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A FREE $\mathbb{Z}_p$-ACTION AND THE SEIBERG-WITTEN INVARIANTS

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ABSTRACT. We consider the situation that $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ acts freely on a closed oriented 4-manifold X with $b_2^+ \geq 2$. In this situation, we study the relation between the Seiberg-Witten invariants of X and those of the quotient manifold $X/\mathbb{Z}_p$. We prove that the invariants of X are equal to those of $X/\mathbb{Z}_p$ modulo p.

1. Introduction

Let us consider the situation that $\mathbb{Z}_p$ acts freely on a closed oriented 4-dimensional manifold X with $b_2^+ \geq 2$. Here $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and p is a prime number and b_2^+ is the dimension of a maximal subspace of $H_2(X;\mathbb{Z})$ on which the intersection pairing is positive definite. (When $p = 2$, we suppose that $\mathbb{Z}_2$-action is orientation preserving.) In such a situation, we study the relations between the Seiberg-Witten invariants of X and those of the quotient manifold $X/\mathbb{Z}_p$. We prove that the invariants of X are equal to those of $X/\mathbb{Z}_p$ modulo p.

The Seiberg-Witten invariants were introduced by Witten [8]. A brief description of these invariants is given in Section 2 (also [5]). The invariants can be defined for a compact, oriented 4-dimensional manifold X with $b_2^+ \geq 2$.

Roughly speaking, the Seiberg-Witten invariants are defined by counting (in a suitable sense) the solutions to a system of differential equations, called the Seiberg-Witten equations, on X which are defined using a Spin^c-structure. In general, we need to perturb the equations to get invariants. When b_2^+ is greater than 1, these invariants are diffeomorphism invariants and constitute a map, SW_X, from the set of equivalence classes, $\mathcal{S}(X)$, of Spin^c-structures on X to $\mathbb{Z}$. Note that the set $\mathcal{S}(X)$ has

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a structure of an affine space modeled on $H^2(X;\mathbb{Z})$, i.e., the difference of two Spin^c-structures corresponds to an element of $H^2(X;\mathbb{Z})$.

As mentioned above, we will prove that the Seiberg-Witten invariant of X is equal modulo p to that of the quotient manifold $\bar{X} = X/\mathbb{Z}_p$. Let us explain this more precisely. Fix a Spin^c-structure on $\bar{X}$ and consider the Seiberg-Witten equations associated to it. Then, lift this Spin^c-structure of $\bar{X}$ with the equations to the covering manifold X. Next we try to perturb the lifted equations in $\mathbb{Z}_p$-invariant way so that we can project down the perturbation onto the quotient manifold $\bar{X}$. Then we see that such perturbation is sufficient to satisfy the transversality, i.e., by such perturbation, the moduli space of solutions becomes a finite dimensional, oriented, compact manifold. The moduli space obtained by such perturbation splits into two parts: $\mathbb{Z}_p$-invariant solutions and not invariant ones. Solutions which come from $\bar{X}$ are $\mathbb{Z}_p$-invariant, and a $\mathbb{Z}_p$-invariant solution is the lift of a solution on some Spin^c-structure of $\bar{X}$. On the other hand, if there is one solution which is not $\mathbb{Z}_p$-invariant, we have other $(p-1)$ solutions (also not-$\mathbb{Z}_p$-invariant) which are the image of the given solution by the $\mathbb{Z}_p$-action. That is, the number of solutions which are not $\mathbb{Z}_p$-invariant is a multiple of p. For simplicity, suppose now that this moduli space is 0-dimensional. In this case, the moduli space is a finite union of signed points, and the Seiberg-Witten invariant is the sum over these points of the corresponding ± 1's. Then we can see that the contribution of not-$\mathbb{Z}_p$-invariant solutions to the Seiberg-Witten invariant is congruent to 0 modulo p because the number of such solutions is a multiple of p. As for $\mathbb{Z}_p$-invariant solutions, we can also see that the contribution to the invariant of a $\mathbb{Z}_p$-invariant solution is ± 1 by using the Kuranishi model. Thus we can expect that the Seiberg-Witten invariant of a Spin^c-structure on $\bar{X}$ is equal modulo p to that of the lifted Spin^c-structure on X.

Now note that it is possible that two or more distinct Spin^c-structures on $\bar{X}$ are lifted to the same Spin^c-structure on X. In fact, if the difference of two Spin^c-structures on $\bar{X}$ is represented by an element of $H^2(\bar{X};\mathbb{Z})$ whose lift to $H^2(X;\mathbb{Z})$ vanishes, then the lifts of these Spin^c-structures to X coincide. The precise statement is as follows:

THEOREM 1.1. *Let p be a prime number, and X be a closed oriented 4-manifold with $b_2^+ \geq 2$. Suppose that $\mathbb{Z}_p$ acts on X freely. Let $\bar{X}$ be the quotient manifold $X/\mathbb{Z}_p$. When $p = 2$, suppose further that the $\mathbb{Z}_2$-action is orientation-preserving and X has "simple type" in that only the dimension zero Seiberg-Witten invariants are non-zero. Fix a*

Spin^c-structure $\bar{c}$ on $\bar{X}$ and let c be its lift to X. Then

$$\mathrm{SW}_X(c) \equiv \sum_{\bar{a}\in\ker\pi^*} \mathrm{SW}_{\bar{X}}(\bar{c}+\bar{a}) \pmod p, \tag{1.2}$$

where $\pi\colon X \to \bar{X}$ is the projection and π^* is the induced homomorphism on the 2nd cohomology group and $\bar{c}+\bar{a}$ is the Spin^c*-structure of which the difference from $\bar{c}$ is represented by the cohomology class $\bar{a}$.*

REMARK 1.3. We assume an appropriate orientation convention of moduli spaces to hold (1.2). The convention is explained in Section 4.3.

