A FREE BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION*

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1. Introduction. In this paper we will prove the existence of a solution of a free boundary value problem for the heat equation. We will accomplish this by demonstrating the existence of a solution to a non-linear integro-differential equation.

Let D be the domain $0 \leq t, 0 \leq x \leq R(t), R(0) = A$, indicated in Fig. 1. The boundary value problem is

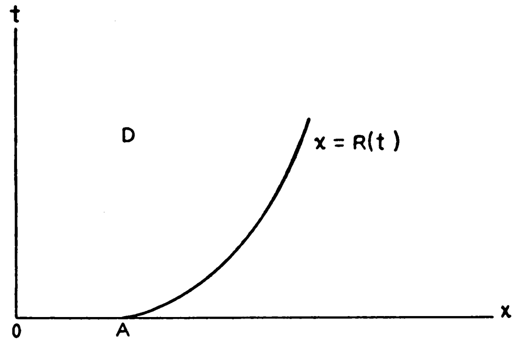

FIG. 1.

$$
\begin{aligned}
&u_t = \alpha^2 u_{xx}(x, t) \epsilon D, \\
&u_x(0, t) = -G(t) \quad G(t) > 0, \\
&u[R(t), t] = T_c = \text{constant}, \\
&u(x, 0) = F(x) \geq T_c \quad 0 \leq x \leq A, \\
&u_x[R(t), t] = B - CR_t(t) \quad B \geq 0, C > 0 \text{ are constants.}
\end{aligned} \tag{1}
$$

Differentiation is denoted by a subscript whether it is a partial derivative of a function of two variables or an ordinary derivative of a function of one variable.

This problem with $A = 0$ has been discussed by several authors, see for example [1, 2, 6], but thus far no existence proof has been established**. The problem describes the physical phenomena of evaporation, fusion, sublimation, etc. For example with $B = 0$, (1) could refer to the following situation. A long metal rod insulated at the sides has begun to melt at one end ($x = 0$). The layer of liquid metal is A units deep and has some initial temperature distribution, $F(x)$. The critical temperature T_c is the melting point of the metal. At $x = 0$ heat is applied to the rod at a known rate proportional

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**For our proof it is essential that $A \neq 0$. See, however, the remark at the end of the existence proof below.

to $G(t)$. As the process continues the interface, $R(t)$, between liquid and solid advances down the rod.

This problem is called a free boundary problem since the part $R(t)$ of the boundary of D is unknown. The additional boundary condition (5) which would over-determine the problem were $R(t)$ known compensates for the free boundary.

We set $B = 0$ to achieve clarity in our discussion, and we introduce the transformation

$$\begin{aligned} t' &= (A^2/\alpha^2)t, \\ x' &= (A/\alpha^2)t, \\ a' &= a - T_c\,, \\ R' &= (A^2/\alpha^4)R, \\ g &= G, \\ f &= F, \\ D' &= D. \end{aligned} \tag{2}$$

Upon dropping the primes the free boundary value problem then becomes

$$u_t = u_{xx}\,,\ (x,\,t)\epsilon D, \tag{3}$$

$$u_x(0,\,t) = -g(t) < 0, \tag{4}$$

$$u[R(t),\,t] = 0, \tag{5}$$

$$u(x,\,0) = f(x) \geq 0, \tag{6}$$

$$u_x[R(t),\,t] = -R_t(t). \tag{7}$$

We require that $g(t)$ and $f(x)$ be continuous and have the following additional properties: $g(t)$ is differentiable; $f(x)$ is continuously differentiable for $0 < x < A$; $f(x) = f(-x)$; $f(x) = 0$ for $x > A$; $f_x(A) < 0$; and $f(0) = -g(0)$. An $f(x)$ with these properties is

$$f(x) = \frac{g(0)}{A}x^2 - g(0)\,|\,x\,| + g(0)A - A^2, \qquad |\,x\,| < A,$$

$f(x) = 0$, otherwise.

