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A Free Boundary Problem for the p-Laplacian: Uniqueness, Convexity, and Successive Approximation of Solutions *

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Abstract

We prove convergence of a trial free boundary method to a classical solution of a Bernoulli-type free boundary problem for the p-Laplace equation, $1 < p < \infty$. In addition, we prove the existence of a classical solution in N dimensions when $p = 2$ and, for $1 < p < \infty$, results on uniqueness and starlikeness of the free boundary and continuous dependence on the fixed boundary and on the free boundary data. Finally, as an application of the trial free boundary method, we prove (also for $1 < p < \infty$) that the free boundary is convex when the fixed boundary is convex.

1 Introduction

We will develop methods for the successive approximation of solutions of the following free boundary problem originating with a power-law generalization of various well-known linear flow laws, such as Ohm's law for electrical current, Fourier's law for heat transfer, or Darcy's law for fluid flow through a porous medium.

Problem 1.1 *Given $1 < p < \infty$, a positive function $a(x)$, and a bounded C^2-domain D^* in $\mathbb{R}^N, N \geq 2$, (with $\Gamma^* = \partial D^*$), we seek a domain $D \supset Cl(D^*)$ such that*

$$|\nabla U(x)| = a(x)$$

on $\Gamma = \partial D$, where U denotes the p-capacitary potential in $\Omega := D \backslash Cl(D^*)$.

In the case where $p = 2$ and $a(x) = \text{constant}$, the first author [A5] has shown that the solution of Problem 1.1 can be interpreted in terms of minimization of heat flow through an annular domain with one fixed boundary component subject to a volume constraint. In a mathematically similar problem with different physical

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implications, Lacey and Shillor [LS] have shown that Γ can be interpreted as the equilibrium surface resulting from an electrochemical machining process in which there is a threshold of current (corresponding to $|\nabla U| = a$) below which etching does not occur.

In each of the above problems, a model based on a linear flow law is fundamental to the analysis. For example, for electric current through resistance, Ohm's law states that $J = -C\nabla U$, where C is the conductivity. The same relationship, this time called Fourier's law, applies in the heat flow minimization problem, where U now denotes the steady-state temperature and J is the heat flow. This law leads naturally to harmonic flow potentials in both cases. Clearly, Ohm's law and Fourier's law are approximate, empirical laws in which the assumption of linearity achieves maximum simplicity of the analysis. From the perspective of the study of nonlinear flow laws, it is natural to consider power-law flows as the next approximation. We define a power-law flow to be one for which the flow vector is given by $J = -C|\nabla U|^{p-2}\nabla U$, where p is a constant satisfying $p > 1$. This means that the magnitude of the flow vector is given by $|J| = C|\nabla U|^{p-1}$. Power-law flows have been previously studied in the context of p-diffusion (see Philip [P]) and deformation plasticity (see Atkinson and Champion [AtC]). For the case of steady-state power-law flows, the flow potential is p-harmonic, and the corresponding flow through the annular domain is given by the p-Dirichlet integral. Thus, one is led to Problem 1.1 from the perspectives of both heat flow minimization and electrochemical machining.

We show for arbitrary $p > 1$ that in essentially the starlike case, the solution is unique, starlike, and continuously dependent on the data. To the authors' knowledge, the existence question has not been examined to this generality in the literature; in fact, even for $p = 2$, there is no existence proof for a classical solution valid in higher dimensions. Early existence results due to Beurling [B], Daniljuk [D], and Lavrent'ev [LV] apply only for $p = 2$ and $N = 2$. The well-known existence results of Alt and Caffarelli [AC] are applicable only for $p = 2$, and these solutions are not necessarily classical for $N \geq 3$. An existence theorem for classical solutions in the starlike case for $p = 2$, $a(x) =$ constant, and $N \geq 2$ was stated by Lacey and Shillor [LS], but their proof is not valid for $N \geq 3$, because it is actually an argument by reference to Beurling's methods, which have never been generalized beyond $N = 2$. In §3, we validate the claims of Lacey and Shillor by proving the existence of a classical solution in the starlike case when $p = 2$, $N \geq 2$, and $a(x)$ is a real analytic function satisfying the same monotonicity condition required for uniqueness.

For arbitrary $1 < p < \infty$, but again in the starlike case, we obtain a global convergence proof for a particular analytical trial free boundary method for the successive approximation of the (classical) solution. To the authors' knowledge, this is the only approximation procedure available for this problem for which there is a known proof of convergence. This trial free boundary process consists of repeated application of a particular monotone operator which preserves star likeness and even convexity (under appropriate additional assumptions). The "operator method" was introduced by the first author [A1] in the case where $p = 2, N = 2$, and the domain lies between the graphs of periodic real functions of a real variable. An important aspect of the present study is the generalization of the operator properties discussed in [A1] to the situation described in Problem 1.1. It will be seen that the success of this generalization depends on a well-known homogeneity property of the p-Laplacian which is not shared by other divergence-form operators (see §2).

As an application of the operator method, we prove that if D^* is convex, then the solution of Problem 1.1 is convex under suitable conditions on the function $a(x)$ (see §5). This result, which adds to a growing literature concerning the convexity of free boundaries (see [Tp], [CS], [A2], [A3], [A4], [A6], [A7], [A8], [A9], and [APP]), specifically generalizes [A4], Lemma 2, to arbitrary dimensions and to a more general class of functions $a(x)$. It is an interesting fact that a modification of the definition of the operator permits this more general statement (see Remark 5.2). We remark that there is substantial recent literature on the closely related problem of the convexity of level surfaces of solutions of elliptic partial differential equations in convex annular domains (see [CF], [CS], [KL], and [L1]), of which the work of Lewis [L1] is pertinent to the present study; in fact, our result follows by combining Lewis's result with certain aspects of the operator method.

Remark 1.2 An additional interpretation of our model arises in the study of fluid flow through porous media. The linear flow law in this case is called Darcy's law, stating that $J = -C\nabla U$, where J is velocity, and U is pressure. Consider the case where two reservoirs of fluid (at different constant pressure) are separated by the homogeneous porous medium occupying the annular region Ω, through which the fluid flows by virtue of the pressure difference. If we choose the same power law generalization considered above, the free boundary in Problem 1.1 can then be interpreted as a surface on which the flow magnitude is given by a specified function of position (assuming, of course, that the other boundary has been specified).

2 The p-Laplace Equation

Here, we summarize a few relevant results from the literature. The p-Lapace equation is the quasilinear (degenerate) partial differential equation:

$$\triangle_p u = \text{ div } (|\nabla u|^{p-2}\nabla u) = 0 \ (1 < p < \infty)$$

In its weak form, the equation is

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \eta \, dx = 0 \tag{1}$$

for all $\eta \in H_0^{1,p}(\Omega)$. (Here, $|\nabla u|^{p-2}\nabla u$ is understood to equal 0 at all points where $\nabla u = 0$.) If $p = 2$, the p-Laplace equation is just the Laplace equation. For the annular domain $\Omega = D \backslash Cl(D^*)$, where D and D^* are bounded domains in $\mathbb{R}^N$ with $Cl(D^*) \subset D$, we define the p-capacitary potential in Ω to be the weak solution of the Dirichlet problem

$$\triangle_p u = 0 \text{ in } \Omega, u = 1 \text{ on } \partial D^*, u = 0 \text{ on } \partial D.$$

Regularity of weak solutions of the p-Laplace equation (p-harmonic functions) was studied in [DiB], [L2], and [T2]. The best regularity result for general bounded domains in $\mathbf{R}^N$ is that if $u \in H^{1,p}(\Omega)$ is a weak solution of the p-Laplace equation in Ω, then $u \in C^{1,\alpha}_{loc}(\Omega)$, where $\alpha = \alpha(p, N) > 0$. In [L1], it is seen that u is real analytic away from zeros of ∇u and that $\inf\{|\nabla u(x)| : x \in \Omega\} > 0$ if Ω is an annular domain with convex boundaries. Furthermore, it is shown in [LB] that, when $\partial\Omega \in C^{1,\alpha}$, there exists a β such that $u \in C^{1,\beta}(Cl(\Omega))$.

The p-Laplace operator has the following homogeneity property, which is necessary for the proof of convergence of the trial free boundary method:

$$\triangle_p(\lambda u(x)) = \lambda^{p-1} \triangle_p u(x).$$

Furthermore, if $u(x)$ satisfies the p-Laplace equation in a bounded domain Ω, then $u(x/\lambda)$ satisfies the p-Laplace equation in $\lambda\Omega = \{\lambda x : x \in \Omega\}$. Also essential in the convergence proof is the fact that weak solutions of the p- Laplace equation satisfy maximum and comparison principles. (See [T1], Lemma 3.1 and Proposition 2.3.3.)

