A FREE BOUNDARY PROBLEM AND AN EXTENSION OF MUSKAT'S MODEL

BY

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1. Introduction

It is the purpose of this paper to derive and solve a mathematical model for the following physical problem. Suppose that in a homogeneous compressible porous medium one incompressible fluid is displacing another. The problem is to describe the motion of the fluids, in particular, the motion of the interface between the fluids, if the initial velocity distribution, or equivalently, the initial pressure distribution, of the fluids is given, together with appropriate boundary data.

We assume the flow to be in the horizontal x-direction, say, and neglect gravitational effects. We further assume the two fluids are immiscible so that for each time t there is a well defined interface between the fluids whose location is given by $x=\varrho(t)$. To the left of $\varrho(t)$ we denote the velocity of the fluid by $u(x, t)$ and its pressure by $p(x, t)$, and to the right we denote velocity and pressure by $v(x, t)$ and $q(x, t)$ respectively. The pressures and velocities are related by Darcy's law:

$$u(x, t) = -a\partial p(x, t)/\partial x, \quad v(x, t) = -b\partial q(x, t)/\partial x, \tag{1.1}$$

where a and b are positive quantities which depend on the physical properties of the fluid in question and of the porous medium and which we take to be constant. Since the fluids are incompressible, their densities are constant and the continuity equations take the form

$$\begin{cases} \partial\varphi/\partial t + \partial u/\partial x = 0 & x < \varrho(t), \\ \partial\varphi/\partial t + \partial v/\partial x = 0 & x > \varrho(t), \end{cases} \tag{1.2}$$

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where φ represents the porosity of the medium (see Scheidegger [21] for a precise definition), which in our case is a function of pressure alone. Following Scheidegger [21], p. 105, or the original work of Sčelkačev [19], [20], we assume that φ is a slowly increasing function of the pressure, and that, for small variations in pressure, the derivative of φ (with respect to pressure) may be taken to be a positive constant. This, together with (1.1) and (1.2), implies

$$\begin{cases} \dfrac{\partial p}{\partial t} = \alpha \dfrac{\partial^2 p}{\partial t^2} & x < \varrho(t) \\ \dfrac{\partial q}{\partial t} = \beta \dfrac{\partial^2 q}{d t^2} & x > \varrho(t), \end{cases} \tag{1.3}$$

where α and β are again positive constants. Since the fluids are assumed to be flowing in contact we must have

$$u(\varrho(t), t) = v(\varrho(t), t), \tag{1.4}$$

and further, since the velocity of the interface is both $\partial \varrho / \partial t$ and $u(\varrho(t), t)$ we must have

$$\frac{d\varrho}{dt} = u(\varrho, t) \tag{1.5}$$

Finally, we assume, with Muskat [13], that the pressure is continuous across the interface so that

$$p(\varrho(t), t) = q(\varrho(t), t). \tag{1.6}$$

These last three equations describe the interface conditions which we treat.

This interface problem was originally formulated by Muskat [13] in three dimensions. Our formulation coincides with his, except for the simplifications arising from the one dimensionality of our simpler situation, and except that Muskat assumes that the terms $\partial p/\partial t$ and $\partial q/\partial t$ are negligible in (1.3) so that each of these equations reduces, in his case, to Laplace's equation. He then formulates the problem entirely in terms of the pressures. His problem (Muskat's model) has apparently remained unsolved except for a few special cases, where the shape of the interface was predetermined by symmetry considerations. (See Scheidegger [21].)

Our approach is to rephrase the problem entirely in terms of the velocities, eliminating p and q and to solve the resulting mathematical model. In fact, by (1.1) and (1.3) u and v satisfy the differential equations

$$\begin{cases} \alpha u_{xx} = u_t & x < \varrho(t), \\ \beta v_{xx} = v_t & x > \varrho(t). \end{cases} \tag{1.7}$$

Next we assume $p(\varrho(t), t)$ and $q(\varrho(t), t)$ are differentiable functions of t and that p and q

satisfy (1.3) on the interface $x=\varrho(t)$, both assumptions being justified by the solution we obtain. Then from (1.6) we have

$$\frac{d}{dt}\,p(\varrho(t),t)=\frac{d}{dt}\,q(\varrho(t),t),$$

and by (1.1) and (1.5) we compute

$$\frac{d}{dt}\,p(\varrho,t)=p_x(\varrho,t)\,\varrho'+p_t(\varrho,t)=-\frac{1}{a}\,u^2(\varrho,t)+\alpha p_{xx}(\varrho,t)=-\frac{1}{a}\,u^2(\varrho,t)-\frac{\alpha}{a}\,u_x(\varrho,t).$$

Making a similar calculation on q and equating the results leads to

$$Ku^2(\varrho(t),t)+\gamma u_x(\varrho(t),t)=\lambda v_x(\varrho(t),t), \tag{1.8}$$

where $K=(a^{-1}-b^{-1})$, $\gamma=\alpha/a$, and $\lambda=\beta/b$.

On the other hand, if we have functions $u(x,t)$, $v(x,t)$, $\varrho(t)$ satisfying (1.7), (1.4), (1.5) and (1.8), we can find functions $p(x,t)$, $q(x,t)$, $\varrho(t)$ satisfying (1.3), (1.1), (1.4), (1.5) and (1.6). In fact, define $U(x,t)$ to be $a^{-1}u(x,t)$ for $x\leqslant\varrho(t)$, $t>0$ and to be $b^{-1}v(x,t)$ for $x\geqslant\varrho(t)$, $t>0$. Letting $x_0<\varrho(t)$ be fixed, define $\Phi(x,t)=\int_{x_0}^{x}U(\xi,t)\,d\xi+\varphi(t)$. Then if $\varphi(t)\equiv\alpha\cdot a^{-1}\int_0^t(\partial/\partial x)\,u(x_0,\tau)\,d\tau$, one may readily verify that $\Phi(x,t)$ is continuous for $-\infty<x<\infty$. Letting $p(x,t)\equiv\Phi(x,t)$, $x\leqslant\varrho(t)$, $t>0$, $q(x,t)\equiv\Phi(x,t)$, $x\geqslant\varrho(t)$, $t>0$, one can now show that these functions satisfy (1.3), (1.1), (1.4), (1.5) and (1.6).

In many applications (1.8) can be simplified to

$$\gamma u_x(\varrho(t),t)=\lambda v_x(\varrho(t),t).$$

There is reason to expect a u^2 (or v^2) term in a more precise formulation of Darcy's law, which implicitly assumes small velocities. Hence neglecting the u^2 term in (1.8) would be consistent with the use of Darcy's law in the form (1.1). In the problem which motivated this investigation, the displacement of oil by water, the velocities encountered are small, being of the order of a few centimeters per day.

We can (and do) treat a non-linear term in this free boundary condition of greater complexity than the Ku^2 term which arises from the physical problem, and it seems to be of some mathematical interest to consider such a generalization.

We are now in a position to formulate our first problem:

PROBLEM I. *Given positive constants $\alpha,\beta,\gamma,\lambda$ and A, and three functions: f defined on $\{-\infty<x\leqslant0\}$, g on $\{0\leqslant x<\infty\}$, and H on $R^n\otimes[0,A)$ we seek three functions u, v, and ϱ such that*

(I1) *$\varrho(t)$ is defined and continuous for $0 \leqslant t < A$, with $\varrho(0) = 0$.*

(I2) *$u(x,t)$ is defined and continuous for $x \leqslant \varrho(t)$, $0 \leqslant t < A$, with $u_x(x, t)$ continuous for $x \leqslant \varrho(t)$, $0 < t < A$, with $u_{xx}(x, t)$ and $u_t(x, t)$ continuous for $x < \varrho(t)$, $0 < t < A$ and satisfying $\alpha u_{xx}(x, t) = u_t(x, t)$, $u(x, 0) = f(x)$.*

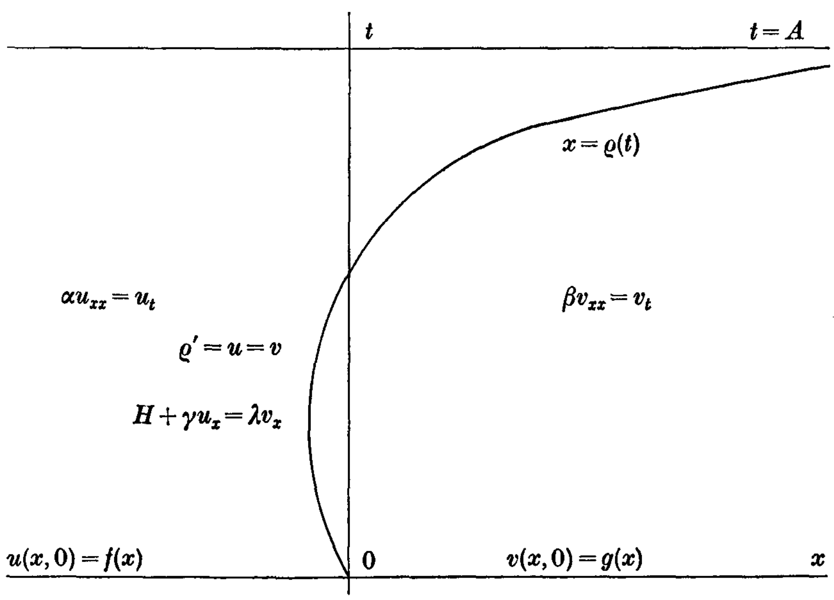
(I3) *$v(x, t)$ is defined and continuous for $x \geqslant \varrho(t)$, $0 \leqslant t < A$, with $v_x(x, t)$ continuous for $x \geqslant \varrho(t)$, $0 < t < A$, with $v_{xx}(x, t)$ and $v_t(x, t)$ continuous for $x > \varrho(t)$, $0 < t < A$, and satisfying $\beta v_{xx}(x, t) = v_t(x, t)$, $v(x, 0) = g(x)$.*

(I4) *$u(\varrho(t), t) = v(\varrho(t), t)$, $0 \leqslant t < A$.*

(I5) *$H(u_{(1)}, u_{(2)}, \ldots, u_{(n)}, t) + \gamma u_x(\varrho(t), t) = \lambda v_x(\varrho(t), t)$, $0 \leqslant t < A$ where $u_{(1)} = u(\varrho(t), t)$, $u_{(j+1)} = \int_0^t u_{(j)}(\tau)\, d\tau$, $j = 1, 2, \ldots, n-1$.*

(I6) *$\varrho'(t) = u(\varrho(t), t)$, $0 \leqslant t < A$.*

The problem is represented schematically in the diagram below:



We shall give a solution to this problem under sufficient smoothness and growth conditions on f, g, and H, together with certain compatibility restrictions. Specifically we assume the following:

(A1) f is twice continuously differentiable on $\{-\infty < x \leqslant 0\}$, and g is twice continuously differentiable on $\{0 \leqslant x < \infty\}$.

(A2) For some $M > 0$

$|f(x)|$, $|f'(x)|$, $|f''(x)|$ are all bounded by $M \exp [x^2/4\alpha A]$

$|g(x)|$, $|g'(x)|$, $|g''(x)|$ are all bounded by $M \exp [x^2/4\beta A]$.

(A3) H is continuously differentiable on $R^n \otimes [0, A)$ and its partial derivatives satisfy a uniform Lipschitz condition on each compact subset of $R^n \otimes [0, A)$.

(A4) $f(0) = g(0)$ and $H(f(0), 0, 0, 0, \ldots, 0) + \gamma f'(0) = \lambda g'(0)$.

Under these conditions we shall show that Problem I has a solution, and that, within the class of functions subject to certain standard exponential growth conditions at infinity related to (A2), the solution is unique.

As a by-product of this investigation we obtain an existence and uniqueness theorem for the case where $\varrho(t)$ is given and we drop the condition $\varrho' = u$. This we formulate as our second problem.