REMARK 1.4. The assumption that X has "simple type" when $p = 2$ is slightly strong. If all non-trivial Seiberg-Witten invariants of X have dimensions congruent to 0 modulo 4, then the theorem holds. Even if not, (1.2) holds for a Spin^c-structure $\bar{c}$ on $\bar{X}$ whose moduli is even dimensional (Remark 4.2).

This paper is organized as follows. In Section 2, we give a brief review on the Seiberg-Witten invariants. In Section 3, we clarify the $\mathbb{Z}_p$-action on the moduli space induced from the action on X. In Section 4, we prove Theorem 1.1. In Section 5, we give an example.

2. The Seiberg-Witten invariants

In this section, we give a brief review on the Seiberg-Witten invariants for preparing the notations we need later.

Let X be a closed, oriented, smooth 4-manifold. Fixing a Riemannian metric on X determines an SO(4)-frame bundle on X. A Spin^c-structure c on X is a lift of the SO(4)-frame bundle to a $\mathrm{Spin}^c(4)$-bundle on X. Each Spin^c-structure gives a pair of unitary $\mathbb{C}^2$ bundles S^+ and S^-, (the bundles of *positive* and *negative spinors*), equipped with a Clifford multiplication

$$\gamma\colon S^+ \otimes T^*X \to S^-.$$

As mentioned in Section 1, the set of equivalence classes, $\mathcal{S}(X)$, of Spin^c-structures on X becomes an affine space modeled on $H^2(X,\mathbb{Z})$. Let c and c' be two Spin^c-structures on X. Then the bundles S^+ of these Spin^c-structures are related by $S^+(c') \cong S^+(c)\otimes E$, where E is a complex line bundle. The difference of c and c' corresponds to $c_1(E) \in H^2(X,\mathbb{Z})$.

For a given Spin^c-structure c, let L be the determinant line bundle of S^+, (i.e. $L = \det(S^+)$), $\mathcal{A}$ be the space of U(1)-connections on L, $\Gamma(S^+)$ be the space of sections of S^+, and $\mathcal{C}$ be the "configuration space"

$\mathcal{A} \times \Gamma(S^+)$. The Seiberg-Witten equations for a configuration (A, ψ) of $\mathcal{C} = \mathcal{A} \times \Gamma(S^+)$ are given by

$$D_A\psi = 0,\qquad F_A^+ = q(\psi), \tag{2.1}$$

where D_A denotes the Spin^c Dirac operator defined by a connection A on L and the Levi-Civita connection of the given Riemannian metric on X, F_A^+ denotes the self-dual part of the curvature of A, and $q(\psi)$ denotes the trace-free part of $(\psi \otimes \psi^*)$ interpreted as an endomorphism of S^+ (and this endomorphism is identified with an imaginary-valued self-dual 2-form via the Clifford multiplication.)

The gauge transformation group $\mathcal{G} \equiv \mathrm{Map}(X; S^1)$ acts on $\mathcal{C}$; a map $u \in \mathcal{G}$ sends (A, ψ) to $(A - 2u^{-1}du, u\psi)$. This group acts freely at (A, ψ) where ψ is not identically zero. A configuration (A, ψ) with $\psi \equiv 0$ is called *reducible.* A configuration (A, ψ) which is not reducible is *irreducible.* Note that the stabilizer of a reducible is $U(1)$. Put $\mathcal{B} = \mathcal{C}/\mathcal{G}$.

The Seiberg-Witten equations (2.1) are $\mathcal{G}$-equivariant. The quotient space of solutions to (2.1) by $\mathcal{G}$ is denoted by $\mathcal{M}$. The space $\mathcal{M}$ is called *the moduli space.*

The equations (2.1) give a $\mathcal{G}$-equivariant map $\tilde{\Psi}\colon \mathcal{C} \to \Gamma(S^-) \times i\Omega^+$ which is defined by

$$\tilde{\Psi}(A, \psi) = (D_A\psi, F_A^+ - q(\psi)),$$

where Ω^+ is the space of self-dual 2-forms.

Let us consider a fibration:

$$\mathcal{V} \overset{\text{def.}}{=} \mathcal{C} \times_{\mathcal{G}} (\Gamma(S^-) \times i\Omega^+) \to \mathcal{B}.$$

Dividing $\tilde{\Psi}$ by $\mathcal{G}$, we get a section $\Psi\colon \mathcal{B} \to \mathcal{V}$. In this view point, $\mathcal{M} = \Psi^{-1}(0)$. Let $\mathcal{C}^* \subset \mathcal{C}$ be the set of irreducibles. Set $\mathcal{B}^* := \mathcal{C}^*/\mathcal{G}$. Then $\mathcal{C}^* \to \mathcal{B}^*$ is a smooth $\mathcal{G}$ principal bundle, and $\mathcal{V} \to \mathcal{B}^*$ is an associated vector bundle.

We have to work in the complete Banach space with suitable Sobolev norm. By [5], we can work with the L_2^2-completions of $\mathcal{C}$ and the L_3^2-completion of $\mathcal{G}$. Then the map Ψ is Fredholm, and the index d of Ψ is given as

$$d = \frac{1}{4}[c_1(L)^2 - (2\chi(X) + 3\,\mathrm{sign}(X))], \tag{2.2}$$

where $\chi(X)$ is the Euler characteristic of X, and $\mathrm{sign}(X)$ is the signature of X.

The next task is to perturb Ψ to make the zero-set of Ψ transversal. Then the zero-set becomes a d-dimensional manifold.

2.1. Perturbation I

We can perturb Ψ in several ways. The first type of perturbation is done by adding a self-dual 2-form μ to the second equation as follows:

$$\begin{aligned} D_A\psi &= 0, \\ F_A^+ &= q(\psi) + \mu. \end{aligned} \tag{2.3}$$

Or equivalently, the map $\tilde{\Psi}$ is perturbed by

$$\tilde{\Psi}(A, \psi) = (D_A\psi, F_A^+ - q(\psi) - \mu).$$

We get a section $\Psi\colon \mathcal{B} \to \mathcal{V}$ from $\tilde{\Psi}$.