3 The Free Boundary Problem: Existence and Uniqueness of Solutions

Throughout the remainder of this paper, we will require the following assumptions on the data in Problem 1.1:

Assumption 3.1 *The given domain $D^* \subset \mathbb{R}^N$ is starlike with respect to all points in the ball $B_\delta(0)$.*

Assumption 3.2 *The function $a(x)$ is continuous and has positive uniform upper and lower bounds in $\mathbb{R}^N$. Moreover, the function $ta(x_0 + t(x - x_0))$ is increasing in $t > 0$ for any $x \in \mathbb{R}^N$ and $x_0 \in B_\delta(0)$.*

Definition 3.3 (Classical Solution) *By a classical solution of Problem 1.1, we mean a domain $D \subset \mathbb{R}^N$ such that the p-capacitary potential U in $\Omega := D\backslash Cl(D^*)$ is in $C^2(\Omega) \cap C^1(Cl(\Omega))$ and satisfies the free boundary condition $|\nabla U(x)| = a(x)$ for every $x \in \partial D$.*

Theorem 3.4 (Existence) *Let $a(x)$ be a C^∞-function; then, for $p = 2$, Problem 1.1 has a classical solution D, which has a C^∞-surface and is starlike with respect to all points in $B_\delta(0)$.*

In order to prove Theorem 3.4, we consider the following variational problem.

Problem 3.5 *For the function $a(x)$ and domain D^* of Problem 1.1, we seek a minimizer of the functional*

$$J(v) := \int_{\mathbf{R}^N} (|\nabla v|^2 + a^2 I_{\{v>0\}})dx$$

over the set $K = \{v \in L^1_{loc}(\mathbb{R}^N), \nabla v \in L^2(\mathbb{R}^N), v = 1 \text{ on } \partial D^*\}$.

Lemma 3.6 *Problem 3.5 has a solution, U, which is Lipschitz continuous and has compact support on $\Omega^* = \mathbb{R}^N \backslash Cl(D^*)$ and satisfies $0 \leq U \leq 1$ in Ω^* and $\triangle U = 0$ in $\{U > 0\}$. Furthermore, U satisfies the free boundary condition of Problem 1.1 in a certain weak sense.*

Proof. See [AC], Theorems 1.3 and 3.3, Lemmas 2.8, 2.3, and 2.4, and Theorem 2.5. □

Lemma 3.7 *If D^* is starlike with respect to all points in $B_\delta(0)$, then so is $\{U > \varepsilon\}$ for all $0 \leq \varepsilon < 1$, where U denotes a solution of Problem 3.5.*

Proof. For $r > 1$, let $a_r(x) = (1/r)a(x/r)$, $U_r(x) = U(x/r)$, $U_r^+(x) = \max(U(x), U_r(x))$, and $U_r^-(x) = \min(U(x), U_r(x))$. Define the functional

$$J(\omega; r; v) := \int_\omega (|\nabla v|^2 + a_r^2 I_{\{v>0\}})dx$$

over the set $K(\omega) = \{v \in L^1_{loc}(\omega), \nabla v \in L^2(\omega)\}$, and define $\Omega^* = \mathbb{R}^N \backslash Cl(D^*)$ and $r\Omega^* = \{rx : x \in \Omega^*\}$. Following the proof of Lemma 3.4 in [A3], we show that $J(r\Omega^*; 1; U) = J(r\Omega^*; 1; U_r^+)$ and $J(r\Omega^*; 1; U_r) = J(r\Omega^*; 1; U_r^-)$ and conclude that $U(x) \leq U_r(x)$ in $r\Omega^*$. Thus, U is non-increasing with increasing $|x|$ along radial lines in $r\Omega^*$, and $\{U > \varepsilon\}$ is starlike with respect to the origin. This proof can be repeated, with the origin replaced by any $x_0 \in B_\delta(0)$, to show that $\{U > \varepsilon\}$ is starlike with respect to all points in $B_\delta(0)$. □

Lemma 3.8 *Let U be a solution of Problem 3.5. Then the free boundary, $\partial\{U > 0\}$, does not intersect $\Gamma^* = \partial D^*$, and, therefore, U is continuous on $Cl(\Omega)$.*

Proof. Let a_0 be a constant such that $a(x) < a_0$ in $\mathbb{R}^N$, and let D_0^* be an interior tangent ball to Γ^*. Define

$$J_0(v) := \int_{\mathbf{R}^N} (|\nabla v|^2 + a_0^2 I_{\{v>0\}})dx$$

over the set $K_0 = \{v \in L^1_{loc}(\mathbb{R}^N), \nabla v \in L^2(\mathbb{R}^N), v = 1 \text{ on } D_0^*\}$. The functional J_0 has a radially symmetric minimizer, U_0 (see [Ba], Corollary 2.1), and the free boundary, $\partial\omega_0$ (where $\omega_0 = \{U_0 > 0\}$), does not intersect $\Gamma_0^* = \partial D_0^*$. In $\mathbb{R}^N$, we define $U^+(x) = \max(U(x), U_0(x))$, $U^-(x) = \min(U(x), U_0(x))$, $\omega = \{U > 0\}$, and $\omega^\pm = \{U^\pm > 0\}$. Note that, since $0 \leq U_0 \leq 1$, we have $U^+ = 1$ on D^* so that U^+ is in K. Also, since $U^+ \geq U_0$, we have $\omega_0 \subset \omega^+$, and $\partial\omega^+$ lies outside ω_0. We claim that $J(U^+) \leq J(U)$, with strict inequality if $\omega^+\backslash\omega$ has positive measure. In terms of the notation: $R = \int_{\mathbf{R}^N} |\nabla U|^2\,dx$, $R_0 = \int_{\mathbf{R}^N} |\nabla U_0|^2\,dx$, $R^\pm = \int_{\mathbf{R}^N} |\nabla U^\pm|^2\,dx$, $|\omega| = \int_\omega a^2(x)\,dx$, $|\omega|_0 = \int_\omega a_0^2(x)\,dx$, our claim is that $R^+ + |\omega^+\backslash\omega| \leq R$, with strict inequality if $\omega^+\backslash\omega$ has positive measure. Toward the proof, we observe that U_0 minimizes J_0 over K_0, and that $U^- = 1$ in D_0^*, since $D_0^* \subset D^*$. Thus, we have $R_0 + |\omega_0|_0 \leq R^- + |\omega^-|_0$. In view of the fact that $R_0 + R = R^- + R^+$ (see [ACF1], §2), we conclude that

$$R^+ + |\omega_0\backslash\omega^-|_0 \leq R. \tag{2}$$

On the other hand, we have $E := \omega^+\backslash\omega = \omega_0\backslash\omega^-$. Thus, since $0 < a(x) < a_0$ in $\mathbf{R}^N$, we have

$$|\omega^+\backslash\omega| \leq |\omega_0\backslash\omega^-|_0, \tag{3}$$

where the inequality is strict if E has positive measure. Inequalities (2) and (3) imply our claim. A consequence of our claim is that the set of points inside ω^+ but outside ω must have measure zero in order to avoid contradicting the minimality of U. Since $\omega_0\backslash\omega \subset \omega^+\backslash\omega$, it follows that the set of points inside ω_0 but outside ω has measure zero. Now U and U_0 are both continuous in Ω, and D_0^* is an arbitrary interior tangent ball to Γ^*, so we conclude that $\partial\omega$ does not intersect Γ^* anywhere. □

Proof of Theorem 3.4. By Lemma 3.7, $\{U > 0\}$ is starlike with respect to all points in $B_\delta(0)$; thus, $\partial\{U > 0\}$ is locally the graph of a Lipschitz continuous function, where the coordinate system is chosen so that the radial direction is the coordinate axis of the dependent variable. Also, as shown by Caffarelli in [C2] (see "Application" and Lemma A1), U satisfies the free boundary condition $|\nabla U(x)| = a(x)$ in a certain weak sense defined in [C2], §1; therefore, it follows from [C1] that $\partial\{U > 0\}$ is a $C^{1,\alpha}$-surface, and, by the results of Kinderlehrer and Nirenberg in [KN], $\partial\{U > 0\}$ is a C^∞-surface on which the free boundary condition holds in a classical sense. □

Further definitions. In the paragraphs below, we will use the following notation: For $E \subset \mathbb{R}^N$ and $\lambda > 0$, $\lambda E = \{\lambda x : x \in E\}$. For $i = 1, 2$, let Γ_i be the boundary of D_i, a bounded, simply connected domain in $\mathbb{R}^N$ which contains the origin. In the family of all such surfaces, we will define the metric $\triangle$, where

$$\triangle(\Gamma_1, \Gamma_2) = \sup\{|\ln \lambda| : \lambda\Gamma_1 \cap \Gamma_2 \neq \emptyset\}.$$

We say that $\Gamma_1 \leq \Gamma_2$ if $D_1 \subset D_2$, and $\Gamma_1 < \Gamma_2$ if $Cl(D_1) \subset D_2$. If Γ is in this family of surfaces, $D(\Gamma)$ denotes the interior complement of Γ, and, for surfaces $\Gamma_1 < \Gamma_2$ in this family, $\Omega(\Gamma_1, \Gamma_2) = D(\Gamma_2) \backslash Cl(D(\Gamma_1))$.