2. Method of solution. Our method of solution will be to apply the method of I. Kolodner [4] and derive a functional equation for $R(t)$. We will show that the existence of a solution with certain properties of the functional equation implies the existence of a solution to the free-boundary problem. We will solve the functional equation by the method of contracting maps (Picard iterations).

The method of Kolodner: Let $x = \rho(t)$ be a continuously differentiable function and such that $\rho(0) = A$. Consider the function $u^\rho(x,\,t)$ defined as

$$u^\rho = v^\rho + w^\rho + f^\rho, \tag{2.1}$$

where

$$v^\rho = -\tfrac{1}{2}\pi^{-1/2}\int_0^t (t-\tau)^{-1/2}\rho_\tau(\tau)\exp\{-[\tfrac{1}{2}[x-\rho(\tau)](t-\tau)^{-1/2}]^2\}\,d\tau, \tag{2.2}$$

$$w^\rho = -\tfrac{1}{2}\pi^{-1/2}\int_0^t (t-\tau)^{-1/2}\rho_\tau(\tau)\exp\{-[\tfrac{1}{2}[x+\rho(\tau)](t-\tau)^{-1/2}]^2\}\,d\tau \tag{2.3}$$
$$+\pi^{-1/2}\int_0^t (t-\tau)^{-1/2}g(\tau)\exp\{-[\tfrac{1}{2}x(t-\tau)^{-1/2}]^2\}\,d\tau,$$

and

$$f^\rho = \tfrac{1}{2}(\pi t)^{-1/2}\int_0^t f(\xi)\exp\{-[\tfrac{1}{2}(x-\xi)t^{-1/2}]^2\}\,d\tau. \tag{2.4}$$

This function is a solution of the heat equation in the domain of Fig. 1 with $\rho(t)$ replacing $R(t)$. We will now calculate the same boundary and initial conditions of $u^\rho(x, t)$ from 2.1 which are prescribed for $u(x, t)$ in the statement of the problem (3)–(7). That is by computing u_x^ρ from 2.1 and then letting (x, t) approach the boundaries of the domain of Fig. 1 we shall obtain the boundary conditions in question for u^ρ and u_x^ρ.

The arguments used in doing this are lengthy and are based upon well-known formulas for the integral solutions of the heat equation (see [3]).

The results are

$$u_x^\rho(\rho(t)+0, t) - u_x^\rho[\rho(t)-0, t] = \rho_t(t), \tag{2.5}$$

$$u_x^\rho(0+, t) = -g(t), \tag{2.6}$$

$$u^\rho(x, 0+) = f^\rho(x, 0+) = f(x). \tag{2.7}$$

(2.6) makes use of the evenness of $f(x)$.

(2.6) and (2.7) show that $u^\rho(x, t)$ satisfies the boundary and initial condition (4) and (6) which are required of $u(x, t)$. (2.5) almost does the same for (7). If in (2.5) we require that

$$u_x^\rho(\rho(t)+0, t) = 0 \tag{2.8}$$

we will have

$$u_x^\rho[\rho(t)-0, t] = -\rho_t(t) \tag{2.9}$$

which is the condition (7) for u. We will see that (2.8) is an integro-differential equation for $\rho(t)$. To show that a solution to this integro-differential equation furnishes a solution to the free boundary problem, we need only apply Green's formula,

$$2\iint_D (u_x)^2\,dx\,dt = \oint u^2\,dx + 2uu_x\,dt \tag{2.10}$$

to the domain D_α of Fig. 2 and to the function u^ρ.

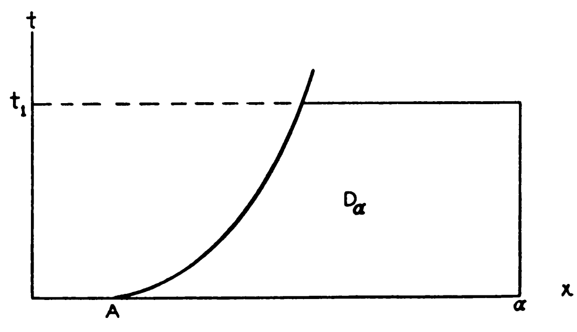

FIG. 2.