For this perturbation, the following are known:

(1) For any μ, the space $\mathcal{M} = \Psi^{-1}(0)$ is compact.
(2) For μ outside an affine space of codimension b_2^+ in Ω^+, there are no reducible solutions to (2.3).
(3) For a generic choice of μ, the zeros of Ψ is transversal. Here, generic means a Baire set of Ω^+. Then the space $\mathcal{M}$ becomes a smooth manifold whose dimension is d in (2.2).

2.2. Perturbation II

We use the Kuranishi model to construct another type of perturbation. First, we introduce an elliptic complex, called the *deformation complex*, which is naturally associated to a solution to the Seiberg-Witten equations (see [5]). This complex incorporates the linearization of the $\mathcal{G}$-action and the linearization of the Seiberg-Witten equations. Let (A, ψ) be a solution to the Seiberg-Witten equation. The deformation complex associated to (A, ψ) is given as:

$$0 \longrightarrow \mathrm{Map}(X; i\mathbb{R}) \xrightarrow{a} \Gamma(S^+) \oplus i\Omega^1 \xrightarrow{b} \Gamma(S^-) \oplus i\Omega^+ \longrightarrow 0,$$

where a is the linearization of $\mathcal{G}$-action at (A, ψ), and b is the linearization of the Seiberg-Witten equations at (A, ψ), i.e., $b = (d\tilde{\Psi})_{(A,\psi)}$. For simplicity, we write this complex simply as

$$0 \longrightarrow \Gamma(E_0) \xrightarrow{a} \Gamma(E_1) \xrightarrow{b} \Gamma(E_2) \longrightarrow 0,$$

and write the i-th cohomology group of the complex as H^i.

Note that the following properties of this complex:

(1) The 0-th cohomology H^0 is the Lie algebra of the stabilizer of (A, ψ). Thus $H^0 = 0$ if and only if (A, ψ) is irreducible.

(2) $H^2 = 0$ if and only if $d\tilde{\Psi}$ is transversal at (A, ψ).
(3) If $H^0 = H^2 = 0$, then $\mathcal{M}$ is smooth at (A, ψ) and H^1 is the tangent space of $\mathcal{M}$ at (A, ψ).

Suppose now that $H^0 = 0$. Then, by the Fredholm property, Ψ is locally right equivalent to a map of the form,

$$\Psi' : U \times H^1 \to V \times H^2, \quad \Psi'(\xi, v) = (L(\xi), \alpha(\xi, v)),$$

where L is a linear isomorphism from U to V, and the differential of α vanishes at 0. ([2, 4.2.19].) Considering the map, called the "Kuranishi map",

$$f : H^1 \to H^2, f(v) = \alpha(0, v),$$

we get a local finite-dimensional model for the moduli space. The moduli space $\mathcal{M} = \Psi^{-1}(0)$ is locally diffeomorphic to the zeros of f. (Note that $(df)_0 = 0$.)

If we take a map $g : H^1 \to H^2$ whose differential at 0 is surjective, by adding g to f and using a cut-off function, we can perturb Ψ to make it transversal at (A, ψ). Let $\mathcal{O}_{(A,\psi)}$ be a small neighborhood of (A, ψ). Then Ψ is written as $\Psi(\xi, v) = (L(\xi), \alpha(\xi, v))$ with $\alpha(0, v) = f(v)$. Take g as above, and let ρ be a cut-off function of $\mathcal{O}_{(A,\psi)}$. Then Ψ can be perturbed to be transversal at (A, ψ) as

$$\Psi' = \Psi + \rho g = (L, \alpha + \rho g).$$

2.3. Definition of the Seiberg-Witten invariants

Suppose that Ψ is transversal on $\mathcal{M} = \Psi^{-1}(0)$ by the perturbation of Section 2.1 or Section 2.2 and suppose also that $\mathcal{M}$ has no reducibles. Then the moduli space $\mathcal{M}$ becomes a compact smooth manifold of whose dimension is d in (2.2). It is known that the orientation of $\mathcal{M}$ is given by a choice of an orientation for the 1-dimensional linear space

$$\det(H^0(X;\mathbb{R}) \otimes H^1(X;\mathbb{R}) \otimes H^{2+}(X;\mathbb{R})). \tag{2.4}$$

To define the invariant, we need the following line bundle $\mathcal{L}$. Fix a base point in X and let $\mathcal{G}_0$ be the subgroup of $\mathcal{G}$ which consists of maps which map the base point to 1. Let $\mathcal{B}_0^*$ be the quotient space of the space $\mathcal{C}^*$ by $\mathcal{G}_0$. The projection $\mathcal{B}_0^* \to \mathcal{B}^*$ defines a S^1-bundle. Let $\mathcal{L}$ be the associated complex line bundle of $\mathcal{B}_0^*$, i.e.,

$$\mathcal{L} = \mathcal{B}_0^* \times_{S^1} \mathbb{C}. \tag{2.5}$$

Here is the definition of the Seiberg-Witten invariant:

DEFINITION 2.6. Let X be a compact, oriented 4-manifold with $b_2^+ \geq 1$ and $c \in \mathcal{S}(X)$ be a Spin^c-structure on X. Fix an orientation for (2.4) and let d be as in (2.2). Perturb Ψ so that Ψ is transversal on $\mathcal{M}$, and $\mathcal{M}$ has no reducibles (when $d \geq 0$). The Seiberg-Witten invariant $\mathrm{SW}_X(c)$ for c is defined as follows:

(1) When $d < 0$ or d is odd, then $\mathrm{SW}_X(c) = 0$.
(2) When $d \geq 0$ and d is even, then $\mathcal{M}$ has a fundamental class $[\mathcal{M}]$. Let $\mathcal{L}$ be as in (2.5). The invariant is given as $\mathrm{SW}_X(c) = c_1(\mathcal{L})^{d/2}[\mathcal{M}]$.

