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A FREE BOUNDARY PROBLEM FOR A PREDATOR-PREY MODEL WITH NONLINEAR PREY-TAXIS

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Abstract. This paper deals with a reaction-diffusion system modeling a free boundary problem of the predator-prey type with prey-taxis over a one-dimensional habitat. The free boundary represents the spreading front of the predator species. The global existence and uniqueness of classical solutions to this system are established by the contraction mapping principle. With an eye on the biological interpretations, numerical simulations are provided which give a real insight into the behavior of the free boundary and the stability of the solutions.

Keywords: prey-predator model; prey-taxis; free boundary; classical solutions; global existence

MSC 2010: 35K57, 35K55, 35R35, 92B05

1. Introduction

The mathematical modeling of the two-species predator-prey ecological system has received increasing attention during the past few decades. Investigation of the spatial and temporal behavior of the predator and prey in an ecological system is an important issue in population ecology.

Various types of mathematical models have been proposed to study predator-prey systems [18], [11], [17]. These studies provide a theoretical framework for understanding the complex spatio-temporal dynamics observed in real ecological systems. Such models are mathematically interesting and rigorous mathematical analysis of these models, such as global existence, uniqueness and stability of solutions has drawn increasing attention, [9], [12], [13], [22], [24]. For instance, global stability of a class of predator-prey systems is shown in [9] by constructing Liapunov functions and the

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influence of color noise on the pattern formation in a predator-prey model has been investigated in [12], [13].

In this paper we investigate a system of reaction-diffusion equations describing a type of predator-prey model with prey-taxis and a free boundary. Let Σ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Sigma$ and $T > 0$ be a fixed time. Consider the following equations for predator-prey model with prey-taxis:

$$\begin{aligned} &u_t - d_1\Delta u + \nabla\cdot(u\chi(u)\nabla v) = -au + bg(v)u, \quad x \in \Sigma\times(0,T),\ t>0,\\ &v_t - d_2\Delta v = k(v) - g(v)u, \quad x\in\Sigma\times(0,T),\ t>0, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ denote the population densities of the predator and prey, respectively, at position x and time t. Parameters $d_1, d_2 > 0$ are the diffusion rates of the predator and prey, respectively, and $a > 0$ is the natural exponential decay of the predator population. The function $g(v) = b_1v/(1+b_2v)$ with $b_1, b_2 > 0$ is the predation rate, where $1/b_2$ is the time spent by the predator to catch the prey, and b_2/b_1 is the manipulation time which is a saturation effect for large densities of the prey. Moreover, b is the conversion rate from the prey to the predator. The growth rate of the prey is governed by $k(v) = rv(1-(v/K))$, the Pearl-Verhulst equation, where $r > 0$ is the natural growth of the prey, and K is the carrying capacity. The predators are attracted by the preys and χ denotes their prey-tactic sensitivity.

Taxis is achieved when the individuals change their pattern of movement or kinesis in response to a stimulus. Indeed, the purposes of the taxis range from movement toward food and avoidance of noxious substances to large-scale aggregations for survival [20]. Prey-taxis is thus defined as the movement of predators controlled by prey density. Actually, the prey-taxis mechanism in the model means a direct movement of the predator u in response to a variation of the prey v, and we assume that $\chi(u) \in C^{1,1}([0,\infty))$ and moreover,

$$\begin{aligned} &\chi(u) = 0 \text{ for } u \geqslant u_{\rm m} \text{ and}\\ &\chi'(u) \text{ is Lipschitz continuous, i.e.}\\ &|\chi'(u_1) - \chi'(u_2)| \leqslant L|u_1 - u_2| \quad \forall\, u_1, u_2 \in [0,\infty), \end{aligned} \tag{1.2}$$

where $u_{\rm m}$ and L are two positive constants. Here the assumption $\chi(u) = 0$ for $u > u_{\rm m}$ says that there is a threshold value $u_{\rm m}$ for the accumulation of u over which the prey tactic cross-diffusion $\chi(u)$ vanishes [1]. This is motivated by the "volume-filling" mechanism [21], [8]. For the sake of simplicity, we use notation $\eta(u) = u\chi(u)$ hereafter. For initial values, throughout this paper we also assume that

$$0 \leqslant u_0(x), \quad 0 \leqslant v_0(x) \leqslant K, \quad u_0(x), v_0(x) \in C^{2+\alpha}[0,h_0), \tag{1.3}$$

where α and h_0 are some constants such that $0 < \alpha < 1$ and $h_0 > 0$.

Equation (1.1) has been proposed in [1] and it has been analyzed mathematically in [1], [23], [27], [14] for fixed habitat domain Σ. The existence of a classical and a weak solution for this system has been proved in a fixed domain [1], [23].

In many realistic modeling situations, both the prey and the predator have a tendency to emigrate from the boundary to obtain their new habitat and to improve the living environment. Then it is more reasonable to consider the domain Σ with a moving free boundary. We may consider that the free boundary is caused only by the predator and the spreading front expands at a speed proportional to the predator's population gradient on the boundary. To be more specific, let us investigate the one-dimensional case and we assume that the prey and the predator species migrate in the habitat $\Sigma_t = (0, h(t))$. Indeed, the species can only invade further into the new environment from the right boundary and we suppose that $h'(t) = -\beta u_x$, where β is a positive constant. The reader can refer to [3], [5] for further information about the ecological background and the derivation of the free boundary conditions.

Motivated by the above explanations, we study the following system with moving free boundary denoted by $x = h(t)$:

$$
\begin{aligned}
&u_t - d_1 u_{xx} + (u\chi(u)v_x)_x = -au + bg(v)u, \quad 0 < x < h(t),\ t > 0,\\
&v_t - d_2 v_{xx} = k(v) - g(v)u, \quad 0 < x < h(t),\ t > 0,\\
&u(h(t),t) = 0,\ v(h(t),t) = 0, \quad t > 0,\\
&h'(t) = -\beta u_x(h(t),t),\ h(0) = h_0 > 0, \quad x = h(t),\\
&u_x(0,t) = v_x(0,t) = 0, \quad t > 0,\\
&u(x,0) = u_0(x), v(x,0) = v_0(x), \quad 0 < x < h_0.
\end{aligned}
\tag{1.4}
$$

Free boundary problems described by partial differential equations which have a different feature, namely, that geometric information is an inherent part of the solution. Typically, the solution of a free boundary problem consists of one or more functions and a set (the so called free boundary) on which certain conditions on the unknown functions are prescribed [16]. Obviously, the mathematical literature of free boundary problems is vast. We refer to book [6] for a review of basic analytical tools and for further reference, for numerical simulation see [2].

Although predator-prey models with free boundary have been studied, such equations does not contain the pray-taxis mechanism [15], [19], [28], [25], [26]. To the best of our knowledge, reaction-diffusion systems modeling predator-prey with prey-taxis and a free boundary have never been considered before. These types of models are complicated and challenging because of nonlinear term $\eta(u)$ for prey-taxis and the free boundary.

The main contribution of this paper is the establishment of the existence and uniqueness of a classical solution to (1.4). Applying the contraction mapping theorem, we prove a local existence result. Moreover, we establish that the local solution is unique. Then it is shown that such solution can be extended to all $t > 0$ as a unique global solution. At last, we provide some numerical results implemented for the simulation of prey-predators interactions.

2. Local existence and uniqueness

In this section, we prove a local existence result invoking the contraction mapping theorem. Then we show that the local solution is unique.

