A FREE BOUNDARY PROBLEM ARISING IN A KINEMATIC WAVE MODEL OF CHANNEL FLOW WITH INFILTRATION*

By

B. SHERMAN

New Mexico Institute of Mining and Technology, Socorro

Abstract. Water flowing down a dry channel and infiltrating into the channel bed constitutes a free boundary problem. The free boundary is the time history of the water edge or front. In this paper we discuss a kinematic wave model of the problem. The problem is formulated in Sec. 1 and the results summarized in Sec. 2. In Secs. 3 and 4 the mathematical details are carried out, and in Sec. 5 a model using the continuity and momentum equations of hydraulics is discussed.

1. Formulation of the problem. The problem of irrigation has been studied for many years; we list, in the references, some of the more recent papers. We are dealing, essentially, with water flowing down a channel and infiltrating into the channel bed. The time history $x = s(t)$ of the front of the water, i.e. the interface between the covered and uncovered part of the channel, is a free boundary which has to be determined along with the velocity $u(x, t)$. This problem has been treated at various levels of mathematical complexity. We choose here a kinematic wave model that has been discussed in [1, 3, 7, 8, 9, 11].

Let x be the distance along a channel of uniform cross-section, $u(x, t)$ the velocity, $h(x, t)$ the depth, $q(x, t)$ the lateral inflow rate, and $f(x, t)$ the infiltration rate. The latter two are in volume per unit area per unit time. We can interpret q as rainfall. Further, let S be the slope of the channel, assumed constant, and S_f the friction slope. For S_f we take the Chézy formula $S_f = u^m/Ch$, m and C positive constants. The continuity and momentum equations are [10, Chapter 11]

$$h_t + (uh)_x = q - f,$$
$$u_t + uu_x + gh_x = g(S - S_f) - qu/h. \tag{1}$$

Conditions under which various terms in the momentum equation can be omitted have been discussed in the literature; we refer here to [4, 5, 6]. The kinematic wave model is obtained by omitting all but the terms $S - S_f$. Thus we get

$$u = \alpha h^{1/m}, \qquad \alpha = (SC)^{1/m}. \tag{2}$$

We assume now that $q = 0$ and that f is a function of t only. Then, writing $n = 1 + m^{-1}$, we get, from (1) and (2),

$$h_t + (\alpha h^n)_x = -f(t). \tag{3}$$

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Considering now water advancing down a dry channel, we assume that the front face of the water advances as a shock. This assumption is, of course, not justified by observation, but calculations based on it may not be grossly inaccurate. It would be necessary to compare this model with the model, discussed in Sec. 5 of this paper, using the complete momentum equation. Let $x = s(t)$ or $t = \zeta(x)$ be this advancing shock. Then, using (2), we get

$$s'(t) = \alpha h^{n-1}(s(t), t), \qquad s(0) = 0, \tag{4}$$

which we can also write

$$\zeta'(x) = [\alpha h^{n-1}(x, \zeta(x))]^{-1}, \qquad \zeta(0) = 0. \tag{5}$$

In the context of our discussion the $f(t)$ on the right of (3) has to be replaced by $f(t - \zeta(x))$, since infiltration at x does not begin until the time, $t = \zeta(x)$, at which the water reaches x. Thus we get

$$h_t + (\alpha h^n)_x = -f(t - \zeta(x)). \tag{6}$$

Finally, to fully define the problem, we have to specify h at $x = 0$:

$$h(0, t) = g(t). \tag{7}$$

Thus, in (5), (6), and (7), we have a free boundary problem; in the (x, t) plane (6) is satisfied in the domain D in the first quadrant which lies between the t axis and $t = \zeta(x)$, the time history of the advancing front. The free boundary $\zeta(x)$ is subject to (5). We make the additional assumption that $g'(t) \le 0$, since otherwise we may get shock formation in D.