When $b_2^+ \geq 2$, SW_X does not depend on the choice of metric and perturbation, i.e., SW_X is a diffeomorphism invariant of X. When $b_2^+ = 1$, SW_X depends on the choice of metric and perturbation. In particular, it depends on the chamber (see [8], [5]).

Suppose c be a Spin^c-structure whose moduli space has positive and even dimension d. By definition, the Seiberg-Witten invariant of c is given as $\mathrm{SW}_X(c) = c_1(\mathcal{L})^{d/2}[\mathcal{M}]$. But this is rather abstract. There is a way to express this quantity more geometrically called *cutting down the moduli space.* Originally, $\mathcal{L}$ is $\mathcal{B}_0^* \times_{S^1} \mathbb{C}$ and this is isomorphic to $\mathcal{C}^* \times_{\mathcal{G}} \mathbb{C}$ where the $\mathcal{G}$-action on $\mathbb{C}$ is defined via evaluating $u \in \mathcal{G}$ at the base point x_0 chosen for the definition of $\mathcal{B}_0^*$. Then the Poincaré dual of $c_1(\mathcal{L})$ is represented as the zero locus of a section of $\mathcal{C} \times_{\mathcal{G}} \mathbb{C}$. The section is constructed as follows. Fix a 1-dimensional complex subspace A_0 of the fiber $S^+_{x_0}$ of the spinor bundle S^+ at the base point x_0 and let $P\colon S^+_{x_0} \to A_0$ be the projection. Define the section $s\colon \mathcal{B}^* \to \mathcal{L}$ by

$$s(A,\psi) := ((A,\psi), P\psi(x_0)).$$

Choosing $d/2$-distinct base points $x_1, \ldots, x_{d/2}$ and complex lines $A_1, \ldots, A_{d/2}$ in fibers on these base points, we obtain sections $s_1, \ldots, s_{d/2}$ in similar way. Set $\mathcal{W}_i := s_i^{-1}(0)$ for $i = 1, \ldots d/2$. Assuming appropriate transversality, we see that $c_1(\mathcal{L})^{d/2}[\mathcal{M}]$ is equal to the intersection number of $\mathcal{W}_1, \ldots, \mathcal{W}_{d/2}$ and $\mathcal{M}$.

We may consider a section Ψ_C of a $\mathcal{G}$-bundle $\mathcal{V}_C$ instead of Ψ and $\mathcal{V}$, where

$$\mathcal{V}_C := \mathcal{C}^* \times_{\mathcal{G}} (\Gamma(S^-) \times i\Omega^+ \times A_1 \times \cdots \times A_{d/2}) \to \mathcal{B}^*, \tag{2.7}$$

and

$$\Psi_C(A,\psi) := (D_A\psi, F_A^+ - q(\psi) - \mu, P\psi(x_1), \ldots, P\psi(x_{d/2})). \tag{2.8}$$

If Ψ_C is transversal, then $\mathcal{M}_C := \Psi_C^{-1}(0)$ is a 0-dimensional manifold. Finally, we see that $\mathrm{SW}_X(c) = c_1(\mathcal{L})^{d/2}[\mathcal{M}]$ is equal to the sum over $\mathcal{M}_C$ of ± 1's according to the orientation.

3. The $\mathbb{Z}_p$-action on the moduli space

The purpose of this section is to clarify the $\mathbb{Z}_p$-action on the moduli space and to prove the lifting map of solutions is injective. From now on, let X and $\bar{X}$ be the ones as in Theorem 1.1, and use the following notations: $\mathcal{M}(X)$, $\mathcal{B}^*(X)$,$\mathcal{M}(X, c)$, $\mathcal{B}^*(X, c)$ etc.

First, let us determine how many Spin^c-structures on $\bar{X}$ lift to same Spin^c-structure on X. Let $\pi \colon X \to \bar{X} = X/\mathbb{Z}_p$ be the projection, and let $\pi^* \colon H^2(\bar{X};\mathbb{Z}) \to H^2(X;\mathbb{Z})$ be the homomorphism induced by π. Noticing that $\mathcal{S}(\bar{X})$ becomes an affine space modeled on $H^2(\bar{X};\mathbb{Z})$, we can easily see the following: if $\bar{c}$, $\bar{c}'$ are distinct Spin^c-structures of $\bar{X}$ such that $\bar{c} - \bar{c}' \in \ker \pi^*$, then their lifts to X coincide, i.e., $\pi^*\bar{c} \cong \pi^*\bar{c}'$.

Now we define the $\mathbb{Z}_p$-action on $\mathcal{B}^*(X)$ and $\mathcal{M}^*(X)$. Let $\pi \colon X \to \bar{X}$ be the projection and fix a Spin^c-structure $\bar{c}$ on $\bar{X}$. The spinor bundle associated to $\bar{c}$ is denoted by $\bar{S}^+$, and its determinant line bundle by $\bar{L} = \det(\bar{S}^+)$. Next let us lift $\bar{c}$ to the Spin^c-structure c on X, i.e., $c = \pi^*\bar{c}$, $S^+ = \pi^*\bar{S}^+$, $L = \pi^*\bar{L}$. The lifting map $\pi^* \colon \mathcal{C}(\bar{X}, \bar{c}) \to \mathcal{C}(X, c)$ is also defined. Then the $\mathbb{Z}_p$-action on X naturally induces the $\mathbb{Z}_p$-action on $\mathcal{C}(X, c)$ such that the elements in $\pi^*\mathcal{C}(\bar{X}, \bar{c})$ are $\mathbb{Z}_p$-invariant: for each $\sigma \in \mathbb{Z}_p$, since $S^+ = \pi^*\bar{S}^+$, we have a bundle isomorphism $\phi \colon \sigma^*S^+ \to S^+$. Then we get a lift $\tilde{\sigma} \colon S^+ \to S^+$ of $\sigma \colon X \to X$ by $\tilde{\sigma} := \phi \circ \sigma^*$. Define the action $\tilde{\sigma}$ on $\mathcal{C}$ by $\tilde{\sigma}(A, \psi) := (\tilde{\sigma}^*A, \tilde{\sigma}_*\psi)$.

