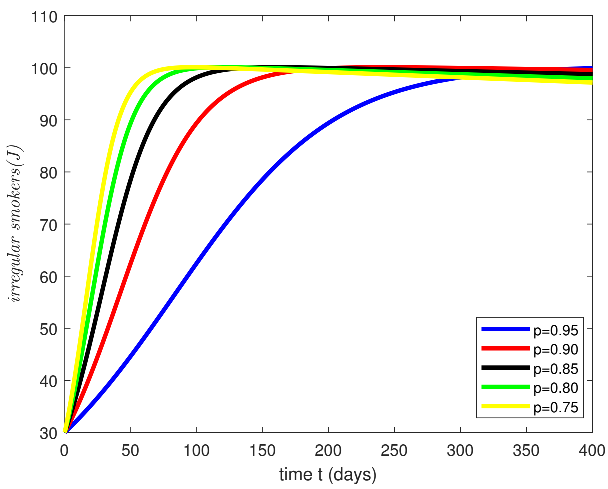


Figure 3. Adaptive nature of the approximated result for the irregular smokers $J(t)$ class of the considered model at different arbitrary fractional orders.

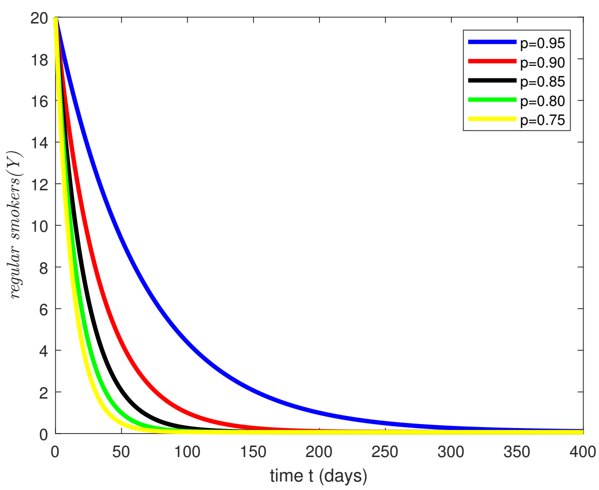


Figure 4. Adaptive nature of the approximated result for the regular smokers $Y(t)$ class of the proposed model at different arbitrary fractional orders.

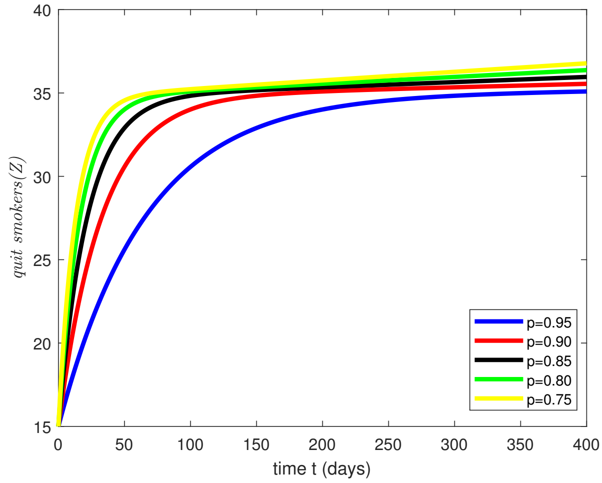


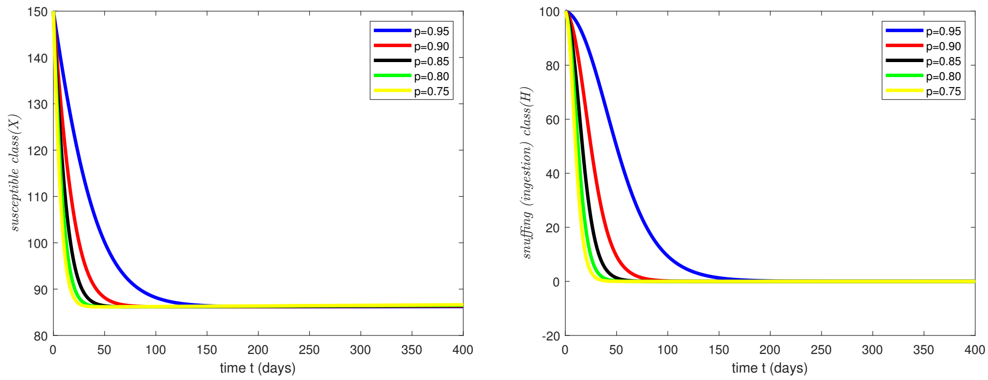
Figure 5. Adaptive nature of the approximated result for the quitter smokers $Z(t)$ class of the considered model at different arbitrary fractional orders.