If we introduce the new variable $\tau = t - \zeta(x)$ in place of t, then the free boundary maps onto $\tau = 0$ in the (x, τ) plane; $k(x, \tau) = h(x, \tau + \zeta(x))$ and $\zeta(x)$ are subject to

$$\begin{aligned} [1 - n\alpha k^{n-1}\zeta'(x)]k_\tau + n\alpha k^{n-1}k_x = -f(\tau), \qquad k(0, \tau) = g(\tau), \\ \zeta'(x) = [\alpha k^{n-1}(x, 0)]^{-1}, \qquad \zeta(0) = 0. \end{aligned} \tag{8}$$

We can write (8) as follows:

$$\left[1 - n\left(\frac{k(x, \tau)}{k(x, 0)}\right)^{n-1}\right]k_\tau + n\alpha k^{n-1}k_x = -f(\tau), \qquad k(0, \tau) = g(\tau). \tag{9}$$

Eq. (9) is a partial differential-difference equation; while k is specified along the τ axis it is not specified along the x axis, but it is this $k(x, 0)$ which appears in (9). If we specify $k(x, 0) = \psi(x)$, $\psi(0) = g(0)$, on the x axis then we have

$$\left[1 - n\left(\frac{k(x, \tau)}{\psi(x)}\right)^{n-1}\right]k_\tau + n\alpha k^{n-1}(x, \tau)k_x = -f(\tau), \qquad k(x, 0) = \psi(x). \tag{10}$$

Eq. (10) is a quasilinear first-order equation with data $\psi(x)$ specified on the x axis; $\psi(x)$ is the depth of the front wall of water. We wish to determine $\psi(x)$ so that the solution $k(x, \tau)$ of (10) has the property $k(0, \tau) = g(\tau)$. Let

$$x = x(\sigma, \xi; \psi), \qquad \tau = \tau(\sigma, \xi; \psi), \qquad k = k(\sigma, \xi; \psi), \tag{11}$$

be the characteristic of (10) passing through the point $(\xi, 0, \psi(\xi))$ of (x, τ, k) space. Here σ is the running parameter along the characteristic; if $\sigma = 0$ in (11) we get

$$x(0, \xi; \psi) = \xi, \qquad \tau(0, \xi; \psi) = 0, \qquad k(0, \xi; \psi) = \psi(\xi). \tag{12}$$

The characteristic equations of (10) are

$$\frac{dx}{d\sigma} = n\alpha k^{n-1}, \qquad \frac{d\tau}{d\sigma} = 1 - n\left(\frac{k}{\psi(x)}\right)^{n-1}, \qquad \frac{dk}{d\sigma} = -f(\tau), \tag{13}$$

and these are subject to the initial conditions (12). If we solve the first two equations of (11) for σ and ξ in terms of x and τ and insert in the third equation of (11) we get the solution of (10). To this end we note that the Jacobian

$$J(\sigma, \xi; \psi) = x_\sigma \tau_\xi - x_\xi \tau_\sigma$$

has the value, at $\sigma = 0$, $J(0, \xi; \psi) = n - 1 > 0$. If $k(0, \tau; \psi)$ is defined for some interval $0 \le \tau \le \tau_0$, then $k(0, \tau; \psi) = g(\tau)$ constitutes an equation for the determination of $\psi(x)$ on some interval $0 \le x \le x_0$. The $k(x, \tau)$ corresponding to this $\psi(x)$ satisfies (9).

We can simplify the problem by replacing the parameter σ by x. We note, from the first equation of (13), that, since $k > 0$, x is an increasing function of σ. Thus we have two functions

$$\tau = \tau(x, \xi; \psi), \qquad k = k(x, \xi; \psi), \tag{14}$$

such that

$$\frac{d\tau}{dx} = \frac{1 - n(k/\psi(x))^{n-1}}{n\alpha k^{n-1}}, \qquad \frac{dk}{dx} = \frac{f(\tau)}{n\alpha k^{n-1}}, \tag{15}$$

subject to the initial conditions

$$\tau(\xi, \xi; \psi) = 0, \qquad k(\xi, \xi; \psi) = \psi(\xi). \tag{16}$$

τ and k are defined on $S(\xi_0) = \{0 \le x \le \xi, 0 \le \xi \le \xi_0\}$, where $\psi(\xi) > 0$ on $0 \le \xi \le \xi_0$. If $\tau_\xi > 0$ on $S(\xi_0)$ then (14) determines a function $k(x, \tau; \psi)$ satisfying (10). We wish to determine $\psi(x)$ so that $k(0, \tau; \psi) = g(\tau)$; in terms of the functions (14) this can be expressed

$$k(0, \xi; \psi) = g(\tau(0, \xi; \psi)). \tag{17}$$

The solution $\psi(x)$ of (17) determines $k(x, \tau)$ satisfying (9).

