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A frequency space for the Heisenberg group

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A FREQUENCY SPACE FOR THE HEISENBERG GROUP

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ABSTRACT. — We revisit the Fourier analysis on the Heisenberg group $\mathbb{H}^d$. Starting from the so-called Schrödinger representation and taking advantage of the projection with respect to the Hermite functions, we look at the Fourier transform of an integrable function f, as a function $\widehat{f}_{\mathbb{H}}$ on the set $\widetilde{\mathbb{H}}^d \overset{\text{def}}{=} \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{R} \setminus \{0\}$. After observing that $\widehat{f}_{\mathbb{H}}$ is uniformly continuous on $\widetilde{\mathbb{H}}^d$ equipped with an appropriate distance $\widehat{d}$, we extend the definition of $\widehat{f}_{\mathbb{H}}$ to the completion $\widehat{\mathbb{H}}^d$ of $\widetilde{\mathbb{H}}^d$. This new point of view provides a simple and explicit description of the Fourier transform of integrable functions, when the "vertical" frequency parameter tends to 0. As an application, we prepare the ground for computing the Fourier transform of functions on $\mathbb{H}^d$ that are independent of the vertical variable.

RÉSUMÉ. — On propose une nouvelle approche de la théorie de Fourier sur le groupe de Heisenberg. En utilisant la représentation de Schrödinger et la projection sur les fonctions de Hermite, la transformée de Fourier d'une fonction intégrable est définie comme une fonction sur l'ensemble $\widetilde{\mathbb{H}}^d \overset{\text{def}}{=} \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{R} \setminus \{0\}$. Cette fonction étant uniformément continue sur $\widetilde{\mathbb{H}}^d$ muni d'une distance adéquate, on peut l'étendre par densité sur le complété $\widehat{\mathbb{H}}^d$ de $\widetilde{\mathbb{H}}^d$. Ce nouveau point de vue fournit une description simple de la limite de la transformée de Fourier des fonctions intégrables lorsque la « fréquence verticale » tend vers 0. On dispose ainsi d'un cadre adéquat pour calculer par exemple la transformée de Fourier d'une fonction indépendante de la variable verticale.

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1. Introduction

Fourier analysis on locally compact Abelian groups is by now classical matter that goes back to the first half of the 20th century (see e.g. [16] for a self-contained presentation).

Consider a locally compact Abelian group $(G, +)$ endowed with a Haar measure μ, and denote by $(\widehat{G}, \cdot)$ the dual group of $(G, +)$ that is the set of characters on G equipped with the standard multiplication of functions. By definition, the Fourier transform of an integrable function $f : G \to \mathbb{C}$ is the continuous and bounded function $\widehat{f} : \widehat{G} \to \mathbb{C}$ (also denoted by $\mathcal{F}f$) defined by

$$(1.1) \qquad \forall \, \gamma \in \widehat{G}, \; \widehat{f}(\gamma) = \mathcal{F}f(\gamma) \overset{\text{def}}{=} \int_G f(x) \, \overline{\gamma(x)} \mathrm{d}\mu(x).$$

Being also a locally compact Abelian group, the "frequency space" $\widehat{G}$ may be endowed with a Haar measure $\widehat{\mu}$. Furthermore, one can normalize $\widehat{\mu}$ so that the following *Fourier inversion formula* holds true for, say, all function f in $L^1(G)$ with $\widehat{f}$ in $L^1(\widehat{G})$:

$$(1.2) \qquad \forall \, x \in G, \; f(x) = \int_{\widehat{G}} \widehat{f}(\gamma) \, \gamma(x) \mathrm{d}\widehat{\mu}(\gamma).$$

As a consequence, we get the Fourier–Plancherel identity

$$(1.3) \quad \int_G |f(x)|^2 \mathrm{d}\mu(x) = \int_{\widehat{G}} |\widehat{f}(\gamma)|^2 \mathrm{d}\widehat{\mu}(\gamma) \quad \text{for all } \; f \; \text{ in } \; L^1(G) \cap L^2(G).$$

The Fourier transform on locally compact Abelian groups has a number of noteworthy properties that we do not wish to enumerate here. Let us just recall that it changes convolution products into products of functions, namely

$$(1.4) \qquad \forall \, f \in L^1(G), \, \forall \, g \in L^1(G), \, \mathcal{F}(f \star g) = \mathcal{F}f \cdot \mathcal{F}g.$$

In the Euclidean case $G = \mathbb{R}^n$ the dual group may be identified to $(\mathbb{R}^n)^\star$ through the map $\xi \mapsto \mathrm{e}^{\mathrm{i}\langle \xi, \cdot \rangle}$ (where $\langle \cdot, \cdot \rangle$ stands for the duality bracket between $(\mathbb{R}^n)^\star$ and $\mathbb{R}^n$), and the Fourier transform of an integrable function f may thus be seen as the function on $(\mathbb{R}^n)^\star$ (usually identified to $\mathbb{R}^n$) given by

$$(1.5) \qquad \mathcal{F}(f)(\xi) = \widehat{f}(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle \xi, x \rangle} \, f(x) \mathrm{d}x.$$

Of course, we have (1.4) and, as is well known, if one endows the frequency space $(\mathbb{R}^n)^\star$ with the measure $\frac{1}{(2\pi)^n} d\xi$ then the inversion and Fourier–Plancherel formulae (1.2) and (1.3) hold true. Among the numerous

additional properties of the Fourier transform on $\mathbb{R}^n$, let us just underline that it allows to "diagonalize" the Laplace operator, namely for all smooth compactly supported functions, we have

$$(1.6) \qquad \mathcal{F}(\Delta f)(\xi) = -|\xi|^2 \widehat{f}(\xi).$$

For noncommutative groups, Fourier theory gets wilder, for the dual group is too "small" to keep the definition of the Fourier transform given in (1.1) and still have the inversion formula (1.2). Nevertheless, if the group has "nice" properties (that we are not going to list here) then one can work out a consistent Fourier theory with properties analogous to (1.2), (1.3) and (1.4) (see e.g. [1, 5, 6, 12, 15, 17, 18, 20] and the references therein for the case of nilpotent Lie groups). In that context, the classical definition of the Fourier transform amounts to replacing characters in (1.1) with suitable families of irreducible representations that are valued in Hilbert spaces (see e.g. [6, 9] for a detailed presentation). Consequently, the Fourier transform is no longer a complex valued function but rather a family of bounded operators on suitable Hilbert spaces. It goes without saying that within this approach, the notion of "frequency space" becomes unclear, which makes Fourier theory much more involved than in the Abelian case.