PROBLEM II. *Given $\alpha, \beta, \gamma, \lambda$, and A as before, and given four functions: f, g, and H as before and ϱ defined on $[0, A)$, we seek two functions u and v satisfying*

(II1) *$u(x, t)$ is defined and continuous for $x \leqslant \varrho(t)$, $0 \leqslant t < A$, with $u_x(x, t)$ continuous for $x \leqslant \varrho(t)$, $0 < t < A$, with $u_{xx}(x, t)$ and $u_t(x, t)$ continuous for $x < \varrho(t)$, $0 < t < A$ and satisfying $\alpha u_{xx}(x, t) = u_t(x, t)$, $u(x, 0) = f(x)$.*

(II2) *$v(x, t)$ is defined and continuous for $x \geqslant \varrho(t)$, $0 \leqslant t < A$, with $v_x(x, t)$ continuous for $x \geqslant \varrho(t)$, $0 < t < A$, with $v_{xx}(x, t)$ and $v_t(x, t)$ continuous for $x > \varrho(t)$, $0 < t < A$ and satisfying $\beta v_{xx}(x, t) = v_t(x, t)$, $v(x, 0) = g(x)$.*

(II3) *$u(\varrho(t), t) = v(\varrho(t), t)$, $0 \leqslant t < A$.*

(II4) *$H(u_{(1)}, u_{(2)}, \ldots, u_{(n)}, t) + \gamma u_x(\varrho(t), t) = \lambda v_x(\varrho(t), t)$, $0 \leqslant t < A$ where $u_{(1)} = u(\varrho(t), t)$, $u_{(j+1)} = \int_0^t u_{(j)}(\tau)\, d\tau$, $j = 1, 2, \ldots, n-1$.*

We solve this problem under (A1)–(A4) and

(A5) $\varrho(t)$ is twice continuously differentiable on $\{0 \leqslant t < A\}$ with $\varrho(0) = 0$.

Work on problems similar to Problem II for the heat equation and more general parabolic equations has been done by several authors beginning apparently with Dacev [1], [2]. (See also Žitarašu [26] and his bibliography.) However, this work seems to be limited to the cases where ϱ is constant and $H \equiv 0$.

The free boundary problem (Problem I) we consider differs in many aspects from the Stefan type problems which have been considered by many previous authors. In particular, Cannon and Hill [3], Douglas [4], Friedman [5], Kamenomostskaya [8], Kolodner [9], Kyner [10], Li-Shang [11], [12], Oleinik [14], Quilghini [15], [16], Sestini [22] and others have contributed to that problem in recent years. A recent book by Rubinštein [18] surveys that problem to the year 1967.(1) We are indebted to these writers only for the spirit

(1) *Added in proof.* Two papers by Friedman, The Stefan problem in several space variables and One dimensional Stefan problems with non-monotone free boundary, appearing in Trans. Amer. Math. Soc. (133) 1968, should be added to this list.

of our approach in this paper. In fact, of this previous work we use only a refinement due to Friedman [5] of Holmgren's early analysis of thermal potentials. (For a discussion of Holmgren's work see Goursat [7].)

We remark that the techniques used for solving problems I and II may also be used to solve problems on finite x-intervals similar to I and II. Rather than giving growth conditions on the solutions at infinity, however, it is necessary to give the values of the functions or their derivatives, or a linear combination of them at the endpoints of the intervals in question. The reduction of these problems to equivalent integral equations proceeds in much the same manner as in our case, the only difference being that appropriate Green's or Neumann functions must be used in place of the fundamental solution which we use.

We use two standard notations for partial derivatives. Thus $k_x(x, t)$ $k_{xx}(x, t)$ and $k_t(x, t)$ mean, as usual the first partial of k with respect to x, the second partial of k with respect to x, and the first partial of k with respect to t. Also we use $k_1(x-\varrho(t), \alpha t-\alpha\tau)$, $k_{11}(x-\varrho(t), \alpha t-\alpha\tau)$, and $k_2(x-\varrho(t), \alpha t-\alpha\tau)$. Here, again as usual, these mean partials with respect to the arguments. Thus the subscript 1 means the first partial with respect to the first argument, etc. Finally, in the way of notation, if I is any interval (open, closed, half-open) on the real line, the differentiability classes $C^k(I)$, $k \geq 0$ an integer, are introduced in the standard way.

2. The Poisson integral and Dirichlet problems

The two problems we formulated in the introduction, or at least our solutions of them, turn primarily on smoothness properties of the solutions of certain Dirichlet problems, which in turn depend primarily on the properties of the Poisson integral

$$\tilde{w}(x,t) \equiv \int_{-\infty}^{\infty} k(x-\xi, \alpha t)\, w(\xi)\, d\xi \tag{2.1}$$

and of the double layer thermal potential

$$\hat{\psi}(\alpha; x, t) \equiv 2\alpha \int_0^t k_1(x-\varrho(\tau), \alpha t-\alpha\tau)\, \psi(\tau)\, d\tau, \tag{2.2}$$

where
$$k(x, t) \equiv (4\pi t)^{-\frac{1}{2}} \exp(-x^2/4t) \tag{2.3}$$

is the fundamental solution of the heat equation.

In this section we are concerned with a description of the properties of these integrals, and of the Dirichlet problems mentioned above, which we will find useful in the later sections. The basic information we are interested in is that if the boundary and the Dirichlet data are sufficiently smooth, then the solution to the Dirichlet problem is smooth up to and on the boundary, and the differential equation continues to be satisfied on the boundary. This result, along with certain concomitant estimates, lies at the heart of our solutions.

Related results—the Schauder estimates on the boundary—have been known for some time for both elliptic and parabolic equations. (See e.g. Friedman [5] Chapter 4 for references.) However, results of the form we use seem not to be in print, and so we outline in this section the main results that we need.

For future reference we list here the following standard estimate. For any integer $n \geq 0$, and any real $h > 1$ and $\alpha > 0$

$$\left| \frac{\partial^n}{\partial x^n} k(x, \alpha t) \right| \leq M t^{-n/2} k(x, \alpha h t), \tag{2.4}$$

where M is a constant depending only on n, h, and α. This follows by observing that if c and C are positive constants then $z^c \exp(-Cz)$ is bounded for $z \geq 0$ by a constant depending only on c and C.

If w is locally integrable on the real line and if there is a constant $A > 0$ for which $w(x) \exp(-x^2/4\alpha A)$ is bounded, then the Poisson integral $\tilde{w}$, given by (2.1) is a solution of the heat equation $\alpha \tilde{w}_{xx} = \tilde{w}_t$, for $0 < t < A$, and all real x. Further, $\tilde{w}(x, t) \to w(x_0)$ as $(x, t) \to (x_0, 0+)$ at each point x_0 where w is continuous, and uniformly on any closed bounded interval on which w is continuous. These are standard elementary facts.

We are interested in the behavior of $\tilde{w}(x, t)$ on a curve $x = \varrho(t)$ where

$$\varrho(t) = Bt + o(t) \quad \text{as } t \to 0+, \tag{2.5}$$

with B constant, and we suppose

$$w(x) = A_0 + A_1 x + A_2 x^2 + o(x^2) \quad \text{as } x \to 0 \tag{2.6}$$

If $w(x) \equiv 0$ for $x > 0$, then

$$\tilde{w}(\varrho(t), t) = A_0/2 - (A_0 B + 2\alpha A_1)\, t^{\frac{1}{2}} / \sqrt{4\pi\alpha} + (A_1 B + 2\alpha A_2)\, t/2 + o(t) \tag{2.7}$$

as $t \to 0+$, and if $w(x) \equiv 0$ for $x < 0$, then the sign on the $t^{\frac{1}{2}}$ term in (2.7) is changed. Adding these results gives

$$\tilde{w}(\varrho(t), t) = A_0 + (A_1 B + 2\alpha A_2) t + o(t) \quad \text{as } t \to 0+. \tag{2.8}$$

If in (2.6) the x^2 and $o(x^2)$ terms are replaced by $o(x)$, then the t and $o(t)$ terms in (2.7) and (2.8) are replaced by $o(t^{\frac{1}{2}})$ and if $w(x)\to A_0$ as $x\to x_0$, then (2.7) and (2.8) reduce to $w(\varrho(t),t)\to A_0/2$ and $w(\varrho(t),\,t)\to A_0$ respectively. If w is smooth enough, and if its derivatives satisfy the same exponential estimate, these formulas can be seen by integrating by parts, and this is really all that is needed for the later sections of this paper. The general case can be established by computing (2.1) for the special case $w(x)=A_0+A_1x+A_2x^2$, and then estimating the difference between (2.1) and the results of this special case

Suppose further that ϱ is continuously differentiable on the closed interval $[0,\,T]$ for some positive $T<A$. Then $\tilde{w}(\varrho(t),\,t)$ is continuously differentiable on $[0,\,T]$. Clearly the only point in question is at $t=0$. But (2.8) ensures the existence of the derivative at 0, and one easily shows its continuity at that point, based on the limits just established.

We now look briefly at two auxilliary Dirichlet problems. For this purpose we need the following mild sharpening of a lemma of Friedman [5] which is itself a sharpening of a classical result of Holmgren (see Goursat [7], sec. 544). We state the lemma without proof, for the uniformities we seek are apparent from a reading of Friedman's proof.

LEMMA 2A. *Let ψ be continuous on $[0,\,T]$, and let ϱ satisfy a uniform Lipschitz condition there. Then*

$$\lim_{x\to\varrho(t)\pm 0}\hat{\psi}(\alpha;x,t)=\mp\psi(t)+2\,\alpha\int_0^t k_1(\Delta_t\varrho,\alpha\Delta t)\,\psi(\tau)\,d\tau,$$

where $\hat{\psi}$ is defined by (2.2), $\Delta t=t-\tau$, $\Delta_t\varrho=\varrho(t)-\varrho(\tau)$ and where, for each $\varepsilon>0$, $0<\varepsilon<T$, the limit is achieved uniformly in $[\varepsilon,\,T]$, and if $\psi(0)=0$, the limit is achieved uniformly in $[0,\,T]$.

We now formulate the Dirichlet problems. They are formulated under heavier hypotheses than is necessary for their solutions, but these additional restrictions enable us to discuss the smoothness questions we need to consider. Thus for the rest of this section we assume that μ and ν are continuously differentiable and ϱ is twice continuously differentiable on $\{0\leqslant t<A\}$, that f and g are twice continuously differentiable on $\{-\infty<x\leqslant 0\}$ and $\{0\leqslant x<\infty\}$ respectively, and that they and their first two derivatives are bounded by $M\exp(x^2/4\alpha A)$ and $M\exp(x^2/4\beta A)$ respectively, i.e., f and g satisfy (A2) of the introduction. We further assume $\mu(0)=f(0)$ and $\nu(0)=g(0)$.

PROBLEM $\mathrm{D_L}$: *Find a function $u(x,t)$ continuous in $\{-\infty<x\leqslant\varrho(t),\ 0\leqslant t<A\}$ with u_{11}, u_2 continuous in $\{-\infty<x<\varrho(t),\ 0<t<A\}$ satisfying $\alpha u_{11}=u_2$ there, with $u(x;0)=f(x)$, $u(\varrho(t),\,t)=\mu(t)$.*

PROBLEM $\mathrm{D_R}$: *Find a function $v(x,t)$ continuous in $\{\varrho(t)\leqslant x<\infty,\ 0\leqslant t<A\}$ with u_{11}, u_2 continuous in $\{\varrho(t)<x<\infty,\ 0<t<A\}$ satisfying $\beta u_{11}=u_2$, there, with $v(x,0)=g(x)$, $v(\varrho(t),\,t)=\nu(t)$.*

These two problems are of course equivalent, but both are formulated for reasons of symmetry in their later application to Problem I and Problem II.

These Dirichlet problems have been studied for a very long time (see e.g. Gevrey [6]) and their solutions known under a variety of hypotheses on ϱ, f, g, μ and ν. Solutions satisfy the following bounds: Given any $T<A$ there are constants $m>0$ and $a>0$ such that $|u(x,t)|\leqslant me^{ax^2}$, $|v(x,t)|\leqslant me^{ax^2}$, $0\leqslant t\leqslant T$, and are unique among the class of functions satisfying such bounds. (See Widder [24], [25] where these uniqueness arguments are given, and Tychonoff [13].)