We can achieve further simplifications by introducing

$$F(x, \xi; \psi) = k^n(x, \xi; \psi); \qquad \phi(x) = \psi^n(x), \qquad \gamma(\tau) = g^n(\tau).$$

Then τ and F satisfy

$$\begin{aligned} \frac{d\tau}{dx} &= \frac{1}{n\alpha} F^{-(n-1)/n} - \frac{1}{\alpha}[\phi(x)]^{-(n-1)/n}, \qquad \tau(\xi, \xi; \phi) = 0, \\ \frac{dF}{dx} &= -\frac{1}{\alpha} f(\tau), \qquad F(\xi, \xi; \phi) = \phi(\xi). \end{aligned} \tag{18}$$

Eq. (17) becomes

$$F(0, \xi; \phi) = \gamma(\tau(0, \xi; \phi)). \tag{19}$$

We can write (18) in the following integral form (here we write $\tau(x, \xi)$ and $F(x, \xi)$, omitting ϕ):

$$\tau(x, \xi) = -\frac{1}{n\alpha}\int_x^\xi [F(\sigma, \xi)]^{-(n-1)/n}\, d\sigma + \frac{1}{\alpha}\int_x^\xi [\phi(\sigma)]^{-(n-1)/n}\, d\sigma,$$
$$F(x, \xi) = \phi(\xi) + \frac{1}{\alpha}\int_x^\xi f(\tau(\sigma, \xi))\, d\sigma. \tag{20}$$

Using the second equation of (20), we can write (19) as

$$\phi(\xi) = -\frac{1}{\alpha}\int_0^\xi f(\tau(\sigma, \xi))\, d\tau + \gamma(\tau(0, \xi)). \tag{21}$$

Eqs. (20) and (21) constitute three integral equations for the functions τ, F, and ϕ. The solution of (8) is equivalent to the solution of (20) and (21); if $\tau = \tau(x, \xi)$ is solved for ξ, $\xi = \xi(x, \tau)$, then the solution of (8) is

$$k(x, \tau) = [F(x, \xi(x, \tau))]^{1/n}, \qquad \zeta(x) = \frac{1}{\alpha}\int_0^x \frac{d\sigma}{[\phi(\sigma)]^{(n-1)/n}}. \tag{22}$$

The solution of (5), (6), and (7) is given by $\zeta(x)$ in (22) and by $h(x, t) = k(x, t - \zeta(x))$.

2. Discussion of the results. In Sec. 3 we discuss the case $f(\tau) = f$ = constant and in Sec. 4 the case $f(\tau)$ not constant. When $f(\tau)$ is constant we consider two cases, $g(\tau) = g$ = constant and $g'(\tau) < 0$. When $f(\tau)$ and $g(\tau)$ are both constant we have explicit solutions, as is evident from (20) and (21). The free boundary $t = \zeta(x)$ is given by (23), $0 \le x \le \alpha g^n/f$, and $h(x, t)$ by (24). It is clear that there is a steady state after $t = ng/f$ with $h(x, t)$ given by (24) and $h(\alpha g^n/f, t) = 0$, $t \ge ng/f$. When $g(\tau)$ is not constant we get the differential equation (27) for $\phi(\xi)$; in (27) Γ is the function inverse to γ. This differential equation has a solution on some interval $0 \le \xi \le \xi_0$, on which $\phi(\xi)$ is a decreasing function with $\phi(\xi_0) = 0$. The free boundary $t = \zeta(x)$ is then given by (22). For further discussion of this case we refer to [7, 8]. There is also, in [7, 8], a discussion of the case $f(\tau)$ constant, $g(\tau) = g$ = constant on $0 \le t \le T$, $g(\tau) = 0$, $t > T$.