To this end, we provide an important lemma which will be needed for the main result. Consider the following problem:

$$\begin{aligned} &u_t + A_1(x,t)u_{xx} + A_2(x,t)u_x \\ &\qquad\qquad + A_3(x,t)(\eta(u)v_x)_x = A_4(x,t)u, \quad 0 < x < \varrho,\ t > 0, \\ &u_x(0,t) = 0, u(\varrho,t) = 0, \quad t > 0, \\ &u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant \varrho, \end{aligned} \tag{2.1}$$

where A_i, $i = 1,\ldots,4$, are functions depending on x and t and ϱ is a positive constant. We establish that there is a unique solution to equation (2.1).

Hereafter, the generic constant M_0 denotes a constant which is independent of T.

Lemma 2.1. *Let* $S_T = (0,\varrho) \times [0,T)$, $v(x,t) \in C^{2+\alpha,1+\alpha/2}(S_T)$, $A_i(x,t) \in C^{\alpha,\alpha/2}(S_T)$ *for* $i = 1,\ldots,4$, *and*

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(S_T)} \leqslant M_0, \quad \|A_i\|_{C^{\alpha,\alpha/2}(S_T)} \leqslant M_0. \tag{2.2}$$

Then under assumptions (1.2) *and* (1.3), *there exists a unique non-negative solution* $u(x,t) \in C^{2+\alpha,1+\alpha/2}(S_T)$ *of the nonlinear problem* (2.1) *for small* $T > 0$, *which depends on* $\|u_0(x)\|_{C^{2+\alpha}(0,\varrho)}$.

Proof. We show that there is a solution by a fixed-point argument. At first, we introduce the set

$$X_{B_0} = \{u \in C^{1+\alpha,\alpha/2}(S_T);\ u(y,0) = u_0(y), \|u\|_{C^{1+\alpha,\alpha/2}(S_T)} \leqslant B_0\},$$

where $B_0 = \|u_0(x)\|_{C^{2+\alpha}} + 1$.

Next, we define the map F as follows. For $u \in X_{B_0}$ define the corresponding map $F(u) = \bar{u}$, where $\bar{u}$ satisfies the equations

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} &+ A_1(x,t)\bar{u}_{xx} + A_2(x,t)\bar{u}_x - A_4(x,t)\bar{u} \\ &= -A_3(x,t)\eta(u)v_{xx} - A_3(x,t)\eta'(u)v_x u_x, \quad 0 < x < \varrho,\ t > 0, \\ &\bar{u}_x(0,t) = 0, \quad \bar{u}(\varrho,t) = 0,\ t > 0, \\ &\bar{u}(x,0) = u_0(x), \quad 0 \leqslant x \leqslant \varrho.\end{aligned} \tag{2.3}$$

By (1.2), (2.2), and the fact that u is in X_{B_0}, we observe

$$-A_3(x,t)\eta(u)v_{xx} - A_3(x,t)\eta'(u)v_x u_x \in C^{\alpha,\alpha/2}. \tag{2.4}$$

Applying (1.3), (2.2), and (2.4) and then in view of the parabolic Schauder theory (see [10]), one can infer that there exits a unique solution $\bar{u}$ to (2.3) and

$$\|\bar{u}\|_{C^{2+\alpha,1+\alpha/2}(S_T)} \leqslant \|\bar{u}(x,0)\|_{C^{2+\alpha}(0,\varrho)} + M_1(B_0) \leqslant B_0 + M_1(B_0) = M_2(B_0), \tag{2.5}$$

where M_1 and M_2 are some constants which depend only on B_0. Such $\bar{u}(x,t)$ satisfies

$$\|\bar{u}(x,t) - \bar{u}(x,0)\|_{C^{1+\alpha,\alpha/2}(S_T)} \leqslant M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\}\|\bar{u}\|_{C^{2+\alpha,1+\alpha/2}(S_T)}. \tag{2.6}$$

To see estimate (2.6) we refer to [23]. Using (2.5) and (2.6), it is concluded that if T is sufficiently small, then we have

$$\begin{aligned}&\|\bar{u}(x,t)\|_{C^{1+\alpha,\alpha/2}(S_T)} \\ &\quad\leqslant \|\bar{u}(x,0)\|_{C^{1+\alpha,\alpha/2}(S_T)} + M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\}\|\bar{u}\|_{C^{2+\alpha,1+\alpha/2}(S_T)} \\ &\quad\leqslant \|\bar{u}(x,0)\|_{C^{1+\alpha,\alpha/2}(S_T)} + M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\}M_2(B_0) \\ &\quad\leqslant \|u_0(x)\|_{C^{2+\alpha}(0,\varrho)} + 1 = B_0,\end{aligned} \tag{2.7}$$

and so $\bar{u} \in X_{B_0}$ and it means that F is a map from X_{B_0} to X_{B_0}.

Next, we show that F is contractive. To this end, consider u_1, u_2 in X_{B_0} and set $F(u_1) = \bar{u}_1$ and $F(u_2) = \bar{u}_2$ and $\bar{s} = \bar{u}_1 - \bar{u}_2$. Then $\bar{s}$ satisfies the equation

$$\bar{s}_t + A_1(x,t)\bar{s}_{xx} + A_2(x,t)\bar{s}_x - A_4(x,t)\bar{s} = A_3(x,t)f,$$

where

$$f = -\eta(u_1)v_{xx} - \eta'(u_1)v_x u_{1x} + \eta(u_2)v_{xx} + \eta'(u_2)v_x u_{2x}. \tag{2.8}$$

Now, we rewrite (2.8) in the following form:

$$\begin{aligned}f &= -(\eta(u_1)-\eta(u_2))v_{xx} - (\eta'(u_1)u_{1x} - \eta'(u_2)u_{2x})v_x\\ &= -(\eta(u_1)-\eta(u_2))v_{xx} - (\eta'(u_1)-\eta'(u_2))u_{2x}v_x - \eta'(u_1)(u_1-u_2)_x v_x.\end{aligned}$$

Then by a similar argument as in the proof of Lemma 2.1 of [23], it can be shown that

$$\|\bar{u}_1-\bar{u}_2\|_{C^{1+\alpha,\alpha/2}(S_T)} \leqslant M_0T^{\alpha/2}\|u_1-u_2\|_{C^{1+\alpha,\alpha/2}(S_T)}.$$

This shows that for a small T, the map F is contractive mapping on X_{B_0}. By the contraction mapping theorem, F has a unique fixed point u, which is the unique solution of (2.1). Invoking the parabolic Schauder estimates, we have a regularity assertion for u that it belongs to $C^{2+\alpha,1+\alpha/2}(S_T)$. □

We return now to system (1.4) and prove the local existence and uniqueness of solutions to this system.

Theorem 2.2. *Let $Q_T=(0,h(t))\times[0,T)$. Under assumptions* (1.2) *and* (1.3), *system* (1.4) *admits a unique solution*

$$U=(u,v,h)\in C^{2+\alpha,1+\alpha/2}(Q_T)\times C^{2+\alpha,1+\alpha/2}(Q_T)\times C^{1+\alpha/2}[0,T)$$

for some small $T>0$ which only depends on $\|u(\cdot,0)\|_{C^{2+\alpha}(0,h_0)}$, $\|v(\cdot,0)\|_{C^{2+\alpha}(0,h_0)}$, h_0 and $h_d=h'(0)$.

Proof. In order to deal with the free boundary, we employ the idea which has been used to prove the main result of [4]. The idea is that we first make a change of variable to straighten the free boundary.