Solutions can be constructed as follows. Let $F(x)$ be given by $f(x)$ for $x\leqslant 0$ and by $f(0)+f'(0)x+f''(0)x^2/2$ for $x>0$, and $G(x)$ by $g(x)$ for $x\geqslant 0$ and by $g(0)+g'(0)+g''(0)x^2/2$ for $x<0$. Then F and G are twice continuously differentiable on the reals. By increasing, if necessary, the value of M we observe that $F(x)$, $F'(x)$, and $F''(x)$ are bounded by $M\exp(x^2/4\alpha A)$ and $G(x)$, $G'(x)$, and $G''(x)$ are bounded by $M\exp(x^2/4\beta A)$.

We define U and V by

$$\begin{cases} U(x,t)=\int_{-\infty}^{\infty} k(x-\xi,\alpha t)\,F(\xi)\,d\xi, & U(x,0)=F(x),\\ V(x,t)=\int_{-\infty}^{\infty} k(x-\xi,\beta t)\,G(\xi)\,d\xi, & V(x,0)=G(x),\end{cases}\tag{2.9}$$

and seek solutions to D_L and D_R, respectively, by

$$\begin{cases} u(x,t)=U(x,t)+\hat{\psi}(\alpha;x,t)\\ v(x,t)=V(x,t)+\hat{\chi}(\beta;x,t),\end{cases}\tag{2.10}$$

where ψ and χ are to be determined and $\hat{\psi}$ and $\hat{\chi}$ are defined by (2.2). Assuming ψ and χ to be continuous we apply Lemma 2A to get

$$\psi=\varphi-K_\alpha\psi;\ \chi=\zeta+K_\beta\chi,\tag{2.11}$$

where $\varphi(t)\equiv\mu(t)-U(\varrho(t),t)$ with clearly $\varphi(0)=0$, $\psi(t)\equiv\nu(t)-V(\varrho(t),t)$ with $\psi(0)=0$, and where the integral operator K_a is defined by

$$K_a h(t)\equiv 2a\int_0^t k_1(\Delta_t\varrho,a\Delta t)\,h(\tau)\,d\tau.\tag{2.12}$$

As before and in the sequel $\Delta_t\varrho=\varrho(t)-\varrho(\tau)$, $\Delta t=t-\tau$.

These integral equations are solvable by iteration. The solution is based on the following easily established estimate.

LEMMA 2B. *Suppose* (1) $h \in C[0, T]$ *with* $h(0)=0$ *and* $|h(t)| \leqslant Nt^p$, $N>0$, $p \geqslant 0$; (2) $\varrho \in C^1[0, T]$ *with* $\varrho(0)=0$ *and* $|\varrho'(t)| \leqslant \tilde{N}$, $\tilde{N}>0$. *Then* $|K_a h(t)| \leqslant NBt^{p+\frac{1}{2}}\Gamma(p+1)/\Gamma(p+3/2)$, $0 \leqslant t \leqslant T$, *where* B *is a constant depending only on* $\tilde{N}$ *and* a.

The solution of the equations (2.11) are then

$$\psi = \sum_{n=0}^{\infty} (-1)^n K_\alpha^n \varphi; \quad \chi = \sum_{n=0}^{\infty} K_\beta^n \zeta, \tag{2.13}$$

respectively, where of course K_a^n represents K_a iterated n times. These series converge uniformly on $[0, T]$ for each positive $T<A$. The solutions of $\mathrm{D_L}$ and $\mathrm{D_R}$ are then given by (2.10), using ψ and χ as determined by (2.13).

Our primary interest being in the smoothness of these solutions, we now turn to such matters. Suppose a positive $T<A$ is given, and $\delta = T - A > 0$. If μ' and ν' are bounded by N, and ϱ' and ϱ'' are bounded by $\tilde{N}$ on $\{0 \leqslant t \leqslant T\}$, then one easily verifies, with the aid of the remark on the differentiability of $\tilde{w}(\varrho(t), t)$, that both φ and ζ are continuously differentiable and that there is a constant $\tilde{N}$, depending only on M, A, T, N, $\tilde{N}$ and δ (and also α and β of course) for which

$$|\varphi'(t)| \leqslant \tilde{N}, \quad |\zeta'(t)| \leqslant \tilde{N}; \quad 0 \leqslant t \leqslant T. \tag{2.14}$$

Parenthetically, we remark that in such estimate we will generally suppress the dependence on α and β, and, later, on γ and λ. We treat δ as an independent parameter since we will later be performing translations which will change A and T but will leave δ unchanged.

The differentiability of ψ and χ depends upon our ability to differentiate the integral operators in (2.13). This is covered by the following

LEMMA 2C. *Suppose* (1) $h \in C^1[0, T]$ *with* $h(0)=0$ *and* $|h'(t)| \leqslant \tilde{N}t^p$, $\tilde{N}>0$, $p \geqslant 0$; (2) $\varrho \in C^2[0, T]$ *with* $\varrho(0)=0$ *and* $|\varrho'(t)| \leqslant \tilde{N}$, $|\varrho''(t)| \leqslant \tilde{N}$, $\tilde{N}>0$. *Then* $K_a h(t)$ *is continuously differentiable for* $0 \leqslant t \leqslant T$ *and there is a constant* $B = B(T)$, *such that*

$$\left| \frac{d}{dt} K_a h(t) \right| \leqslant \tilde{N} B(\tilde{N}^3 + \tilde{N})\, t^{p+1} \Gamma(p+1)/\Gamma(p+\tfrac{3}{2}).$$

To establish this we observe that $|h(t)| \leqslant \tilde{N}t$, $|\varrho(t)| \leqslant \tilde{N}t$, and $|\Delta_t \varrho| \leqslant \tilde{N}\Delta t$. Then

$$K_a h(t) = (4\pi a)^{-\frac{1}{2}} \int_0^t \frac{h\tau}{(\Delta t)^{\frac{1}{2}}} \frac{\Delta_t \varrho}{\Delta t} \{1 - \exp(-(\Delta_t \varrho)^2/4a\Delta t]\}\, d\tau$$
$$+ (4\pi a)^{-\frac{1}{2}} \int_0^t \frac{h(\tau)}{(\Delta t)^{\frac{1}{2}}} \left[\varrho'(t) - \frac{\Delta_t \varrho}{\Delta t}\right] d\tau - (4\pi a)^{-\frac{1}{2}} \varrho'(t) \int_0^t \frac{h(\tau)\, d\tau}{(\Delta t)^{\frac{1}{2}}}.$$

Clearly the only difficulty is with the last term. We write the integral in that term as

$$\int_0^t \tau^{-\frac{1}{2}} h(t-\tau)\, d\tau$$

and observe that, since $h(0)=0$, its derivative is

$$\int_0^t \tau^{-\frac{1}{2}} h'(t-\tau)\, d\tau = \int_0^t \frac{h'(\tau)\, d\tau}{(\Delta t)^{\frac{1}{2}}}.$$

The differentiability then established, one can differentiate the last expression for $K_a h(t)$, differentiating under the integral signs of the first two integrals. The resulting expression can then be estimated, somewhat tediously to be sure, to obtain the stated estimate. The inequality (2.4) and $|e^{-a}-e^{-b}| \leqslant |a-b|$ for $a \geqslant 0$, $b \geqslant 0$ are useful in these calculations.

From Lemma 2C, applied successively to the terms of the series in (2.13), it follows that there is a constant $B = B(M, A, T, N, \bar{N}, \delta)$ such that

$$|\psi'(t)| \leqslant B, \quad |\chi'(t)| \leqslant B; \quad 0 \leqslant t \leqslant T, \tag{2.15}$$

from which we have immediately,

$$|\psi(t)| \leqslant Bt, \quad |\chi(t)| \leqslant Bt; \quad 0 \leqslant t \leqslant T, \tag{2.16}$$

We now consider u and v, given by (2.10). For $x < \varrho(t)$ we have

$$u_1(x,t) = U_1(x,t) + 2\alpha \int_0^t k_{11}(x-\varrho(\tau), \alpha\Delta t)\, \psi(\tau)\, d\tau.$$

By use of the formula

$$\frac{\partial}{\partial \tau} k(x-\varrho(\tau), \alpha\Delta t) = k_1(x-\varrho(\tau), \alpha\Delta t)\, \varrho'(\tau) - \alpha k_2(x-\varrho(t), \alpha\Delta t) \tag{2.17}$$

we deduce, since $\psi(0)=0$,

$$u_1(x,t) = U_1(x,t) - 2\int_0^t k_1(x-\varrho(t), \alpha\Delta t)\, \varrho'(\tau)\, \psi(\tau)\, d\tau + 2\int_0^t k(x-\varrho(\tau), \alpha\Delta t)\, \psi'(\tau)\, d\tau.$$

As $x \to \varrho(t) - 0$, $u_1(x,t)$ converges uniformly in $\{0 \leqslant t \leqslant T\}$, for each positive $T < A$, to

$$u_1(\varrho(t),t) = U_1(\varrho(t),t) - 2\int_0^t k_1(\Delta_t\varrho, \alpha\Delta t)\, \varrho'(\tau)\, \psi(\tau)\, d\tau$$
$$- \varrho'(t)\, \psi(t) + 2\int_0^t k(\Delta_t\varrho, \alpha\Delta t)\, \psi'(\tau)\, d\tau, \tag{2.18}$$

and so $u_1(x,t)$ is continuous in $\{0 \leqslant x \leqslant \varrho(t),\ 0 \leqslant t < A\}$.

Similarly $v_1(x, t)$ is continuous in $\{\varrho(t) \leqslant x < \infty,\ 0 \leqslant t < A\}$ and

$$v_1(\varrho(t), t) = V_1(\varrho(t), t) - 2\int_0^t k_1(\Delta_t\varrho, \beta\Delta t)\,\varrho'(\tau)\,\chi(\tau)\,d\tau$$
$$+\,\varrho'(t)\chi(t) + 2\int_0^t k(\Delta_t\varrho, \beta\Delta t)\,\chi'(\tau)\,d\tau \tag{2.19}$$

By differentiating the above expression for $u_1(x, t)$, making the same substitution for k_{11} as before, we can see that u_{11} is continuous in $\{0 \leqslant x \leqslant \varrho(t),\ 0 \leqslant t < A\}$ except at the origin, and, in fact, if $\mu'(0) = f'(0)\varrho'(0) + \alpha f''(0)$, i.e. $\psi'(0) = 0$, then u_{11} is continuous at the origin as well.

From this it follows immediately that the differential equation $\alpha u_{11} = u_2$ is satisfied on $x = \varrho(t)$, $t > 0$, and even at $t = 0$ provided again that $\mu'(0) = f'(0)\varrho'(0) + \alpha f''(0)$. Further we observe that $\mu'(t) = u_1(\varrho(t), t)\varrho'(t) + \alpha u_{11}(\varrho(t), t)$ for $0 < t < A$. This follows by differentiating $u(\varrho(t) - \varepsilon, t)$ and letting $\varepsilon \to 0$. We also note that as $t \to 0$ we have

$$\begin{aligned} u_1(\varrho(t), t) &= f'(0) + 2(\alpha\pi)^{-\frac12} t^{\frac12}\psi'(0) + o(t^{\frac12}) \\ &= f'(0) + 2(\alpha\pi)^{-\frac12} t^{\frac12}[\mu'(0) - f'(o)\varrho'(0) - \alpha f''(0)] + o(t^{\frac12}) \end{aligned}$$

and

$$\begin{aligned} v_1(\varrho(t), \varrho) &= g'(0) - 2(\alpha\pi)^{-\frac12} t^{\frac12}\chi'(0) + o(t^{\frac12}) \\ &= g'(0) - 2(\alpha\pi)^{-\frac12} t^{\frac12}[\nu'(0) - g'(o)\varrho'(0) - \beta g''(0)] + o(t^{\frac12}). \end{aligned}$$

These formulas follow by estimating the terms of (2.17) and (2.18)

Finally we take note of the following estimates.