In the case $f(\tau)$ not constant we use a fixed-point argument. We prove, in Sec. 4, that there is a $\xi_0 > 0$ such that (20) and (21) have a unique solution in $S(\xi_0)$. But this is only a local existence theorem. It is plausible, however, that $\phi(\xi)$ exists on $0 \le \xi \le \xi_0$, is a decreasing function, and $\phi(\xi_0) = 0$. We give a brief discussion of the existence of τ_ξ, F_ξ and ϕ' and the inequalities $\tau_\xi(x, \xi) > 0$, $\phi'(\xi) < 0$; again all of this is local.

In Sec. 5 we formulate a model based on the full momentum equation.

3. The case $f(\tau) = f$ = constant. We assume first that $g(\tau) = g$ = constant also. Then, from (20) and (21) we get

$$\phi(\xi) = \gamma - (f\xi/\alpha), \qquad F(x, \xi) = \gamma - (fx/\alpha),$$
$$\tau(x, \xi) = \frac{n-1}{f}\left[\left(\gamma - \frac{fx}{\alpha}\right)^{1/n} - \left(\gamma - \frac{f\xi}{\alpha}\right)^{1/n}\right],$$

so

$$\psi(\xi) = \left[g^n - \frac{f\xi}{\alpha}\right]^{1/n}, \qquad k(x, \xi) = \left[g^n - \frac{fx}{\alpha}\right]^{1/n},$$

$$\tau(x, \xi) = \frac{n-1}{f}\left[\left(g^n - \frac{fx}{\alpha}\right)^{1/n} - \left(g^n - \frac{f\xi}{\alpha}\right)^{1/n}\right].$$

We note that in this case $\xi_0 = \alpha g^n/f$ since the depth $\psi(\xi) > 0$ on $0 \le \xi < \alpha g^n/f$ and $\psi(\alpha g^n/f) = 0$. From (22) we get, since $F(x, \xi)$ is independent of ξ,

$$k(x, \tau) = \left[g^n - \frac{fx}{\alpha}\right]^{1/n}, \qquad \zeta(x) = \frac{n}{f}\left[g - \left(g^n - \frac{fx}{\alpha}\right)^{1/n}\right], \tag{23}$$

and, from (23), since $k(x, \tau)$ is independent of τ,

$$h(x, t) = \left[g^n - \frac{fx}{\alpha}\right]^{1/n}. \tag{24}$$

The free boundary beyond $t = ng/f$ is $x = \alpha g^n/f$ [7]; $h(\alpha g^n/f, t) = 0$, $t > ng/f$. $h(x, t)$ above $t = \zeta(x)$ is given by (24).

In the case $f(\tau) = f =$ constant and $g'(\tau) < 0$ we get from (20) and (21)

$$\begin{gathered}\phi(\xi) = -\frac{f\xi}{\alpha} + \gamma(\tau(0, \xi)), \qquad F(x, \xi) = -\frac{fx}{\alpha} + \gamma(\tau(0, \xi)), \\ \tau(x, \xi) = \frac{1}{f}[\phi(\xi)]^{1/n} - \frac{1}{f}\left[\phi(\xi) + \frac{f}{\alpha}(\xi - x)\right]^{1/n} + \frac{1}{\alpha}\int_x^\xi [\phi(\sigma)]^{-(n-1)/n}\, d\sigma.\end{gathered} \tag{25}$$

Let Γ be the function inverse to γ. Then, from (25),

$$\Gamma\left(\phi + \frac{f\xi}{\alpha}\right) = \frac{1}{f}\phi^{1/n} - \frac{1}{f}\left[\phi + \frac{f\xi}{\alpha}\right]^{1/n} + \frac{1}{\alpha}\int_0^\xi [\phi(\sigma)]^{-(n-1)/n}\, d\sigma. \tag{26}$$

From (26) we get the differential equation

$$\Gamma'\left(\phi + \frac{f\xi}{\alpha}\right)\left(\phi' + \frac{f}{\alpha}\right) = \frac{1}{nf}\phi^{-(n-1)/n}\phi' - \frac{1}{nf}\left[\phi + \frac{f\xi}{\alpha}\right]^{-(n-1/n}\left(\phi' + \frac{f}{\alpha}\right) + \frac{1}{\alpha}\phi^{-(n-1)/n}.$$