The dynamical behavior of the susceptible class $X(t)$ is depicted in Figure 1 at different fractional orders. For the lower fractional order, the drop is noticeable, while for the higher fractional order, it is less noticeable. Furthermore, after 250 days, it becomes approximately constant. Figure 2 presents the behavior of the snuffing class $H(t)$ at various fractional orders. It grows at high fractional order and for low fractional order, it grows first and then going to decrease after 150 days.

In Figure 3, the irregular class $J(t)$ grows rapidly for low fractional order and then grows slowly for high fractional order. The dynamical behavior of regular class $Y(t)$ is shown in Figure 4. The results show that for the lower fractional order, the drop is remarkable, while for the higher order, it is less remarkable.

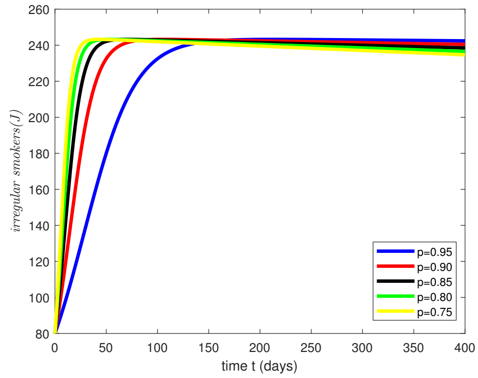
Similarly, Figure 5 shows that the growth in quitter class $Z(t)$ grows rapidly for low fractional order and grows slowly for high fractional order.

In the next figure, Figure 6, we present the dynamical behavior of the five compartments of the considered model by taking different values for the initial conditions along with the same fractional orders.

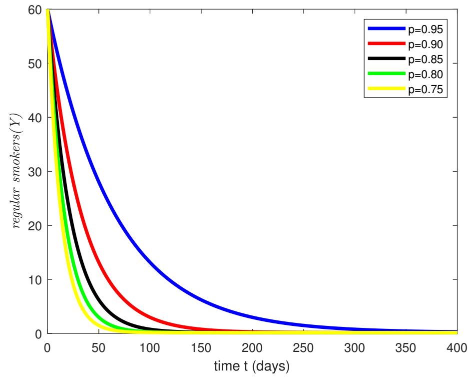


(**a**) (**b**)

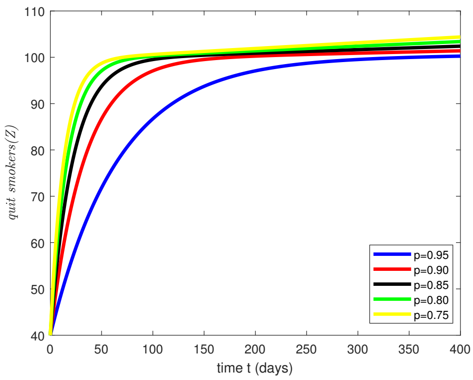
Figure 6. *Cont.*



(**c**)



(**d**)



(**e**)

Figure 6. Fractional dynamical behavior of all the compartments of the considered model having different initial conditions along with the same fractional orders. (**a**) Dynamical behavior of class $X(t)$ at $t = 150$; (**b**) Dynamical behavior of class $H(t)$ at $t = 100$; (**c**) Dynamical behavior of class $J(t)$ at $t = 80$; (**d**) Dynamical behavior of class $Y(t)$ at $t = 60$; (**e**) Dynamical behavior of class $X(t)$ at $t = 40$.

Next, we are intended to present a comparison of the numerical solutions of distinct compartments of the studied model via both the suggested method of HPM and LADM for the first few terms that display simulation similarities as given from Figures 7–11.

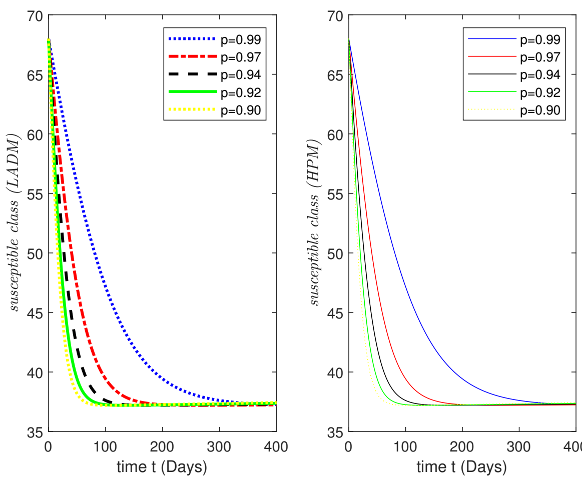


Figure 7. Comparison of the approximate solution for $X(t)$ class of the suggested model at different arbitrary fractional orders by LADM and HPM.