THEOREM 2 D. *Given a positive $T < A$, there are positive constants $M' = M'(M, A, T, N, \bar{N}, \delta)$ and $A' = \delta/2$ such that u, u_x, u_{xx}, v, v_x, v_{xx}, are all bounded uniformly by $M' \exp[(x - \varrho(t))^2/4\alpha A']$ and $M' \exp[(x - \varrho(t))^2/4\beta A']$, respectively for $0 \leqslant t \leqslant T$.*

To see that these estimates hold, one can use (2.10) and the formulas one gets by differentiating (2.10). The terms arising from U (or V) are easily estimated in the stated form, and the integrals arising from the double layer potential can be shown to be bounded by the techniques Friedman [5] uses in his proof of our Lemma 2A.

3. The integral equations

We assume that Problems I and II have solutions, and that the functions u and v satisfy exponential bounds of the following form: for each $T < A$ there exist constants $m > 0$ and $a > 0$ such that

$$|u(x, t)| \leqslant m e^{ax^2}, \quad |v(x, t)| \leqslant m e^{ax^2}; \quad 0 \leqslant t \leqslant T. \tag{3.1}$$

In both cases we denote the common value of $u(\varrho(t), t)$ and $v(\varrho(t), t)$ by μ:

$$\mu(t) = u(\varrho(t), t) = v(\varrho(t), t) \tag{3.2}$$

and we proceed to derive an integral equation for μ, assuming $\mu \in C^1[0, A)$.

The calculations involved in the derivation are the same for both problems. It is just that in the one case ϱ is a given function and in the other it is determined from μ by the equation

$$\varrho(t) = \int_0^t \mu(\tau)\, d\tau. \tag{3.3}$$

Clearly a knowledge of μ is sufficient to determine the solution, for in either case ϱ is known, and then u and v are given by (2.10) with ψ and χ being given by (2.13), with v taken as μ. Furthermore by (2.18) and (2.19) $u_1(\varrho(t), t)$ and $v_1(\varrho(t), t)$ are known in terms of μ. We will then denote $\gamma u_1(\varrho(t), t)$ by $\mathcal{L}\mu(t)$ and $\lambda v_1(\varrho(t), t)$ by $\mathcal{R}\mu(t)$ so that

$$\mathcal{L}\mu(t) = \gamma U_1(\varrho(t), t) - 2\gamma \int_0^t k_1(\Delta_t \varrho, \alpha\Delta t)\, \varrho'(\tau)\, \psi(\tau)\, d\tau$$
$$- \gamma\varrho'(t)\, \psi(t) + 2\gamma \int_0^t k(\Delta_t \varrho, \alpha\Delta t)\, \psi'(\tau)\, dt, \tag{3.4}$$

and

$$\mathcal{R}\mu(t) = \lambda V_1(\varrho(t), t) - 2\lambda \int_0^t k_1(\Delta_t \varrho, \beta\Delta t)\, \varrho'(\tau)\, \chi(\tau)\, d\tau$$
$$+ \lambda\varrho'(t)\, \chi(t) + 2\lambda \int_0^t k(\Delta_t \varrho, \beta\Delta t)\, \chi'(\tau)\, d\tau. \tag{3.5}$$

Let (x, t) be a fixed point in $\{-\infty < x < \varrho,\ 0 < t < A\}$, and denote $k(x-\xi, \alpha\Delta t)$ by k, and $u(\xi, \tau)$ by u. We integrate Green's identity

$$\alpha \frac{\partial}{\partial \xi}\left(k \frac{\partial u}{\partial \xi} - u \frac{\partial k}{\partial \xi}\right) - \frac{\partial}{\partial \tau}(uk) \equiv 0$$

over $\{-R \leqslant \xi \leqslant \varrho(\tau),\ 0 \leqslant \tau \leqslant t - \varepsilon$, and let $R \to \infty$ and then $\varepsilon \to 0$. From the δ-function property of k, we get

$$u(x, t) = \int_{-\infty}^0 k(x-\xi, \alpha t)\, f(\xi)\, d\xi + \int_0^t k(x - \varrho(\tau), \alpha\Delta t)\, \mu(\tau)\, \varrho(\tau)\, d\tau$$
$$+ \frac{\alpha}{\gamma} \int_0^t k(x - \varrho(\tau), \alpha\Delta t)\, \mathcal{L}\mu(\tau)\, d\tau + \alpha \int_0^t k_1(x - \varrho(\tau), \alpha\Delta t)\, \mu(\tau)\, d\tau, \tag{3.6}$$

where $u_1(\varrho(\tau), \tau)$ has been replaced by $\mathcal{L}\mu(\tau)/\gamma$.

This equation can be differentiated with respect to x. After substituting for the resulting k_{11} from the identity (2.17), integrating a couple of integrals by parts, using $f(0)=\mu(0)$, and letting $x\to\varrho(t)-0$ we get

$$\frac{1}{2}u_1(\varrho(t),t)=\gamma\int_{-\infty}^{t}k(\varrho(t)-\xi,\alpha t)f'(\xi)\,d\xi+\alpha\int_0^t k_1(\Delta_t\varrho,\alpha\Delta t)\,\mathfrak{L}\mu(\tau)\,d\tau$$
$$+\gamma\int_0^t k(\Delta_t\varrho,\alpha\Delta t)\,\mu'(\tau)\,d\tau. \tag{3.7}$$

A similar calculation on v leads to

$$\frac{1}{2}v_1(\varrho(t),t)=\lambda\int_0^{\infty}k(\varrho(t)-\xi,\beta t)g'(\xi)\,d\xi-\beta\int_0^t k_1(\Delta_t\varrho,\beta\Delta t)\,\mathfrak{R}\mu(\tau)\,d\tau$$
$$-\lambda\int_0^t k(\Delta_t\varrho,\beta\Delta t)\,\mu'(\tau)\,d\tau. \tag{3.8}$$

Substituting these expression into (I5) or II4) leads, after some rearranging, to

$$[\lambda(4\alpha)^{-\frac{1}{2}}+\gamma(4\beta)^{-\frac{1}{2}}]\frac{1}{\sqrt{\pi}}\int_0^t\frac{\mu'(t)}{(t-\tau)^{\frac{1}{2}}}\,d\tau=-\tfrac{1}{2}H(\mu_{(1)}(t),\mu_{(2)}(t),\ldots,\mu_{(n)}(t),t)$$
$$+\lambda\int_0^{\infty}k(\varrho(t)-\xi,\beta t)\,g'(\xi)\,d\xi-\gamma\int_{-\infty}^{0}k(\varrho(t)-\xi,\alpha t)f'(\xi)\,d\xi$$
$$-\alpha\int_0^t k_1(\Delta_t\varrho,\alpha\Delta t)\,\mathfrak{L}\mu(\tau)\,d\tau-\beta\int_0^t k_1(\Delta_t\varrho,\beta\Delta t)\,\mathfrak{R}\mu(\tau)\,d\tau$$
$$+\gamma(4\pi\alpha)^{-\frac{1}{2}}\int_0^t[1-e^{-(\Delta_t\varrho)^2/4\alpha\Delta t}]\frac{\mu'(\tau)}{(\Delta t)^{\frac{1}{2}}}\,d\tau$$
$$+\lambda(4\pi\beta)^{-\frac{1}{2}}\int_0^t[1-e^{-(\Delta_t\varrho)^2/4\beta\Delta t}]\frac{\mu'(\tau)}{(\Delta t)^{\frac{1}{2}}}\,d\tau, \tag{3.9}$$

where $\mu_{(1)}(t)=\mu(t)$, $\mu_{(j+1)}(t)=\int_0^t\mu_{(j)}(s)\,ds$, $j=1,\ldots,n-1$.

We now form the Riemann–Liouville integral of order $\frac{1}{2}$ of both sides of this last equation. (See, for example Riesz [17].) Since $\mu(0)=a$ we get

$$\frac{1}{2}\left[\frac{\gamma}{\alpha^{\frac{1}{2}}}+\frac{\lambda}{\beta^{\frac{1}{2}}}\right](\mu(t)-a)=-\frac{1}{2\sqrt{\pi}}\int_0^t(t-\vartheta)^{-\frac{1}{2}}H\,d\vartheta$$
$$+\frac{1}{\sqrt{\pi}}\int_0^t\frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}}\left[\lambda\int_0^{\infty}k(\varrho(\vartheta)-\xi,\beta\vartheta)\,g'(\xi)\,d\xi-\int_{-\infty}^{0}k(\varrho(\vartheta)-\xi,\alpha\vartheta)f'(\xi)\,d\xi\right]$$
$$-\frac{\alpha}{\sqrt{\pi}}\int_0^t\frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}}\int_0^{\vartheta}k_1(\Delta_\vartheta\varrho,\alpha\Delta\vartheta)\,\mathfrak{L}\mu(\tau)\,d\tau$$

$$-\frac{\beta}{\sqrt{\pi}}\int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}}\int_0^\vartheta k_1(\Delta_\vartheta\varrho, \alpha\Delta\vartheta)\,\mathcal{R}\mu(\tau)\,d\tau$$

$$+\frac{\gamma}{2\sqrt{\pi}\,\alpha^{\frac{1}{2}}}\int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}}\int_0^\vartheta [1-e^{-(\Delta_\vartheta\varrho)^2/4\alpha\Delta\vartheta}]\,\frac{\mu'(\tau)}{(\Delta\vartheta)^{\frac{1}{2}}}\,d\tau$$

$$+\frac{\lambda}{2\sqrt{\pi}\,\beta^{\frac{1}{2}}}\int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}}\int_0^\vartheta [1-e^{-(\Delta_\vartheta\varrho)^2/4\beta\Delta\vartheta}]\,\frac{\mu'(\tau)}{(\Delta\vartheta)^{\frac{1}{2}}}\,d\tau.$$

We examine the first two integrals on the right of this equation in order to write them in a more tractable form. We note that, by (2.7), or rather the comments following that formula, at $\vartheta=0$ the bracket in the second integral has the value $\frac{1}{2}[\lambda g'(0)-\gamma f'(0)]=\frac{1}{2}H(a, 0, \ldots, 0)$. We can therefore integrate these two integrals by parts then, and the integrated terms cancel. Further integrations by parts with respect ξ leads to

$$\frac{1}{2}\left(\frac{\gamma}{\alpha^{\frac{1}{2}}}+\frac{\lambda}{\beta^{\frac{1}{2}}}\right)(\mu(t)-a=\frac{2\lambda g'(0)}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}\varrho'(\vartheta)\,k(\varrho(\vartheta),\beta\vartheta)\,d\vartheta$$

$$+\frac{2\lambda\beta g'(0)}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}k_1(\varrho(\vartheta),\beta\vartheta)\,d\vartheta+\frac{2\gamma f'(0)}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}\varrho'(\vartheta)k(\varrho(\vartheta),\alpha\vartheta)\,d\vartheta$$

$$+\frac{2\alpha\gamma f'(0)}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}k_1(\varrho(\vartheta),\alpha\vartheta)\,d\vartheta+\frac{2\lambda}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}\varrho'(\vartheta)\,d\vartheta\int_0^t k(\varrho(\vartheta)-\xi,\beta\vartheta)g''(\xi)\,d\xi$$

$$+\frac{2\lambda\beta}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}\,d\vartheta\int_0^\infty k_1(\varrho(\vartheta)-\xi,\beta\vartheta)\,g''(\xi)\,d\xi$$

$$-\frac{2\gamma}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}\varrho'(\vartheta)\,d\vartheta\int_{-\infty}^0 k(\varrho(\vartheta)-\xi,\alpha\vartheta)\,f''(\xi)\,d\xi$$

$$-\frac{2\alpha\gamma}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}\,d\vartheta\int_{-\infty}^0 k_1(\varrho(\vartheta)-\xi,\alpha\vartheta)\,f''(\xi)\,d\xi-\frac{1}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}\frac{d}{d\vartheta}H\,d\vartheta$$

$$-\frac{\alpha}{\sqrt{\pi}}\int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}}\int_0^\vartheta k_1(\Delta_\vartheta\varrho,\alpha\Delta\vartheta)\,\mathcal{L}\mu(\tau)\,d\tau-\frac{\beta}{\sqrt{\pi}}\int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}}\int_0^\vartheta k_1(\Delta_\vartheta\varrho,\beta\Delta\vartheta)\,\mathcal{R}\mu(\tau)\,d\tau$$

$$+\frac{\gamma}{2\pi\alpha^{\frac{1}{2}}}\int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}}\int_0^\vartheta [1-e^{-(\Delta_\vartheta\varrho)^2/4\alpha\Delta\vartheta}]\,\frac{\mu'(\tau)}{(\Delta\vartheta)^{\frac{1}{2}}}\,d\tau$$

$$+\frac{\lambda}{2\pi\beta^{\frac{1}{2}}}\int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}}\int_0^\vartheta [1-e^{-(\Delta_\vartheta\varrho)^2/4\beta\Delta\vartheta}]\,\frac{\mu'(\tau)}{(\Delta\vartheta)^{\frac{1}{2}}}\,d\tau.$$

Then our integral equation takes the form

$$\mu(t) = a + \frac{2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}}{\gamma\beta^{\frac{1}{2}} + \lambda\alpha^{\frac{1}{2}}} \sum_{1}^{13} I_n, \tag{3.10}$$

where the I_n's are given, in order, by the integrals standing on the right side of the previous equation.