Let $\zeta(y)$ be a function in $C^\infty(\mathbb{R})$ satisfying

$$\zeta(y)=1 \text{ if } |y-h_0|<\frac{h_0}{4},\ \zeta(y)=0 \ \text{ if } \ |y-h_0|>h_0,\ |\zeta'(y)|<\frac{2}{h_0}.$$

We introduce a transformation that will straighten the free boundary. It is

$$(x,t)\to(y,t),\ \text{where } x=y+\zeta(y)(h(t)-h_0),\ 0\leqslant y<\infty. \tag{2.9}$$

Notice that as long as

$$|h(t)-h_0|<\frac{h_0}{4},$$

transformation (2.10) is a diffeomorphism from $[0,\infty)$ onto $[0,\infty)$, since $\partial x/\partial y > 1/2$, and

$$0\leqslant x\leqslant h(t)\Longleftrightarrow 0\leqslant y\leqslant h_0,\quad x=h(t)\Longleftrightarrow y=h_0.$$

It is easy to see that

$$
\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{1}{1+\zeta'(y)(h(t)-h_0)} = \sqrt{A(h(t),y)},\\
\frac{\partial^2 y}{\partial x^2} &= \frac{-\zeta''(y)(h(t)-h_0)}{[1+\zeta'(y)(h(t)-h_0)]^3} = B(h(t),y),\\
-\frac{1}{h'(t)}\frac{\partial y}{\partial t} &= \frac{\zeta(y)}{1+\zeta'(y)(h(t)-h_0)} = C(h(t),y).
\end{aligned} \tag{2.10}
$$

In this setting we define

$$
p(y,t) = u(x,t), \quad q(y,t) = v(x,t),
$$

and problem (1.4) becomes

$$
\begin{aligned}
& p_t - d_1 A p_{yy} - (d_1 B + h'(t)C)p_y + (Aq_{yy} + Bp_y)\eta(p)\\
&\qquad\qquad + A\eta'(p)q_y p_y = -ap + bg(q)p, \quad 0 < y < h_0,\ t > 0,\\
& q_t - d_2 A q_{yy} - (d_2 B + h'(t)C)q_y = k(q) - g(q)p, \quad 0 < y < h_0,\ t > 0,\\
& p(h_0,t) = q(h_0,t) = 0, \quad t > 0,\\
& h'(t) = -\beta p_y(h_0,t), h(0) = h_0, \quad t > 0,\\
& p_y(0,t) = q_y(0,t) = 0, \quad t > 0,\\
& p(y,0) = u_0(y), q(y,0) = v_0(y), \quad 0 \leqslant y \leqslant h_0.
\end{aligned} \tag{2.11}
$$

Hereafter, we use system (2.11) in the rest of the proof.

We shall prove the local existence by a fixed-point argument again. We introduce

$$
\begin{aligned}
X_{1T} &= \{p \in C^{\alpha,\alpha/2}(Q_T);\ p(y,0) = u_0(y),\ \|p\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant B_1\},\\
X_{2T} &= \{q \in C^{\alpha,\alpha/2}(Q_T);\ q(y,0) = v_0(y),\ \|q\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant B_2\},\\
X_{3T} &= \{h \in C^{1+\alpha/2}([0,T]);\ h(0) = h_0, h'(0) = h_d,\ \|h'\|_{C^{\alpha/2}([0,T])} \leqslant B_3\},
\end{aligned}
$$

where

$$
h_d = -\beta u_0'(h_0), \quad B_1 = \|u_0\|_{C^{2+\alpha}(0,h_0)} + 1, \quad B_2 = \|v_0\|_{C^{2+\alpha}(0,h_0)} + 1, \quad B_3 = |h_d| + 1,
$$

and T such that

$$
0 < T < \frac{h_0}{4B_3}. \tag{2.12}
$$

Then we set $X_T := X_{1T} \times X_{2T} \times X_{3T}$, which is a complete metric space with the metric

$$
\begin{aligned}
&d((p_1, q_1, h_1), (p_2, q_2, h_2)) \\
&\qquad = \|p_1 - p_2\|_{C^{\alpha,\alpha/2}(Q_T)} + \|q_1 - q_2\|_{C^{\alpha,\alpha/2}(Q_T)} + \|h_1' - h_2'\|_{C^{\alpha/2}([0,T])}.
\end{aligned}
$$

First, we observe that due to our choice of T in (2.12) for a given $U = (p, q, h) \in X_T$ we have

$$
|h(t) - h_0| \leqslant B_3 T \leqslant \frac{h_0}{4}.
$$

Therefore, the transformation $(x, t) \to (y, t)$ introduced in (2.10) is well defined.

For a given

$$
U = (p, q, h) \in X_T \tag{2.13}
$$

we define a corresponding function $\overline{U} = G(U)$ by $\overline{U} = (\bar{p}, \bar{q}, \bar{h})$, where $\overline{U}$ satisfies the equations

$$
\begin{aligned}
&\bar{p}_t - d_1 A \bar{p}_{yy} - (d_1 B + h'(t) C - B\eta(p))\bar{p}_y + A(\eta(\bar{p})\bar{q}_y)_y = (-a + bg(q))\bar{p}, \\
&\bar{p}(h_0, t) = 0, \quad \bar{p}_y(0, t) = 0, \quad \bar{p}(y, 0) = u_0(y)
\end{aligned} \tag{2.14}
$$

for $0 < y < h_0$, $t > 0$, and

$$
\begin{aligned}
&\bar{q}_t - d_2 A \bar{q}_{yy} - (d_2 B + h'(t) C)\bar{q}_y = \Big(r - \frac{r}{K} q - \frac{b_1 p}{1 + b_2 q}\Big)\bar{q}, \\
&\bar{q}(h_0, t) = 0, \quad \bar{q}_y(0, t) = 0, \quad \bar{q}(y, 0) = v_0(y)
\end{aligned} \tag{2.15}
$$

for $0 < y < h_0, t > 0$ with

$$
\begin{aligned}
&\bar{h}(t) = h_0 - \int_0^t \beta \bar{p}_y(h_0, \tau)\, \mathrm{d}\tau, \quad t > 0, \\
&\bar{h}(0) = h_0.
\end{aligned} \tag{2.16}
$$

It is clear that $\bar{h}'(0) = -\beta u_0'(h_0) = h'(0)$.

From (1.3), (2.13), (2.15), and the parabolic Schauder theory, we have that there exists a unique solution $\bar{q}$ to (2.15),

$$
\|\bar{q}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant \|\bar{q}(x,0)\|_{C^{2+\alpha}(0,h_0)} + M_3(B_2) \leqslant B_2 + M_3(B_2) = M_4(B_2), \tag{2.17}
$$

where M_3 and M_4 are some constants which depend only on B_2. Indeed,

$$
\|\bar{q}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant M_4(B_2). \tag{2.18}
$$

For such function $\bar{q}(y,t)$ we have

$$\|\bar{q}(y,t)-\bar{q}(0,t)\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\}\|\bar{q}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)}. \tag{2.19}$$

Using (2.18) and (2.19), we conclude that there is a T_1 sufficiently small such that

$$\begin{aligned}\|\bar{q}(y,t)\|_{C^{\alpha,\alpha/2}(Q_T)} &\leqslant \|\bar{q}(0,t)\|_{C^{2+\alpha}(Q_T)} \\ &\quad + M_0 \max\{T_1^{\alpha/2}, T_1^{1-\alpha/2}\}\|\bar{q}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \\ &\leqslant \|\bar{q}_0\|_{C^{2+\alpha}(Q_T)} + 1 = B_2.\end{aligned} \tag{2.20}$$