Thus

$$\frac{d\phi}{d\xi} = \frac{-nf^2\Gamma'\left(\phi + \frac{f\xi}{\alpha}\right) - f\left(\phi + \frac{f\xi}{\alpha}\right)^{-(n-1)/n} + nf\phi^{-(n-1)/n}}{n\alpha f\Gamma'\left(\phi + \frac{f\xi}{\alpha}\right) - \alpha\phi^{-(n-1)/n} - \alpha\left(\phi + \frac{f\xi}{\alpha}\right)^{-(n-1)/n}}. \tag{27}$$

The initial condition is $\phi(0) = \gamma(0) = g^n(0)$. The denominator on the right of (27) is negative (since $\Gamma' < 0$ and $\phi > 0$) and the numerator is positive. Thus $\phi(\xi)$ is a decreasing function of ξ. The solution of (27) exists (at least) on $0 \le \xi \le \xi_0$, where $\phi(\xi) > 0$ on $0 \le \xi \le \xi_0$ and $\phi(\xi_0) = 0$.

4. The case $f(\tau)$ not constant. In this case we use a fixed-point argument. We assume $0 < f(\tau) \le b$. Let $T\phi$ be the right side of (21). Specifying the positive continuous function $\phi(\xi)$ on $0 \le \xi \le \xi_0$, ξ_0 to be determined, we calculate $F(x, \xi)$ and $\tau(x, \xi)$ by (20), or equivalently (18). Then $F(x, \xi)$ and $\tau(x, \xi)$ are defined and continuous on $S(\xi_0)$, and

$\phi^*(\xi) = T\phi$ is defined and continuous on $0 \leq \xi \leq \xi_0$. We note, from (18), that $F(x, \xi)$ is a decreasing function of x on $0 \leq x \leq \xi$. From (21) we see that $\phi(\xi) \leq \gamma(0)$ on $0 \leq \xi \leq \xi_0$ implies $\phi^*(\xi) \leq \gamma(0)$ on $0 \leq \xi \leq \xi_0$; this follows from $f(\tau) \geq 0$, $\gamma'(\tau) \leq 0$. Suppose $\phi(\xi) \geq a > 0$ on $0 \leq \xi \leq \xi_0$. Then, since F is positive on $S(\xi_0)$, we get, from (20),

$$\tau(0, \xi) < \frac{1}{\alpha}\int_0^\xi [\phi(\sigma)]^{-(n-1)/n}\, d\sigma \leq \frac{\xi_0}{\alpha} a^{-(n-1)/n} \tag{28}$$

on $0 \leq \xi \leq \xi_0$. From (28) and $f(\tau) \leq b$ we get

$$\phi^*(\xi) = T\phi \geq -\frac{b\xi_0}{\alpha} + \gamma\left(\frac{\xi_0}{\alpha} a^{-(n-1)/n}\right). \tag{29}$$

The function of ξ_0 on the right side of (29) is decreasing, has the value $\gamma(0)$ for $\xi_0 = 0$, and is negative for sufficiently large ξ_0. Therefore, if $a < \gamma(0)$,

$$-\frac{b\xi_0}{\alpha} + \gamma\left(\frac{\xi_0}{\alpha} a^{-(n-1)/n}\right) = a \tag{30}$$

has a unique root $\xi_0(a)$. Thus if $\xi_0 \leq \xi_0(a)$, (29) implies $\phi^*(\xi) \geq a$ on $0 \leq \xi \leq \xi_0$. Then, if $\xi_0 \leq \xi_0(a)$, the class $B_1(\xi_0)$ of continuous functions $\phi(\xi)$ satisfying $a \leq \phi(\xi) \leq \gamma(0)$ on $0 \leq \xi \leq \xi_0$ is carried, by T, into the class of continuous functions $\phi^*(\xi)$ satisfying $a \leq \phi^*(\xi) \leq \gamma(0)$. If $B(\xi_0)$ is the Banach space of continuous functions $f(\xi)$ on $0 \leq \xi \leq \xi_0$ with norm $\|f\| = \max|f(\xi)|$, $0 \leq \xi \leq \xi_0$, then $B_1(\xi_0)$ is a closed subset of $B(\xi_0)$ and is, therefore, a complete metric space. Subject to the choice $\xi_0 \leq \xi_0(a)$, T maps $B_1(\xi_0)$ into itself. If we can further restrict ξ_0 so that T is a contraction on $B_1(\xi_0)$, then $\phi^* = T\phi$ has a unique fixed point; i.e., the equation $\phi = T\phi$, which is (21), has a unique solution.