LEMMA 3.1. *Let (A, ψ) be irreducible, i.e., (A, ψ) be in $\mathcal{C}^*(X)$.*

(1) *If $\tilde{\sigma}(A, \psi) = u \cdot (A, \psi)$ in $\mathcal{C}^*(X)$ for some $u \in \mathcal{G}$, then there exists another action $\tilde{\sigma}'$ on $\mathcal{C}^*(X)$ covering σ which satisfies $\tilde{\sigma}'(A, \psi) = (A, \psi)$.*
(2) *If there are two actions $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ on $\mathcal{C}^*(X)$ covering $\sigma \in \mathbb{Z}_p$ and satisfy $\tilde{\sigma}_i(A, \psi) = (A, \psi)$ for $i = 1, 2$, then $\tilde{\sigma}_1 = \tilde{\sigma}_2$.*
(3) *$\pi^* \colon \mathcal{B}^*(\bar{X}, \bar{c}) \to \mathcal{B}^*(X, c)$ is well-defined and injective.*

Proof. To prove the Assertion (1), it is enough that we define $\tilde{\sigma}'$ by $\tilde{\sigma}' := u^{-1} \circ \tilde{\sigma}$. (Note that $(u^{-1} \circ \tilde{\sigma})^p = \mathrm{id} \in \mathcal{G}$ from the irreducibility of (A, ψ).)

To prove the Assertion (2), consider $g = (\tilde{\sigma}_2)^{-1} \circ \tilde{\sigma}_1$. Note that $g \in \mathcal{G}$. Then the irreducibility of (A, ψ) implies $g = 1$, i.e., $\tilde{\sigma}_1 = \tilde{\sigma}_2$.

Let us prove the Assertion (3). Let $\mathcal{G}^{\mathbb{Z}_p}$ be the $\mathbb{Z}_p$-equivariant gauge transformations. Then $\mathcal{B}^*(\bar{X},\ \bar{c}) \cong \pi^*\mathcal{C}^*\ (\bar{X}, \bar{c})\ /\mathcal{G}^{\mathbb{Z}_p}$, and the well-definedness is obvious. We prove the injectivity. Take $(\bar{A}, \bar{\psi}), (\bar{A}', \bar{\psi}') \in \mathcal{C}^*(\bar{X})$ and let $(A, \psi) = \pi^*(\bar{A}, \bar{\psi})$ and $(A', \psi') = \pi^*(\bar{A}', \bar{\psi}')$. Suppose that (A, ψ) and (A', ψ') are gauge equivalent, i.e., $u \cdot (A, \psi) = (A', \psi')$ for some $u \in \mathcal{G}$. For any $\sigma \in \mathbb{Z}_p$, the commutator $[\tilde{\sigma}, u] \in \mathcal{G}$ satisfies $[\tilde{\sigma}, u](A, \psi) = (A, \psi)$. The irreducibility of (A, ψ) implies $[\tilde{\sigma}, u] = 1$, i.e., $u \in \mathcal{G}^{\mathbb{Z}_p}$. Thus we see that $(\bar{A}, \bar{\psi})$ and $(\bar{A}', \bar{\psi}')$ are gauge equivalent. □

If $\tilde{\sigma}(A, \psi) = u \cdot (A, \psi)$ in $\mathcal{C}^*(X, c)$ for some $u \in \mathcal{G}$ which is not $\mathbb{Z}_p$-equivariant, then (A, ψ) is not in $\pi^*\mathcal{C}^*(\bar{X}, \bar{c})$. But, by the Assertion (1) and (2) in the lemma, we see that (A, ψ) is the lift from another Spin^c-structure $\bar{c}'$ on $\bar{X}$, and the possibility of $\bar{c}'$ is parametrized by $\ker \pi^*$. Thus we have the following.

COROLLARY 3.2. *Let c be a* Spin^c*-structure on X which is the lift of a* Spin^c*-structure $\bar{c}$ on $\bar{X}$ and $\mathcal{B}^*(c)^{\mathbb{Z}_p}$ be the $\mathbb{Z}_p$-invariant set of $\mathcal{B}^*(c)$. Then*

$$\pi^* \colon \coprod_{\bar{a} \in \ker \pi^*} \mathcal{B}^*(\bar{c} + \bar{a}) \to \mathcal{B}^*(c)^{\mathbb{Z}_p}, \tag{3.3}$$

is bijective.

REMARK 3.4. By using the argument in [3], we can prove that the bijective correspondence (3.3) is a homeomorphism.

4. Proof of Theorem 1.1

In this section, we prove our main theorem. To prove the theorem, we will prove the following.