Employing (2.14), (2.13), and Lemma 2.1, a similar proof yields that there exists a unique solution $\bar{p}$ of (2.14) such that

$$\|\bar{p}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant M_5(B_1), \tag{2.21}$$

where M_5 is a constants which depends only on B_1. It is easily verified that $\bar{p}(y,t)$ satisfies

$$\|\bar{p}(y,t)-\bar{p}(0,t)\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\}\|\bar{p}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)}. \tag{2.22}$$

Applying (2.21) and (2.22), we conclude that there is a T_2 sufficiently small such that

$$\begin{aligned}\|\bar{p}(y,t)\|_{C^{\alpha,\alpha/2}(Q_T)} &\leqslant \|\bar{p}(0,t)\|_{C^{2+\alpha}(Q_T)} \\ &\quad + M_0 \max\{T_2^{\alpha/2}, T_2^{1-\alpha/2}\}\|\bar{p}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \\ &\leqslant \|\bar{p}_0\|_{C^{2+\alpha}(Q_T)} + 1 = B_1.\end{aligned} \tag{2.23}$$

Now, we return to the dynamic of h, i.e.

$$\bar{h}'(t) = -\beta \bar{p}_y(h_0,t). \tag{2.24}$$

Using the norm $\|\cdot\|_{C^{\alpha/2}[0,T]}$, we have

$$\|\bar{h}'\|_{C^{\alpha/2}[0,T]} = \beta\|\bar{p}_y(h_0,t)\|_{C^{\alpha/2}[0,T]}. \tag{2.25}$$

Also note that

$$h_d = \frac{\mathrm{d}\bar{h}(0)}{\mathrm{d}t} = -\beta\bar{p}_y(h_0,0), \quad \bar{p}_y(h_0,0) = -\frac{h_d}{\beta}. \tag{2.26}$$

For function $\bar{p}_y(h_0,t)$ we have

$$\|\bar{p}_y(h_0,t)-\bar{p}_y(h_0,0)\|_{C^{\alpha/2}[0,T]} \leqslant T\|\bar{p}_y(h_0,.)\|_{C^{1+\alpha/2}[0,T]}. \tag{2.27}$$

Now, from (2.26) and (2.27), there is a T_3 sufficiently small such that

$$\begin{aligned}\|\bar{p}_y(h_0,t)\|_{C^{\alpha/2}[0,T]} &\leqslant T_3\|\bar{p}_y(h_0,\cdot)\|_{C^{1+\alpha/2}[0,T]} + \|\bar{p}_y(h_0,0)\|_{C^{1+\alpha/2}[0,T]}\\ &\leqslant \frac{1}{\beta}+\frac{|h_d|}{\beta}.\end{aligned} \tag{2.28}$$

From (2.25) we have

$$\|\bar{h}'\|_{C^{\alpha/2}[0,T]} \leqslant |h_d|+1 = B_3. \tag{2.29}$$

Taking $T = \min\{T_1,T_2,T_3\}$, we conclude from (2.20), (2.23), and (2.29) that $\overline{U}\in X_T$.

Then we prove that G is contractive on X_T. First of all, take

$$(p_1,q_1,h_1),(p_2,q_2,h_2)\in X_T \tag{2.30}$$

and set $\overline{w}=\bar{p}_1-\bar{p}_2$ and $\overline{z}=\bar{q}_1-\bar{q}_2$. Then it follows from (2.14) that $\overline{w}$ solves the problem which consists of the equation

(2.31)

$$\begin{aligned}&\overline{w}_t - d_1A(h_1,y)\overline{w}_{yy} - (d_1A(h_1,y)-d_1A(h_2,y))\bar{p}_{2_{yy}} - d_1B(h_1,y)\overline{w}_y\\
&\quad - (d_1B(h_1,y)-d_1B(h_2,y))\bar{p}_{2_y} - (h_1'(t)C(h_1,y))\overline{w}_y\\
&\quad - (C(h_1,y)-C(h_2,y))h_1'(t)\bar{p}_{2_y} - (h_1'(t)-h_2'(t))C(h_2,y)\bar{p}_{2_y}\\
&\quad + B(h_1,y)\eta(p_1)\overline{w}_y + (B(h_1,y)-B(h_2,y))\eta(p_1)\bar{p}_{2_y} + (\eta(p_1)-\eta(p_2))B(h_2,y)\bar{p}_{2_y}\\
&\quad - (-a+bg(q_1))\overline{w} - b(g(q_1)-g(q_2))\bar{p}_2\\
&= - A(h_1,y)\eta(p_1)z_{yy} - (A(h_1,y)-A(h_2,y))\eta(p_1)q_{2_{yy}}\\
&\quad - (\eta(p_1)-\eta(p_2))A(h_2,y)q_{2_{yy}} - A(h_1,y)\eta'(p_1)q_{1_y}w_y - A(h_1,y)\eta'(p_1)p_{2_y}z_y\\
&\quad - (A(h_1,y)-A(h_2,y))\eta'(p_1)p_{2_y}q_{2_y} - (\eta'(p_1)-\eta'(p_2))A(h_2,y)p_{2_y}q_{2_y},\end{aligned}$$

and it follows from (2.15) that $\overline{z}$ solves the problem which consists of the equation

$$\begin{aligned}&\overline{z}_t - d_2A(h_1,y)\overline{z}_{yy} - (d_2A(h_1,y)-d_2A(h_2,y))\bar{q}_{2_{yy}} - d_2B(h_1,y)\overline{z}_y\\
&\quad - (d_2B(h_1,y)-d_2B(h_2,y))\bar{q}_{2_y} - (h_1'(t)C(h_1,y))\overline{z}_y\\
&\quad - (C(h_1,y)-C(h_2,y))h_1'(t)\bar{q}_{2_y} - (h_1'(t)-h_2'(t))C(h_2,y)\bar{q}_{2_y}\\
&= \Big(r-\frac{r}{K}v_1-\frac{b_1p_1}{1+b_2q_1}\Big)\overline{z} - \frac{r}{K}(q_1-q_2)\bar{q}_2\\
&\quad - (p_1-p_2)\frac{b_1\bar{q}_2}{1+b_2q_1} - \Big(\frac{1}{1+b_2q_1}-\frac{1}{1+b_2q_2}\Big)b_1p_2\bar{q}_2.\end{aligned} \tag{2.32}$$

From (2.16) we have

$$\bar{h}_1'(t) - \bar{h}_2'(t) = -(\beta \bar{p}_{1_y}(h_0,t) - \beta \bar{p}_{2_y}(h_0,t)). \tag{2.33}$$