Theorem 3.9 (Uniqueness, Starlikeness, Continuous Dependence)
(i) If a classical solution of Problem 1.1 exists, for any $1 < p < \infty$, then it is unique, and it is starlike with respect to all points in $B_\delta(0)$.

(ii) Suppose Γ^* and $\tilde{\Gamma}^*$ are the fixed boundaries in Problem 1.1 with Γ and $\tilde{\Gamma}$ the corresponding free boundaries; then

$$\Gamma^* \leq \tilde{\Gamma}^* \text{ implies that } \Gamma \leq \tilde{\Gamma}, \tag{4}$$

and

$$\triangle(\Gamma, \tilde{\Gamma}) \leq \triangle(\Gamma^*, \tilde{\Gamma}^*). \tag{5}$$

(iii) Suppose $a(x)$ and $\tilde{a}(x)$ satisfy Assumption 3.2 and that Γ and $\tilde{\Gamma}$ are the corresponding free boundaries; then

$$\tilde{a}(x) < a(x) \text{ in } \mathbb{R}^N \text{ implies that } \Gamma < \tilde{\Gamma}; \tag{6}$$

Furthermore, the solution Γ depends continuously on $a(x)$ in the following sense: If $a(x)$ and $\tilde{a}(x)$ are any functions satisfying Assumption 3.2 and the additional condition that $a(\lambda x)$ and $\tilde{a}(\lambda x)$ are nondecreasing in $\lambda > 0$ for all $x \in \mathbb{R}^N$ (including the case where $a(x)$ and $\tilde{a}(x)$ are identically constant), we have

$$\triangle(\Gamma, \tilde{\Gamma}) \leq \sup\{|\ln(a(x)/\tilde{a}(x))| : x \in \mathbb{R}^N\}, \tag{7}$$

More generally, we have that

$$\triangle(\Gamma, \tilde{\Gamma}) = \ln \lambda \leq \ln(1 + B_0\delta), \tag{8}$$

provided that $a(x)$ and $\tilde{a}(x)$ satisfy only Assumption 3.2, where $|a(x) - \tilde{a}(x)| \leq \delta$ in $\mathbf{R}^N$ and B_0 is a uniform constant to be determined.

The proof of our theorem requires a well-known comparison principle due to Lavrent'ev (see [LV], Theorem 1.1), which is generalized to solutions of the p-Laplace equation using the weak comparison principle for p-harmonic functions. (Observe that no special smoothness assumptions are made about the entire boundary surfaces.)

Lemma 3.10 (Lavrent'ev Principle) *Let $\Gamma^*, \Gamma, \tilde{\Gamma}^*,$ and $\tilde{\Gamma}$ be $(N-1)$-dimensional hypersurfaces which are boundaries of bounded, simply connected domains in $\mathbb{R}^N$ with $\Gamma^* < \Gamma$ and $\tilde{\Gamma}^* < \tilde{\Gamma}$. Let Ω (resp. $\tilde{\Omega}$) be the annular domain whose boundaries are Γ^* and Γ (resp. $\tilde{\Gamma}^*$ and $\tilde{\Gamma}$), and let U (resp. $\tilde{U}$) be the p-capacitary potential in Ω (resp. $\tilde{\Omega}$). Let $\lambda \geq 1$ be a value such that $\Gamma^* \leq \lambda\tilde{\Gamma}^*$ and $\Gamma \leq \lambda\tilde{\Gamma}$ (where $\lambda\tilde{\Gamma}^* \cap \Gamma$ may be nonempty). If $\lambda x \in \Gamma \cap \lambda\tilde{\Gamma}$ (resp. $\lambda x^* \in \Gamma^* \cap \lambda\tilde{\Gamma}^*$), and if $|\nabla\tilde{U}(x)|$ and $|\nabla U(x)|$ (resp. $|\nabla\tilde{U}(x^*)|$ and $|\nabla U(x^*)|$) both exist, then*

$$|\nabla\tilde{U}(x)| \geq \lambda|\nabla U(\lambda x)| \ (\textit{resp. } |\nabla\tilde{U}(x^*)| \leq \lambda|\nabla U(\lambda x^*)|).$$

Proof. See [M], Lemma 3.3.1. □

Proof of part (i) of Theorem 3.9. To prove that the free boundary is unique, we assume that Γ and $\tilde{\Gamma}$ are solutions to Problem 1.1 with $\Gamma \neq \tilde{\Gamma}$. Let $\ln(\lambda_0) = \triangle(\Gamma, \tilde{\Gamma})$, with $\lambda_0 > 1$; then $\Gamma \leq \lambda_0\tilde{\Gamma}$, $\tilde{\Gamma} \leq \lambda_0\Gamma$, and one of the intersections $\Gamma \cap \lambda_0\tilde{\Gamma}$ or $\lambda_0\Gamma \cap \tilde{\Gamma}$ is nonempty. We assume that there is a point $\lambda_0 x_0 \in \Gamma \cap \lambda_0\tilde{\Gamma}$. Furthermore, Assumption 3.1 implies that $\Gamma^* \leq \lambda_0\Gamma^*$, where $\Gamma^* = \partial D^*$. Thus, Lemma 3.10 implies that

$$a(x_0) = |\nabla U(\tilde{\Gamma}; x_0)| \geq \lambda_0|\nabla U(\Gamma; \lambda_0 x_0)| = \lambda_0 a(\lambda_0 x_0), \tag{9}$$

which contradicts Assumption 3.2. (We reach a similar contradiction with $\lambda_0 x_0 \in \lambda_0\Gamma \cap \tilde{\Gamma}$.) The same application of Lemma 3.10, with $\tilde{\Gamma} = \Gamma$, proves that the free boundary is starlike with respect to the origin. This argument may be repeated, with the origin replaced by any point in $B_\delta(0)$, to show that the free boundary is starlike with respect to all points in $B_\delta(0)$.

Proof of part (ii) of Theorem 3.9. Essentially the same argument as that used to prove the assertions in part (i) proves the assertions in part (ii). Specifically, for the proof of (4), suppose that $\Gamma^* \leq \tilde{\Gamma}^*$ but that $\tilde{\Gamma} < \Gamma$. Then, for some $\lambda_0 > 1$, we have $\Gamma \leq \lambda_0\tilde{\Gamma}$, and there is a point $\lambda_0 x_0 \in \Gamma \cap \lambda_0\tilde{\Gamma}$. By Assumption 3.1, $\Gamma^* \leq \tilde{\Gamma}^* \leq \lambda_0\tilde{\Gamma}^*$, and Lemma 3.10 may be applied as before to obtain (9) and contradict Assumption 3.2. To prove (5), we assume that $\triangle(\Gamma, \tilde{\Gamma}) > \triangle(\Gamma^*, \tilde{\Gamma}^*)$. Then, with $\ln(\lambda_0) = \triangle(\Gamma, \tilde{\Gamma})$, we have $\Gamma^* \leq \lambda_0\tilde{\Gamma}^*$, $\Gamma \leq \lambda_0\tilde{\Gamma}$, and a point $\lambda_0 x_0 \in \Gamma \cap \lambda_0\tilde{\Gamma}$, and Lemma 3.10 may again be applied to obtain (9) and a contradiction to Assumption 3.2.