Using (2.7) and the fact that $f(x) = 0$ for $x > A$ and also using (2.8) we have

$$2 \iint_{D_\alpha} (u_x^\rho)^2 \, dx \, dt = 2 \int_0^{t_1} u^\rho(\alpha, t) u_x^\rho(\alpha, t) \, dt - \int_{\rho(t)}^{\alpha} (u^\rho)^2(x, t_1) \, dx - \int_0^{t_1} (u^\rho)^2[\rho(t), t] \rho_t \, dt. \tag{2.11}$$

If we now let $\alpha \to \infty$ and observe that (2.1)–(2.4) imply that $u^\rho(\alpha, t)$ and $u_x^\rho(\alpha, t) \to 0$, then (2.11) becomes

$$2 \iint_{D_\infty} (u_x^\rho)^2 \, dx \, dt + \int_{\rho(t)}^{\infty} (u^\rho)^2(x, t_1) \, dx = -\int_0^t (u^\rho)^2[\rho(t), t] \rho_t(t) \, dt. \tag{2.12}$$

If we require that

$$\rho_t \geq 0 \tag{2.13}$$

then the right-hand side of (2.12) is not positive and (2.12) shows that $u^\rho = 0$ for $x > \rho(t)$ and all t. This in turn implies that $u^\rho(\rho + 0, t) = 0$, and since u^ρ is continuous along $x = \rho(t)$ that $u^\rho(\rho, t) = 0$.

Thus the requirement (2.13) makes μ^ρ satisfy the condition (5) for u. We see that if $\rho(t)$ is a smooth function for which $\rho(0) = A$ and $p_t \geq 0$ and such that $u_x^\rho(\rho(t) + 0, t) = 0$, then the function u^ρ solves the boundary value problem (3)–(7). The condition (2.8) yields from (2.1)–(2.4) the following integro-differential equation for ρ

$$\begin{aligned}
\rho_t(t) &= \rho(t)\pi^{-1/2} \int_0^t g(\tau)(t - \tau)^{-3/2} \exp \{-[\tfrac{1}{2}\rho(t)(t - \tau)^{-1/2}]^2\} \, d\tau \\
&- \tfrac{1}{2}\pi^{-1/2} \int_0^t \rho_\tau(\tau)(t - \tau)^{-3/2}[\rho(t) - \rho(\tau)] \exp \{-[\tfrac{1}{2}[\rho(t) - \rho(\tau)](t - \tau)^{-1/2}]^2\} \, d\tau \\
&- \tfrac{1}{2}\pi^{-1/2} \int_0^t \rho_\tau(\tau)(t - \tau)^{-3/2}[\rho(t) + \rho(\tau)] \exp \{-[\tfrac{1}{2}[\rho(t) + \rho(\tau)](t - \tau)^{-1/2}]^2\} \, d\tau \\
&+ \tfrac{1}{2}\pi^{-1/2}t^{-3/2} \int_0^t f(\xi)[\rho(t) - \xi] \exp \{-[\tfrac{1}{2}[\rho(t) - \xi]t^{-1/2}]^2\} \, d\tau,
\end{aligned} \tag{2.14}$$

$$\rho(0) = A.$$

We introduce the abbreviation

$$\rho_t(t) = F(\rho, \rho_t, g, f, A, t) = F(\rho) \tag{2.15}$$

for (2.14).

If we can solve this equation and if its solution $\rho(t)$, has a non-negative derivative then $\rho(t)$ is $R(t)$, the free boundary, and u^ρ is a solution of the boundary value problem.

3. Properties of $\rho_t^\nu(t)$. In this section we deduce some properties of the integro-differential equation (2.14) and of the free boundary which will be of use in our existence proof.