Using (2.31), (2.32), and (2.33), by a direct calculation we conclude that

$$\begin{aligned} &\overline{w}_t - d_1 A(h_1,y)\overline{w}_{yy} - (d_1 B(h_1,y) + h_1'(t)C(h_1,y) \\ &\qquad\qquad - B(h_1,y)\eta(p_1))\overline{w}_y - (-a + bg(q_1))\overline{w} = I_1, \\ &\overline{z}_t - d_2 A(h_1,y)\overline{z}_{yy} - (d_2 B(h_1,y) + h_1'(t)C(h_1,y))\overline{z}_y \\ &\qquad\qquad = \Big(r - \frac{r}{K}q_1 - \frac{b_1 p_1}{1+b_2 q_1}\Big)\overline{z} + I_2, \\ &\bar{h}_1'(t) - \bar{h}_2'(t) = -(\beta \bar{p}_{1_y}(h_0,t) - \beta \bar{p}_{2_y}(h_0,t)), \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} I_1 = {} & (d_1 A(h_1,y) - d_1 A(h_2,y))\bar{p}_{2_{yy}} + (d_1 B(h_1,y) - d_1 B(h_2,y))\bar{p}_{2_y} \\ & + (C(h_1,y) - C(h_2,y))h_1'(t)\bar{p}_{2_y} + (h_1'(t) - h_2'(t))C(h_2,y)\bar{p}_{2_y} \\ & - (B(h_1,y) - B(h_2,y))\eta(p_1)\bar{p}_{2_y} - (\eta(p_1) - \eta(p_2))B(h_2,y)\bar{p}_{2_y} \\ & + b(g(q_1) - g(q_2))\bar{p}_2 - A(h_1,y)\eta(p_1)z_{yy} - (A(h_1,y) - A(h_2,y))\eta(p_1)q_{2_{yy}} \\ & - (\eta(p_1) - \eta(p_2))A(h_2,y)q_{2_{yy}} - A(h_1,y)\eta'(p_1)q_{1_y}w_y - A(h_1,y)\eta'(p_1)p_{2_y}z_y \\ & - (A(h_1,y) - A(h_2,y))\eta'(p_1)p_{2_y}q_{2_y} - (\eta'(p_1) - \eta'(p_2))A(h_2,y)p_{2_y}q_{2_y} \end{aligned}$$

and

$$\begin{aligned} I_2 = {} & (d_2 A(h_1,y) - d_2 A(h_2,y))\bar{q}_{2_{yy}} + (d_2 B(h_1,y) - d_2 B(h_2,y))\bar{q}_{2_y} \\ & + (C(h_1,y) - C(h_2,y))h_1'(t)\bar{q}_{2_y} - (h_1'(t) - h_2'(t))C(h_2,y)\bar{q}_{2_y} \\ & - \frac{r}{K}(q_1 - q_2)\bar{q}_2 - (p_1 - p_2)\frac{b_1 \bar{q}_2}{1+b_2 q_1} - \Big(\frac{1}{1+b_2 q_1} - \frac{1}{1+b_2 q_2}\Big)b_1 p_2 \bar{q}_2. \end{aligned}$$

To simplify the calculations, we can rewrite system (2.34) as

$$\begin{aligned} &\overline{w}_t - \bar{d}_1 \overline{w}_{yy} - \overline{f}_1 \overline{w}_y = \overline{f}_2 \overline{w} + I_1, \\ &\overline{w}(h_0,t) = 0, \quad \overline{w}_y(0,t) = 0, \quad \overline{w}(y,0) = 0, \end{aligned} \tag{2.35}$$

$$\begin{aligned} &\overline{z}_t - \bar{d}_2 \overline{z}_{yy} - \overline{g}_1 \overline{z}_y = \overline{g}_2 \overline{z} + I_2, \\ &\overline{z}(h_0,t) = 0, \quad \overline{z}_y(0,t) = 0, \quad \overline{z}(y,0) = 0, \end{aligned} \tag{2.36}$$

$$\begin{aligned} &\bar{h}_1'(t) - \bar{h}_2'(t) = -(\beta \bar{p}_{1_y}(h_0,t) - \beta \bar{p}_{2_y}(h_0,t)), \\ &(\bar{h}_1 - \bar{h}_2)(0) = 0, \end{aligned} \tag{2.37}$$

where

$$\bar{d}_1 = d_1 A(h_1, y), \quad \bar{d}_2 = d_2 A(h_1, y),$$
$$\overline{f}_1 = d_1 B(h_1, y) + h_1'(t) C(h_1, y) - B(h_1, y)\eta(p_1), \quad \overline{f}_2 = -a + bg(q_1),$$
$$\overline{g}_1 = d_2 B(h_1, y) + h_1'(t) C(h_1, y), \quad \overline{g}_2 = r - \frac{r}{K} q_1 - \frac{b_1 p_1}{1 + b_2 q_1}.$$

In view of (2.10) and (2.30), we see

$$\|\overline{f}_i\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant M_0, \quad \|\overline{g}_i\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant M_0, \quad \|\bar{d}_i\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant M_0. \tag{2.38}$$

We set $\delta = \|U_1 - U_2\|_{X_T}$. Regarding the definition of I_1 and I_2, we have

$$\|I_1\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant M_0 \delta, \tag{2.39}$$
$$\|I_2\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant M_0 \delta. \tag{2.40}$$

By (2.35), (2.36), L^p-theory and the Schauder parabolic estimate, we obtain

$$\|\bar{p}_1 - \bar{p}_2\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant M_0 \|I_1\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant M_0 \delta, \tag{2.41}$$
$$\|\bar{q}_1 - \bar{q}_2\| C^{2+\alpha,1+\alpha/2}(Q_T) \leqslant M_0 \|I_2\|_{C^{\alpha,\alpha/2}(Q_T)} \leqslant M_0 \delta. \tag{2.42}$$

From (2.37) we conclude that

$$\begin{aligned} \|\bar{h}_1'(t) - \bar{h}_2'(t)\|_{C^{\alpha/2}([0,T])} &= \beta \|\bar{p}_{1_y}(h_0, t) - \bar{p}_{2_y}(h_0, t)\|_{C^{\alpha/2}([0,T])} \\ &\leqslant M_0 \|\bar{p}_1 - \bar{p}_2\|_{C^{1+\alpha,\alpha/2}(Q_T)}. \end{aligned} \tag{2.43}$$

Invoking a similar argument as in the achievement of (2.19), we have

$$\begin{aligned} &\|\bar{p}_1 - \bar{p}_2\|_{C^{\alpha,\alpha/2}(Q_T)} \\ &\quad \leqslant M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\} \|\bar{p}_1 - \bar{p}_2\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant M_0 \delta, \end{aligned} \tag{2.44}$$
$$\begin{aligned} &\|\bar{q}_1 - \bar{q}_2\|_{C^{\alpha,\alpha/2}(Q_T)} \\ &\quad \leqslant M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\} \|\bar{q}_1 - \bar{q}_2\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant M_0 \delta. \end{aligned} \tag{2.45}$$

Then, applying (2.43), the following estimate is obtained:

$$\begin{aligned} &\|\bar{h}_1'(t) - \bar{h}_2'(t)\|_{C^{\alpha/2}([0,T])} \\ &\quad \leqslant M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\} \|\bar{p}_1 - \bar{p}_2\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant M_0 \delta. \end{aligned} \tag{2.46}$$

Now, (2.44), (2.45), and (2.46) yield

$$\begin{aligned} \|\overline{U}_1 - \overline{U}_2\|_{X_T} &\leqslant M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\} \delta \\ &\leqslant M_0 \max\{T^{\alpha/2}, T^{1-\alpha/2}\} \|U_1 - U_2\|_{X_T}. \end{aligned} \tag{2.47}$$

Using estimate (2.47), we conclude that if T is sufficiently small (T depends only on M_0), then G is a contractive map from X_T to X_T. By the contraction mapping theorem, G has a unique fixed point U, which is the unique solution of (2.11). Furthermore, we can raise the regularity of U to $C^{2+\alpha,1+\alpha/2}(Q_T)$ employing the parabolic Schauder estimates. □

3. Global classical solution

In this section we will show that system (1.4) has a global solution. It is observed from Theorem 2.2 that there exists a unique local solution of system (1.4). To show that such local solutions can be extended to all $t > 0$ as the global solution, we need some prior estimates. In what follows, we introduce some lemmas to obtain the estimates.