Proof of part (iii) of Theorem 3.9. For the proof of (6), we assume that $\tilde{a}(x) < a(x)$ on $\mathbb{R}^N$ but that the corresponding free boundaries satisfy $\tilde{\Gamma} \leq \Gamma$. Then there is a $\lambda_0 \geq 1$ and a point $\lambda_0 x_0$ such that $\Gamma \leq \lambda_0\tilde{\Gamma}$ and $\lambda_0 x_0 \in \Gamma \cap \lambda_0\tilde{\Gamma}$. Since the fixed boundary satisfies $\Gamma^* \leq \lambda_0\Gamma^*$ by Assumption 3.1, Lemma 3.10 may again be applied to obtain the inequality

$$\tilde{a}(x_0) = |\nabla U(\tilde{\Gamma}; x_0)| \geq \lambda_0|\nabla U(\Gamma; \lambda_0 x_0)| = \lambda_0 a(\lambda_0 x_0). \tag{10}$$

Since $\lambda_0 a(\lambda_0 x_0) \geq a(x_0)$, this contradicts the assumption that $\tilde{a}(x) < a(x)$ in $\mathbb{R}^N$. Now for the proof of (7), let $\ln(\lambda_0) = \triangle(\Gamma, \tilde{\Gamma})$. Assuming that there is a point $\lambda_0 x_0 \in \Gamma \cap \lambda_0 \tilde{\Gamma}$, an application of Lemma 3.10 gives (10), and the additional assumption on $a(x)$ implies that $\tilde{a}(x_0) \geq \lambda_0 a(x_0)$. Similarly, if we assume that $\lambda_0 x_0 \in \lambda_0 \Gamma \cap \tilde{\Gamma}$, we obtain $a(x_0) \geq \lambda_0 \tilde{a}(x_0)$, and, in either case, we have

$$\ln(\lambda_0) = \triangle(\Gamma, \tilde{\Gamma}) \leq |\ln a(x_0) - \ln \tilde{a}(x_0)| \leq \sup\{|\ln a(x) - \ln \tilde{a}(x)| : x \in \mathbb{R}^N\}.$$

Finally, to prove (8), let $\ln \lambda = \triangle(\Gamma, \tilde{\Gamma})$, and let $x_0 \in ((1/\lambda)\Gamma \cap \tilde{\Gamma})$. Let $0 < a_0 = \min\{a(x), \tilde{a}(x) : x \in \mathbb{R}^N\}$, and let Γ_0 solve Problem 1.1 with free boundary condition $|\nabla U(\Gamma_0; x)| = a_0$ on Γ_0; then, by part (ii) of this theorem, $\Gamma \leq \Gamma_0$, and $\tilde{\Gamma} \leq \Gamma_0$. Choose R so large that $\partial B_R(0) > \Gamma_0$; then $\lambda \leq \lambda_{\max}$, where $\ln(\lambda_{\max}) = \triangle(\partial B_R, \Gamma^*)$. (Note that R and $\lambda_{\max}$ depend only on $a(x), \tilde{a}(x)$, and Γ^*.) Choose $r > 1$ so that $r\Gamma^* < \Gamma$ and $r\Gamma^* < \tilde{\Gamma}$, and define $\Gamma_{\lambda,r} = \partial((1/\lambda)D \cup rD^*)$. Observe that $\Gamma_{\lambda,r} > \Gamma^*$, $\Gamma_{\lambda,r} \geq (1/\lambda)\Gamma$, and $x_0 \in \Gamma_{\lambda,r}$. The function $V(x) := U(\lambda x)$ is the p-capacitary potential in $\Omega((1/\lambda)\Gamma^*, (1/\lambda)\Gamma)$.

$$|\nabla V(x)| = \lambda |\nabla U(\lambda x)| = \lambda a(\lambda x) \geq a(x). \tag{11}$$

Let $U_{\lambda,r}(x)$ be the p-capacitary potential in $\Omega((1/\lambda)\Gamma^*, \Gamma_{\lambda,r})$. By Lemma 3.10,

$$|\nabla U_{\lambda,r}(x)| \geq |\nabla V(x)| \text{ for all } x \in \Gamma_{\lambda,r} \cap (1/\lambda)\Gamma. \tag{12}$$

It can be shown, by a comparison argument (similar to the proof of Lemma 4.4 below) involving $U_{\lambda,r}$ and the p-capacitary potentials in $\Omega((1/\lambda)\Gamma^*, \partial B_R)$ and $\Omega((1/\lambda_{\max})\Gamma^*, \partial B_R)$, that there is a positive constant C_0, independent of λ, such that

$$U_{\lambda,r}(x) \leq 1 - C_0(\lambda - 1) \text{ for all } x \in \Gamma^*. \tag{13}$$

Since $U_{\lambda,r} = 0$ on $\Gamma_{\lambda,r}$, (13) and the weak comparison principle imply that $U_{\lambda,r}/(1 - C_0(\lambda - 1)) \leq \tilde{U}$ in $\Omega(\Gamma^*, \Gamma_{\lambda,r})$ and, as in Lemma 3.10,

$$|\nabla \tilde{U}| \geq \frac{|\nabla U_{\lambda,r}|}{1 - C_0(\lambda - 1)} \text{ on } \Gamma_{\lambda,r} \cap \tilde{\Gamma}. \tag{14}$$

Combining (11), (12), and (14), which all hold at x_0, yields

$$\tilde{a}(x_0) = |\nabla \tilde{U}(x_0)| \geq \frac{|\nabla V(x_0)|}{1 - C_0(\lambda - 1)} \geq \frac{a(x_0)}{1 - C_0(\lambda - 1)},$$

and

$$\triangle(\Gamma, \tilde{\Gamma}) = \ln \lambda \leq \ln\left(1 + \frac{1}{C_0}\left(1 - \frac{a(x_0)}{\tilde{a}(x_0)}\right)\right).$$

The assertion follows, where $B_0 = 1/(a_0 C_0)$. □

4 The Trial Free Boundary Method

Let $\mathbf{X}$ be the set of all $(N-1)$-dimensional surfaces of the form $\Gamma = \partial D$, where $D = D(\Gamma)$ denotes a bounded domain in $\mathbb{R}^N$ which is starlike with respect to all points in a fixed ball $B_\delta(0), \delta > 0$. (Observe that the surfaces in $\mathbf{X}$ are not necessarily C^1-surfaces.) The set $\mathbf{X}$ is complete with respect to the metric $\triangle$ defined in §3. Given $\Gamma^* \in \mathbf{X}$, let $\mathbf{Y} = \{\Gamma \in \mathbf{X} : \Gamma > \Gamma^*\}$. For any Γ in $\mathbf{Y}$, let $\Omega(\Gamma) = D \backslash Cl(D^*)$ = the annular domain whose boundary is $\Gamma \cup \Gamma^*$, and let $S(\Gamma)$ denote the complement of D. We use the notation $U(\Gamma; x)$ to denote the p-capacitary potential in $\Omega(\Gamma)$ (see §2).

We will use a family of operators $T_\varepsilon : \mathbf{Y} \to \mathbf{Y}$, $\varepsilon \in (0,1)$, defined as the composition of auxiliary operators ϕ_ε and ψ_ε. For $\Gamma \in \mathbf{Y}$, we define

$$\phi_\varepsilon(\Gamma) = \partial\{x \in D(\Gamma) : U(\Gamma; x) > \varepsilon\},$$

and

$$\psi_\varepsilon(\Gamma) = \{x \in S(\Gamma) : \frac{\varepsilon}{d(x,\Gamma)} = a(x)\}.$$

Then

$$T_\varepsilon(\Gamma) = \psi_\varepsilon(\phi_\varepsilon(\Gamma)).$$

It will be shown later that $T_\varepsilon : \mathbf{Y} \to \mathbf{Y}$ and that T_ε is a monotone operator in the sense that $T_\varepsilon(\Gamma_1) \leq T_\varepsilon(\Gamma_2)$ whenever $\Gamma_1 \leq \Gamma_2$ in $\mathbf{Y}$. It is also possible to find C^2-surfaces $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, with $\tilde{\Gamma}_1 < \tilde{\Gamma}_2$, such that the set $\tilde{\mathbf{Y}} := \{\Gamma \in \mathbf{Y} : \tilde{\Gamma}_1 \leq \Gamma \leq \tilde{\Gamma}_2\}$ has the property that $T_\varepsilon : \tilde{\mathbf{Y}} \to \tilde{\mathbf{Y}}$ for all $\varepsilon \in (0,1)$. Observe that the surfaces in $\tilde{\mathbf{Y}}$ are not necessarily C^2-surfaces; the proof of Theorem 4.1 below requires only that the inner surface $\tilde{\Gamma}_1$ and the outer surface $\tilde{\Gamma}_2$ be C^2-surfaces.