LEMMA 4.1. (A) *Let d be the virtual dimension of $\mathcal{M}(X, c)$ and $\bar{d}$ be that of $\mathcal{M}(\bar{X}, \bar{c})$. Then $d = p\bar{d}$.*
(B) *When $\bar{d} = 0$, by the $\mathbb{Z}_p$-invariant perturbation, we get moduli spaces $\mathcal{M}(X, c)$ and $\mathcal{M}(\bar{X}, \bar{c} + \bar{a})$ for $\bar{a} \in \ker \pi^*$ on which the corresponding sections are transverse. Furthermore, the following hold:*

1. *There is an isomorphism $\mathcal{M}(X, c)^{\mathbb{Z}_p} \cong \coprod_{\bar{a} \in \ker \pi^*} \mathcal{M}(\bar{X}, \bar{c} + \bar{a})$ as oriented 0-dimensional manifolds.*
2. *The number of elements in a 0-dimensional manifold $\mathcal{M}(X, c) \setminus \mathcal{M}(X, c)^{\mathbb{Z}_p}$ is a multiple of p. Furthermore, when $p \neq 2$, it is divided into some groups of p points which have the same orientation.*

(C) *When* $\bar{d} > 0$ *and even, by the* $\mathbb{Z}_p$*-invariant perturbation, we get cut-down moduli spaces* $\mathcal{M}_C(X, c)$ *and* $\mathcal{M}_C(\bar{X}, \bar{c} + \bar{a})$ *for* $\bar{a} \in \ker \pi^*$ *on which the corresponding sections are transverse. Furthermore,* (1) *and* (2) *hold for* $\mathcal{M}_C(X, c)$ *and* $\mathcal{M}_C(\bar{X}, \bar{c} + \bar{a})$*'s.*

First, we prove Theorem 1.1 from Lemma 4.1.

Proof of Theorem 1.1. Theorem 1.1 immediately follows from Lemma 4.1 except the case when $p = 2$ and $\bar{d}$ is odd. In this case, the Seiberg-Witten invariant for $\bar{X}$ vanishes by definition, but $d = 2\bar{d}$ is even, so it is possible that the corresponding invariant for X is non-trivial. However, this is excluded by the assumption that X has simple type. Thus the theorem is proved. □

REMARK 4.2. From above proof, we see that we can weaken the assumption that X has simple type when $p = 2$ as in Remark 1.4.

REMARK 4.3. The fact that the moduli space splits into $\mathbb{Z}_p$-invariant part and not-$\mathbb{Z}_p$-invariant one as in Lemma 4.1, or the formula (1.2) in Theorem 1.1 might be considered as a baby version of a nonlinearization of the Lefshetz fixed point formula for elliptic operators.

To prove Lemma 4.1, we perturb the equations or the section Ψ in $\mathbb{Z}_p$-invariant way and prove this perturbation is enough to make Ψ transversal at zeros.

The perturbation is done by two steps. The first step is to perturb around $\mathbb{Z}_p$-invariant solutions by using the Kuranishi model. The second step is to perturb around the others by adding some self-dual 2-forms.

4.1. Perturbation around $\mathbb{Z}_p$-invariant solutions

First, we perturb Ψ around $\mathbb{Z}_p$-invariant solutions. Beforehand, choose $\mu \in \Omega^+(\bar{X})$ in (2.3) such that each $\mathcal{M}(\bar{X}, \bar{c}+\bar{a})$ for $\bar{a} \in \ker \pi^*$ is a smooth manifold, and lift the equations or the sections Ψ to X.

Let $(\bar{A}, \bar{\psi})$ be an element of $\mathcal{M}(\bar{X}, \bar{c} + \bar{a})$, and $\{\Gamma(\bar{E}_i)\}$ be the deformation complex of $(\bar{A}, \bar{\psi})$. Then lift this complex to X and write the lifted complex as $\{\Gamma(E_i)\}$. Obviously $\{\Gamma(E_i)\}$ is the deformation complex associated to the lift (A, ψ) of $(\bar{A}, \bar{\psi})$. Let us apply to $\{\Gamma(E_i)\}$ the Lefschetz fixed point formula for elliptic complex [1]. We use the notations: $H^i := H^i(\Gamma(E))$, $\bar{H}^i := H^i(\Gamma(\bar{E}))$.

Let σ be a generator of $\mathbb{Z}_p$. Then an endomorphism T of the complex $\Gamma(E)$ is induced by σ in similar way as in Section 3. Since $\Gamma(E_i)$ is the lift of $\Gamma(\bar{E}_i)$, we have a natural bundle isomorphism $\phi_i\colon \sigma^* E_i \to E_i$. Then we can define $T_i\colon \Gamma(E_i) \to \Gamma(E_i)$ as the composition

$$\Gamma(E_i) \xrightarrow{\sigma^*} \Gamma(\sigma^* E_i) \xrightarrow{\Gamma(\phi_i)} \Gamma(E_i).$$

Since each one of $\sigma, \sigma^2, \dots, \sigma^{(p-1)}$ has no fixed point, the Lefschetz fixed point formula asserts that

$$L(T^n) := \sum_{i=0}^{2} (-1)^i \operatorname{Trace} H^i T^n = 0, \tag{4.4}$$

for $n = 1, \dots, p-1$ where $H^i T^n$ is a homomorphism on H^i induced by $(T_i)^n$.

Since it is easier to handle complex representations, let us complexify H^i and $H^i T^n$. Since $T^p \equiv \mathrm{id}$, $H^i \otimes \mathbb{C}$ splits into eigenspaces as follows:

$$H^i \otimes \mathbb{C} = H^i_{(0)} \oplus H^i_{(1)} \oplus \cdots \oplus H^i_{(p-1)}, \tag{4.5}$$

where $H^i_{(n)}$ is the eigenspace of the eigenvalue $(\omega)^n = (\exp(2\pi i/p))^n$. Note that $H^i_{(0)} \cong \bar{H}^i \otimes \mathbb{C}$. By (4.4), we obtain

$$\begin{aligned} 0 &= L(T^1) = C_0 + \omega C_1 + \omega^2 C_2 + \cdots + \omega^{p-1} C_{p-1}, \\ 0 &= L(T^2) = C_0 + \omega^2 C_1 + \omega^4 C_2 + \cdots + \omega^{p-2} C_{p-1}, \\ &\cdots \\ 0 &= L(T^{p-1}) = C_0 + \omega^{p-1} C_1 + \omega^{p-2} C_2 + \cdots + \omega C_{p-1}, \end{aligned} \tag{4.6}$$

where $C_n = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{C}} H^i_{(n)}$. Solving (4.6), we have

$$C_0 = C_1 = \cdots = C_{p-1}. \tag{4.7}$$

Lemma 4.8. *Set $h^i_{(n)} := \dim_{\mathbb{C}} H^i_{(n)}$. If $(\bar{A}, \bar{\psi})$ is an irreducible solution on $\bar{X}$ and the transversality holds at $(\bar{A}, \bar{\psi})$, then we have the following.*