Lemma 3.1. *Let* $e(x,t) \in W_p^{2,1}(Q_T)$ *and*

$$\|e(x,t)\|_{W_p^{2,1}(Q_T)} \leqslant E_1,$$

where E_1 *is a constant depending on* T *and* $p > 5$. *Then* $e(x,t) \in C^{1+\alpha,\alpha/2}$ *and*

$$\|e(x,t)\|_{C^{1+\alpha,\alpha/2}} \leqslant E_2, \tag{3.1}$$

where E_2 *is a constant depending on* T.

Proof. Recall that $W_p^{2,1}(Q_T) \hookrightarrow C^{1+\gamma,(1+\gamma)/2}$ for $p > 5$, $\gamma = 1 - 5/p$ (see [10], Lemma 3.3, page 80).

If we take $\gamma > \alpha$ (it is possible when choosing p sufficiently large in the relation $\gamma = 1 - 5/p$), then we find that

$$\begin{aligned}\|e(x,t)\|_{C^{1+\alpha,\alpha/2}} &= \|e(x,t)\|_{C^{1,0}} + \|e(x,t)\|_{C^{\alpha,0}} + \|e(x,t)\|_{C^{0,\alpha/2}} \\ &\quad + \|D_x e(x,t)\|_{C^{\alpha,0}} + \|D_x e(x,t)\|_{C^{0,\alpha/2}} \\ &\leqslant \|e(x,t)\|_{C^{1,0}} + M\|e(x,t)\|_{C^{\gamma,0}} + T^{(1+\gamma-\alpha)/2}\|e(x,t)\|_{C^{0,(1+\gamma)/2}} \\ &\quad + M\|D_x e(x,t)\|_{C^{\gamma,0}} + T^{(1+\gamma-\alpha)/2}\|D_x e(x,t)\|_{C^{0,(1+\gamma)/2}} \\ &\leqslant (M + T^{(1+\gamma-\alpha)/2})\|e(x,t)\|_{C^{1+\gamma,(1+\gamma)/2}}.\end{aligned}$$

□

Lemma 3.2. *Let* (u, v, h) *be a solution of* (1.4) *for* $t \in [0, T]$ *and* $T > 0$. *Assume* $ab_2 > bb_1$. *Then*

$$0 < u(x,t) \leqslant L_1,$$
$$0 < v(x,t) \leqslant L_2,$$
$$0 < h'(t),$$

where L_i, $i = 1, 2$, *are two positive constants independent of* T.

Proof. Applying the maximum principle [10] and assumption (1.3), it can be deduced that

$$0 < v(x,t), \quad 0 < u(x,t).$$

The proof of the boundedness of v is straightforward and is based on the comparison principle of ODEs. We refer to the proof of Lemma 3.1 in [23] for the details. Also the boundedness of u has been established in [7].

The strong maximum principle yields the inequality $u_x(h(t), t) < 0$ in $(0, T]$. Hence, $h'(t) = -\beta u_x(h(t), t) > 0$ in $(0, T]$. Hereafter, we have generic constants M which depend on time T. □

Lemma 3.3. *Assume that*

$$(u, v, h) \in C^{2,1}(Q_T) \times C^{2,1}(Q_T) \times C^1[0, T)$$

is a solution to system (1.4). *Then we have*

$$\|u\|_{W_p^{2,1}(Q_T)} \leqslant M, \quad \|v\|_{W_p^{2,1}(Q_T)} \leqslant M$$

for any $p > 5$.

Proof. Consider the equation corresponding to v in system (1.4). Then it can be rewritten as

$$v_t - d_2 v_{xx} - \Big(r - \frac{r}{K}v - \frac{b_1}{1 + b_2 v}u\Big)v = 0. \tag{3.2}$$

In view of Lemma 3.2, we obtain

$$\Big\|r - \frac{r}{K}v - \frac{b_1}{1 + b_2 v}u\Big\|_{L^\infty(Q_T)} \leqslant M. \tag{3.3}$$

By (3.2), (3.3), and the parabolic L^p-estimate we have

$$\|v\|_{W_p^{2,1}(Q_T)} \leqslant M. \tag{3.4}$$

From (3.4) and Lemma 3.1 we have

$$\|v_x\|_{L^\infty(Q_T)} \leqslant M. \tag{3.5}$$

The equation associated to u in system (1.4) can be rewritten in the following form

$$u_t - d_1 u_{xx} + \eta'(u) v_x u_x = -\eta(u) v_{xx} - au + bg(v). \tag{3.6}$$

Then we have

$$\|\eta'(u) v_x\|_{L^\infty(Q_T)} \leqslant M, \quad \| - \eta(u) v_{xx} - au + bg(v)\|_{L^p(Q_T)} \leqslant M. \tag{3.7}$$

Using (3.5), (3.6) and (3.7), and the parabolic L^p-estimate, we deduce

$$\|u\|_{W_p^{2,1}(Q_T)} \leqslant M. \tag{3.8}$$

□

Lemma 3.4. *Assume that*

$$(u, v, h) \in C^{2,1}(Q_T) \times C^{2,1}(Q_T) \times C^1[0, T)$$

is a solution to system (1.4). *Then*

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant M, \quad \|v\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant M, \quad \|h\|_{C^{1+\alpha/2}} \leqslant M.$$

Proof. By Lemmas 3.1 and 3.3 we see

$$\|u\| \leqslant M, \quad \|v\|_{C^{\alpha,\alpha/2}} \leqslant M. \tag{3.9}$$

Using Lemma 3.2, inequality (3.9), and the parabolic Schauder estimate for the following equation

$$v_t - d_2 v_{xx} = k(v) - g(v)u, \tag{3.10}$$

we find that

$$\|v\|_{C^{2+\alpha,1+\alpha/2}} \leqslant M. \tag{3.11}$$

Furthermore, using Lemma 3.2, (3.9), (3.11), and the parabolic Schauder estimate for the following equation

$$u_t - d_1 u_{xx} + \eta'(u) v_x u_x = -\eta(u) v_{xx} - au + bg(v), \tag{3.12}$$

we have

$$\|u\|_{C^{2+\alpha,1+\alpha/2}} \leqslant M. \tag{3.13}$$

In view of (3.13), one can infer from equation

$$\frac{\mathrm{d}\overline{h(t)}}{\mathrm{d}t} = -\beta u_x(h(t),t) \tag{3.14}$$

that

$$\|h\|_{C^{1+\alpha/2}} \leqslant M. \tag{3.15}$$

This completes the proof. □

Now, we are prepared for the main result of this paper.

Theorem 3.5. *There exists a unique solution of system* (1.4) *for all* $t > 0$.

Proof. Suppose that $[0,T)$, $T < \infty$, is the maximum time interval for the existence of the solution. Consider $0 < \varepsilon < T$ as an arbitrary constant. We take $U(x, T-\varepsilon)$ as a new initial value. Then we can extend the solution to $Q_{(T-\varepsilon)+\tau}$ for small $\tau > 0$ regarding Theorem 2.2. Furthermore, Theorem 2.2 tells that τ depends only on an upper bound of $\|u(x,T-\varepsilon)\|_{C^{2+\alpha}[0,h(t)]}$, $\|v(x,T-\varepsilon)\|_{C^{2+\alpha}[0,h(t)]}$, $h(T-\varepsilon)$, and $h'(T-\varepsilon)$. By the estimate in Lemma 3.4, we find that τ depends on a constant $M(T)$. Recall that τ is independent of ε, i.e. $\tau = \tau(T)$. If we take $\varepsilon < \tau(T)$, then we get

$$(T-\varepsilon) + \tau > T,$$

which contradicts the assumption that $[0,T)$ is the maximum time interval for the existence of the solution. Therefore, the maximum time interval for the existence of the solution is $[0,\infty)$. □

4. Numerical study

In this section we provide some numerical results implemented for the simulation of prey-predators interactions. With an eye on the biological interpretations, the numerical simulations give a real insight into the behavior of the free boundary and the stability of the solutions.