Theorem 4.1 (Convergence of the operator method) *In Problem 1.1, assume that Γ^* is a C^2-surface. Then T_ε is a contraction on $\tilde{\mathbf{Y}}$ for any $\varepsilon \in (0,1)$; in other words, there exists a value $\alpha = \alpha(\varepsilon)$, $0 \leq \alpha < 1$, such that*

$$\triangle(T_\varepsilon(\Gamma_1), T_\varepsilon(\Gamma_2)) \leq \alpha \triangle (\Gamma_1, \Gamma_2) \text{ for all } \Gamma_1, \Gamma_2 \in \tilde{\mathbf{Y}}. \tag{15}$$

Thus, by the Banach fixed point theorem, T_ε has a unique "fixed point" $\tilde{\Gamma}_\varepsilon \in \tilde{\mathbf{Y}}$ which can be obtained by successive approximations in the sense that, for any $\Gamma \in \tilde{\mathbf{Y}}, n \in \mathbf{N}$,

$$\triangle(T_\varepsilon^n(\Gamma), \tilde{\Gamma}_\varepsilon) \leq \frac{\alpha^n}{1-\alpha} \triangle (T_\varepsilon(\Gamma), \Gamma). \tag{16}$$

Moreover,

$$\triangle(\tilde{\Gamma}_\varepsilon, \tilde{\Gamma}) \to 0 \text{ as } \varepsilon \to 0^+, \tag{17}$$

where $\tilde{\Gamma}$ is a classical solution to Problem 1.1.

Theorem 4.2 (Convexity of the free boundary) *If Γ^* is a convex C^2-surface, and if $1/a(x)$ is concave in a neighborhood of the annular domain bounded by $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, then $\tilde{\Gamma}$ is convex.*

Lemma 4.3 (Properties of the operators) *The operator T_ε has the following properties:*
(i) $T_\varepsilon : \mathbf{Y} \to \mathbf{Y}$.
(ii) T_ε is a monotone operator.

Proof of (i). It is clear from the definition of ϕ_ε that $\phi_\varepsilon(\Gamma) > \Gamma^*$ for every $\varepsilon \in (0,1)$ and $\Gamma \in \mathbf{Y}$; thus, it only remains to be shown that $\phi_\varepsilon(\Gamma)$ is starlike with respect to all points in $B_\delta(0)$. This will follow from the fact that $U(\Gamma; x)$ is non-increasing with increasing $|x-x_0|$ along any radial line in $\Omega(\Gamma)$ originating at a point $x_0 \in B_\delta(0)$. We first show that $U(\Gamma; x)$ is nonincreasing with increasing $|x|$ along radial lines in $\Omega(\Gamma)$ originating at the origin. Observe that, for $\lambda \geq 1$, $V(x) := U(\Gamma; (1/\lambda)x)$ is the p-capacitary potential in $\lambda\Omega(\Gamma)$, the annular domain whose boundary is $\lambda\Gamma^* \cup \lambda\Gamma$. Since $U(\Gamma; x) \leq V(x) = 1$ on $\lambda\Gamma^*$ and $0 = U(\Gamma; x) \leq V(x)$ on Γ, the comparison principle for weak solutions of the p-Laplace equation implies that $U(\Gamma; x) \leq V(x) = U(\Gamma; (1/\lambda)x)$ in the intersection of $\Omega(\Gamma)$ with $\lambda\Omega(\Gamma)$. This proof may be repeated, with the origin replaced by any point $x_0 \in B_\delta(0)$. To show that the surface $\psi_\varepsilon(\Gamma)$ is starlike with respect to the origin, let $x \in S(\Gamma)$, let $\lambda > 1$, and define $f(x) = |x|((\varepsilon/d(x,\Gamma) - a(x))$; then, since $\lambda a(\lambda x) > a(x)$ and $d(\lambda x, \Gamma) > d(\lambda x, \lambda\Gamma) = \lambda d(x, \Gamma)$, we have $f(\lambda x) = \lambda|x|((\varepsilon/d(\lambda x, \Gamma)) - a(\lambda x)) < f(x)$. Thus, the function $f(x)$ is strictly decreasing with increasing $|x|$ along radial lines in $S(\Gamma)$. Again, the same argument can be repeated, with the origin replaced by any point $x_0 \in B_\delta(0)$, to show that $f(x) := |x-x_0|((\varepsilon/d(x,\Gamma) - a(x))$ is monotone decreasing with increasing $|x-x_0|$ along all radial lines originating at a point $x_0 \in B_\delta(0)$.

Proof of (ii). We will show that ϕ_ε and ψ_ε are monotone operators; for then $\Gamma_1 \leq \Gamma_2$ in $\mathbf{Y}$ implies that $\phi_\varepsilon(\Gamma_1) \leq \phi_\varepsilon(\Gamma_2)$, and $T_\varepsilon(\Gamma_1) = \psi_\varepsilon(\phi_\varepsilon(\Gamma_1)) \leq \psi_\varepsilon(\phi_\varepsilon(\Gamma_2)) = T_\varepsilon(\Gamma_2)$. Let $\Gamma_1 \leq \Gamma_2$ in $\mathbf{Y}$. By the weak comparison principle, $U(\Gamma_1; x) \leq U(\Gamma_2; x)$ in $\Omega(\Gamma_1)$; thus, $\{U(\Gamma_1; x) > \varepsilon\} \subset \{U(\Gamma_2; x) > \varepsilon\}$, which means that $\phi_\varepsilon(\Gamma_1) \leq \phi_\varepsilon(\Gamma_2)$. Furthermore, if $x \in S(\Gamma_2)$, then $d(x, \Gamma_2) \leq d(x, \Gamma_1)$ and $|x|((\varepsilon/d(x,\Gamma_2)) - a(x)) \geq |x|((\varepsilon/d(x,\Gamma_1)) - a(x))$. Thus, if $x \in \psi_\varepsilon(\Gamma_2)$, then $|x|((\varepsilon/d(x,\Gamma_1)) - a(x)) \leq 0$, and, by the monotonicity of $f(x) = |x|((\varepsilon/d(x,\Gamma_1) - a(x))$, we have $x \in S(\psi_\varepsilon(\Gamma_1))$, which implies that $\psi_\varepsilon(\Gamma_1) \leq \psi_\varepsilon(\Gamma_2)$. $\square$

Outline of the proof that the operator is a contraction. For the proof that T_ε is a contraction, $\varepsilon \in (0,1)$ is fixed. Choose $\tilde{\lambda} > 1$ such that $\ln(\tilde{\lambda}) = \triangle(\tilde{\Gamma}_1, \tilde{\Gamma}_2)$. Then the operator ψ_ε is non-expanding in the sense that

$$\psi_\varepsilon(\lambda\Gamma) \leq \lambda\psi_\varepsilon(\Gamma) \tag{18}$$

for all $\Gamma \in \tilde{\mathbf{Y}}$ and $1 \leq \lambda \leq \tilde{\lambda}$. For the proof, choose $x \in \psi_\varepsilon(\Gamma)$; then $\lambda x \in S(\lambda\Gamma)$, and by the monotonicity of the function $f(x) = |x|((\varepsilon/d(x,\Gamma)) - a(x))$, it follows that $f(\lambda x) \leq f(x) = 0$. Thus, $\lambda x \in S(\psi_\varepsilon(\lambda\Gamma))$, which implies (18). Due to the monotonicity of the operators, it suffices to show that there exists $\alpha = \alpha(\varepsilon)$, with $0 \le \alpha < 1$, such that

$$\phi_\varepsilon(\lambda\Gamma) \le \lambda^\alpha \phi_\varepsilon(\Gamma) \text{ for all } \Gamma \in \tilde{\mathbf{Y}} \text{ and } 1 \le \lambda \le \tilde{\lambda}. \tag{19}$$

This result follows immediately from Lemmas 4.4 through 4.7; in Lemmas 4.4 and 4.5, we show that there exists an α such that $\phi_\varepsilon(\lambda\Gamma) \le \lambda^\alpha \phi_\varepsilon(\Gamma)$, and Lemmas 4.6 and 4.7 together show that $\alpha = \alpha(\varepsilon)$ and $0 \le \alpha < 1$. □

Lemma 4.4 *If Γ^* is a C^2-surface, then there exists a constant $C > 0$ such that*

$$U(\lambda\Gamma; \lambda x) \le (1 - C(\lambda - 1))\varepsilon \tag{20}$$

uniformly for all $\Gamma \in \tilde{\mathbf{Y}}, x \in \phi_\varepsilon(\Gamma)$ and $\lambda \in [1, \tilde{\lambda}]$.