LEMMA 1. *If $r(t)$ is a continuously differentiable function for $t \geq 0$ then*

$$\rho_t(0) = \lim_{t \to 0} F(r) = -f_x(A). \tag{3.1}$$

Proof. The proof proceeds according to the methods of [3]. The first three integrals on the right in (2.14) tend to zero when t tends to zero while the fourth tends to $-f_x(A)$.

LEMMA 2. If $R_t(t)$ exists and is continuous then $R_t(t) > 0$.

Proof. From (6) and (7) we see that $R_t(0) = -f_x(A) > 0$. Suppose to the contrary that $R_t(t) \not> 0$. Let t' be the smallest positive value of t for which $R_t(t) = 0$. Let $D_{t'}$ be that part of D where $t \leq t'$. u_x is a solution of the heat equation in $D_{t'}$. Since $g(t)$, $f_x(x)$, and $R_t(t)$ are continuous and $f_x(0) = -g(0)$ and $-f_x(A) = R_t(0)$, u_x is continuous on that part of the boundary of $D_{t'}$ which is not on the line $t = t'$. By the maximum principle [5], the maximum of u_x occurs on this part of the boundary of $D_{t'}$. Since $-g < 0$, $f'(x) < 0$, and $R_t(t) < 0$ for $0 \leq t < t'$, this maximum must occur at $x = R(t')$, $t = t'$. Thus $u_x \leq 0$ everywhere in $D_{t'}$. Thus since $u = 0$ for $x = R(t)$, we have that $u \geq 0$ in $D_{t'}$. Then in the closure of $D_{t'}$, every point on the free boundary is a minimum point of u. Now at a minimum point of u, the outward drawn derivative in a characteristic direction must be strictly negative*. That is, u_x must be strictly negative along the free boundary. But at $t = t'$, $u_x = -R_t = 0$. This contradiction implies the result.

4. Existence. To demonstrate the existence of a solution to the free boundary problem (3)–(7), we must show that the integro-differential equation (2.14) or (2.15) possesses a solution $\rho(t)$ for which $\rho_t(t) > 0$. To do this we will use the principle of contracting mappings. We will first obtain the existence of a solution $\rho(t)$ with $\rho_t(t) > 0$ in the small and then using Lemma 2 of Sec. 3, show that this solution exists for all $t > 0$.

Existence in the small. Let B be the Banach space of continuously differentiable functions $\{\rho(t)\}$, $0 \leq t \leq T$ for some fixed T to be specified. Let the norm be

$$\| \rho(t) \| = | \rho(0) | + \operatorname*{lub}_{0 \leq t \leq T} | \rho_t(t) |. \tag{4.1}$$

Thus convergence in B is uniform convergence of the function and its continuous derivative.

Let G be the closed set in B whose elements satisfy the following properties

$$\begin{aligned} &(a) \quad \rho(0) = A, \\ &(b) \quad \rho_t(0) = -f_x(A), \\ &(c) \quad 0 < l \leq \rho_t(t) \leq K, \end{aligned} \tag{4.2}$$

where $l < -f_x(A)$ and $K > -f_x(A)$ are to be specified.

Now consider the map $\rho_t' = F(\rho)$. We will show that if ρ_1 and $\rho_2 \in G$, then ρ_1' and $\rho_2' \in G$ and $\| \rho_1' - \rho_2' \| \leq C(T) \| \rho_1 - \rho_2 \|$, where $0 < C(T) < 1$ for T sufficiently small. These statements show that $\rho_t' = F(\rho)$ is a map of G into G which is continuous and moreover contracting. Thus by the principle of contracting mappings $\rho_t' = F(\rho)$ has a fixed point in G. This fixed point is our solution and possesses by [4.2(c)] a positive derivative. We now proceed with the proof.

If $\rho(0) = A$ and $\rho_t(0) = -f_x(A)$ then the same is true for $\rho'(0)$ and $\rho_t'(0)$. The first since $\rho_t' = F(\rho)$ is an integro-differential equation and we may arbitrarily require $\rho_t'(0) = A$ and the second is the assertion of Lemma 1 of Sec. 3.