To prove that ξ_0 can be chosen so that T on $B_1(\xi_0)$ is a contraction let

$$A = \sup|f'(\tau)|, \qquad B = \sup|\gamma'(\tau)|, \qquad \tau \geq 0.$$

We assume A and B are finite. For a continuous function $f(x, \xi)$ defined on $S(\xi_0)$ we define $\|f\| = \max|f(x, \xi)|$ on $S(\xi_0)$. Let ϕ_1 and ϕ_2 be two functions in $B_1(\xi_0)$ with (τ_1, F_1) and (τ_2, F_2) the corresponding functions determined by (18). Then, from (20),

$$\begin{aligned} |F_1(x, \xi) - F_2(x, \xi)| &\leq |\phi_1(\xi) - \phi_2(\xi)| + \frac{1}{\alpha}\int_x^\xi A|\tau_1(\sigma, \xi) - \tau_2(\sigma, \xi)|\, d\sigma \\ &\leq \|\phi_1 - \phi_2\| + \frac{A\xi_0}{\alpha}\|\tau_1 - \tau_2\|, \end{aligned}$$

and therefore

$$\|F_1 - F_2\| \leq \|\phi_1 - \phi_2\| + \frac{A\xi_0}{\alpha}\|\tau_1 - \tau_2\|. \tag{31}$$

Also from (20) we get

$$\begin{aligned} |\tau_1(x, \xi) - \tau_2(x, \xi)| &\leq \frac{1}{n\alpha}\int_x^\xi \frac{n-1}{n} a^{-(2n-1)/n}|F_1(\sigma, \xi) - F_2(\sigma, \xi)|\, d\sigma \\ &\quad + \frac{1}{\alpha}\int_x^\xi \frac{n-1}{n} a^{-(2n-1)/n}|\phi_1(\sigma) - \phi_2(\sigma)|\, d\sigma \\ &\leq D\xi_0\left(\frac{1}{n}\|F_1 - F_2\| + \|\phi_1 - \phi_2\|\right), \end{aligned}$$

where

$$D(a) = \frac{n-1}{n\alpha} a^{-(2n-1)/n},$$

so

$$\|\tau_1 - \tau_2\| \le D\xi_0\left(\frac{1}{n}\|F_1 - F_2\| + \|\phi_1 - \phi_2\|\right). \tag{32}$$

From (31) and (32) we get

$$\|\tau_1 - \tau_2\| \le D\xi_0\left(1 + \frac{1}{n}\right)\|\phi_1 - \phi_2\| + \frac{D\xi_0^2 A}{n\alpha}\|\tau_1 - \tau_2\|,$$

from which we get, assuming $D\xi_0^2 A < n\alpha$,

$$\|\tau_1 - \tau_2\| \le \frac{D\xi_0 \alpha(n+1)}{n\alpha - D\xi_0^2 A}\|\phi_1 - \phi_2\|. \tag{33}$$

From $\phi^* = T\phi$ we get

$$\|\phi_1^* - \phi_2^*\| \le \left(\frac{A\xi_0}{\alpha} + B\right)\|\tau_1 - \tau_2\|. \tag{34}$$

Combining (33) and (34) we get

$$\|\phi_1^* - \phi_2^*\| \le G(\xi_0, a)\|\phi_1 - \phi_2\|, \tag{35}$$

where

$$G(\xi_0, a) = \left(\frac{A\xi_0}{\alpha} + B\right)\left(\frac{D\xi_0\alpha(n+1)}{n\alpha - D\xi_0^2 A}\right).$$