Proof. For any $\Gamma \in \tilde{\mathbf{Y}}$ and $1 \le \lambda \le \tilde{\lambda}$, the weak comparison principle implies that

$$U(\lambda\Gamma; \lambda x) \le (\max\{U(\lambda\Gamma; y) : y \in \lambda\Gamma^*\})U(\Gamma; x)$$

in $\Omega(\Gamma)$. This is because both $U(\lambda\Gamma; \lambda x)$ and $(\max\{U(\lambda\Gamma; y) : y \in \lambda\Gamma^*\})U(\Gamma; x)$ are weak solutions of the p-Laplace equation in $\Omega(\Gamma)$, and, for $x \in \Gamma$, $U(\Gamma; x) = U(\lambda\Gamma; \lambda x) = 0$, while, on Γ^*, $U(\Gamma; x) = 1$ and $U(\lambda\Gamma; \lambda x) \le \max\{U(\lambda\Gamma; y) : y \in \lambda\Gamma^*\}$. Also, since $\lambda\Gamma \le \tilde{\lambda}\tilde{\Gamma}_2$, we have that $U(\lambda\Gamma; x) \le U(\tilde{\lambda}\tilde{\Gamma}_2; x)$ in $\Omega(\lambda\Gamma)$ by the weak comparison principle. Therefore,

$$U(\lambda\Gamma; \lambda x) \le \max\{U(\tilde{\lambda}\tilde{\Gamma}_2; y) : y \in \lambda\Gamma^*\}U(\Gamma; x) \text{ in } \Omega(\Gamma), \tag{21}$$

and the assertion will follow from an estimate of $\max\{U(\tilde{\lambda}\tilde{\Gamma}_2; y) : y \in \lambda\Gamma^*\}$. To obtain this estimate, we observe that, since the annular domain $\Omega = \Omega(\tilde{\lambda}\tilde{\Gamma}_2)$ is starlike with respect to all points in $B_\delta(0)$, and since $\partial\Omega$ is a C^2-surface, it follows from [LB], Theorem 1, and from a proof of Lewis in [L1], §3, that $U(\tilde{\lambda}\tilde{\Gamma}_2; x) \in C^1(Cl(\Omega))$ and that $\inf\{|\nabla U(\tilde{\lambda}\tilde{\Gamma}_2; x)| : x \in \Omega\} > 0$. Under these conditions, the function $v(x) := \nabla U(\tilde{\lambda}\tilde{\Gamma}_2; x) \cdot x$ satisfies a uniformly elliptic partial differential equation in Ω and, therefore, we have that

$$-\nabla U(\tilde{\lambda}\tilde{\Gamma}_2; x) \cdot x \ge \min\{(-\nabla U(\tilde{\lambda}\tilde{\Gamma}_2; y) \cdot y) : y \in \partial\Omega\} = C > 0,$$

by the maximum principle. Now for any $y \in \lambda\Gamma^*$, the Mean Value Theorem implies that

$$U(\tilde{\lambda}\tilde{\Gamma}_2; y) - U(\tilde{\lambda}\tilde{\Gamma}_2; (1/\lambda)y)) = \nabla U(\tilde{\lambda}\tilde{\Gamma}_2; (1/\lambda^*)y) \cdot (y - (1/\lambda)y),$$

for some $1 < \lambda^* < \lambda$, so that, for any $y \in \lambda\Gamma^*$,

$$\begin{aligned} U(\tilde{\lambda}\tilde{\Gamma}_2; y) &\le U(\tilde{\lambda}\tilde{\Gamma}_2; (1/\lambda)y) + \nabla U(\tilde{\lambda}\tilde{\Gamma}_2; (1/\lambda^*)y) \cdot (1/\lambda^*)y(\lambda - 1) \\ &\le 1 - C(\lambda - 1). \end{aligned} \tag{22}$$

By combining (22) with (21), one sees that

$$U(\lambda\Gamma;\lambda x) \le (1 - C(\lambda-1))U(\Gamma;x) \tag{23}$$

uniformly for all $x \in \Omega(\Gamma)$, $\Gamma \in \tilde{\mathbf{Y}}$, and $1 \le \lambda \le \tilde{\lambda}$. Since $U(\Gamma;x) = \varepsilon$ on $\phi_\varepsilon(\Gamma)$, (20) holds uniformly for all $x \in \phi_\varepsilon(\Gamma)$, $\Gamma \in \tilde{\mathbf{Y}}$, and $1 \le \lambda \le \tilde{\lambda}$. □

Lemma 4.5 *There exists a constant* $\alpha \in (0,1)$ *such that* $\phi_\varepsilon(\lambda\Gamma) \le \lambda^\alpha \phi_\varepsilon(\Gamma)$ *uniformly for all* $\Gamma \in \tilde{\mathbf{Y}}$ *and* $1 \le$ *lambda* $\le \tilde{\lambda}$. *Specifically, we have*

$$\alpha = 1 - \frac{C\varepsilon}{\tilde{\lambda}\tilde{R}M(\lambda\Gamma)},$$

where $\tilde{R} = \max\{|x| : x \in \tilde{\Gamma}_2\}$, $E(\lambda\Gamma)$ *denotes the annular domain bounded by the surfaces* $\phi_\varepsilon(\lambda\Gamma)$ *and* $\lambda\phi_\varepsilon(\Gamma)$, *and* $M(\lambda\Gamma) = \max\{|\nabla U(\lambda\Gamma;x)| : x \in E(\lambda\Gamma)\}$.

Proof. By Lemma 4.4, we have that $U(\lambda\Gamma;\lambda x) \le (1 - C(\lambda-1))\varepsilon < \varepsilon$ for all x in $\phi_\varepsilon(\Gamma)$, $\Gamma \in \tilde{\mathbf{Y}}$ and $1 \le \lambda \le \tilde{\lambda}$. Thus, $\phi_\varepsilon(\lambda\Gamma) < \lambda\phi_\varepsilon(\Gamma)$, and $E(\lambda\Gamma)$ exists. Now for any $x \in \phi_\varepsilon(\lambda\Gamma)$, choose $r > 1$ such that $rx \in \lambda\phi_\varepsilon(\Gamma)$. By the Mean Value Theorem,

$$\begin{aligned} |U(\lambda\Gamma;x) - U(\lambda\Gamma;rx)| &\le \max\{|\nabla U(\lambda\Gamma;y) : y \in E(\lambda\Gamma)\}|x|(r-1) \\ &\le M(\lambda\Gamma)\tilde{\lambda}\tilde{R}(r-1). \end{aligned} \tag{24}$$

Since $U(\lambda\Gamma;x) = \varepsilon$, and since $rx = \lambda y$ for some $y \in \phi_\varepsilon(\Gamma)$, we have $U(\lambda\Gamma;rx) \le (1 - C(\lambda-1))\varepsilon$. Therefore, it follows from (24) that $\varepsilon - (1 - C(\lambda-1))\varepsilon \le M(\lambda\Gamma)\tilde{\lambda}\tilde{R}(r-1)$, whence $r \ge 1 + C\varepsilon(\lambda-1)/(\tilde{\lambda}\tilde{R}M(\lambda\Gamma))$. Since this estimate holds for all $x \in \phi_\varepsilon(\lambda\Gamma)$, we conclude that

$$\phi_\varepsilon(\lambda\Gamma) \le \frac{\lambda}{1 + (C\varepsilon/(\tilde{\lambda}\tilde{R}M(\lambda\Gamma)))(\lambda-1)}\phi_\varepsilon(\Gamma) \le \lambda^\alpha\phi_\varepsilon(\Gamma),$$

where $\alpha = 1 - C\varepsilon/(\tilde{\lambda}\tilde{R}M(\lambda\Gamma))$. □

Lemma 4.6 *For any* $\varepsilon \in (0,1)$, *there exists a positive value* $r_0 = r_0(\varepsilon)$ *such that* $d(\Gamma;\phi_\varepsilon(\Gamma)) \ge r_0$ *uniformly for all* $\Gamma \in \tilde{\mathbf{Y}}$.