(1) *$h^0_{(0)} = h^0_{(1)} = \cdots = h^0_{(p-1)} = 0$ and $h^2_{(0)} = 0$.*

(2) *The virtual dimension $\bar{d}$ of $\mathcal{M}(\bar{X})$ is equal to $h^1_{(0)}$, and the virtual dimension d of $\mathcal{M}(X)$ is equal to $ph^1_{(0)}$. Thus we have $d = p\bar{d}$.*

Proof. Recall that $H^i_{(0)} \cong \bar{H}^i \otimes \mathbb{C}$. Then the irreduciblity of $(\bar{A}, \bar{\psi})$ implies $H^0_{(0)} = \bar{H}^0 = 0$, and the transversality at $(\bar{A}, \bar{\psi})$ implies $H^2_{(0)} = \bar{H}^2 = 0$. Since the lift of an irreducible $(\bar{A}, \bar{\psi})$ is also irreducible, we also have $H^0_{(1)} = H^0_{(2)} = \cdots = H^0_{(p-1)} = 0$. Thus we have the Assertion (1).

Since $\bar{H}^0 = \bar{H}^2 = 0$, the virtual dimension $\bar{d}$ of $\mathcal{M}(\bar{X})$ is equal to $\dim_{\mathbb{R}} \bar{H}^1 = \dim_{\mathbb{C}} H^1_{(0)} = h^1_{(0)}$.

Let us calculate the virtual dimension d of $\mathcal{M}(X)$. By (4.7) and the Assertion (1), we have

$$(4.9) \qquad h^1_{(0)} = h^1_{(n)} - h^2_{(n)} \quad \text{for } n = 1, \dots, p-1.$$

With this understood, we have $d = \operatorname{ind} \Gamma(E) = ph^1_{(0)}$. □

(1) *The case when* $\bar{d} = 0$.

In the neighborhood of (A, ψ), the Kuranishi map is of the form $f \colon \mathcal{O} \subset H^1 \to H^2$. Recall that $df|_0 = 0$.

Let us perturb $\Psi \colon \mathcal{B} \to \mathcal{V}$ around (A, ψ). By the assumption $\bar{d} = h^1_{(0)} = 0$, (4.9) implies $\dim H^1 = \dim H^2$. Furthermore we can take a T-equivariant linear isomorphism $g \colon H^1 \to H^2$. And take a T-invariant cut-off function ρ of a small neighborhood $\mathcal{O}_{(A,\psi)}$ of (A, ψ). Recall that, in $\mathcal{O}_{(A,\psi)}$, Ψ is of the form $\Psi(\xi, v) = (L(\xi), \alpha(\xi, v))$ with $f(v) = \alpha(0, v)$. Perturb Ψ with g and ρ as

$$\Psi' := \Psi + \rho \cdot g = (L, \alpha + \rho \cdot g).$$

Then Ψ' is transverse at (A, ψ). Since we use a linear isomorphism for perturbation, we see that the contribution of (A, ψ) to the invariant is ± 1. Note that the manipulations above are T-equivariant.

Similarly, we perturb Ψ around all $\mathbb{Z}_p$-invariant solutions. (Since $\mathcal{M}(\bar{X})$ is compact and 0-dimensional, the number of $\mathbb{Z}_p$-invariant solutions is finite.)

(2) *The case when* $\bar{d} > 0$.

Recall that $d (=$ virtual dimension of $\mathcal{M}(X)) = p\bar{d}$.

Assume that $\bar{d}$ is even. Then we need to cut down the moduli space.

Choose distinct base points $\bar{x}^1, \dots, \bar{x}^{\bar{d}/2}$ on $\bar{X}$ and complex lines in fibers at $\bar{x}^1, \dots, \bar{x}^{\bar{d}/2}$. We obtain a section $\bar{\Psi}_C$ as (2.8).

Lifting $\bar{\Psi}_C$ to X, we obtain Ψ_C of the form

$$(4.10) \quad \Psi_C(A, \psi) = \Big(D_A\psi, F_A^+ - q(\psi) - \mu, P\psi(x^1_{(0)}), P\psi(x^1_{(1)}), \dots, P\psi(x^1_{(p-1)}), P\psi(x^2_{(0)}), \dots, P\psi(x^{\bar{d}/2}_{(p-1)}) \Big),$$

where $x^k_{(0)}, \dots, x^k_{(p-1)}$ are the lifts of $\bar{x}^k$.

The deformation complex associated to Ψ_C is different from that of Ψ. But the Lefschetz number of this complex does not change because, for each $n = 1, \dots, p-1$, T^n acts on $(P\psi(x^1_{(0)}), \dots, P\psi(x^{\bar{d}/2}_{(p-1)}))$-part essentially as permutation which does not affect the trace of H^iT^n. In this way we can reduce the argument for the case that $\bar{d}$ is positive and even to the one which is used to deal with the case of $\bar{d} = 0$.

4.2. Perturbation around not-$\mathbb{Z}_p$-invariant solutions

Next we will perturb around not-$\mathbb{Z}_p$-invariant solutions. Let Ψ be the section perturbed around all $\mathbb{Z}_p$-invariant solutions by Section 4.1. (When $d > 0$, we use Ψ_C instead of Ψ.) Then, note that Ψ has the following form,

$$\Psi(A, \psi) = (D_A\psi, F_A^+ - q(\psi) - \mu) + \sum_j \rho^j \cdot g^j,$$

where $\mu \in \Omega^+(X)$ is the term come from the perturbation on $\bar{X}$.