In Appendix 1, we show that

$$\| \rho_1' - \rho_2' \| \leq C(T) \| \rho_1 - \rho_2 \|, \qquad \rho_1, \rho_2 \in G, \tag{4.3}$$

*This is an unpublished result due to L. Nirenberg.

where $C(T) = C(T, l, K, A, g, f) < 1$ for T sufficiently small. In Appendix 2 we show that ρ_t' is continuous. In Appendix 3 we show that

$$| \rho_t'(t) - [-f_x(A)] | \leq C_1 t^{1/2}, \tag{4.4}$$

uniformly for $\rho \varepsilon G$.

Thus we fix T so that $C(T) < 1$ and

$$C_1 T^{1/2} < \max \{| l - [-f_x(A)] |, | K - [-f_x(A)] |\}.$$

Our map is then into and contracting and a solution $\rho(t)$ of (2.14) exists up to time T. Moreover this solution has a positive derivative.

Existence in the large. The proof of existence in the large proceeds as the above proof. The Banach space is now B_1 the set of continuously differentiable functions $\rho(t)$, $0 \leq t \leq T_1$ where $T_1 > T$ is to be specified. We use the same norm as in B. G is replaced by G_1, those functions in B_1 which up to time T are equal to the solution $R(t)$ which we have just shown to exist. A is replaced by $R(T)$ and $-f_x(A)$ by $R_t(T)$. l and K are replaced by two other numbers l_1 and K_1 with $0 < l_1 < R_t(T)$ and $K_1 > R_t(T)$.

With this setup the requirements of the principle of contracting mappings are satisfied here in essentially the same way as above. Thus we produce a T_1 such that for $T_1 - T > 0$ and sufficiently small, we have existence of $R(T)$ up to time T_1 and moreover $R_t(t) > 0$ for $0 \leq t \leq T_1$. We see that we may iterate this procedure and produce a sequence of T_i, $i = 1, 2, \cdots$, such that $R(t)$ exists and $R_t(t) > 0$ for $t \leq T_i$. There remains only to show that $T_i \rightarrow \infty$. From the form of the estimates in the appendices and the definitions of the sets G_i we see that this will be the case if $R_t(t)$ never vanishes for then we may always extend our solution slightly further. But the vanishing of $R_t(t)$ is ruled out by Lemma 2 of Sec. 3. Thus $T_i \rightarrow \infty$ and $R(t)$ exists for all time.

Remark. We have mentioned that for our proof it is essential that $A \neq 0$. A passage to the limit as $A \rightarrow 0$ is suggested to obtain the solution with $A = 0$. This limit procedure would be legitimate if for some sequence $R^A(t)$ with A tending to zero, the slopes, $R_t^A(t)$, have a positive lower bound. For in this event the set of functions $R^A(t)$ are equi-continuously differentiable and a simple compactness argument justifies the passage to the limit. If the condition (7) is changed to $u_x [R(t), t] = B - R_t(t), B > 0$, then an obvious extension of Lemma 2 yields a positive lower bound for the slopes $R_t^A(t)$ and in this case our method yields existence for the case $A = 0$.

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Appendix 1. *Continuity and contracting of the map* $\rho_t' = F(\rho)$. In this appendix we show that the map $\rho_t' = F(\rho)$ is continuous for $\rho \varepsilon G$ is contracting. Let $u(t)$ and $v(t)$ be two generic elements in G. Let $u_t' = F(u)$ and $v_t' = F(v)$. We have

$$|| u' - v' || = \max_{0 \leq t \leq T} || F(u) - F(v) ||$$

since $u'(0) = v'(0) = A$.

The computation of the right-hand side of $(A1)$ is divided into four parts corresponding to each of the four integrals occurring in F. The computations for the first three integrals are so alike that we illustrate the computation involving the third and fourth integrals only.