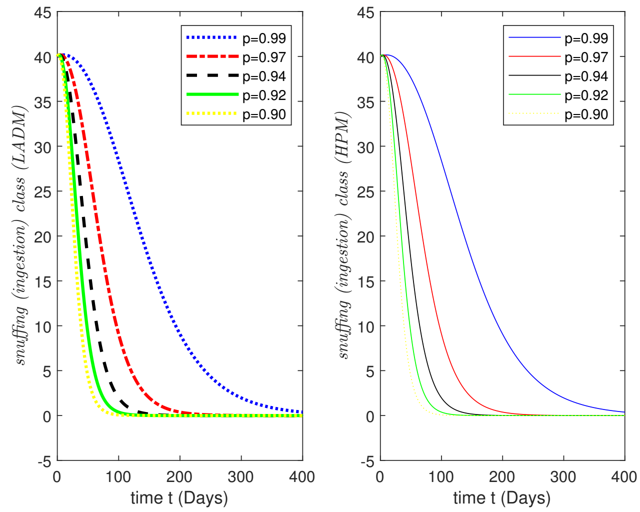


Figure 8. Comparison of the approximate solution for $H(t)$ class of the proposed model at different arbitrary fractional orders by LADM and HPM.

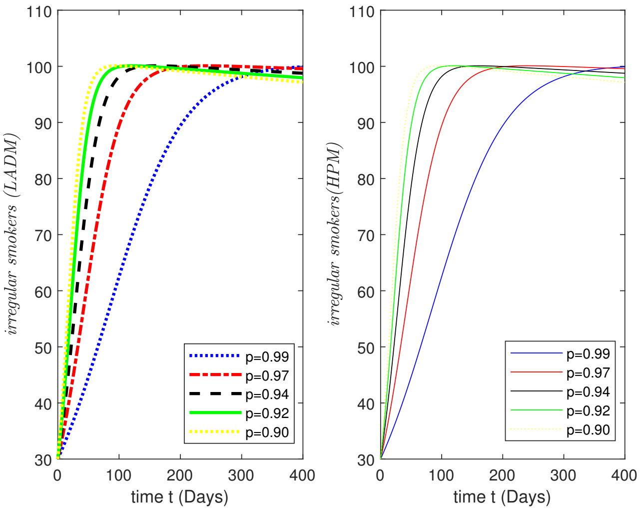


Figure 9. Comparison of the approximate solution for $J(t)$ class of the proposed model at different arbitrary fractional orders by LADM and HPM.

Figures 7–11 show the comparison of the numerical simulations of the solutions of all the compartments of the considered model. Furthermore, it is noted that both the techniques LADM and HPM have the same results, so the graphical representation is also the same for each compartment. The susceptible class $X(t)$ and the regular smokers class $Y(t)$ are decreasing with time, the snuffing class $H(t)$ first increases with time and then decreases after some time, and the irregular smokers class $J(t)$ are rapidly increasing over time.

Moreover, the below Figures 12–16 present the comparison of LADM and HPM by taking different values of the initial conditions but having the same fractional order values.

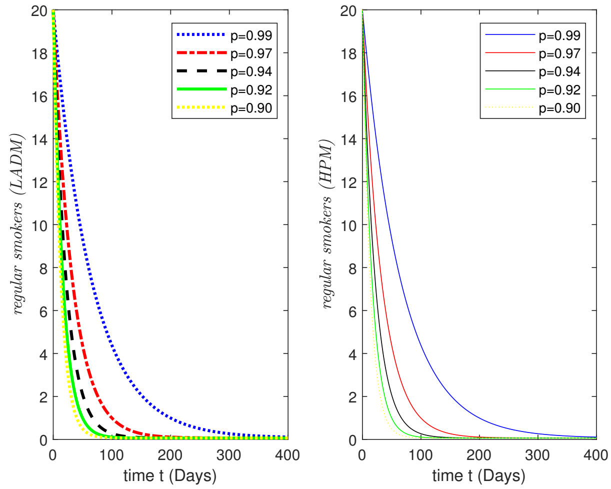


Figure 10. Comparison of the approximate solution for $Y(t)$ class of the suggested model at different arbitrary fractional orders by LADM and HPM.