Fix $\sigma \in \mathbb{Z}_p$. By the compactness of the moduli space, we can choose a covering $\{\mathcal{O}^k_{(0)}, \mathcal{O}^k_{(1)}, \dots \mathcal{O}^k_{(p-1)}\}_{k=1,\dots,N}$ and cut-off functions $\rho^k_{(0)}, \dots,$ $\rho^k_{(p-1)}$ $(k = 1, \dots, N)$ which have the following properties:

(1) $\{\mathcal{O}^k_{(0)}, \dots \mathcal{O}^k_{(p-1)}\}_{k=1,\dots,N}$ covers all of not-σ-invariant solutions which are not transverse. Furthermore, no $\mathcal{O}^k_{(n)}$ contains any σ-invariant solution.

(2) For $k = 1, \dots N$, $\mathcal{O}^k_{(n)} = \sigma^n \mathcal{O}^k_{(0)}$, and when $m \neq n$, $\mathcal{O}^k_{(m)} \cap \mathcal{O}^k_{(n)} = \emptyset$

(3) $\operatorname{Supp} \rho^k_{(n)} \subset \mathcal{O}^k_{(n)}$ and $\bigcup_{n,k} \operatorname{Supp} \rho^k_{(n)}$ contains all of not-σ-invariant solutions which are not transverse. Furthermore, $\rho^k_{(n)}$ is the pull-back of $\rho^k_{(0)}$ by $(\sigma^n)^{-1}$.

Choosing $\mu^1, \dots, \mu^N \in \Omega^+$, we perturb Ψ as

$$\begin{aligned} \Psi(A, \psi) &= \Big(D_A\psi, F_A^+ - q(\psi) - \mu + \sum_{k=1}^{N} \sum_{n=0}^{p-1} \rho^k_{(n)} \sigma^n(\mu^k)\Big) \\ &\quad + \sum_j \rho^j \cdot g^j. \end{aligned} \tag{4.11}$$

By Section 2.1, we see that generic choice of $\mu^1, \dots, \mu^N$ makes Ψ transversal.

4.3. Completion of the proof of Lemma 4.1

In the previous section, we have shown Lemma 4.1 up to the consideration of the orientation of the moduli space. Now, we complete the proof.

After perturbing Ψ by Section 4.1 and Section 4.2, $\mathcal{M}(X,c) = \Psi^{-1}(0)$ or $\mathcal{M}_C(X,c) = \Psi_C^{-1}(0)$ becomes a 0-dimensional manifold.

The Assertion (A) is proved in Lemma 4.8.

The proofs of (B) and (C) are same. To prove the Assertion (1), we determine the orientation convention of moduli spaces. First, fix the orientation of $\mathcal{M}(\bar{X})$. This is done by choosing a homology orientation of $\bar{X}$ in (2.4). Next, determine the orientation of $\mathcal{M}(X)$ as follows. Determine the $\mathbb{Z}_p$-invariant part by lifting the orientation of $\mathcal{M}(\bar{X})$. Then choose the rest such that the orientation does not change. Furthermore, when $p = 2$, take an orientation preserving map as the linear isomorphism g in Section 4.1 (1). (When $p \neq 2$, T-equivariant linear isomorphism is always orientation preserving.) Then, the Assertion (1) is obvious by Corollary 3.2 and construction.

The Assertion (2) follows from the fact that the perturbation is completely $\mathbb{Z}_p$-equivariant and that $\mathbb{Z}_p$-action on $\mathcal{B}$ or $\mathcal{M}$ is orientation preserving when p is odd.

5. Example

In this section, we give an example.

EXAMPLE 5.1. (This is due to Prof. Ue.) Let Y be a $K3$ surface and let $L(p,q)$ be a lens space of type (p,q). Surgery on $L(p,q) \times S^1$ killing the generator of $\pi_1(S^1)$ produce a rational homology 4-sphere N with $\pi_1(N) = \mathbb{Z}_p$. Let us consider $X = Y \sharp N$ and its p fold covering $\tilde{X} = pY \sharp \tilde{N}$ where $\tilde{N}$ is the universal covering of N. Obviously, $\mathbb{Z}_p$ acts freely on $\tilde{X}$ and $X = \tilde{X}/\mathbb{Z}_p$. Let us examine Theorem 1.1.

It is well-known that the canonical Spin^c-structure c_0 on a $K3$ surface is the only Spin^c-structure which has nontrivial Seiberg-Witten invariant and its value is 1. Thus we have $\mathrm{SW}_Y(c_0) = 1$. In this case, $\ker \pi^*$ is the set of p-tortions in $H^2(X;\mathbb{Z})$ and this is isomorphic to $H^2(N,\mathbb{Z}) \cong \mathbb{Z}_p$. By [6] or [4], we have

$$\mathrm{SW}_X(c_0 + a) = 1 \tag{5.2}$$

for $a \in \ker \pi^*$.

On the other hand, $\tilde{X}$ is of the form $\tilde{X} = pY \sharp \tilde{N}$. The fact that $b_2^+(Y) \geq 1$ and Witten's vanishing theorem [8] implies that $SW_{\tilde{X}} \equiv 0$.

Thus Theorem 1.1 holds. (Similar examples are found in [6], [7]. See also [4].)

REMARK 5.3. In Example 5.1, Theorem 1.1 claims the existence of solutions on $\tilde{X}$ although the invariant is trivial. In fact, for the Spin^c-structure $\pi^* c_0$ on $\tilde{X}$, there are p solutions which are $\mathbb{Z}_p$-invariant and also p solutions which are not $\mathbb{Z}_p$-invariant with opposite sign.

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