(i) THIRD INTEGRAL. We are led to consider

$$I_1 + I_2 = \tfrac{1}{2}\pi^{-1/2} \left| \int_0^t u_\tau(\tau)(t-\tau)^{-3/2}[u(t) + u(\tau) - v(t) - v(\tau)] \cdot \exp\{-[\tfrac{1}{2}[u(t) + u(\tau)](t-\tau)^{-1/2}\} \, d\tau \right|$$
$$+ \tfrac{1}{2}\pi^{-1/2} \left| \int_0^t [v_\tau(\tau) - u_\tau(\tau)][v(t) + v(\tau)](t-\tau)^{-3/2} \cdot \exp\{-[\tfrac{1}{2}[v(t) + v(\tau)](t-\tau)^{-1/2}]^2\} \, d\tau \right| .$$

By the law of the mean, we have

$$I_1 \leq \tfrac{1}{2}K\pi^{-1/2} \int_0^t |u(t) + u(\tau) - v(t) - v(\tau)| \, (t-\tau)^{-3/2} \, |e^{-z^2}(1 - 2z^2)| \, d\tau,$$

where z is $[\frac{1}{2}[\rho(t) + \rho(\tau)] \, (t-\tau)^{-1/2}]$ for some $\rho \varepsilon B$. Thus

$$I_1 \leq Kt\pi^{-1/2} \, || u - v || \int_0^t (t-\tau)^{-3/2} \, |e^{-z^2}(1 - 2z^2)| \, d\tau.$$

Now

$$\frac{2A + lt}{2(t-\tau)^{1/2}} \leq \frac{\rho(t) + \rho(\tau)}{2(t-\tau)^{1/2}} \leq \frac{A + Kt}{(t-\tau)^{1/2}} \quad \text{for} \quad \rho \varepsilon B.$$

Then

$$I_1 \leq Kt\pi^{-1/2} \, || u - v || \int_0^t (t-\tau)^{-3/2}[1 + 2(A + Kt)^2(t-\tau)^{-1}] \cdot \exp\{-[\tfrac{1}{2}(2A + lt)(t-\tau)^{-1/2}]^2\} \, d\tau.$$

Let

$$\sigma = \tfrac{1}{2}(2A + lt)(t-\tau)^{-1/2}, \qquad 2\, d\sigma = \tfrac{1}{2}(2A + lt)(t-\tau)^{-3/2} \, d\tau.$$

Then

$$I_1 \leq 4K\pi^{-1/2} \, || u - v || \, t(2A + lt)^{-1} \int_{2A + lt/2t^{1/2}}^{\infty} e^{-\sigma^2}[1 + 8\sigma^2(A + Kt)^2(2A + lt)^{-2}] \, d\sigma.$$

For I_2 we have

$$I_2 \leq \tfrac{1}{2}\pi^{-1/2} \, || u - v || \int_0^t [v(t) + v(\tau)](t-\tau)^{-3/2} \exp\{-[\tfrac{1}{2}[v(t) + v(\tau)](t-\tau)^{-1/2}]^2\} \, d\tau$$
$$\leq \pi^{-1/2} \, || u - v || \, (A + Kt) \int_0^t (t-\tau)^{-3/2} \exp\{-[\tfrac{1}{2}(2A + lt)(t-\tau)^{-1/2}]^2\} \, d\tau$$
$$= 4\pi^{-1/2} \, || u - v || \, \frac{A + Kt}{2A + lt} \int_{2A + lt/2t^{1/2}}^{\infty} e^{-\sigma^2} \, d\sigma$$
$$\leq 4\left(\frac{t}{\pi}\right)^{1/2} || u - v || \, (A + Kt)(2A + lt)^{-2} \exp\{-\tfrac{1}{4}(2A + lt)^2 t^{-1}\}.$$

ii. Fourth integral. For this integral we have

$$I_3 + I_4 = \tfrac{1}{2}\pi^{-1/2}t^{-3/2}\left|\int_{-A}^{A} f(\xi)[u(t) - v(t)] \exp\{-\tfrac{1}{4}[u(t) - \xi]^2 t^{-1}\}\, d\xi\right|$$