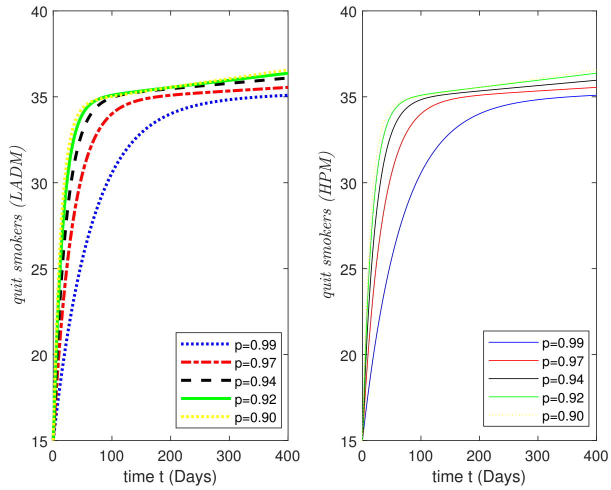


Figure 11. Comparison of the approximate solution for $Z(t)$ class of the proposed model at different arbitrary fractional orders by LADM and HPM.

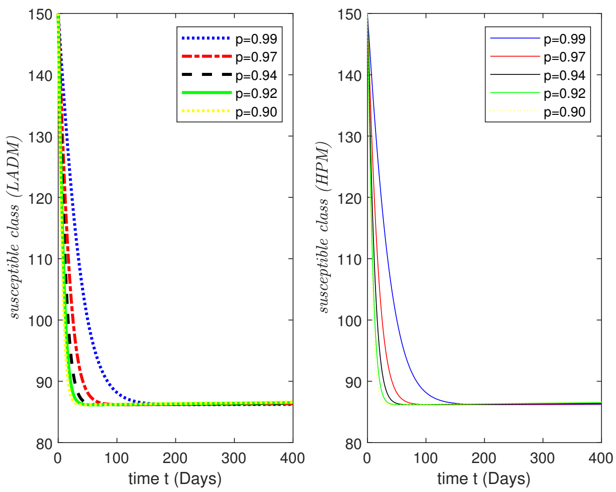


Figure 12. Comparison of the approximate solution for $X(t)$ class of the proposed model having initial condition 150.

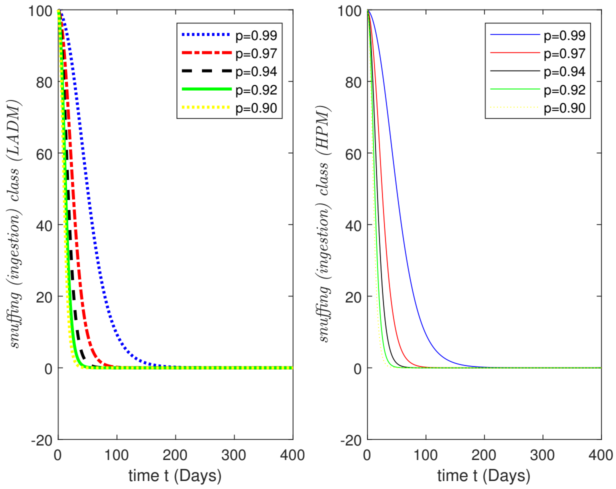


Figure 13. Comparison of the approximate solution for $H(t)$ class of the proposed model having initial condition 100.

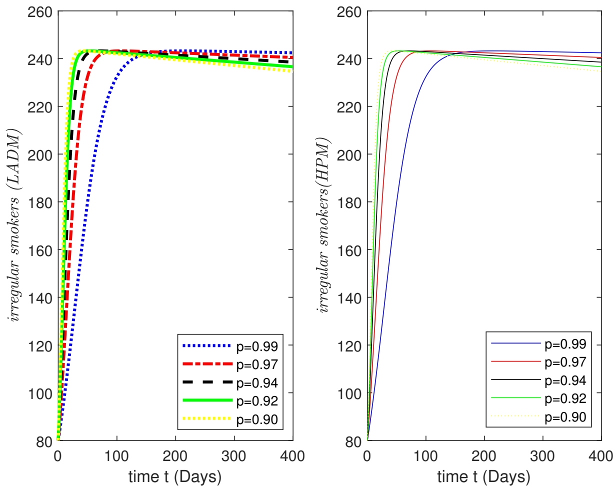


Figure 14. Comparison of the approximate solution for $J(t)$ class of the proposed model having initial condition 80.