In the present paper, we want to focus on the Heisenberg group which, to some extent, is the simplest noncommutative nilpotent Lie group and comes into play in various areas of mathematics, ranging from complex analysis to geometry or number theory, probability theory, quantum mechanics and partial differential equations (see e.g. [3, 7, 17, 18]). As several equivalent definitions coexist in the literature, let us specify the one that we shall adopt throughout.

DEFINITION 1.1. — Let $\sigma(Y, Y') = \langle \eta, y' \rangle - \langle \eta', y \rangle$ be the canonical symplectic form on $T^\star \mathbb{R}^d$. The Heisenberg group $\mathbb{H}^d$ is the set $T^\star \mathbb{R}^d \times \mathbb{R}$ equipped with the product law

$$w \cdot w' \overset{\text{def}}{=} \bigl(Y + Y', s + s' + 2\sigma(Y, Y')\bigr) = \bigl(y + y', \eta + \eta', s + s' + 2\langle \eta, y' \rangle - 2\langle \eta', y \rangle\bigr)$$

where $w = (Y, s) = (y, \eta, s)$ and $w' = (Y', s') = (y', \eta', s')$ are generic elements of $\mathbb{H}^d$.

As regards topology and measure theory on the Heisenberg group, we shall look at $\mathbb{H}^d$ as the set $\mathbb{R}^{2d+1}$, after identifying (Y, s) in $\mathbb{H}^d$ to (y, η, s) in $\mathbb{R}^{2d+1}$. With this viewpoint, the _Haar measure_ on $\mathbb{H}^d$ is just the Lebesgue measure on $\mathbb{R}^{2d+1}$. In particular, one can define the following convolution

product for any two integrable functions f and g:

$$(1.7) \qquad f \star g(w) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w \cdot v^{-1}) g(v) \mathrm{d}v = \int_{\mathbb{H}^d} f(v) g(v^{-1} \cdot w) \mathrm{d}v.$$

Even though convolution on the Heisenberg group is noncommutative, if one defines the *Lebesgue spaces* $L^p(\mathbb{H}^d)$ to be just $L^p(\mathbb{R}^{2d+1})$, then one still gets the classical Young inequalities in that context.

As already explained above, as $\mathbb{H}^d$ is noncommutative, in order to have a good Fourier theory, one has to resort to more elaborate irreducible representations than characters. In fact, the group of characters on $\mathbb{H}^d$ is isometric to the group of characters on $T^\star \mathbb{R}^d$. Roughly, if one defines the Fourier transform according to (1.1) then the information pertaining to the vertical variable s is lost.

There are essentially two (equivalent) approaches, that are based either on the *Bargmann representation* or on the *Schrödinger representation* (see [6]). For simplicity, let us just recall the second one which is the family of group homomorphisms $w \mapsto U_w^\lambda$ (with $\lambda \in \mathbb{R} \setminus \{0\}$) between $\mathbb{H}^d$ and the unitary group $\mathcal{U}(L^2(\mathbb{R}^d))$ of $L^2(\mathbb{R}^d)$, defined for all $w = (y, \eta, s)$ in $\mathbb{H}^d$ and u in $L^2(\mathbb{R}^d)$ by

$$U_w^\lambda u(x) \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\lambda(s+2\langle \eta, x-y \rangle)} u(x - 2y).$$

The classical definition of Fourier transform of integrable functions on $\mathbb{H}^d$ reads as follows:

DEFINITION 1.2. — *The* Fourier transform *of an integrable function f on $\mathbb{H}^d$ is the family $(\mathfrak{F}_{\mathbb{H}}(f)(\lambda))_{\lambda \in \mathbb{R} \setminus \{0\}}$ of bounded operators on $L^2(\mathbb{R}^d)$ given by*

$$\mathfrak{F}_{\mathbb{H}}(f)(\lambda) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w) U_w^\lambda \mathrm{d}w.$$

In the present paper, we strive for another definition of the Fourier transform, that is as similar as possible to the one for locally compact groups given in (1.1). In particular, we want the Fourier transform to be a complex valued function defined on some "explicit" frequency space that can be endowed with a structure of a locally compact and complete metric space, and to get formulae similar to (1.2), (1.3), (1.4) together with a diagonalization of the Laplace operator (for the Heisenberg group of course) analogous to (1.6).

There is a number of motivations for our approach. An important one is that, having an explicit frequency space will allow us to get elementary proofs of the basic results involving the Fourier transform, just by mimicking the corresponding ones of the Euclidean setting. In particular, we

expect our setting to open the way to new results for partial differential equations on the Heisenberg group. Furthermore, our definition will enable us to get an explicit (and comprehensible) description of the range of the Schwartz space by the Fourier transform. As a consequence, extending the Fourier transform to the set of tempered distributions will become rather elementary (see more details in our forthcoming paper [2]).

In the present paper, we will give two concrete applications of our approach. First, in Theorem 2.7, we will provide an explicit asymptotic description of the Fourier transform when (what plays the role of) the vertical frequency parameter tends to 0. Our second application is the extension (also explicit) of the Fourier transform to functions depending only on the horizontal variable (this is Theorem 2.8).

2. Results

Before presenting the main results of the paper, let us recall how, with the standard definition of the Fourier transform in $\mathbb{H}^d$, Properties (1.2), (1.3) and (1.4) may be stated (the reader may refer to e.g. [3, 4, 7, 8, 9, 10, 12, 13, 17, 18, 20] for more details).