Proof. For any $\Gamma \in \tilde{Y}$ and $x_0 \in \Gamma$, let

$$K(x_0) = \{x \in \mathbb{R}^N : |x_0||x - x_0|(1 - (\delta/\tilde{R})^2)^{1/2} \le x_0 \cdot (x - x_0) \le |x_0|\},$$

and let $\omega(x_0) = N_\mu(K(x_0))/K(x_0)$, where $\mu = d(\Gamma^*, \tilde{\Gamma}_1)/2$ and $N_\mu(\cdot)$ denotes the μ-neighborhood of a set. Observe that $K(x_0)$ is a right circular come with vertex x_0 such that $K(x_0) \subset S(\Gamma)$ and $\omega(x_0) \cap Cl(D^*) = \emptyset$. Also observe that the annular regions $\omega(x_0), x_0 \in \Gamma$, are all congruent. Define $u(x_0;x) = 1 - v(x_0;x)$, where $v(x_0;x)$ denotes the p-capacitary potential in $\omega(x_0)$. Also define $\gamma_\varepsilon(x_0) = \partial\{x \in \omega(x_0) : u(x_0;x) > \varepsilon\}$, observing that $r_0 := d(x_0, \gamma_\varepsilon(x_0)) > 0$ is a positive value which is independent of x_0, due to the congruity of these configurations. Observe that $U(\Gamma;x) = 0$ on $\Gamma \cap Cl(\omega(x_0))$, where $0 \le u(x_0;x) \le 1$, and that $0 \le U(\Gamma;x) \le 1$ in $Cl(\Omega(\Gamma)) \cap \partial(K(x_0) \cup \omega(x_0))$, where $u(x_0;x) = 1$. Therefore, $U(\Gamma;x) \le u(x_0;x)$ in $\omega(x_0) \cap \Omega(\Gamma)$, by the weak comparison principle. Thus, $\{x \in \omega(x_0) \cap \Omega(\Gamma) : U(\Gamma;x) > \varepsilon\} \subset \{x \in \omega(x_0) \cap \Omega(\Gamma) : u(x_0;x) > \varepsilon\}$, whence $d(\phi_\varepsilon(\Gamma), x_0) \ge d(x_0, \gamma_\varepsilon(x_0)) = r_0$. The assertion follows, since the above argument applies to all $\Gamma \in \tilde{\mathbf{Y}}$ and $x_0 \in \Gamma$. □

Lemma 4.7 *There exists a constant M_0 such that $M(\lambda\Gamma) \le M_0$ uniformly for all $\Gamma \in \tilde{\mathbf{Y}}$ and all $\lambda \in [1, \tilde{\lambda}]$.*

Proof. For all $\Gamma \in \tilde{\mathbf{Y}}$, $1 \le \lambda \le \tilde{\lambda}$, and $x \in E(\lambda\Gamma)$, we have that $d(x, \Gamma^*) \ge d(\phi_\varepsilon(\lambda\Gamma), \Gamma^*) \ge d(\phi_\varepsilon(\tilde{\Gamma}_1), \Gamma^*) > 0$, uniformly for all $\Gamma \in \tilde{Y}$, $1 \le \lambda \le \tilde{\lambda}$, and $x \in E(\lambda\Gamma)$, since $\lambda\Gamma \ge \tilde{\Gamma}_1$ and, therefore, $\phi_\varepsilon(\lambda\Gamma) \ge \phi_\varepsilon(\tilde{\Gamma}_1) > \Gamma^*$. Also, $d(x, \lambda\Gamma) \ge d(\lambda\phi_\varepsilon(\Gamma), \lambda\Gamma) = \lambda d(\phi_\varepsilon(\Gamma), \Gamma) \ge d(\phi_\varepsilon(\Gamma), \Gamma) > 0$, due to Lemma 4.6. Therefore, there exists a fixed value $\chi > 0$ such that $B_{3\chi}(x) \subset \Omega(\lambda\Gamma)$ for all $x \in E(\lambda\Gamma), \Gamma \in \tilde{\mathbf{Y}}$, and $1 \le \lambda \le \tilde{\lambda}$. By a gradient bound given by Tolksdorf (see [T2], Theorem 1) and the preceding discussion, there exist constants $c > 0$ and $\alpha > 0$, depending only on χ, N and p, such that

$$|\nabla U(\lambda\Gamma;x)| \le c\chi^{\alpha-1} \text{ for all } x \in E(\lambda\Gamma),$$

so that $M(\lambda\Gamma) \le M_0 = c\chi^{\alpha-1} < \infty$ for all $\Gamma \in \tilde{\mathbf{Y}}$ and $1 \le \lambda \le \tilde{\lambda}$. □

Convergence of fixed points to the free boundary. For any $\varepsilon \in (0,1)$, we define $\gamma_+(\varepsilon) = \max\{\ln\lambda : x \in \tilde{\Gamma}, \lambda x \in \tilde{\Gamma}_\varepsilon, \lambda > 0\}$ and $\gamma_-(\varepsilon) = \max\{\ln\lambda : x \in \tilde{\Gamma}_\varepsilon, \lambda x \in \tilde{\Gamma}, \lambda > 0\}$. We also define $E_\pm = \{\varepsilon \in (0,1) : \gamma_\pm(\varepsilon) \ge 0\}$. Since $\triangle(\tilde{\Gamma}_\varepsilon, \tilde{\Gamma}) = \max\{\gamma_+(\varepsilon), \gamma_-(\varepsilon)\}$, it suffices to show that $\limsup_{\varepsilon\to 0^+} \gamma_\pm(\varepsilon) = 0$.

"+" case. Let $\varepsilon \in E_+$ and $\gamma_+ = \gamma_+(\varepsilon)$; then $\tilde{\Gamma}_\varepsilon \le (\exp(\gamma_+))\tilde{\Gamma}$ and there is a point $x_0 = x_0(\varepsilon) \in \tilde{\Gamma}_\varepsilon \cap (\exp(\gamma_+))\tilde{\Gamma}$. Since $T_\varepsilon(\tilde{\Gamma}_\varepsilon) = \tilde{\Gamma}_\varepsilon$, it follows that $\varepsilon/d(\phi_\varepsilon(\tilde{\Gamma}_\varepsilon), x) = a(x)$ for all $x \in \tilde{\Gamma}_\varepsilon$, and it is possible to choose $x_1 = x_1(\varepsilon) \in \phi_\varepsilon(\tilde{\Gamma}_\varepsilon)$ so that

$$d\left(\frac{x_0}{e^{\gamma_+}}, \frac{x_1}{e^{\gamma_+}}\right) = \frac{1}{e^{\gamma_+}} d(x_0, x_1) = \frac{\varepsilon}{(e^{\gamma_+})a(x_0)} \le \frac{\varepsilon}{a(x_0/e^{\gamma_+})}. \tag{25}$$

Since $\tilde{\Gamma}$ is the solution of Problem 1.1,

$$||\nabla U(\tilde{\Gamma};x)| - a(x)| \le \sigma(d(\tilde{\Gamma}, x)) \text{ for } x \in \Omega(\tilde{\Gamma}), \tag{26}$$

where $\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. By the weak comparison principle and (23) with $\lambda = \exp(\gamma_+)$,

$$\varepsilon = U(\tilde{\Gamma}; x_1) \le U((e^{\gamma_+})\tilde{\Gamma}; x_1) \le (1 - C(e^{\gamma_+} - 1))U(\tilde{\Gamma}; x_1/e^{\gamma_+}) \tag{27}$$

for every sufficiently small $\varepsilon \in E_+$. Using the Mean Value Theorem, (25), and (26), we have, for some z on the line segment joining $x_0/\exp(\gamma_+)$ to $x_1/\exp(\gamma_+)$,

$$U\left(\tilde{\Gamma}; \frac{x_1}{e^{\gamma_+}}\right) \leq |\nabla U(\tilde{\Gamma}; z)| d\left(\frac{x_0}{e^{\gamma_+}}, \frac{x_1}{e^{\gamma_+}}\right) \leq (a(x_0/e^{\gamma_+}) + \sigma(\varepsilon)) \frac{\varepsilon}{a(x_0/e^{\gamma_+})}.$$

Combining this inequality with (27), and using the fact that $a(x)$ has a positive lower bound, a_0, we see that

$$\frac{\varepsilon}{1 - C(e^{\gamma_+} - 1)} \leq \varepsilon(1 + \sigma(\varepsilon)/a_0),$$

or

$$\gamma_+ \leq \ln\left(1 + \frac{1}{C}\left(1 - \frac{1}{1 + \sigma(\varepsilon)/a_0}\right)\right).$$

Thus, $\limsup_{\varepsilon \to 0^+} \gamma_+(\varepsilon) = 0$.