$$+ \tfrac{1}{2}\pi^{-1/2}t^{-3/2}\left|\int_{-A}^{A} f(\xi)[v(t) - \xi][\exp\{-\tfrac{1}{4}[u(t) - \xi]^2 t^{-1}\}\right.$$

$$\left. - \exp\{-\tfrac{1}{4}[u(t) - \xi]^2 t^{-1}\}]\, d\xi\right|$$

$$I_3 \leq \tfrac{1}{2}(\pi t)^{-1/2} \| u - v \| \int_{-A}^{A} | f(\xi) | \exp\{-\tfrac{1}{4}[u(t) - \xi]^2 t^{-1}\}^2\, d\xi$$

$$= \tfrac{1}{2}(\pi t)^{-1/2} \| u - v \| \int_{-A}^{\lambda} + \int_{\lambda}^{A} | f(\xi) | \exp\{-\tfrac{1}{4}[u(t) - \xi]^2 t^{-1}\}^2\, d\xi$$

$$= I_5 + I_6 .$$

where $\lambda = \lambda(t)$

$$I_5 \leq \tfrac{1}{2}(\pi t)^{-1/2} \| u - v \| (A - \lambda) \max_{A \leq \xi \leq \lambda} | f(\xi) |$$

$$I_5 \leq \tfrac{1}{2}(\pi t)^{-1/2}(A - \lambda)^2 \max_{A \leq \xi \leq \lambda} | f_x(\xi) | \tag{A1}$$

since $f(A) = 0$.

$$I_6 \leq \tfrac{1}{2}(\pi t)^{-1/2}(\lambda + A) \exp\{-\tfrac{1}{4}(A + lt - \lambda)^2 t^{-1}\} \max_{-A \leq \xi \leq \lambda} | f(\xi) |. \tag{A2}$$

From $(A1)$ and $(A2)$ we see that if as $t \to 0$ $\lambda(t) \to A$ faster than $t^{1/4}$ but slower than $t^{1/2}$ then

$$I_5 + I_6 \leq \| u - v \| \text{ const } \; o(1).$$

The computation involving I_4 is similar to the one just conducted and will be omitted.

Appendix 2. *Continuous differentiability of an image under F.* In this appendix we show that if $\rho_t' = F(\rho)$, $\rho \in G$ then ρ_t' is a continuous function of t. The computations are slight variations of the computations in Appendix 1. Therefore we will carry them out only for the first integral in $F(\rho)$.

Let $a \geq 0$ and consider

$$\pi^{-1/2}\rho(a + \Delta t) \int_0^{a+\Delta t} g(\tau)(a + \Delta t - \tau)^{-3/2} \exp\{-[\tfrac{1}{2}\rho(a + \Delta t)(a + \Delta t - \tau)^{-1/2}]^2\}\, d\tau$$

$$- \pi^{-1/2}\rho(a) \int_0^{a} g(\tau)(a - \tau)^{-3/2} \exp\{-[\tfrac{1}{2}\rho(a)(a - \tau)^{-1/2}]^2\}\, d\tau$$

$$= \pi^{-1/2}\rho(a + \Delta t) \int_0^{a+\Delta t} g(\tau)(a + \Delta t - \tau)^{-3/2} \exp\{-[\tfrac{1}{2}\rho(a + \Delta t)(a + \Delta t - \tau)^{-1/2}]^2\}\, d\tau$$

$$+ \pi^{-1/2} \int_0^{a} g(\tau)[\rho(a + \Delta t)(a + \Delta t - \tau)^{-3/2} \exp\{-[\tfrac{1}{2}\rho(a + \Delta t)(a + \Delta t - \tau)^{-1/2}]^2\}$$

$$- \rho(a)(a - \tau)^{-3/2} \exp\{-[\tfrac{1}{2}\rho(a)(a - \tau)^{-1/2}]^2\}]\, d\tau$$

$$= I_1 + I_2 .$$

In what follows, we use the fact that $\rho \in G$ implies that $\rho(a)$, $\rho_t(a)$, $\rho(a + \Delta t)$, $\rho_t(a + \Delta t) > 0$.