THEOREM 2.1. — Let f be an integrable function. Then we have

$$(2.1) \qquad \forall\, \lambda \in \mathbb{R} \setminus \{0\}\,,\ \|\mathfrak{F}_{\mathbb{H}}(f)(\lambda)\|_{\mathcal{L}(L^2)} \leqslant \|f\|_{L^1(\mathbb{H}^d)}$$

and, for any function u in $L^2(\mathbb{R}^d)$, the map $\lambda \mapsto \mathfrak{F}_{\mathbb{H}}(f)(\lambda)(u)$ is continuous from $\mathbb{R} \setminus \{0\}$ to $L^2(\mathbb{R}^d)$. For any function f in the Schwartz space $\mathcal{S}(\mathbb{H}^d)$ (which is the classical Schwartz space on $\mathbb{R}^{2d+1}$), we have the inversion formula:

$$(2.2) \qquad \forall\, w \in \mathbb{H}^d,\ f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\mathbb{R}} \operatorname{tr}\big(U_{w^{-1}}^{\lambda} \mathfrak{F}_{\mathbb{H}} f(\lambda)\big) |\lambda|^d \mathrm{d}\lambda\,,$$

where $\operatorname{tr}(A)$ denotes the trace of the operator A. Moreover, if f belongs to $L^1(\mathbb{H}^d) \cap L^2(\mathbb{H}^d)$ then $\mathfrak{F}_{\mathbb{H}}(f)(\lambda)$ is an Hilbert–Schmidt operator for any λ in $\mathbb{R} \setminus \{0\}$, and we have

$$(2.3) \qquad \|f\|_{L^2(\mathbb{H}^d)}^2 = \frac{2^{d-1}}{\pi^{d+1}} \int_{\mathbb{R}} \|\mathfrak{F}_{\mathbb{H}}(f)(\lambda)\|_{HS}^2 |\lambda|^d \mathrm{d}\lambda$$

where $\|\cdot\|_{HS}$ stands for the Hilbert–Schmidt norm.

We also have an analogue of the convolution identity (1.4). Indeed, as the map $w \mapsto U_w^{\lambda}$ is a homomorphism between $\mathbb{H}^d$ and $\mathcal{U}(L^2(\mathbb{R}^d))$, we get for any integrable functions f and g,

$$(2.4) \qquad \mathfrak{F}_{\mathbb{H}}(f \star g)(\lambda) = \mathfrak{F}_{\mathbb{H}}(f)(\lambda) \circ \mathfrak{F}_{\mathbb{H}}(g)(\lambda).$$

Let us next recall the definition of the (sub-elliptic) Laplacian on the Heisenberg group, that will play a fundamental role in our approach. Being a real Lie group, the Heisenberg group may be equipped with a linear space of *left invariant* vector fields, that is vector fields commuting with any left translation $\tau_w(w') \overset{\text{def}}{=} w \cdot w'$. It is well known that this linear space has dimension $2d + 1$ and is generated by the vector fields

$$S \overset{\text{def}}{=} \partial_s , \quad \mathcal{X}_j \overset{\text{def}}{=} \partial_{y_j} + 2\eta_j \partial_s \quad \text{and} \quad \Xi_j \overset{\text{def}}{=} \partial_{\eta_j} - 2y_j \partial_s , \ 1 \leqslant j \leqslant d.$$

The *Laplacian* associated to the vector fields $(\mathcal{X}_j)_{1\leqslant j\leqslant d}$ and $(\Xi_j)_{1\leqslant j\leqslant d}$ reads

$$(2.5) \qquad\qquad \Delta_{\mathbb{H}} \overset{\text{def}}{=} \sum_{j=1}^{d} (\mathcal{X}_j^2 + \Xi_j^2).$$

As in the Euclidean case (see Identity (1.6)), the Fourier transform allows to diagonalize Operator $\Delta_{\mathbb{H}}$, a property that is based on the following relation that holds true for all functions f and u in $\mathcal{S}(\mathbb{H}^d)$ and $\mathcal{S}(\mathbb{R}^d)$, respectively (see e.g. [11, 14]):

$$(2.6) \quad \mathfrak{F}_{\mathbb{H}}(\Delta_{\mathbb{H}}f)(\lambda) = 4\mathfrak{F}_{\mathbb{H}}(f)(\lambda) \circ \Delta_{\text{osc}}^{\lambda}$$

$$\text{with} \quad \Delta_{\text{osc}}^{\lambda} u(x) \overset{\text{def}}{=} \sum_{j=1}^{d} \partial_j^2 u(x) - \lambda^2 |x|^2 u(x).$$

This prompts us to take advantage of the spectral structure of the harmonic oscillator to get an analog of Formula (1.6). To this end, we need to introduce the family of Hermite functions $(H_n)_{n\in\mathbb{N}^d}$ defined by

$$H_n \overset{\text{def}}{=} \left(\frac{1}{2^{|n|}n!} \right)^{\frac{1}{2}} C^n H_0 \quad \text{with} \quad C^n \overset{\text{def}}{=} \prod_{j=1}^{d} C_j^{n_j} \quad \text{and} \quad H_0(x) \overset{\text{def}}{=} \pi^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}}$$

where $C_j \overset{\text{def}}{=} -\partial_j + M_j$ stands for the *creation operator* with respect to the j-th variable and M_j is the multiplication operator defined by $M_j u(x) \overset{\text{def}}{=} x_j u(x)$. As usual, $n! \overset{\text{def}}{=} n_1! \cdots n_d!$ and $|n| \overset{\text{def}}{=} n_1 + \cdots + n_d$.

It is well known that the family $(H_n)_{n\in\mathbb{N}^d}$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^d)$. In particular,

$$(2.7) \qquad\qquad \forall\, (n, m) \in \mathbb{N}^d \times \mathbb{N}^d, \ (H_n | H_m)_{L^2} = \delta_{n,m},$$

where $\delta_{n,m} = 1$ if $n = m$, and $\delta_{n,m} = 0$ if $n \neq m$.

Besides, we have

$$(2.8) \quad (-\partial_j^2 + M_j^2)H_n = (2n_j + 1)H_n \quad \text{and} \quad -\Delta_{\text{osc}}^1 H_n = (2|n| + d)H_n.$$

For λ in $\mathbb{R} \setminus \{0\}$, we further introduce the rescaled Hermite function

$$H_{n,\lambda}(x) \overset{\text{def}}{=} |\lambda|^{\frac{d}{4}} H_n(|\lambda|^{\frac{1}{2}} x).$$

It is obvious that $(H_{n,\lambda})_{n \in \mathbb{N}^d}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, that

$$(-\partial_j^2 + \lambda^2 M_j^2) H_{n,\lambda} = (2n_j + 1)|\lambda| H_{n,\lambda}$$

and thus

$$(2.9) \qquad -\Delta_{\text{osc}}^{\lambda} H_{n,\lambda} = (2|n| + d)|\lambda| H_{n,\lambda}.$$

We are now ready to give "our" definition[1] of the Fourier transform on $\mathbb{H}^d$.