Thus if ξ_0 is chosen, for a given a, $0 < a < \gamma(0)$, so that

$$0 < \xi_0 \le \xi_0(a), \qquad 0 < G(\xi_0, a) < 1, \tag{36}$$

then T is a mapping of $B_1(\xi_0)$ into itself which is a contraction. The inequalities (36) can be written, after a brief calculation,

$$0 < \xi_0 \le \xi_0(a), \qquad 0 < \xi_0 < H(a),$$

where

$$H(a) = \frac{B\alpha(n+1)}{2A(n+2)}\left[\left(1 + \frac{4An(n+2)}{D(a)B^2\alpha(n+1)^2}\right)^{1/2} - 1\right], \tag{37}$$

or

$$0 < \xi_0 < \min(\xi_0(a), H(a)). \tag{38}$$

When $g(\tau) = g = \text{constant}$, $B = 0$, so we get from (30) and (37)

$$\xi_0(a) = \frac{\alpha}{b}(\gamma - a), \qquad H(a) = n\alpha[A(n+2)(n-1)]^{-1/2} a^{(2n-1)/2n}.$$

It is easily seen, in this case, that the right side of (38) is maximum when a is selected to be the root of $\xi_0(a) = H(a)$.

We give a brief outline of the proof that τ_ξ, F_ξ, and ϕ' exist, and that $\tau_\xi(x, \xi) > 0$ and $\phi'(\xi) < 0$. On assuming the existence of τ_ξ, F_ξ, and ϕ' we get, from (20),

$$\tau_\xi(x, \xi) = \frac{n-1}{n\alpha}[\phi(\xi)]^{-(n-1)/n} + \frac{n-1}{n^2\alpha}\int_x^\xi [F(\sigma, \xi)]^{-(2n-1)/n} F_\xi(\sigma, \xi)\, d\sigma,$$

$$F_\xi(x, \xi) = \phi'(\xi) + \frac{1}{\alpha} f(0) + \frac{1}{\alpha}\int_x^\xi f'(\tau(\sigma, \xi))\tau_\xi(\sigma, \xi)\, d\sigma, \tag{39}$$

and, from (21),

$$\phi'(\xi) = -\frac{1}{\alpha} f(0) - \frac{1}{\alpha}\int_0^\xi f'(\tau(\sigma, \xi))\tau_\xi(\sigma, \xi)\, d\sigma + \gamma'(\tau(0, \xi))\tau_\xi(0, \xi). \tag{40}$$

We may write (39) equivalently as

$$\frac{d\tau_\xi}{dx} = -\frac{n-1}{n^2\alpha} F^{-(2n-1)/n} F_\xi, \qquad \tau_\xi(\xi, \xi) = \frac{n-1}{n\alpha}[\phi(\xi)]^{-(n-1)/n},$$

$$\frac{dF_\xi}{dx} = -\frac{1}{\alpha} f'(\tau)\tau_\xi, \qquad F_\xi(\xi, \xi) = \phi'(\xi) + \frac{1}{\alpha} f(0). \tag{41}$$

We use a fixed-point argument on (20), (21), (39), (40), as follows: writing

$$\phi'(\xi) = \omega(\xi), \qquad \phi(\xi) = \gamma(0) + \int_0^\xi \omega(\sigma)\, d\sigma, \tag{42}$$

we consider the Banach space of continuous vector functions $(\omega_0(\xi), \omega(\xi))$ on $0 \le \xi \le \xi_0$ with norm $\|\omega_0\| + \|\omega\|$, ξ_0 to be determined. The set of vectors (ϕ, ω) is a closed subset $B_1(\xi_0)$ of that space. We may interpret the right sides of (21) and (40) as a mapping T of $B_1(\xi_0)$ into itself. More precisely, if ω is specified then ϕ is known from (42); then τ and F are determined from (18), and τ_ξ and F_ξ from (41), where, in (41), ϕ' is replaced by ω. Thus the right sides of (21) and (40) are determined. With appropriate restrictions of ξ_0, T is a contraction; this requires further regularity conditions on f and γ. It is clear, from the first equation in (39), that the sign of τ_ξ in the neighborhood of the origin is determined by the first term on the right of (39). This term is positive, so $\tau_\xi(x, \xi) > 0$ on $S(\xi_0)$ for ξ_0 sufficiently small. From (40) we see that the sign of $\phi'(\xi)$ in the neighborhood of $\xi = 0$ is determined by the first and third terms on the right of (40). The first term is negative and the third term is ≤ 0. Thus $\phi'(\xi) < 0$ on $0 \le \xi \le \xi_0$ for sufficiently small ξ_0.