"–" case. Let $\varepsilon \in E_-$, so that $\tilde{\Gamma} \leq (\exp(\gamma_-))\tilde{\Gamma}_\varepsilon$, and there is a point $x_0 = x_0(\varepsilon) \in \tilde{\Gamma} \cap (\exp(\gamma_-))\tilde{\Gamma}_\varepsilon$. Since $\tilde{\Gamma}_\varepsilon$ is a fixed point of the operator T_ε, there is a point $x_1 = x_1(\varepsilon) \in (\exp(\gamma_-))\phi_\varepsilon(\tilde{\Gamma}_\varepsilon)$, such that

$$\frac{1}{e^{\gamma_-}} d(x_0, x_1) = d\left(\frac{x_0}{e^{\gamma_-}}, \frac{x_1}{e^{\gamma_-}}\right) = \frac{\varepsilon}{a(x_0/e^{\gamma_-})} \geq \frac{\varepsilon}{(e^{\gamma_-})a(x_0)}. \tag{28}$$

Using the Mean Value Theorem,(26) and (28),

$$U(\tilde{\Gamma}; x_1) \geq (a(x_0) - \sigma(\varepsilon)) \frac{\varepsilon}{a(x_0)} = \varepsilon - \sigma(\varepsilon) \frac{\varepsilon}{a(x_0)}. \tag{29}$$

Also, as in (27), we have

$$U(\tilde{\Gamma}; x_1) \leq U((e^{\gamma_-})\tilde{\Gamma}_\varepsilon; x_1) \leq (1 - C(e^{\gamma_-} - 1)) U\left(\tilde{\Gamma}_\varepsilon; \frac{x_1}{e^{\gamma_-}}\right) = (1 - C(e^{\gamma_-} - 1))\varepsilon.$$

Combining this with (29) and using the fact that $a(x)$ has a positive lower bound, a_0, one concludes that $\ln \gamma_- \leq \ln(1 + \sigma(\varepsilon)/(a_0 C))$, from which it follows that $\limsup_{\varepsilon \to 0^+} \gamma_- = 0$. □

5 Convexity of the Free Boundary

Proof of Theorem 4.2 . We will prove that, if $1/a(x)$ is concave in a neighborhood of the domain bounded by $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, then for every $\varepsilon \in (0, 1)$, we have $T_\varepsilon : \mathbf{Y}_C \to \mathbf{Y}_C$, where

$$\mathbf{Y}_C = \{\Gamma \in \mathbf{Y} : \Gamma \text{ is convex}\}.$$

It follows that $\tilde{\Gamma}_\varepsilon$ is convex for every $\varepsilon \in (0, 1)$ and, therefore, that $\tilde{\Gamma}$ is convex. Lewis proves in [L1], Theorem 1, that, if $\Omega(\Gamma)$ is convex, then, for every $\varepsilon \in$

$(0,1)$, the set $\{x \in \Omega(\Gamma) : U(\Gamma; x) > \varepsilon\}$ is convex; therefore, $\phi_\varepsilon : \mathbf{Y}_C \to \mathbf{Y}_C$, and it only remains for us to show that $\psi_\varepsilon : \mathbf{Y}_C \to \mathbf{Y}_C$ for all $\varepsilon \in (0,1)$. Let $\Gamma \in \mathbf{Y}_C$, let $x_1, x_2 \in \psi_\varepsilon(\Gamma)$, and let $L = \{\lambda x_1 + (1-\lambda)x_2 : 0 \le \lambda \le 1\}$. For any $x \in \mathbb{R}^N$, define

$$f(x) = \frac{\varepsilon}{a(x)d(\Gamma, x)};$$

clearly, $f(x_1) = f(x_2) = 1$. We will show that $f(x) \ge 1$ for all $x \in L$. For $i = 1, 2$, let z_i be the point on Γ such that $f(x_i) = \varepsilon/(a(x_i)d(\Gamma, x_i)) = \varepsilon/(a(x_i)d(z_i, x_i))$, and let l be the line segment joining z_1 to z_2; then, since Γ is convex, $f(x) \ge \varepsilon/(a(x)d(l, x)) =: g(x)$ for any $x \in L$. Now, as in [A2], §4.3, the function $\phi(\lambda) := d(l, \lambda x_1 + (1-\lambda)x_2)$ satisfies $\phi''(\lambda) \ge 0$ for all $0 \le \lambda \le 1$. Therefore, $d(l, \lambda x_1 + (1-\lambda)x_2) \le \lambda d(l, x_1) + (1-\lambda)d(l, x_2)$ for $0 \le \lambda \le 1$, whence

$$g(\lambda x_1 + (1-\lambda)x_2) \ge h(\lambda) := \frac{\varepsilon}{a(\lambda)(\lambda d(z_1, x_1) + (1-\lambda)d(z_2, x_2))},$$

for $0 \le \lambda \le 1$, where $a(\lambda) = a(\lambda x_1 + (1-\lambda)x_2)$. Clearly, $h(0) = h(1) = 1$, and the concavity of $1/a(x)$ implies that $1/a(\lambda)$ is concave in the interval $[0,1]$. We will show, in Lemma 5.1, that the concavity of $1/a(\lambda)$ implies that $h(\lambda) \ge 1$ for all $0 \le \lambda \le 1$, and, therefore, $f(x) \ge 1$ for all $x \in L$. □

Lemma 5.1 *Let* $h(\lambda) = 1/(a(\lambda)(A\lambda + B))$, *where* $1/a(\lambda)$ *is concave and* $A\lambda + B > 0$ *for* $0 \le \lambda \le 1$. *Then, if* $h(0) > 0$ *and* $h(1) > 0$, *then* $h(\lambda) \ge \min(h(0), h(1))$ *for any* $0 \le \lambda \le 1$.

Proof. First, assume that $1/a(\lambda) \in C^2([0,1])$, and $(1/a(\lambda))'' < 0$ in $[0,1]$. Write $h(\lambda) = f(\lambda)g(\lambda)$, where $f(\lambda) = 1/a(\lambda)$, and $g(\lambda) = 1/(A\lambda + B)$. By assumption, $f''(\lambda) < 0$ in $[0,1]$. Suppose that $h(\lambda)$ attains an absolute minimum at a point $\lambda_0 \in (0,1)$. Then, at λ_0, we have $h' = f'g + fg' = 0$, and $h'' = f''g + 2f'g' + fg'' \ge 0$. By substituting for $f'(\lambda_0)$, and using the fact that $f''(\lambda_0) < 0$, one sees that $0 \le h'' < (f/g)(g''g - 2(g')^2)$ at λ_0. But a direct calculation shows that $g''g = 2(g')^2$ for all $0 \le \lambda \le 1$, so that we have a contradiction in this case.

Now assume only that $f(\lambda) \in C^2([0,1])$ and that $f''(\lambda) = (1/a(\lambda))'' \le 0$ in $[0,1]$. Let $\eta \in C^\infty(\mathbb{R})$ be chosen so that $\eta > 0$ and $\eta'' < 0$ on $[0,1]$; then, for every $t > 0, (f + t\eta)'' < 0$ on $[0,1]$. Let $H_t(\lambda) = (f + t\eta)(\lambda)g(\lambda)$. Since $H_t(0) = (f + t\eta)(0)g(0) > 0$, and $H_t(1) = (f + t\eta)(1)g(1) > 0$, the above argument shows that $H_t(\lambda) \ge \min(H_t(0), H_t(1))$, or $(f + t\eta)(\lambda)g(\lambda) \ge \min((f + t\eta)(0)g(0), (f + t\eta)(1)g(1))$ for every $0 \le \lambda \le 1$. Letting $t \to 0^+$, we see that $h(\lambda) = f(\lambda)g(\lambda) \ge \min(f(0)g(0), f(1)g(1)) = \min(h(0), h(1))$ for every $0 \le \lambda \le 1$.

Finally, assume only that $f(\lambda) = 1/a(\lambda)$ is concave on $[0,1]$. Extend f as a continuous function with compact support in $\mathbb{R}$ such that $[0,1] \subset \text{support}(f)$.

Let $\rho(\lambda) \in C_0^\infty(-1,1)$ such that $\rho(\lambda) \geq 0$ and $\int_{-\infty}^{\infty} \rho(t)dt = 1$. For each $n \in \mathbf{N}$, define

$$F_n(\lambda) = \int_{-\infty}^{\infty} n\rho(n(t-\lambda))f(t)dt = \int_{-\infty}^{\infty} n\rho(nt)f(t-\lambda)dt.$$

Then $F_n \in C^\infty$, and F_n converges almost everywhere to f on $[0,1]$. (See, for example, [Mo], page 20.) Since f is continuous, F_n converges uniformly on $[0,1]$ to f, and, since f is concave on $[0,1], F_n$ is concave on $[0,1]$ for every n. Let $h_n(\lambda) = F_n(\lambda)g(\lambda)$. The argument in the previous paragraph shows that $h_n(\lambda) \geq \min(h_n(0), h_n(1))$ for all $0 \leq \lambda \leq 1$ and $n \in \mathbf{N}$. Since $h_n(\lambda)$ converges uniformly to $h(\lambda)$ on $[0,1]$, we conclude that $h(\lambda) \geq \min(h(0), h(1))$ on $[0,1]$. □

Remark 5.2. The operator ψ_ε may be defined in terms of a generalized distance function as in [A1]. One still obtains convergence of the trial free boundary method, but the proof that this operator preserves convexity is more difficult and has only been carried out in $\mathbb{R}^2$. (See [A4], Lemma 2.)

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