Recall that the habitat is $[0,h(t)]$, $t \geqslant 0$, and we set $h(0) = h_0 = 1$. The zero flux boundary conditions $u_x(0,t) = v_x(0,t) = 0$, $t > 0$, mean that no individuals cross the boundary of the habitat, and the Dirichlet boundary conditions $u(h(t), t) = v(h(t), t) = 0$, $t > 0$, correspond to a completely hostile exterior region.

We use the explicit finite difference method for the numerical solution of the given biological model (1.4). For the explicit scheme, we set $\delta x = 0.02$ being the spatial step size and $\delta t = (\delta x)^2/3$ being the time step size. Few numerical examples are given to illustrate the spatial and temporal behavior of the interacting species. The numerical tests are run forward in time finishing at time $T = 30000\,\delta t$.

Example 1. In this example we take the following ecological parameters

$$a = 2,\ b = 0.5,\ r = 2,\ K = 1,\ b_1 = 1,\ b_2 = 1,\ \beta = 10,\ d_1 = 1,\ d_2 = 1.5.$$

The prey-tactic sensitivity χ is considered in the form

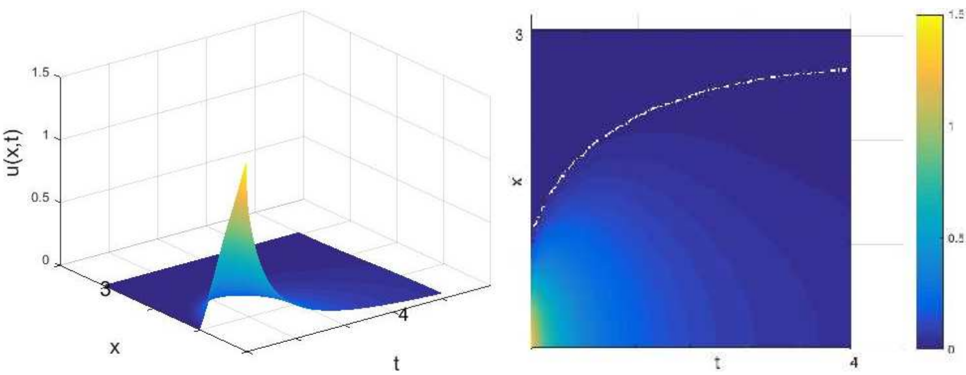
$$\chi(u) = \begin{cases} \alpha\Big(1 - \dfrac{u}{u_{\rm m}}\Big), & 0 \leqslant u \leqslant u_{\rm m}, \\ 0, & u \geqslant u_{\rm m} \end{cases}$$

with $\alpha = 2$ and $u_{\rm m} = 2$. In this example, we set

$$u_0(x) = 1.5(1 - x), \quad v_0(x) = x(1 - x).$$

It is obvious that functions χ, u_0, and v_0 satisfy assumptions (1.2) and (1.3).

In Figures 1 and 2, we can see the spatial and temporal dynamics of the interacting species and the propagation of the free boundary. In Figures 1, it is observed that the predator population disperse through the region regarding the effect of the prey-

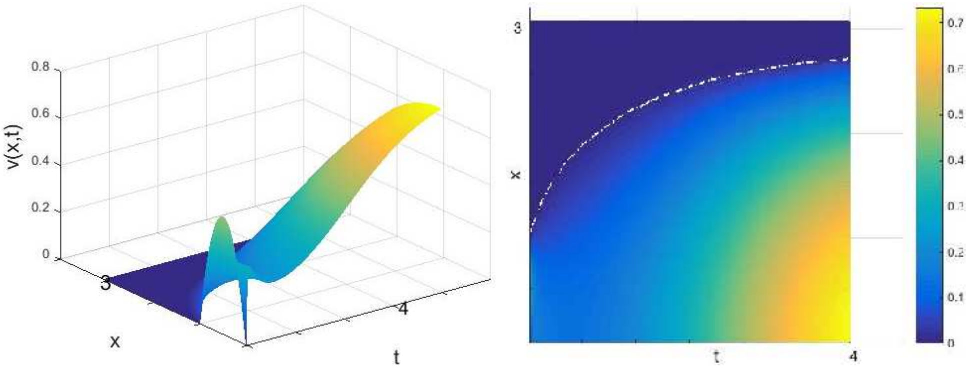


(a) Population density of the predator.

(b) Population density of the predator (from above) and the free boundary in white.

Figure 1. Dynamics of the predator population. The white curve depicts the free boundary.

taxis and the diffusion, and the predator invade further into new environment from the right boundary. The predator population decreases and at last vanishes when $t = T$.



(a) Population density of the prey.

(b) Population density of the prey (from above) and the free boundary in white.

Figure 2. Dynamics of the prey population. The white curve depicts the free boundary.

In view of Figure 2, we see that the preys spread over the area and invade further into new environment from the right boundary. The populations interact and the prey population increases until it reaches a steady state.

The white curve in Figures 1(b) and 6(b) shows the behavior of the free boundary. While u, the population density of the predator, is positive in the area near the free boundary, the free boundary develops and changes gradually over the time steps. The free boundary has stopped evolving after u becomes zero in the vicinity of it.

Indeed, two populations interact and grow or decay until the whole region is at a population's coexistence steady state. Figure 3 illustrates the steady state profile of the prey population.

It is worth noting that if r and K are reduced to 0.5 and 1, respectively, then both populations decay and reach a zero steady state and the free boundary will not propagate anymore, see Figure 4.

E x a m p l e 2. In this example we examine the case when u_0 and v_0 do not satisfy assumption (1.3). Let

$$u_0(x) = \begin{cases} 0.4, & 0.4 \leqslant x \leqslant 0.8, \\ 0, & \text{otherwise}, \end{cases} \qquad v_0(x) = \begin{cases} 0.5, & 0.4 \leqslant x \leqslant 0.8, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$a = 2,\ r = 1,\ K = 0.5,\ \beta = 0.5,\ d_1 = 0.07,\ d_2 = 0.05,\ u_{\rm m} = 0.5,\ \alpha = 0.02,$$

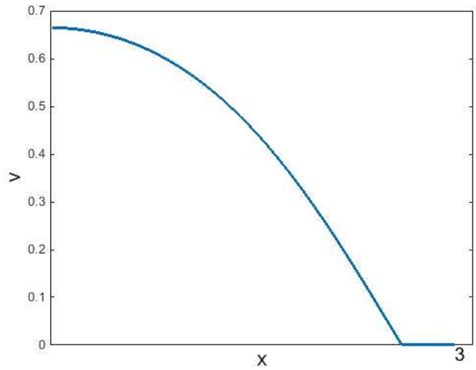
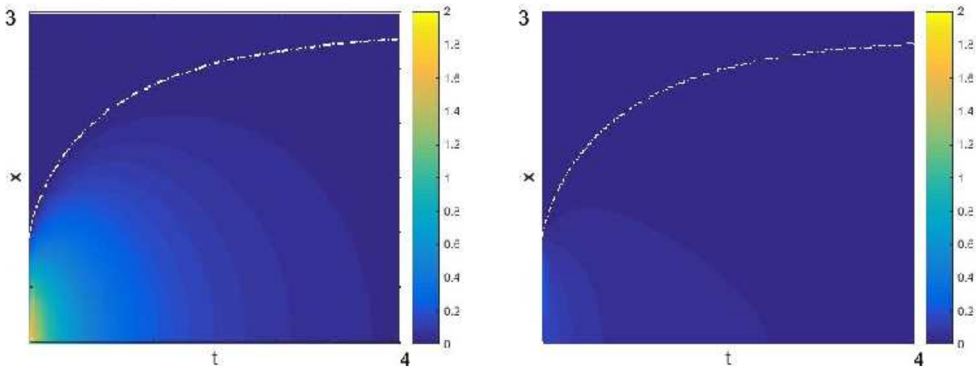


Figure 3. Steady state profile of the prey.