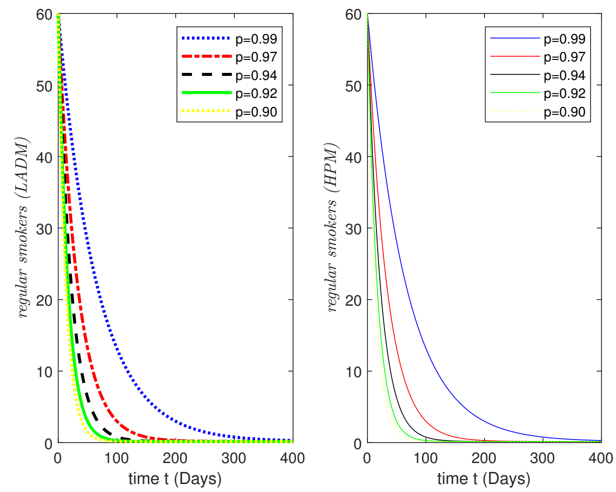


Figure 15. Comparison of the approximate solution for $Y(t)$ class of the proposed model having initial condition 60.

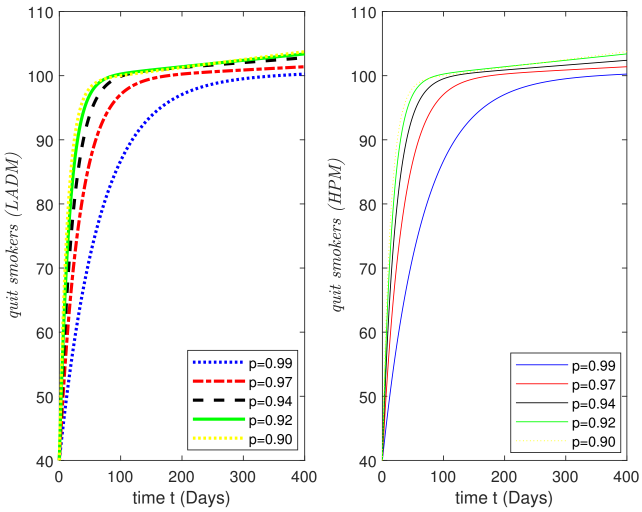


Figure 16. Comparison of the approximate solution for $Z(t)$ class of the proposed model having initial condition 40.

5. Conclusions

In this paper, the dynamical behavior of the smoking mathematical model via the Caputo-Fabrizio differential operator is analyzed and makes use of the utilities of fractional calculus. The considered model has been numerically explored by employing the Homotopy perturbation method (HPM) and the Laplace transform with the Adomian decomposition method (LADM). The numerical results obtained with the hired methods are significantly identical and have provided the finest confirmation for the considered model in arbitrary order derivatives. The outcomes are influenced by various parameters used in the considered model. Both methods converge and are useful for solving nonlinear fractional order differential equations. It is also important to emphasize that, when compared to standard methods, the method can save processing effort while maintaining good numerical accuracy. Moreover, the numerical simulation results of the underlying problem have been facilitated with the aid of Matlab, and the graphical behavior of each compartment of the considered problem at arbitrary order derivatives has been presented. From the presented simulation, we conclude that fractional order systems reveal more prosperous dynamics than the one with the integer order. As well, we anticipate that the existing study will be more beneficial in the account of smoking matter thinking of determination and education. Conceivably, in the future, the gained elements may lead us to do more research on this subject. For example, the mathematical model can be modernized by considering various dynamical structures and can be investigated with different types of derivatives.

Author Contributions: Conceptualization, R.S.; Data curation, Y.N.A.; Formal analysis, I.E.S.; Funding acquisition, I.E.S.; Investigation, Y.N.A.; Resources, I.E.S. and M.u.R.; Software, R.S.; Supervision, M.u.R.; Validation, R.S.; Visualization, M.u.R.; Writing—original draft, S.T.; Writing—review & editing, M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data have been created for this study.

Acknowledgments: The authors would like to acknowledge The University of Lahore and Shanghai Jiao Tong University, for the provision of the research plate form to complete this research work.

Conflicts of Interest: The authors declare no conflict of interest.

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