DEFINITION 2.2. — Let $\widetilde{\mathbb{H}}^d \overset{\text{def}}{=} \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{R} \setminus \{0\}$. We denote by $\widehat{w} = (n, m, \lambda)$ a generic point of $\widetilde{\mathbb{H}}^d$. For f in $L^1(\mathbb{H}^d)$, we define the map $\mathcal{F}_{\mathbb{H}} f$ (also denoted by $\widehat{f}_{\mathbb{H}}$) to be

$$\mathcal{F}_{\mathbb{H}} f : \begin{cases} \widetilde{\mathbb{H}}^d \longrightarrow \mathbb{C} \\ \widehat{w} \longmapsto \big(\mathfrak{F}_{\mathbb{H}}(f)(\lambda) H_{m,\lambda} | H_{n,\lambda}\big)_{L^2}. \end{cases}$$

From now on, we shall use only that definition of the Fourier transform, which amounts to considering the "infinite matrix" of $\mathfrak{F}_{\mathbb{H}} f(\lambda)$ in the orthonormal basis of $L^2(\mathbb{R}^d)$ given by $(H_{n,\lambda})_{n \in \mathbb{N}}$. To emphasize the likeness with the Euclidean case, we shall rewrite $\mathcal{F}_{\mathbb{H}} f$ in terms of the mean value of f modulated by some oscillatory functions which may be seen as suitable Wigner distribution functions of the family $(H_{n,\lambda})_{n \in \mathbb{N}^d, \lambda \neq 0}$, and will play the same role as the characters $e^{i\langle \xi, \cdot \rangle}$ in $\mathbb{R}^n$. Indeed, by definition, we have

$$\mathcal{F}_{\mathbb{H}} f(\widehat{w}) = \int_{\mathbb{H}^d \times \mathbb{R}^d} f(w) \, e^{-is\lambda} \, e^{-2i\lambda\langle \eta, x-y \rangle} \, H_{m,\lambda}(x - 2y) H_{n,\lambda}(x) dw dx.$$

Therefore, making an obvious change of variable, we discover that

$$(2.10) \qquad \mathcal{F}_{\mathbb{H}} f(\widehat{w}) = \int_{\mathbb{H}^d} \overline{e^{is\lambda} \, \mathcal{W}(\widehat{w}, Y)} \, f(Y, s) dY ds \quad \text{with}$$

$$(2.11) \qquad \mathcal{W}(\widehat{w}, Y) \overset{\text{def}}{=} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z \rangle} \, H_{n,\lambda}(y + z) H_{m,\lambda}(-y + z) dz.$$

At this stage, looking at the action of the Laplace operator on $e^{is\lambda} \, \mathcal{W}(\widehat{w}, Y)$ is illuminating. Indeed, easy computations (carried out in Appendix) give

$$(2.12) \qquad (\mathcal{X}_j^2 + \Xi_j^2)\big(e^{is\lambda} \, \mathcal{W}(\widehat{w}, Y)\big) = -4|\lambda|(2m_j + 1) e^{is\lambda} \, \mathcal{W}(\widehat{w}, Y).$$

By summation on $j \in \{1, \ldots, d\}$, we get

$$(2.13) \qquad \Delta_{\mathbb{H}}\big(e^{is\lambda} \, \mathcal{W}(\widehat{w}, Y)\big) = -4|\lambda|(2|m| + d) e^{is\lambda} \, \mathcal{W}(\widehat{w}, Y),$$

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[1] This point of view has been already taken in [19] in connection with uncertainty inequalities on the Heisenberg group.

from which one may deduce that, whenever f is in $\mathcal{S}(\mathbb{H}^d)$ (again, refer to the Appendix),

$$(2.14) \qquad \mathcal{F}_{\mathbb{H}}(\Delta_{\mathbb{H}}f)(\widehat{w}) = -4|\lambda|(2|m| + d)\widehat{f}_{\mathbb{H}}(\widehat{w}).$$

Let us underline the similarity with Relation (1.6) pertaining to the Fourier transform in $\mathbb{R}^n$.

One of the basic principles of the Fourier transform on $\mathbb{R}^n$ is that "*regularity implies decay*". It remains true in the Heisenberg framework, as stated in the following lemma.

LEMMA 2.3. — *For any non negative integer p, there exist an integer N_p and a positive constant C_p such that for any $\widehat{w}$ in $\widetilde{\mathbb{H}}^d$ and any f in $\mathcal{S}(\mathbb{H}^d)$, we have*

$$(2.15) \quad \bigl(1 + |\lambda|(|n| + |m| + d) + |n - m|\bigr)^p|\widehat{f}_{\mathbb{H}}(n, m, \lambda)| \leqslant C_p\|f\|_{N_p,\mathcal{S}},$$

where $\|\cdot\|_{N,\mathcal{S}}$ denotes the classical family of semi-norms of $\mathcal{S}(\mathbb{R}^{2d+1})$, namely

$$\|f\|_{N,\mathcal{S}} \overset{\text{def}}{=} \sup_{|\alpha|\leqslant N} \bigl\|(1 + |Y|^2 + s^2)^{N/2}\, \partial_{Y,s}^{\alpha}f\bigr\|_{L^\infty}.$$

As may be easily checked by the reader, in our setting, there are very simple formulae corresponding to (2.2) and (2.3), if the set $\widetilde{\mathbb{H}}^d$ is endowed with the measure $\mathrm{d}\widehat{w}$ defined by:

$$(2.16) \qquad \int_{\widetilde{\mathbb{H}}^d} \theta(\widehat{w})\mathrm{d}\widehat{w} \overset{\text{def}}{=} \sum_{(n,m)\in\mathbb{N}^{2d}} \int_{\mathbb{R}} \theta(n, m, \lambda)|\lambda|^d\mathrm{d}\lambda.$$

Then Theorem 2.1 recasts as follows:

THEOREM 2.4. — *The following inversion formula holds true for any function f in $\mathcal{S}(\mathbb{H}^d)$:*

$$(2.17) \qquad f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} e^{is\lambda}\, \mathcal{W}(\widehat{w}, Y)\widehat{f}_{\mathbb{H}}(\widehat{w})\mathrm{d}\widehat{w}.$$

Moreover, for any function f in $L^1(\mathbb{H}^d) \cap L^2(\mathbb{H}^d)$, we have

$$(2.18) \qquad \|\widehat{f}_{\mathbb{H}}\|_{L^2(\widetilde{\mathbb{H}}^d)}^2 = \frac{\pi^{d+1}}{2^{d-1}}\|f\|_{L^2(\mathbb{H}^d)}^2.$$

In this new setting, the convolution identity (2.4) rewrites as follows for all integrable functions f and g:

$$(2.19) \quad \mathcal{F}_{\mathbb{H}}(f \star g)(n, m, \lambda) = (\widehat{f}_{\mathbb{H}} \cdot \widehat{g}_{\mathbb{H}})(n, m, \lambda)$$

$$\text{with} \quad (\widehat{f}_{\mathbb{H}} \cdot \widehat{g}_{\mathbb{H}})(n, m, \lambda) \overset{\text{def}}{=} \sum_{\ell\in\mathbb{N}^d} \widehat{f}_{\mathbb{H}}(n, \ell, \lambda)\widehat{g}_{\mathbb{H}}(\ell, m, \lambda).$$

The reader is referred to the appendix for the proof.

Next, we aim at endowing the set $\widetilde{\mathbb{H}}^d$ with a structure of a metric space. According to the decay inequality (2.15), it is natural to introduce the following distance $\widehat{d}$:

$$(2.20) \quad \widehat{d}(\widehat{w}, \widehat{w}') \overset{\text{def}}{=} \big|\lambda(n+m) - \lambda'(n'+m')\big|_1 + \big|(n-m) - (n'-m')\big|_1 + |\lambda - \lambda'|,$$

where $|\cdot|_1$ denotes the ℓ^1 norm on $\mathbb{R}^d$.

At first glance, the metric space $(\widetilde{\mathbb{H}}^d, \widehat{d})$ seems to be the natural frequency space within our approach. However, it fails to be complete, which may be a source of difficulties for further development. We thus propose to work with its completion, that is described in the following proposition.

PROPOSITION 2.5. — *The completion of the set $\widetilde{\mathbb{H}}^d$ for the distance $\widehat{d}$ is the set $\widehat{\mathbb{H}}^d$ defined by $\widehat{\mathbb{H}}^d \overset{\text{def}}{=} (\mathbb{N}^{2d} \times \mathbb{R} \setminus \{0\}) \cup \widehat{\mathbb{H}}_0^d$ with $\widehat{\mathbb{H}}_0^d \overset{\text{def}}{=} \mathbb{R}_{\mp}^d \times \mathbb{Z}^d$ and $\mathbb{R}_{\mp}^d \overset{\text{def}}{=} ((\mathbb{R}_-)^d \cup (\mathbb{R}_+)^d)$. On $\widehat{\mathbb{H}}^d$, the extended distance (still denoted by $\widehat{d}$) is given by*

$$\widehat{d}((n,m,\lambda), (n',m',\lambda')) = \big|\lambda(n+m) - \lambda'(n'+m')\big|_1$$
$$+ \big|(m-n) - (m'-n')\big|_1 + |\lambda - \lambda'| \quad \text{if } \lambda \neq 0 \text{ and } \lambda' \neq 0,$$
$$\widehat{d}((n,m,\lambda), (\dot{x},k)) = \widehat{d}((\dot{x},k), (n,m,\lambda))$$
$$\overset{\text{def}}{=} |\lambda(n+m) - \dot{x}|_1 + |m - n - k|_1 + |\lambda| \quad \text{if } \lambda \neq 0,$$
$$\widehat{d}((\dot{x},k), (\dot{x}',k')) = |\dot{x} - \dot{x}'|_1 + |k - k'|_1.$$

Proof. — Consider a Cauchy sequence $(n_p, m_p, \lambda_p)_{p\in\mathbb{N}}$ in $(\widetilde{\mathbb{H}}^d, \widehat{d})$. If p and p' are large enough, then $|(m_p - n_p) - (m_{p'} - n_{p'})|$ is less than 1, and thus $m_p - n_p$ has to be a constant, that we denote by k. Next, we see that $(\lambda_p)_{p\in\mathbb{N}}$ is a Cauchy sequence of real numbers, and thus converges to some λ in $\mathbb{R}$. If $\lambda \neq 0$ then our definition of $\widehat{d}$ implies that the sequence $(n_p)_{p\in\mathbb{N}}$ is constant after a certain index, and thus converges to some n in $\mathbb{N}^d$. Therefore we have $(n_p, m_p, \lambda_p) \to (n, n+k, \lambda)$.

If $\lambda = 0$ then the Cauchy sequence $\big(\lambda_p(n_p + m_p)\big)_{p\in\mathbb{N}}$ has to converge to some $\dot{x}$ in $\mathbb{R}^d$. By definition of the extended distance, it is clear that

$$(n_p, m_p, \lambda_p)_{p\in\mathbb{N}} \to (\dot{x}, k) \quad \text{in } \widehat{\mathbb{H}}^d.$$

Now, if $\dot{x} \neq 0$ then there exists some index j such that $\dot{x}_j \neq 0$. Because the sequence $\big(\lambda_p(n_{j,p} + m_{j,p})\big)_{p\in\mathbb{N}}$ tends to $\dot{x}_j$ and $n_{j,p} + m_{j,p}$ is positive (for large enough p), we must have $\text{sgn}(\lambda_p) = \text{sgn}(\dot{x}_j)$. Therefore, all the of $\dot{x}$ have the same sign.