(a) Population density of the predator (from above) and the free boundary in white.

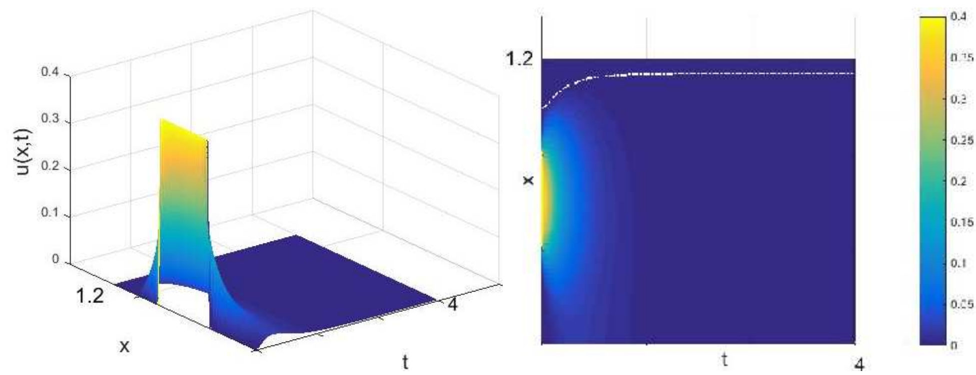
(b) Population density of the prey (from above) and the free boundary in white.

Figure 4. Dynamics of the predator and prey. The white curves depict the free boundaries.

as the ecological parameters. Here we take

$$g(v) = 0.5(v + \sin(v)), \quad \chi(u) = \begin{cases} \alpha\Big(1 - \dfrac{\exp(u)}{\exp(u_{\mathrm{m}})}\Big), & 0 \leqslant u \leqslant u_{\mathrm{m}}, \\ 0, & u \geqslant u_{\mathrm{m}}. \end{cases}$$

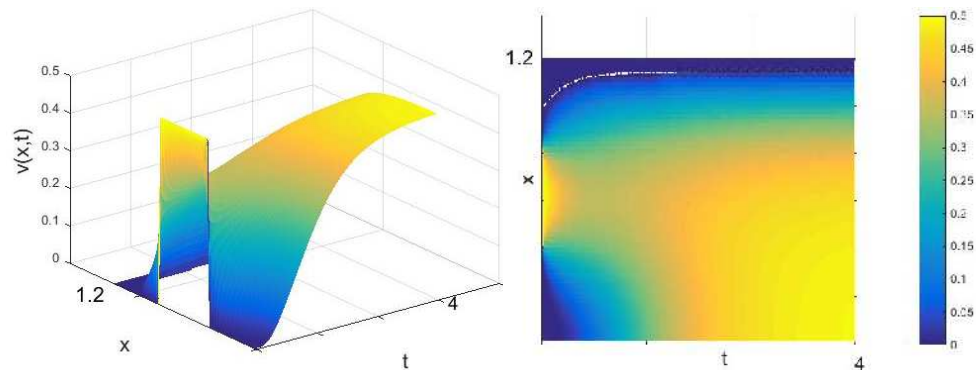
Although functions u_0 and v_0 do not satisfy assumptions (1.3) and χ has a more complicated nonlinear form, the behavior of the two populations and the free boundary is similar in spirit to the previous example, see Figures 5, 6. Another interesting result can be observed from Figures 7, where we set $\beta = -0.5$. In this case the habitat becomes smaller and the populations escape from the right boundary inward the domain. The free boundary evolves until the predator population decreases to zero. After that, the prey population continues to increase until it reaches a steady state.



(a) Population density of the predator.

(b) Population density of the predator (from above) and the free boundary in white.

Figure 5. Dynamics of the predator population. The white curve depicts the free boundary.

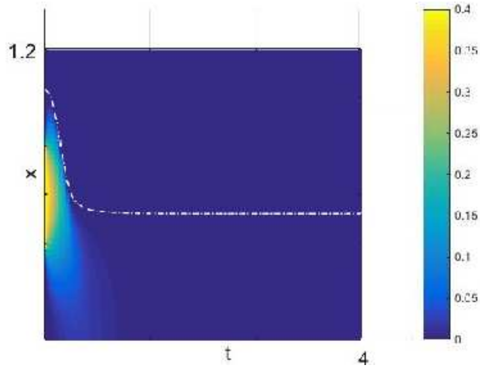


(a) Population density of the prey.

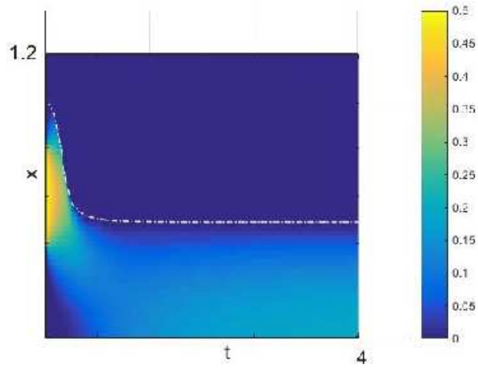
(b) Population density of the prey (from above) and the free boundary in white.

Figure 6. Dynamics of the prey population. The white curve depicts the free boundary.

In summary, it seems that the result of Sections 2 and 3 holds under weaker assumptions on u_0 and v_0 and they are valid for negative β as well.



(a) Population density of the predator (from above) and the free boundary in white.



(b) Population density of the prey (from above) and the free boundary in white.

Figure 7. Dynamics of the predator and prey population. The white curves depict the free boundary.

5. Conclusions

In this paper, we have considered reaction-diffusion system (1.4) modeling a predator-prey problem with prey-taxis and a free boundary over a one-dimensional habitat. Applying the contraction mapping principle, the global existence and uniqueness of the classical solutions to this system have been proved under assumptions (1.2) and (1.3). In view of some experimental situations, the biological equation (1.4) including a free boundary is a more realistic model, since both the prey and the predator have a tendency to emigrate from the boundary to obtain their new habitat and to improve the living environment.

The spatial and temporal behavior of the interacting species and the evolution of the free boundary have been investigated numerically. The numerical study implies that the two populations interact until they reach a steady state. In particular, the free boundary propagates until the predator population converges to zero due to the fact that $h'(t) = -\beta u_x$.

It would be interesting if the question of stability of steady states will be addressed in future studies. In view of the numerical investigations, it seems reasonable to conjecture that equation (1.4) will not reach a steady state solution when u, the predator population, is positive in the area near the free boundary and in the steady-state solution the function u should be zero.

Numerical tests reveal that the results of Sections 2 and 3 hold under milder assumptions on the data.

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