Let us now denote by $C_a^1[0, A)$ the class of functions, each of which is continuously differentiable on $[0, A)$ and has the value a at $t=0$. Then the right-hand side of (3.9) defines a mapping, S, of $C_a^1[0, A)$ into, certainly, the continuous functions on $[0, A)$ with initial value a. We shall show that in fact S maps $C_a^1[0, A)$ into $C_a^1[0, A)$.

THEOREM 3 A. *Let*

$$S\mu(t) \equiv a + \frac{2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}}{\gamma\beta^{\frac{1}{2}} + \lambda\alpha^{\frac{1}{2}}} \sum_{n=1}^{13} I_n, \quad 0 \leqslant t < A.$$

If $\mu \in C_a^1[0, A)$, *then* $S\mu \in C_a^1[0, A)$ *and*

$$\frac{d}{dt} S\mu(t) = \frac{2\alpha^{\frac{1}{2}}\alpha^{\frac{1}{2}}}{\gamma\beta^{\frac{1}{2}} + \lambda\alpha^{\frac{1}{2}}} \sum_{n=0}^{19} J_n, \quad 0 \leqslant t < A,$$

where the J_n*'s are defined below.*

Proof. Each I_n, $n=1$ to $n=9$, is in the form

$$C \int_0^t (t-\vartheta)^{\frac{1}{2}} \Phi(\vartheta)\, d\vartheta,$$

where C is constant and $\Phi(\vartheta)$ is continuous in $[0, A)$, so that each of these I_n's is differentiable, with a continuous derivative in $[0, A)$ of the form

$$\frac{C}{2} \int_0^t (t-\vartheta)^{-\frac{1}{2}} \Phi(\vartheta)\, d\vartheta.$$

We set $J_n \equiv dI_n/dt$, $n=1, \ldots, 9$ and consider the differentiability of $I_{10}, \ldots, I_{13}$.

$$I_{10} = \frac{1}{4\pi\alpha^{\frac{1}{2}}} \int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}} \int_0^\vartheta \frac{\mathfrak{L}\mu(\tau)}{(\Delta\vartheta)^{\frac{1}{2}}} \frac{\Delta_\vartheta \varrho}{\Delta\vartheta} [e^{-(\Delta_\vartheta\varrho)^2/4\alpha\Delta\vartheta} - 1]\, d\tau$$

$$+ \frac{1}{4\pi\alpha^{\frac{1}{2}}} \int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}} \int_0^\vartheta \frac{\mathfrak{L}\mu(\vartheta)}{(\Delta\vartheta)^{\frac{1}{2}}} \left[\frac{\Delta_\vartheta \varrho}{\Delta\vartheta} - \varrho'(\vartheta)\right] d\tau + \frac{1}{4\pi\alpha^{\frac{1}{2}}} \int_0^t \frac{\varrho'(\vartheta)\, d\vartheta}{(t-\vartheta)^{\frac{1}{2}}} \int_0^\vartheta \frac{\mathfrak{L}\mu(\tau)}{(\vartheta-\tau)^{\frac{1}{2}}}\, d\tau$$

$\equiv I_{10a} + I_{10b} + I_{10c}$ respectively.

In I_{10a} and I_{10b} we integrate by parts to get

$$I_{10a}=\frac{1}{2\pi\alpha^{\frac12}}\int_0^t (t-\vartheta)^{\frac12}\,d\vartheta\int_0^{\vartheta}\mathfrak{L}\mu(\tau)\frac{\partial}{\partial\vartheta}\left\{\frac{\Delta_\vartheta\varrho}{(\Delta\vartheta)^{\frac32}}\,(e^{-(\Delta_\vartheta\varrho)^2/4\alpha\Delta\vartheta}-1]\right\}d\tau,$$

$$I_{10b}=\frac{1}{2\pi\alpha^{\frac12}}\int_0^t (t-\vartheta)^{\frac12}\,d\vartheta\int_0^{\vartheta}\mathfrak{L}\mu(\tau)\frac{\partial}{\partial\vartheta}\left\{\frac{1}{(\Delta\vartheta)^{\frac12}}\left[\frac{\Delta_\vartheta\varrho}{\Delta\vartheta}-\varrho'(\vartheta)\right\}d\tau.$$

We handle I_{10c} differently:

$$I_{10c}=\frac{1}{4\pi\alpha^{\frac12}}\int_0^t \mathfrak{L}\mu(\tau)\,d\tau\int_\tau^t\frac{\varrho'(\vartheta)\,d\vartheta}{(t-\vartheta)^{\frac12}(\vartheta-\tau)^{\frac12}}=\frac{1}{4\pi\alpha^{\frac12}}\int_0^t\mathfrak{L}\mu(\tau)\,d\tau\int_0^1\frac{\varrho'(\tau+s(t-\tau))\,ds}{s^{\frac12}(1-s)^{\frac12}}.$$

Each of these terms is clearly continuously differentiable and we thus get

$$\begin{aligned}\frac{d}{dt}I_{10}=&\frac{1}{4\pi\alpha^{\frac12}}\int_0^t\frac{d\vartheta}{(t-\vartheta)^{\frac12}}\int_0^{\vartheta}\mathfrak{L}\mu(\tau)\frac{\partial}{\partial\vartheta}\left\{\frac{\Delta_\vartheta\varrho}{(\Delta\vartheta)^{\frac32}}\,[e^{-(\Delta_\vartheta\varrho)^2/4\alpha\Delta\vartheta}-1]\right\}d\tau\\&+\frac{1}{4\pi\alpha^{\frac12}}\int_0^t\frac{d\vartheta}{(t-\vartheta)^{\frac12}}\int_0^{\vartheta}\mathfrak{L}\mu(\tau)\frac{\partial}{\partial\vartheta}\left\{\frac{1}{(\Delta\vartheta)^{\frac12}}\left[\frac{\Delta_\vartheta\varrho}{\Delta\vartheta}-\varrho'(\vartheta)\right]\right\}d\tau\\&+\frac{1}{4\pi\alpha^{\frac12}}\int_0^t\mathfrak{L}\mu(\tau)\,d\tau\int_0^1\frac{s^{\frac12}\varrho''(\tau+s(t-\tau))\,ds}{(1-s)^{\frac12}}+\frac{1}{4\alpha^{\frac12}}\varrho'(t)\,\mathfrak{L}\mu(t)\\\equiv&J_{10}+J_{11}+J_{12}+J_{13},\ \text{respectively.}\end{aligned}$$

Similarly

$$\begin{aligned}\frac{dI_{11}}{dt}=&\frac{1}{4\pi\beta^{\frac12}}\int_0^t\frac{d\vartheta}{(t-\vartheta)^{\frac12}}\int_0^{\vartheta}\mathfrak{R}\mu(\tau)\frac{\partial}{\partial\vartheta}\left\{\frac{\Delta_\vartheta\varrho}{(\Delta\vartheta)^{\frac32}}\,[e^{-(\Delta_\vartheta\varrho)^2/4\beta\Delta\vartheta}-1]\right\}d\tau\\&+\frac{1}{4\pi\beta^{\frac12}}\int_0^t\frac{d\vartheta}{(t-\vartheta)^{\frac12}}\int_0^{\vartheta}\mathfrak{R}\mu(\tau)\frac{\partial}{\partial\vartheta}\left\{\frac{1}{(\Delta\vartheta)^{\frac12}}\left[\frac{\Delta_\vartheta\varrho}{\Delta\vartheta}-\varrho'(\vartheta)\right]\right\}d\tau\\&+\frac{1}{4\pi\beta^{\frac12}}\int_0^t\mathfrak{R}\mu(\tau)\,d\tau\int_0^1\frac{s^{\frac12}\varrho''(\tau+s(t-\tau))}{(1-s)^{\frac12}}\,ds+\frac{1}{4\beta^{\frac12}}\varrho'(t)\,\mathfrak{R}\mu(t)\\\equiv&J_{14}+J_{15}+J_{16}+J_{17},\ \text{respectively.}\end{aligned}$$

The two remaining terms can be integrated by parts as they stand, then differentiated with respect to t to get

$$J_{18}\equiv\frac{dI_{12}}{dt}=\frac{\gamma}{2\pi\alpha^{\frac12}}\int_0^t\frac{d\vartheta}{(t-\vartheta)^{\frac12}}\int_0^{\vartheta}\mu'(\tau)\frac{\partial}{\partial\vartheta}\left\{\frac{1}{(\Delta\vartheta)^{\frac12}}\,[1-e^{-(\Delta\vartheta)^2/4\alpha\Delta\vartheta}]\right\}d\tau$$

and $$J_{19}\equiv\frac{d}{dt}I_{13}=\frac{\lambda}{2\pi\beta^{\frac12}}\int_0^t\frac{d\vartheta}{(t-\vartheta)^{\frac12}}\int_0^{\vartheta}\mu'(\tau)\frac{\partial}{\partial\vartheta}\left\{\frac{1}{(\Delta\vartheta)^{\frac12}}\,[1-e^{-(\Delta_\vartheta\varrho)^2/4\beta\Delta\vartheta}]\right\}d\tau.$$

This completes the proof of the theorem.

We have seen that any solution to either Problem I or Problem II, satisfying the exponential growth conditions (3.1) leads to a solution μ of the integral equation (3.10).

Conversely, suppose we have a solution μ of the integral equation (3.10). Then, by Theorem 3A, it is differentiable (and in the case of Problem I, defines $\varrho \in C^2[0, A)$, $\varrho(0)=0$). We can operate on (3.10) with the Riemann-Liouville integral of order $\frac{1}{2}$, leading back to (3.9). Further, we can determine ψ and χ from (2.6) (with $\nu=\mu$) and form the functions u and v by (2.5). These functions will then both equal μ on $x=\varrho(t)$, and (3.9) is equivalent to $H(\mu_{(1)}(t), \mu_{(2)}(t), \ldots, \mu_{(n)}(t), t)+\gamma u_1(\varrho(t), t)=\lambda v_1(\varrho(t), t)$. Thus we are able to conclude the equivalence of the integral equation (3.10) with the original problems. In particular, if the integral equation has a unique solution, then the original problems have unique solutions within the class of functions satisyfing exponential bounds of the form (3.1).

4. Existence and uniqueness

We note that in the case of Problem II equation (3.10) is in a sense nearly linear—the only non-linearities arising from the contribution of the boundary function H. However, in the other case it is highly non-linear since then ϱ is the integral of μ. Because of the relative simplicity of the equation for Problem II we confine our attention from here on to Problem I, the modification necessary to adapt the argument being readily apparent. In particular we remark that in the simpler case the strip arguments to which we later resort (Theorem 4Q) are unnecessary.