5. Dynamic models. The model discussed in this section is an extension of the model for the breaking of a dam discussed in [10, p. 313]; in [10] S, S_f, q, and f are all 0. We consider a channel with $q = 0$ for all x, $f = 0$ for $x \le 0$, and f time-dependent but independent of x for $x > 0$. When $x < 0$ a piston fitting the channel and moving with constant velocity $u_0 > 0$; to the left of the piston there is uniform flow with $u = u_0$ and h_0, where

$$S - (u_0^m / Ch_0) = 0. \tag{43}$$

Because of (43), (1) has the solution $h = h_0$, $u = u_0$ in $x < u_0 t$, $t < 0$. At time $t = 0$ the piston is at $x = 0$; for $t > 0$ the piston moves according to $x = s(t)$ so that the height of the water at the piston face is 0, $h(s(t), t) = 0$, and because the water is in contact with the piston face, $u(s(t), t) = s'(t)$. Let $t = \xi(x)$ be inverse to $x = s(t)$. Then for $x < s(t)$, $t > 0$ we have, from (1),

$$h_t + (uh)_x = -f(t - \xi(x)), \qquad u_t + uu_x + gh_x = g\left(S - \frac{u^m}{Ch}\right). \tag{44}$$

To (44) we add the conditions on $t = \xi(x)$

$$h(x, \xi(x)) = 0, \qquad u(x, \xi(x)) = [\xi'(x)]^{-1}, \qquad \xi(0) = 0, \tag{45}$$

and the initial conditions

$$u(x, 0) = u_0, \qquad h(x, 0) = h_0, \qquad x < 0. \tag{46}$$

Eqs. (44), (45) and (46) constitute a free boundary problem for h, u, and ξ. The characteristics of the hyperbolic system (44) are

$$dx/dt = u - (gh)^{1/2}, \qquad dx/dt = u + (gh)^{1/2}. \tag{47}$$

It seems plausible, physically, that if $\xi(x)$ were specified, then (44), (46), and the second equation of (45) would be sufficient to determine u and h. But, because $\xi(x)$ is not known, the extra condition in (45) is necessary in order that (44), (45), and (46) determine u, h, and ξ. Mathematically, we have to consider the characteristics issuing from a point on $t = \xi(x)$ [2, pp. 471–475]. We note that on the free boundary $t = \xi(x)$ the two characteristic directions coincide (because $h = 0$ in $t = \xi(x)$), and it is evident from the second equation of (45) and from (47) that the free boundary is a characteristic. Thus it is not clear mathematically that (44), (45), and (46) is well posed, but it seems plausible on physical grounds that it is.

Assuming the existence of a unique solution of (44), (45), and (46), we know $h(0, t)$. This, we may reasonably suppose, is a decreasing function of t. Using $h(0, t)$ as the $g(t)$ of Sec. 4, we will then be in position to compare the free boundary of the kinematic wave model with the free boundary of the dynamic model (44), (45), and (46). In particular, the comparative dependence on S and C can be investigated.

If f is a positive constant then it is reasonable to expect that the solution of (44), (45), and (46) tends to the solution $u(x)$, $h(x)$ of

$$(uh)_x = -f(x), \qquad uu_x + gh_x = g\left(S - \frac{u^m}{Ch}\right), \tag{48}$$

on $-\infty < x < x_0$, where $f(x) = 0$, $x \le 0$, $f(x) = f$, $x > 0$. We determine x_0 as follows: $uh = u_0 h_0$, $x < 0$, and $uh = u_0 h_0 - fx$, $x > 0$. Since uh cannot be negative we get $x_0 = u_0 h_0 / f$. Referring to Sec. 3 and noting (43), we see that this x_0 is equal to $\alpha h_0^n / f$, which is the position of the free boundary at zero depth in the kinematic wave model when $g(t) = h_0$. Since $h_0 \ge h(0, t) \ge h(0)$, this suggests that if $h_0 - h(0)$ is small and if $g(t)$ in the kinematic wave model satisfies $h_0 \ge g(t) \ge h(0)$ then the kinematic wave model is a reasonable approximation to the dynamic model (44), (45), (46).

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