Conversely, let us prove that any point of $\mathbb{R}_+^d \times \mathbb{Z}^d$ (the case of $\mathbb{R}_-^d \times \mathbb{Z}^d$ being similar) is the limit in the sense of $\widehat{d}$ of some sequence $(n_p, m_p, \lambda_p)_{p \in \mathbb{N}}$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist two families of sequences of positive integers $(a_{j,p})_{p \in \mathbb{N}}$ and $(b_{j,p})_{p \in \mathbb{N}}$ such that

$$\forall\, j \in \{1, \ldots, d\}\,,\ \dot{x}_j = \lim_{p \to \infty} \dot{x}_{j,p}\ \text{ with }\ \dot{x}_{j,p} \overset{\text{def}}{=} \frac{a_{j,p}}{b_{j,p}}\ \text{ and }\ \lim_{p \to \infty} b_{j,p} = \infty.$$

Let us write that

$$\dot{x}_p = 2\lambda_p n_p\ \text{ with }\ \lambda_p \overset{\text{def}}{=} \left(2 \prod_{j=1}^d b_{j,p}\right)^{-1}\ \text{ and }\ n_p \overset{\text{def}}{=} \left(a_{j,p} \prod_{j' \neq j}^d b_{j,p}\right)_{1 \leqslant j \leqslant d}.$$

As $(\lambda_p)_{p \in \mathbb{N}}$ tends to 0, we have that $\lim_{p \to \infty} \widehat{d}\big((n_p, n_p + k, \lambda_p), (\dot{x}, k)\big)$ converges to 0. $\qquad \square$

Remark 2.6. — It is not difficult to check that the closed bounded subsets of $\widehat{\mathbb{H}}^d$ (for the distance $\widehat{d}$) are compact. The details are left to the reader.

The above proposition prompts us to extend the Fourier transform of an integrable function, to the frequency set $\widehat{\mathbb{H}}^d$, that will play the same role as $(\mathbb{R}^n)^\star$ in the case of $\mathbb{R}^n$. With this new point of view, we expect the Fourier transform of any integrable function to be continuous on the whole $\widehat{\mathbb{H}}^d$. This is exactly what is stated in the following theorem.

THEOREM 2.7. — *The Fourier transform $\widehat{f}_{\mathbb{H}}$ of any integrable function on $\mathbb{H}^d$ may be extended continuously to the whole set $\widehat{\mathbb{H}}^d$. Still denoting by $\widehat{f}_{\mathbb{H}}$ (or $\mathcal{F}_{\mathbb{H}} f$) that extension, the linear map $\mathcal{F}_{\mathbb{H}} : f \mapsto \widehat{f}_{\mathbb{H}}$ is continuous from the space $L^1(\mathbb{H}^d)$ to the space $\mathcal{C}_0(\widehat{\mathbb{H}}^d)$ of continuous functions on $\widehat{\mathbb{H}}^d$ tending to 0 at infinity. Moreover, we have for all $(\dot{x}, k)$ in $\widehat{\mathbb{H}}_0^d$,*

$$\tag{2.21} \mathcal{F}_{\mathbb{H}} f(\dot{x}, k) = \int_{T^\star \mathbb{R}^d} \overline{\mathcal{K}}_d(\dot{x}, k, Y) f(Y, s) \mathrm{d}Y \mathrm{d}s \quad \text{with}$$

$$\mathcal{K}_d(\dot{x}, k, Y) = \bigotimes_{j=1}^d \mathcal{K}(\dot{x}_j, k_j, Y_j) \quad \text{and}$$

$$\tag{2.22} \mathcal{K}(\dot{x}, k, y, \eta) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\left(2|\dot{x}|^{\frac{1}{2}}(y \sin z + \eta \, \mathrm{sgn}(\dot{x}) \cos z) + kz\right)} \mathrm{d}z.$$

In other words, for any sequence $(n_p, \lambda_p)_{p \in \mathbb{N}}$ of $\mathbb{N}^d \times (\mathbb{R} \setminus \{0\})$ such that

$$\lim_{p \to \infty} \lambda_p = 0 \quad \text{and} \quad \lim_{p \to \infty} \lambda_p n_p = \frac{\dot{x}}{2},$$

we have

$$\lim_{p \to \infty} \widehat{f}_{\mathbb{H}}(n_p, n_p + k, \lambda_p) = \int_{T^\star \mathbb{R}^d} \overline{\mathcal{K}}_d(\dot{x}, k, Y) f(Y, s) \mathrm{d}Y \mathrm{d}s.$$

Granted with the above result, one can propose a natural extension of the Fourier transform to (smooth) functions on $\mathbb{H}^d$ *independent of the vertical variable s*. This will come up as a consequence of the following theorem.

THEOREM 2.8. — *Let us define the following operator $\mathcal{G}_{\mathbb{H}}$ on $L^1(T^\star\mathbb{R}^d)$:*

$$\mathcal{G}_{\mathbb{H}}g(\dot{x}, k) \stackrel{\text{def}}{=} \int_{T^\star\mathbb{R}^d} \overline{\mathcal{K}}_d(\dot{x}, k, Y)g(Y)\mathrm{d}Y.$$

Then, for any function χ in $\mathcal{S}(\mathbb{R})$ with value 1 at 0 and compactly supported Fourier transform, and any function g in $L^1(T^\star\mathbb{R}^d)$, we have

(2.23)
$$\lim_{\varepsilon\to0} \mathcal{F}_{\mathbb{H}}(g \otimes \chi(\varepsilon\cdot)) = 2\pi(\mathcal{G}_{\mathbb{H}}g)\mu_{\widehat{\mathbb{H}}^d_0}$$

in the sense of measures on $\widehat{\mathbb{H}}^d$, where $\mu_{\widehat{\mathbb{H}}^d_0}$ is the measure (supported in $\widehat{\mathbb{H}}^d_0$) defined for all continuous compactly supported functions θ on $\widehat{\mathbb{H}}^d$ by

(2.24)
$$\langle \mu_{\widehat{\mathbb{H}}^d_0}, \theta \rangle = \int_{\widehat{\mathbb{H}}^d_0} \theta(\dot{x}, k)\mathrm{d}\mu_{\widehat{\mathbb{H}}^d_0}(\dot{x}, k)$$
$$\stackrel{\text{def}}{=} 2^{-d} \sum_{k\in\mathbb{Z}^d} \left(\int_{(\mathbb{R}_-)^d} \theta(\dot{x}, k)\mathrm{d}\dot{x} + \int_{(\mathbb{R}_+)^d} \theta(\dot{x}, k)\mathrm{d}\dot{x} \right).$$

The above theorem allows to give a meaning of the Fourier transform of a smooth function that does not depend on the vertical variable. The next step would be to study whether our approach allows, as in the Euclidean case, to extend the definition of the Fourier transform to a much larger set of functions, or even to tempered distributions. This requires a fine characterization of $\mathcal{F}_{\mathbb{H}}(\mathcal{S}(\mathbb{H}^d))$ the range of $\mathcal{S}(\mathbb{H}^d)$ by $\mathcal{F}_{\mathbb{H}}$, which will be the purpose of our companion paper [2].