We note that the estimates established in section 2 are available with μ and ν identified and with $N=\bar{N}$. (Note that in the simpler case $\bar{N}$, as used in section 2, becomes part of the data of the problem).

It will be convenient to introduce the following family of standard norms in the space $C^1[0, A)$, i.e., the class of functions which are continuously differentiable in the interval $0 \leqslant t < A$. Given any positive $\sigma < A$ and any $\mu \in C^1[0, A)$ we define

$$\|\mu\|_\sigma = \sup_{0\leqslant t\leqslant\sigma} |\mu(t)| + \sup_{0\leqslant t\leqslant\sigma} |\mu'(t)|. \tag{4.1}$$

Clearly $\|\mu\|_\sigma$ is finite for each such σ and μ, and is nondecreasing in σ for each μ.

Throughout this section we will use B as a generic symbol for positive constants which depend on M, A, T, N, δ, and of course α, β, γ, and λ.

We will use b as a generic symbol for positive constants which depend *only* on M, α, β, γ, and λ. We will as before suppress the arguments α, β, γ, and λ.

We are first interested in showing that for N sufficiently large and σ sufficiently small then $\mu \in C^1_a[0, A)$ and $\|\mu\|_\sigma \leqslant N$ implies $\|S\mu\|_\sigma \leqslant N$. To this end we begin with the following

LEMMA 4A. *Let $N>0$ and a positive $T>A$ be given. If $\|\mu\|_T \leqslant N$ then*

$$\sup_{0\leqslant t\leqslant\sigma} |S\mu(t)| \leqslant b + B\sigma,$$

where S is defined in Theorem 3A and where b and B are constants as described above.

Proof. We take $b=|a|=|f(0)|\leqslant M$. It thus suffices to show that each I_n is bounded by $B\sigma$. We give the proof for I_6 and I_9, the others being simpler but similar, though differing in technical detail. We merely remark that I_{10} and I_{11} may be estimated simply if one establishes first that

$$|\mathcal{L}\mu(t)| \leqslant B, \quad |\mathcal{R}\mu(t)| \leqslant B; \quad 0\leqslant t\leqslant T,$$

which one may do by the methods of this section.

For I_6 we find

$$|I_6| \leqslant B\int_0^t (t-\vartheta)^{\frac{1}{2}}\vartheta^{-\frac{1}{2}}\,d\vartheta \int_{-\infty}^{\infty} k(\varrho(\vartheta)-\xi, h\beta\vartheta)\exp(\xi^2/4\beta A)\,d\xi,$$

where we have estimated k_1 by (2.4) and g'' by (A 2) of the introduction. We choose h so that $A-hT=\delta/2=(T-A)/2$. Then

$$|I_6| \leqslant B\int_0^t (t-\vartheta)^{\frac{1}{2}}\vartheta^{-\frac{1}{2}}\exp\left[\varrho^2(\vartheta)/4\beta(A-h\vartheta)\right]d\vartheta.$$

Since $\varrho^2(\vartheta)/4\beta(A-h\vartheta)\leqslant N^2T^2/2\beta\delta$ we conclude

$$|I_6| \leqslant B\sigma, \quad 0\leqslant t\leqslant\sigma\leqslant T<A.$$

For I_9 we remark that $dH/d\vartheta$ is bounded by a bound which depends on N and T, and hence the estimate follows immediately.

LEMMA 4B. *Let $N>0$ and a positive $T<A$ be given. If $\|\mu\|_T\leqslant N$, then*

$$\sup_{0\leqslant t\leqslant\sigma}\left|\frac{d}{dt}S\mu(t)\right| \leqslant b\exp(B\sigma^2) + B\sigma^{\frac{1}{2}}.$$

Proof. Again we do sample calculations. We consider

$$J_1 = \frac{\lambda g'(0)}{\sqrt{\pi}}\int_0^t (t-\vartheta)^{\frac{1}{2}}\mu(\vartheta)\,k(\varrho(\vartheta),\beta\vartheta)\,d\vartheta = \frac{\lambda g'(0)}{2\pi\beta^{\frac{1}{2}}}\int_0^t \frac{(\mu(\vartheta)-a)\exp(-\varrho^2(\vartheta)/4\beta\vartheta)}{(t-\vartheta)^{\frac{1}{2}}\vartheta^{\frac{1}{2}}}\,d\vartheta$$
$$+\frac{\lambda g'(0)\,a}{2\pi\beta^{\frac{1}{2}}}\int_0^t \frac{\exp(-\varrho^2(\vartheta)/4\beta\vartheta)-1}{(t-\vartheta)^{\frac{1}{2}}\vartheta^{\frac{1}{2}}}\,d\vartheta + \frac{\lambda g'(0)\,a}{2\beta^{\frac{1}{2}}}.$$

We estimate $$|J_1| \leqslant b + B\sigma^{\frac{1}{2}}, \quad 0 \leqslant t \leqslant \sigma \leqslant T < A.$$

The others are similarly estimated through J_5, while J_6, J_7, and J_8 are bounded by $b \exp(N^2\sigma^2)$, and J_9, J_{10}, J_{11}, and J_{12} by $B\sigma^{\frac{1}{2}}$. Next

$$J_{13} = \varrho'(t)\,\mathfrak{L}\mu(t)/4\alpha^{\frac{1}{2}} = \frac{\gamma\varrho'(t)}{4\alpha^{\frac{1}{2}}}\,U_1(\varrho(t), t) - \frac{\alpha^{\frac{1}{2}}\gamma\varrho'(t)}{2}\int_0^t k_1(\Delta_t\varrho, \alpha\Delta t)\,\varrho'(\tau)\,\psi(\tau)\,d\tau$$

$$-\frac{\gamma\varrho'^2(t)\,\psi(t)}{4\alpha^{\frac{1}{2}}} + \frac{\gamma\varrho'(t)}{4\alpha^{\frac{1}{2}}}\int_0^t k(\Delta_t\varrho, \alpha\Delta t)\,\psi'(\tau)\,d\tau.$$

By applying (2.8), or more properly, the remarks just following that formula, to U_1 and the estimates of section 2 to the other terms we compute $|J_{13}| \leqslant b + B\sigma^{\frac{1}{2}}$, $0 \leqslant t \leqslant \sigma \leqslant T < A$. The estimates on the remaining J's are similar.

These lemmas immediately imply the following.

THEOREM 4C. *Let $N > 0$ and a positive $T < A$ be given. If $\|\mu\|_T \leqslant N$, then*

$$\|S\mu\|_\sigma \leqslant be^{B\sigma^2} + B\sigma^{\frac{1}{2}}, \quad 0 \leqslant \sigma \leqslant T.$$

From this estimate we get the following

THEOREM 4D. *Given a positive $T < A$, there is an N_1, depending only on M, and a σ_0 depending only on N_1, M, A, T, δ such that $\|\mu\|_\sigma \leqslant N_1$ implies $\|S\mu\|_\sigma \leqslant N_1$ for $0 \leqslant \sigma \leqslant \sigma_0$.*

Proof. Choose $T < A$ and $N_1 \geqslant 3b$ where b is the constant appearing in Theorem 4C. For any positive $\sigma \leqslant T$ and any $\mu \in C^1_\alpha[0, A)$ with $\|\mu\|_\sigma \leqslant N_1$, μ can be redefined, if necessary, in the interval $\sigma < t < A$ so that $\|\mu\|_T \leqslant 2N_1 \equiv N$. Then by Theorem 5C

$$\|S\mu\|_\sigma \leqslant be^{B\sigma^2} + B\sigma^{\frac{1}{2}}, \quad 0 \leqslant \sigma \leqslant T.$$

We choose σ_0 so small that $e^{B\sigma^2} \leqslant 2$ and $B\sigma^{\frac{1}{2}} \leqslant b$. Then for $\sigma \leqslant \sigma_0$ we have

$$\|S\mu\|_\sigma \leqslant 2b + b = 3b \leqslant N_1.$$

Clearly σ_0 depends only on the stated parameters.

We now want to head toward a solution of the integral equation by iteration. The work so far in this section has established that an iteration procedure can be defined, at least in a sufficiently small interval. We now want to establish the convergence of such a procedure. For this purpose we estimate $\|S\mu_1 - S\mu_2\|_\sigma$ in terms of $\|\mu_1 - \mu_2\|_\sigma$.

In the following we will assume that μ_1 and μ_2 are given elements of $C^1_\alpha[0, A)$. Each of these then gives rise to corresponding values of ϱ, φ, ζ, ψ, and χ, which will be designated by ϱ_1, ϱ_2, etc., respectively. We also find it convenient to use the symbol D for $\|\mu_1-\mu_2\|_\sigma$. It follows immediately that

$$\left.\begin{aligned} |\mu_1(t)-\mu_2(t)| &\leqslant Dt, \\ |\varrho_1(t)-\varrho_2(t)| &\leqslant Dt^2, \\ |\Delta_t\varrho_1-\Delta_t\varrho_2| &\leqslant Dt\Delta t, \end{aligned}\right\} \quad 0\leqslant t\leqslant\sigma\leqslant T<A. \tag{4.2}$$

LEMMA 4E. *Let $N>0$ and a positive $T<A$ be given. If $\|\mu_1\|_T\leqslant N$, $\|\mu_2\|_T\leqslant N$ then*

$$\left.\begin{aligned} |\varphi_1(t)-\varphi_2(t)| &\leqslant BDt \\ |\zeta_1(t)-\zeta_2(t)| &\leqslant BDt \end{aligned}\right\} \quad 0\leqslant t\leqslant\sigma\leqslant T<A.$$

Proof.

$$\begin{aligned} |\varphi_1(t)-\varphi_2(t)| &\leqslant |\mu_1(t)-\mu_2(t)|+|U(\varrho_1(t),t)-U(\varrho_2(t),t)| \\ &\leqslant Dt+\int_{-\infty}^{\infty}|k(\varrho_1(t)-\xi,\alpha t)-k(\varrho_2(t)-\xi,\alpha t)|\,|F(\xi)|\,d\xi. \end{aligned}$$

The integral can be estimated by estimating F by (A2), applying the mean value theorem to the difference and estimating the resulting k_1 by (2.4), and choosing h (entering through (2.4)) appropriately.

LEMMA 4F. *Let $N>0$ and a positive $T<A$ be given. If $\|\mu_1\|_T\leqslant N$, $\|\mu_2\|_T\leqslant N$, then, for $0\leqslant t\leqslant\sigma\leqslant T<A$,*

$$2\alpha|k_1(\Delta_t\varrho_1,\alpha\Delta t)-k_1(\Delta_t\varrho_2,\alpha\Delta t)|\leqslant\frac{BD(\Delta t)^{-\frac{1}{2}}}{\sqrt{\pi}}$$

$$2\beta|k_1(\Delta_t\varrho_1,\beta\Delta t)-k_1(\Delta_t\varrho_2,\beta\Delta t)|\leqslant\frac{BD(\Delta t)^{-\frac{1}{2}}}{\sqrt{\pi}}.$$

Proof. We give the proof of the first inequality. The expression on the left of that inequality is

$$\begin{aligned} &\left|\frac{\Delta_t\varrho_1}{\Delta t}\,\frac{e^{-(\Delta_t\varrho_1)^2/4\alpha\Delta t}}{2\sqrt{\pi}\,\alpha^{\frac{1}{2}}(\Delta t)^{\frac{1}{2}}}-\frac{\Delta_t\varrho_2}{\Delta t}\,\frac{e^{-(\Delta_t\varrho_2)^2/4\alpha\Delta t}}{2\sqrt{\pi}\,\alpha^{\frac{1}{2}}(\Delta t)^{\frac{1}{2}}}\right|\leqslant\frac{|\Delta_t\varrho_1-\Delta_t\varrho_2|}{2\sqrt{\pi}\,\alpha^{\frac{1}{2}}(\Delta t)^{\frac{3}{2}}}+\frac{|\Delta_t\varrho_2|\,|e^{-(\Delta_t\varrho_1)^2/4\alpha\Delta t}-e^{-(\Delta_t\varrho_2)^2/4\alpha\Delta t}|}{2\sqrt{\pi}\,\alpha^{\frac{1}{2}}(\Delta t)^{\frac{3}{2}}} \\ &\qquad\leqslant\frac{Dt(\Delta t)^{-\frac{1}{2}}}{2\sqrt{\pi}\,\alpha^{-\frac{1}{2}}}+\frac{N\Delta t}{2\sqrt{\pi}\,\alpha^{\frac{1}{2}}(\Delta t)^{\frac{3}{2}}}\,\frac{|[(\Delta_t\varrho_1)^2-(\Delta_t\varrho_2)^2]|}{4\alpha\Delta t}\leqslant BD(\Delta t)^{-\frac{1}{2}}/\sqrt{\pi}. \end{aligned}$$

The function ψ is related to φ through the operator K_α defined by (2.12). For each μ in $C_a^1[0,A)$ we have a ϱ, and hence by (2.12) a K_α, and from distinct μ's there will in general result distinct K_α's. We dinstinguish the K_α arising from μ_1 from that coming from μ_2 by writing $K_{1\alpha}$ for the first and $K_{2\alpha}$ for the second.