Consider first I_1. Let $\sigma = \rho(a + \Delta t)/2(a + \Delta t - \tau)^{1/2}$. Then

$$I_1 = 4\pi^{-1/2} \int_x^\infty g e^{-\sigma^2}\, dx, \qquad x = \tfrac{1}{2}\rho(a + \Delta t)(\Delta t)^{-1/2}$$

$$< 4M\pi^{-1/2} \int_x^\infty e^{-\sigma^2}\, d\sigma, \qquad M = \max_{0 \le t \le T} |\, g(t)\, |.$$

Since

$$\int_x^\infty e^{-\sigma^2}\, d\sigma < \frac{1}{2x} e^{-x^2}, \qquad x \ge 0$$

we have that

$$I_1 < 4M\pi^{-1/2}(\Delta t)^{1/2}/\rho(a + \Delta t).$$

For I_2 we have on application of the law of the mean of the differential calculus:

$$I_2 \le \pi^{-1/2}\Delta t M \int_0^a \Bigg\{\Bigg[\rho_t(t)(t - \tau)^{-3/2} + \frac{3}{2}\rho(t)(t - \tau)^{-3/2} - \rho(t)(t - \tau)^{-1/2}\left\{\frac{1}{2}\rho_t(t)(t - \tau)^{-1/2} + \frac{1}{4}\rho(t)(t - \tau)^{-3/2}\right\}\Bigg] \cdot \exp\left[-\left[\frac{1}{2}\rho(t)(t - \tau)^{-1/2}\right]^2\right]\Bigg\}\, d\tau,$$

where the quantity in the curly brackets is to be evaluated at some value of t in the open interval $0 \le a < t < a + \Delta t$. The integrand is thus finite and the integral exists.

Appendix 3. *The initial value of* ρ_t'. In this appendix we show that $|\, \rho_t'(t) - [-f_x(A)]\, | \le C_1(t)^{1/2}$. We sketch the ideas of the proof since the computations are variations of those in Appendix 1.

We note first that the first three integrals vanish as $t \to 0$. For the first integral,

$$\pi^{-1/2}\rho(\tau) \int_0^t g(\tau)(t - \tau)^{-3/2} \exp\{-[\tfrac{1}{2}\rho(t)(t - \tau)^{-1/2}]^2\}\, d\tau,$$

the singularity at $t = \tau$ in the exponential $[\rho(0) = A > 0]$ causes the integrand to be bounded in the range of integration. This bound depends on the minimum of $\rho(t) \in G$ and this is bounded below by $A + lT$. Thus the entire integral vanishes with its upper limit uniformly in G.

For the second integral,

$$\tfrac{1}{2}\pi^{-1/2} \int_0^t \rho_\tau(\tau)[\rho(t) - \rho(\tau)](t - \tau)^{-3/2} \exp\{-[\tfrac{1}{2}[\rho(t) - \rho(\tau)](t - \tau)^{-1/2}]^2\}\, d\tau,$$

the term $[\rho(t) - \rho(\tau)]/(t - \tau)$ approaches $\rho_t(0) = -f_x(A)$ uniformly in G. Thus the integral tends to zero like $t^{1/2}$ uniformly in G.

In the third integral,

$$\tfrac{1}{2}\pi^{-1/2} \int_0^t \rho_\tau(\tau)[\rho(t) + \rho(\tau)](t - \tau)^{-3/2} \exp\{-[\tfrac{1}{2}[\rho(t) - \rho(\tau)](t - \tau)^{-1/2}]^2\}\, d\tau,$$

the singularity in the exponent causes the integral to vanish with its upper limit as in the first integral.

We have, referring to [3], observed that the fourth integral tends to $-f_x(A)$ as $t \to 0$. By examining the proof of this process one may observe that the limit is approached like $t^{1/2}$ as $t \to 0$.

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