We end this section with a short description of the structure of the rest of the paper, and of the main ideas of the proofs.

Section 3 is devoted to the proof of the first part of Theorem 2.7. It relies on the fact that the function $\mathcal{W}(\cdot, Y)$ is uniformly continuous (for distance $\widehat{d}$) on bounded sets of $\widehat{\mathbb{H}}^d$, and can thus be extended to the closure $\widetilde{\mathbb{H}}^d$ of $\widehat{\mathbb{H}}^d$. Establishing that property requires our using an explicit asymptotic expansion of $\mathcal{W}$.

Proving Theorem 2.8 is the purpose of Section 4. The main two ingredients are the following ones. First, we show that if ψ is an integrable function on $\mathbb{R}$ with integral 1, then we have

$$\lim_{\varepsilon\to0} \frac{1}{\varepsilon}\psi\left(\frac{\lambda}{\varepsilon}\right)\mathrm{d}\widehat{w} = \mu_{\widehat{\mathbb{H}}^d_0},$$

in the sense of measures on $\widehat{\widetilde{\mathbb{H}}}^d$. That is to say, for any continuous compactly supported function θ on $\widehat{\widetilde{\mathbb{H}}}^d$, we have

(2.25) $$\lim_{\varepsilon \to 0} \int_{\widehat{\widetilde{\mathbb{H}}}^d} \frac{1}{\varepsilon} \psi\left(\frac{\lambda}{\varepsilon}\right) \theta(\widehat{w}) \mathrm{d}\widehat{w} = \langle \mu_{\widehat{\mathbb{H}}_0^d}, \theta_{|\widehat{\mathbb{H}}_0^d} \rangle.$$

Then by a density argument, the proof of Theorem 2.8 reduces to the case when g is in $\mathcal{S}(T^\star \mathbb{R}^d)$.

Section 5 is devoted to computing $\mathcal{K}$. This will be based on the following properties (that we shall first establish):

- $\mathcal{K}(0, k, Y) = \delta_{0,k}$ for all Y in $T^\star \mathbb{R}$;
- The symmetry identities:

(2.26) $$\mathcal{K}(\dot{x}, -k, -Y) = \overline{\mathcal{K}(\dot{x}, k, Y)}, \qquad \mathcal{K}(-\dot{x}, -k, Y) = (-1)^k \mathcal{K}(\dot{x}, k, Y)$$
$$\text{and} \quad \mathcal{K}(-\dot{x}, k, Y) = \overline{\mathcal{K}(\dot{x}, k, Y)};$$

- The identity

(2.27) $$\Delta_Y \mathcal{K}(\dot{x}, k, Y) = -4|\dot{x}|\mathcal{K}(\dot{x}, k, Y);$$

- The relation

(2.28) $$ik\mathcal{K}(\dot{x}, k, Y) = \big(\eta \partial_y \mathcal{K}(\dot{x}, k, Y) - y \partial_\eta \mathcal{K}(\dot{x}, k, Y)\big) \, \mathrm{sgn}(\dot{x});$$

- The convolution property

(2.29) $$\mathcal{K}(\dot{x}, k, Y_1 + Y_2) = \sum_{k' \in \mathbb{Z}} \mathcal{K}(\dot{x}, k - k', Y_1) \mathcal{K}(\dot{x}, k', Y_2);$$

- And finally, the following relation for $\dot{x} > 0$ given by the study of $\mathcal{F}_{\mathbb{H}}(|Y|^2 f)$:

(2.30) $$|Y|^2 \mathcal{K} + \dot{x} \partial_{\dot{x}}^2 \mathcal{K} + \partial_{\dot{x}} \mathcal{K} - \frac{k^2}{4\dot{x}} \mathcal{K} = 0.$$

Let us emphasize that proving first (2.25) is essential to justify rigorously (2.22).

Finally, Section 6 is devoted to the proof of an inversion formula involving Operator $\mathcal{G}_{\mathbb{H}}$. Some basic properties of Hermite functions and of Wigner transform of Hermite functions are recalled in Appendix. There, for the reader convenience, we also prove the decay result stated in Lemma 2.3.

3. The uniform continuity of the Fourier transform of an L^1 function

The key to the proof of Theorem 2.7 is a refined study of the behavior of functions $\mathcal{W}(\,\cdot\,, Y)$ defined by (2.11) on the set $\widetilde{\mathbb{H}}^d$. Of course, a special

attention will be given to the neighborhood of $\widehat{\mathbb{H}}_0^d$. This is the aim of the following proposition.

PROPOSITION 3.1. — Let R_0 be a positive real number, and let

$$\mathcal{B}(R_0) \overset{\text{def}}{=} \left\{ (n, m, \lambda) \in \widetilde{\mathbb{H}}^d, \ |\lambda|(|n + m| + d) + |n - m| \leqslant R_0 \right\}$$
$$\times \left\{ Y \in T^{\star}\mathbb{R}^d, \ |Y| \leqslant R_0 \right\}.$$

The function $\mathcal{W}(\cdot, Y)$ restricted to $\mathcal{B}(R_0)$ is uniformly continuous with respect to $\widehat{w}$, that is for all $\forall\, \varepsilon > 0$, there exists $\alpha_\varepsilon > 0$ such that

$$\forall\, (\widehat{w}_j, Y) \in \mathcal{B}(R_0)\,, \ \widehat{d}(\widehat{w}_1, \widehat{w}_2) < \alpha_\varepsilon \implies \left| \mathcal{W}(\widehat{w}_1, Y) - \mathcal{W}(\widehat{w}_2, Y) \right| < \varepsilon.$$