LEMMA 4G. *Let $N>0$ and a positive $T<A$ be given. If $\|\mu_1\|_T \leqslant N$, $\|\mu_2\|_T \leqslant N$, and if q_1 and q_2 are continuous on $[0, T]$ and satisfy there*

$$|q_2(t)| \leqslant C_1 t^p/\Gamma(p+1)$$

$$|q_1(t)-q_2(t)| \leqslant C_2 t^{p+1}/\Gamma(p+1)$$

then there is a constant $B=B(T, N)$ such that

$$|K_{1\alpha} q_1(t) - K_{2\alpha} q_2(t)| \leqslant B(C_1+C_2) D t^{p+\frac{3}{2}}/\Gamma(p+3/2).$$

$$|K_{1\beta} q_1(t) - K_{2\beta} q_2(t)| \leqslant B(C_1+C_2) D t^{p+\frac{3}{2}}/\Gamma(p+3/2).$$

Proof. Consider

$$\begin{aligned} |K_{1\alpha} q_1(t) - K_{2\alpha} q_2(t)| &\leqslant 2\alpha \int_0^t |k_1(\Delta_t \varrho_1, \alpha\Delta t)\, q_1(\tau) - k_1(\Delta_t \varrho_2, \alpha\Delta t)\, q_2(\tau)|\, d\tau \\ &\leqslant 2\alpha \int_0^t k_1(\Delta_t \varrho_1, \alpha\Delta t)\, |q_1(\tau) - q_2(\tau)|\, d\tau \\ &\quad + 2\alpha \int_0^t |q_2(\tau)|\, |k_1(\Delta_t \varrho_1, \alpha\Delta t) - k_1(\Delta_t \varrho_2, \alpha\Delta t)|\, d\tau. \end{aligned}$$

Then, we estimate the first term by

$$4(\pi\alpha)^{-\frac{1}{2}} \int_0^t \frac{N}{(\Delta t)^{\frac{1}{2}}} \frac{C_2 D \tau^{p+1}}{\Gamma(p+1)}\, d\tau \leqslant \frac{N C_2 D\, t^{p+\frac{3}{2}}}{2\alpha^{\frac{1}{2}} \sqrt{\pi}\, \Gamma(p+1)} \int_0^1 s^{p+1} (1-s)^{-\frac{1}{2}}\, ds \leqslant \frac{N C_2 D t^{p+\frac{3}{2}}}{2\alpha^{\frac{1}{2}}} \frac{1}{\Gamma(p+\frac{3}{2})}.$$

To estimate the second term we use Lemma 4F. This term is then bounded by

$$\int_0^t \frac{C_1 \tau^{p+1}}{\Gamma(p+1)} \frac{BD(\Delta t)^{-\frac{1}{2}}}{\sqrt{\pi}}\, d\tau = \frac{C_1 BD}{\sqrt{\pi}\, \Gamma(p+1)} \int_0^t \tau^{p+1} (t-\tau)^{-\frac{1}{2}}\, d\tau \leqslant \frac{C_1 B D t^{p+\frac{3}{2}}}{\Gamma(p+\frac{3}{2})},$$

and the result follows. The same calculations apply to $|K_{1\beta} q_1(t) - K_{2\beta} q_2(t)|$, with α replaced by β. Then taking as B the larger of those for the two cases gives the result.

LEMMA 4H. *Let $N>0$ and a positive $T<A$ be given. If $\|\mu_1\|_T \leqslant N$, $\|\mu_2\|_T \leqslant N$, then*

$$\left. \begin{aligned} |\psi_1(t) - \psi_2(t)| &\leqslant BDt \\ |\chi_1(t) - \chi_2(t)| &\leqslant BDt \end{aligned} \right\} \quad 0 \leqslant t \leqslant \sigma \leqslant T < A.$$

Proof. We observe that $|\varphi_2(t)| \leqslant B$, and, by Lemma 2 B, $|K_{2\alpha}^n \varphi(t)| \leqslant Bt^{n/2}/\Gamma(1+n/2)$. By induction, based on Lemma 4 G, we compute

$$|K_{1\alpha}^n \varphi_1(t) - K_{2\alpha}^n \varphi_2(t)| \leqslant (n+1)\, B^{n+1} D t^{1+n/2}/\Gamma(1+n/2),$$

from which the first conclusion follows. One estimates $|\chi_1(t) - \chi_2(t)|$ similarly.

LEMMA 4I. *Let $N > 0$ and a positive $T < A$ be given. If $q_1(t)$ and $q_2(t)$ are continuously differentiable in* $[0, T]$ *and satisfy*

$$\left.\begin{array}{r} |q_2'(t)| \leqslant C_1 t^p/\Gamma(p+1) \\ |q_1'(t) - q_2'(t)| \leqslant C_2 D t^p/\Gamma(p+1) \end{array}\right\} \quad 0 \leqslant t \leqslant T,$$

where $p \geqslant 0$, then

$$\left| \frac{d}{dt} K_{1\alpha} q(t) - \frac{d}{dt} K_{2\alpha} q_2(t) \right| \leqslant B(C_1 + C_2)\, D t^{p+\frac{1}{2}}/\Gamma(p+\tfrac{3}{2})$$

$$\left| \frac{d}{dt} K_{2\beta} q_1(t) - \frac{d}{dt} K_{2\beta} q_2(t) \right| \leqslant B(C_1 + C_2)\, D t^{p+1}/\Gamma(p+\tfrac{3}{2}).$$

The proof follows the same outline as that of Lemma 4G. The calculations are long but straightforward, so they are omitted.

LEMMA 4J. *Let $N > 0$ and a positive $T < A$ be given. If $\|\mu_1\|_T \leqslant N$ and $\|\mu_2\|_T \leqslant N$, then*

$$\left.\begin{array}{r} |\psi_1'(t) - \psi_2'(t)| \leqslant BD \\ |\chi_1'(t) - \chi_2'(t)| \leqslant BD \end{array}\right\} \quad 0 \leqslant t \leqslant \sigma \leqslant T < A.$$

Proof. One observes first that $|\varphi_2'(t)| \leqslant B$ and $|\varphi_1'(t) - \varphi_2'(t)| \leqslant BD$. The boundedness of φ_2' is clear, and in the difference a typical estimate is

$$|U_{11}(\varrho_1(t), t) - U_{11}(\varrho_2(t), t)| \leqslant \int_{-\infty}^{\infty} |k_{111}(\bar{\varrho} - \xi, \alpha t)| \exp(\xi^2/4\alpha A)\, d\xi$$

$$\leqslant BDt^2 \int_{-\infty}^{\infty} \frac{k(\bar{\varrho} - \xi, \alpha h t)}{t^{\frac{3}{2}}} \exp(\xi^2/4\alpha A)\, d\xi \leqslant BDT^{\frac{1}{2}} \sqrt{A/(A - hT)} \exp[\bar{\varrho}^2/4\alpha(A - hT)].$$

We choose $h > 1$ so that $A - hT = \delta/2$ and estimate $|\bar{\varrho}(t)| \leqslant Nt \leqslant NT$. One then iterates the application of Lemma 4I to estimate

$$\left| \frac{d}{dt} K_{1\alpha}^n \varphi_1(t) - \frac{d}{dt} K_{2\alpha}^n \varphi_2(t) \right|,$$

and sums to complete the proof.

LEMMA 4K. *Let $N > 0$ and a positive $T < A$, be given.*
If $\|\mu_1\|_T \leqslant N$, $\|\mu_2\|_T \leqslant N$, then

$$\left.\begin{aligned} |\mathfrak{L}\mu_1(t) - \mathfrak{L}\mu_2(t)| &\leqslant BDt \\ |\mathfrak{R}\mu_1(t) - \mathfrak{R}\mu_2(t)| &\leqslant BDt \end{aligned}\right\} \quad 0 \leqslant t \leqslant \sigma \leqslant T < A.$$

With the previous inequalities established these estimates follow simply from (3.4) and (3.5).

THEOREM 4L. *Let $N > 0$ and a positive $T < A$ be given. If $\|\mu_1\|_T \leqslant N$, $\|\mu_2\|_T \leqslant N$, then*

$$|S\mu_1(t) - S\mu_2(t)| \leqslant BDt, \quad 0 \leqslant t \leqslant \sigma \leqslant T < A.$$

We will consider the terms in $S\mu_1$ and $S\mu_2$ as given in Theorem 3A. The terms I_n occurring there are defined in section 3 and of course depend on μ_1 and μ_2 respectively. We distinguish these by superscripts. That is, I_n^1 and I_n^2 will be I_n computed for μ_1 and μ_2 respectively. It is clearly sufficient to show $|I_n^1 - I_n^2| \leqslant BDt$, $n = 1, 2, \ldots, 13$.

We compute a few typical examples, the others being similar. We consider

$$\begin{aligned}
|I_1^1 - I_1^2| &\leqslant \frac{2\lambda|g'(0)|}{\sqrt{\pi}} \int_0^t (t-\vartheta)^{\frac{1}{2}} |\mu_1(\vartheta)\, k(\varrho_1(\vartheta), \beta\vartheta) - \mu_2(\vartheta)\, k(\varrho_2(\vartheta), \beta\vartheta)|\, d\vartheta \\
&\leqslant \frac{2\lambda M}{\sqrt{\pi}} \int_0^t (t-\vartheta)^{\frac{1}{2}} k(\varrho_1(\vartheta), \beta\vartheta)\, |\mu_1(\vartheta) - \mu_2(\vartheta)|\, d\vartheta \\
&+ \frac{2\lambda M}{\sqrt{\pi}} \int_0^t (t-\vartheta)^{\frac{1}{2}} |\mu_2(\vartheta)|\, |k(\varrho_1(\vartheta), \beta\vartheta) - k(\varrho_2(\vartheta), \beta\vartheta)|\, d\vartheta \\
&\leqslant \frac{2\lambda M}{\sqrt{\pi}} \int_0^t (t-\vartheta)^{\frac{1}{2}} \frac{D\vartheta}{(4\pi\beta\vartheta)^{\frac{1}{2}}}\, d\vartheta \\
&+ \frac{2\lambda M N}{\sqrt{\pi}} \int_0^t \frac{(t-\vartheta)^{\frac{1}{2}}\vartheta}{(4\pi\beta\vartheta)^{\frac{1}{2}}} |\exp(-\varrho_1^2(\vartheta)/4\beta\vartheta) - \exp(-\varrho_2^2(\vartheta)/4\beta\vartheta)|\, d\vartheta \\
&\leqslant \frac{\lambda M D}{\pi\beta^{\frac{1}{2}}} \int_0^t (t-\vartheta)^{\frac{1}{2}} \vartheta^{\frac{1}{2}}\, d\vartheta + \frac{\lambda M N}{\pi\beta^{\frac{1}{2}}} \int_0^t (t-\vartheta)^{\frac{1}{2}}\, \vartheta^{\frac{1}{2}}\, \frac{|\varrho_1^2(\vartheta) - \varrho_2^2(\vartheta)|}{4\beta\vartheta}\, d\vartheta \\
&\leqslant BDt + \frac{\lambda M N}{\pi\beta^{\frac{1}{2}}} \int_0^t \frac{(t-\vartheta)^{\frac{1}{2}}\vartheta^{-\frac{1}{2}}|\varrho_1(\vartheta) - \varrho_2(\vartheta)|\, 2N\vartheta}{4\beta\vartheta}\, d\vartheta \\
&\leqslant BDt + \frac{\lambda M N^2}{2\pi\beta^{\frac{3}{2}}} \int_0^t (t-\vartheta)^{\frac{1}{2}} \vartheta^{\frac{3}{2}}\, d\vartheta \leqslant BDt.
\end{aligned}$$

The differences $|I_2^1 - I_2^2|$, $|I_3^1 - I_3^2|$ and $|I_4^1 - I_4^2|$ can be estimated similarly. Of the next four differences we estimate only $|I_6^1 - I_6^2|$ as typical.

$$|I_6^1 - I_6^2| \leqslant \frac{2\lambda\beta}{\sqrt{\pi}} \int_0^t (t-\vartheta)^{\frac{1}{2}} \int_0^\infty |k_1(\varrho_1(\vartheta)-\xi, \beta\vartheta) - k_1(\varrho_2(\vartheta)-\xi, \beta\vartheta)|\, |g''(\xi)|\, d\xi\, d\vartheta$$

$$\leqslant \frac{2M\lambda\beta}{\sqrt{\pi}} \int_0^t (t-\vartheta)^{\frac{1}{2}} D\vartheta^2 \int_{-\infty}^\infty |k_{11}(\bar{\varrho}-\xi, \beta\vartheta)|\, e^{\xi^2/4\alpha A}\, d\xi\, d\vartheta$$

$$\leqslant BD \int_0^t (t-\vartheta)^{\frac{1}{2}} \vartheta^{\frac{3}{2}} \int_{-\infty}^\infty k(\bar{\varrho}-\xi, h'\beta\vartheta)\, e^{\xi^2/4\alpha A}\, d\xi\, d\vartheta$$

(choosing h so that $A - hT = \delta/2$)

$$\leqslant BD \int_0^t (t-\vartheta)^{\frac{1}{2}} \vartheta^{\frac{3}{2}} \sqrt{2A/\delta}\, e^{N^2 T^2/2\delta}\, d\vartheta \leqslant BDt.$$

The difference $|I_9^1 - I_9^2|$ is easily estimated under the condition (A3) of the introduction, and the other differences are estimated as the preceeding ones, the calculations differing only in details.

THEOREM 4M. *Let $N > 0$ and a positive $T > A$ be given.* If $\|\mu_1\|_T \leqslant N$, $\|\mu_2\|_T \leqslant N$ *then*

$$\left| \frac{d}{dt} S\mu_1(t) - \frac{d}{dt} S\mu_2(t) \right| \leqslant BDt, \quad 0 \leqslant t \leqslant \sigma \leqslant T < A.$$

Proof. For this calculation we must estimate $|J_n^1 - J_n^2|$ where J_n^1 and J_n^2 are the terms of $(d/dt)\, S\mu_1(t)$, $(d/dt)\, S\mu_2(t)$, respectively, where the J_n's are defined in section 4.

The difference $|J_n^1 - J_n^2|$ can be estimated as in $|I_n^1 - I_n^2|$, $n = 1$, to 9, by replacing $(t-\vartheta)^{\frac{1}{2}}$ by $(t-\vartheta)^{-\frac{1}{2}}$ in those previous calculations. We proced on to $|J_{10}^1 - J_{10}^2|$:

$$|J_{10}^1 - J_{10}^2| \leqslant \frac{1}{4\pi\alpha^{\frac{1}{2}}} \int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}} \int_0^\vartheta \left| \mathfrak{L}\mu_1(\tau) \frac{\partial}{\partial\vartheta} \left\{ \frac{\Delta_\vartheta \varrho_1}{(\Delta\vartheta)^{\frac{3}{2}}} [e^{-(\Delta_\vartheta \varrho_1)^2/4\alpha\Delta\vartheta} - 1] \right\} \right.$$

$$\left. - \mathfrak{L}\mu_2(\tau) \frac{\partial}{\partial\vartheta} \left\{ \frac{\Delta_\vartheta \varrho_2}{(\Delta\vartheta)^{\frac{3}{2}}} [e^{-(\Delta_\vartheta \varrho_2)^2/4\alpha\Delta\vartheta} - 1] \right\} \right| d\tau$$

$$\leqslant \frac{1}{4\pi\alpha^{\frac{1}{2}}} \int_0^t \frac{d\vartheta}{(t-\vartheta)^{\frac{1}{2}}} \int_0^\vartheta |\mathfrak{L}\mu_1(\tau) - \mathfrak{L}\mu_2(\tau)| \left| \frac{\partial}{\partial\vartheta} \left\{ \frac{\Delta_\vartheta \varrho_1}{(\Delta\vartheta)^{\frac{3}{2}}} [e^{-(\Delta_\vartheta \varrho_1)^2/4\alpha\Delta\vartheta}] \right\} \right| d\tau$$

$$+ \frac{1}{4\pi\alpha^{\frac{1}{2}}} \int_0^t \frac{d\vartheta}{t\vartheta} \int_0^\vartheta |\mathfrak{L}\mu_2(\tau)| \left| \frac{\partial}{\partial\vartheta} \left\{ \frac{\Delta_\vartheta \varrho_1}{(\Delta\vartheta)^{\frac{3}{2}}} [e^{-(\Delta_\vartheta \varrho_1)^2/4\Delta\vartheta} - 1] \right\} \right.$$

$$\left. - \frac{\partial}{\partial\vartheta} \left\{ \frac{\Delta_\vartheta \varrho_2}{(\Delta\vartheta)^{\frac{1}{2}}} [e^{-(\Delta_\vartheta \varrho_2)^2/4\Delta\vartheta} - 1] \right\} \right| d\tau.$$

The first integral can be estimated by replacing $|\mathcal{L}\mu_1(\tau)-\mathcal{L}\mu_2(\tau)|$ by $BD\tau$, and estimating the differentiated term as in the proof of Theorem 4B. The extra τ arising from the difference $|\mathcal{L}\mu_1(\tau)-\mathcal{L}\mu_2(\tau)|$ enables one to get the estimate BDt for this integral. In estimating the second integral one replaces $|\mathcal{L}\mu_2(\tau)|$ by B and differentiates the terms out. These can then be broken up to take advantage of the estimates (4.2), and this second integral can also be estimated by BDt. $|J_{11}^1-J_{11}^2|$ is estimated similarly but more simply, using also $|\mu_1'(t)-\mu_2'(t)|\leqslant D$. $|J_{12}^1-J_{12}^2|$ is still simpler, and $|J_{13}^1-J_{13}^2|$ has already been essentially estimated by Theorem 4K since

$$|J_{13}^1-J_{13}^2|\leqslant\frac{1}{4\alpha^{\frac{1}{2}}}|\mu_1(t)-\mu_2(t)|\,|\mathcal{L}\mu_1(t)|+\frac{1}{4\alpha^{\frac{1}{2}}}|\mu_2(t)|\,|\mathcal{L}\mu_1(t)-\mathcal{L}\mu_2(t)|.$$

The differences $|J_n^1-J_n^2|$, $n=14$ to 17 are handled similarly. Finally the differences $|J_n^1-J_n^2|$ for $n=18$ and 19 are easily estimated by these same arguments.

THEOREM 4N. *Given $N>0$ and a positive $T<A$, if $\|\mu_1\|_T\leqslant N$, $\|\mu_2(t)\|_T\leqslant N$, then*

$$\|S\mu_1-S\mu_2\|_\sigma\leqslant B\sigma\|\mu_1-\mu_2\|_\sigma,\quad 0\leqslant\sigma\leqslant T<A.$$

Proof. From the two previous theorems by estimating t on the right by σ, and taking suprema on the left we get

$$\sup_{0\leqslant t\leqslant\sigma}|S\mu_1(t)-S\mu_2(t)|\leqslant B\sigma\|\mu_1-\mu_2\|_\sigma$$

and

$$\sup_{0\leqslant t\leqslant\sigma}\left|\frac{d}{dt}S\mu_1(t)-\frac{d}{dt}S\mu_2(t)\right|\leqslant B\sigma\|\mu_1-\mu_2\|_\sigma.$$

Adding gives the stated result.

THEOREM 4O. *Given a positive $T<A$, there is $\sigma_1>0$ depending on M, A, T, δ and a $\mu\in C_a^1[0,A)$ for which*

$$\mu(t)=S\mu(t),\quad 0\leqslant t\leqslant\sigma_1$$

and if $\nu\in C_a^1[0,A)$ with $\nu(t)=S\nu(t)$, $0\leqslant t\leqslant\sigma_1$ then $\mu(t)\equiv\nu(t)$, $0\leqslant t\leqslant\sigma_1$

Proof. Choose N_1 and σ_0 so that, by Theorem 4D, $\|\mu\|_{\sigma_0}\leqslant N_1$ implies $\|S\mu\|_{\sigma_0}\leqslant N_1$, choose $\mu_1\in C_a^1[0,A)$ so that $\|\mu_1\|_{\sigma_0}\leqslant N_1$. Then form $\mu_2=S\mu_1$, and in general $\mu_{n+1}=S\mu_n$, $n\geqslant 1$. Then by Theorem 4D $\|S\mu_n\|_{\sigma_0}\leqslant N_1$, $n\geqslant 1$. Each μ_n can be redefined, if necessary, for $\sigma_0<t<A$ so that $\|\mu_n\|_T\leqslant N\equiv 2N_1$, and so

$$\|\mu_{n+1}-\mu_n\|_\sigma\leqslant B\sigma\|\mu_n-\mu_{n-1}\|_\sigma\leqslant(B\sigma)^{n-1}\|\mu_2-\mu_1\|_\sigma$$

Now choose σ_1 so small that (1) $\sigma_1 \leqslant \sigma_0$, (2) $B\sigma \leqslant r < 1$. Then μ_n converges uniformly to

$$\mu \equiv \mu_1 + \sum_{n=1}^{\infty} (\mu_{n+1} - \mu_n)$$

and μ_n' converges uniformly to

$$\mu' \equiv \mu_1' + \sum_{n=1}^{\infty} (\mu_{n+1}' - \mu_n')$$

for $0 \leqslant t \leqslant \sigma_1$, and so

$$\mu(t) = S\mu(t) \quad 0 \leqslant t \leqslant \sigma_1.$$

The uniqueness follows, since $B\sigma < 1$. This immediately leads to the following.

THEOREM 4P. *Problem I has a solution* (ϱ, u, v) *for* $0 \leqslant t \leqslant \sigma_1$, *and only one solution for which* $\varrho \in C^2$, *and for which u and v satisfy bounds of the form* (3.1), *and Problem II has a solution* (u, v) *for* $0 \leqslant t < A$, *and only one solution for which* $\mu \in C^1$ *and for whcih u and v satisfy bounds of the form* (3.1).

We now extend the existence of the solution of Problem I to all positive $t < A$.

THEOREM 4Q. *Problem I has a solution* (ϱ, u, v) *for* $0 \leqslant t < A$ *and only one solution for which* $\varrho \in C^2$ *and u and v satisfy bounds of the form* (3.1).

Proof. Choose T_1 and T so that $0 < T_1 < T < A$. Since T_1 can be chosen arbitrarily close to A it is sufficient to show that the solution exists for $0 \leqslant t \leqslant T_1$. If $\sigma_1 > T_1$ we are finished. If not we can translate the origin to $(\varrho(\sigma_1), \sigma_1)$ and reset the problem and extend the solution. The only question is whether we can get a uniform σ_1 for all the reset problems. But this follows easily from Theorem 2D.

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