Furthermore, for any $(\dot{x}, k)$ in $\widehat{\mathbb{H}}_0^d$, we have

$$\lim_{\widehat{w} \to (\dot{x}, k)} \mathcal{W}(\widehat{w}, Y) = \mathcal{K}_d(\dot{x}, k, Y)$$

where the function $\mathcal{K}_d$ is defined on $\widehat{\mathbb{H}}_0^d \times T^{\star}\mathbb{R}^d$ by

$$(3.1) \quad \mathcal{K}_d(\dot{x}, k, Y) \overset{\text{def}}{=} \sum_{(\ell_1, \ell_2) \in \mathbb{N}^d \times \mathbb{N}^d} \frac{(i\eta)^{\ell_1}}{\ell_1!} \frac{y^{\ell_2}}{\ell_2!} F_{\ell_1, \ell_2}(k) \, (\operatorname{sgn} \dot{x})^{\ell_1} |\dot{x}|^{\frac{\ell_1 + \ell_2}{2}}$$

$$\text{with} \quad F_{\ell_1, \ell_2}(k) \overset{\text{def}}{=} \sum_{\substack{\ell_1' \leqslant \ell_1, \ell_2' \leqslant \ell_2 \\ k + \ell_1 - 2\ell_1' = \ell_2 - 2\ell_2'}} (-1)^{\ell_2 - \ell_2'} \binom{\ell_1}{\ell_1'} \binom{\ell_2}{\ell_2'}.$$

Above, $\operatorname{sgn} \dot{x}$ designates the (common) sign of all components of $\dot{x}$, and $|\dot{x}| \overset{\text{def}}{=} (|\dot{x}_1|, \ldots, |\dot{x}_d|)$.

Proof. — Let us first perform the change of variable $z' = -y + z$ in (2.11) so as to get

$$(3.2) \quad \begin{aligned} \mathcal{W}(\widehat{w}, Y) &= e^{2i\lambda\langle \eta, y \rangle} \widetilde{\mathcal{W}}(\widehat{w}, Y) \quad \text{with} \\ \widetilde{\mathcal{W}}(\widehat{w}, Y) &\overset{\text{def}}{=} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z' \rangle} H_{n,\lambda}(2y + z') H_{m,\lambda}(z') \mathrm{d}z'. \end{aligned}$$

Obviously, the uniform continuity of $\mathcal{W}$ reduces to that of $\widetilde{\mathcal{W}}$. Moreover, as the integral defining $\widetilde{\mathcal{W}}$ is a product of d integrals on $\mathbb{R}$ (of modulus bounded by 1), it is enough to study the one dimensional case.

Let us start with the case where both $\widehat{w}_1 = (n_1, m_1, \lambda_1)$ and $\widehat{w}_2 = (n_2, m_2, \lambda_2)$ are relatively far away from $\widehat{\mathbb{H}}_0^1$. As we need only to consider

the situation where $\widehat{w}_1$ and $\widehat{w}_2$ are close to one another, one may assume that $(n_1, m_1) = (n_2, m_2) = (n, m)$. Then we can write that

$$
\begin{aligned}
\widetilde{\mathcal{W}}(\widehat{w}_1, Y) &- \widetilde{\mathcal{W}}(\widehat{w}_2, Y) \\
&= \int_{\mathbb{R}} \left(\mathrm{e}^{2\mathrm{i}\lambda_1 \eta z} - \mathrm{e}^{2\mathrm{i}\lambda_2 \eta z} \right) H_{n,\lambda_1}(2y + z) H_{m,\lambda_1}(z) \mathrm{d}z \\
&\quad + \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}\lambda_2 \eta z} \left(H_{n,\lambda_1}(2y + z) - H_{n,\lambda_2}(2y + z) \right) H_{m,\lambda_1}(z) \mathrm{d}z \\
&\quad + \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}\lambda_2 \eta z} H_{n,\lambda_2}(2y + z) \left(H_{m,\lambda_1}(z) - H_{m,\lambda_2}(z) \right) \mathrm{d}z.
\end{aligned}
$$

Clearly, we have

$$
\begin{aligned}
(3.3) \quad \left| \int_{\mathbb{R}} \left(\mathrm{e}^{2\mathrm{i}\lambda_1 \eta z} - \mathrm{e}^{2\mathrm{i}\lambda_2 \eta z} \right) H_{n,\lambda_1}(2y + z) H_{m,\lambda_1}(z) \mathrm{d}z \right| \\
\leqslant 2|\lambda_1|^{-\frac{1}{2}} R_0 |\lambda_2 - \lambda_1| \, \|M H_m\|_{L^2}.
\end{aligned}
$$

Next, let us study the continuity of the map $\lambda \longmapsto H_{n,\lambda}$ in the case $d = 1$. One may write

$$
\begin{aligned}
H_{n,\lambda_1}(x) &- H_{n,\lambda_2}(x) \\
&= \left(|\lambda_1|^{\frac{1}{4}} - |\lambda_2|^{\frac{1}{4}} \right) H_n(|\lambda_1|^{\frac{1}{2}} x) \\
&\quad + |\lambda_2|^{\frac{1}{4}} \left(H_n(|\lambda_1|^{\frac{1}{2}} x) - H_n(|\lambda_2|^{\frac{1}{2}} x) \right) \\
&= \left(|\lambda_1|^{\frac{1}{4}} - |\lambda_2|^{\frac{1}{4}} \right) H_n(|\lambda_1|^{\frac{1}{2}} x) \\
&\quad + |\lambda_2|^{\frac{1}{4}} \left(|\lambda_1|^{\frac{1}{2}} - |\lambda_2|^{\frac{1}{2}} \right) x \int_0^1 H_n'\left(\left(|\lambda_2|^{\frac{1}{2}} + t(|\lambda_1|^{\frac{1}{2}} - |\lambda_2|^{\frac{1}{2}}) \right) x \right) \mathrm{d}t.
\end{aligned}
$$

If $\left| |\lambda_1|^{\frac{1}{2}} - |\lambda_2|^{\frac{1}{2}} \right| \leqslant \frac{1}{2} |\lambda_2|^{\frac{1}{2}}$, then the changes of variable $x' = |\lambda_1|^{\frac{1}{2}} x$ and $x' = (|\lambda_2|^{\frac{1}{2}} + t(|\lambda_1|^{\frac{1}{2}} - |\lambda_2|^{\frac{1}{2}})) x$, respectively, together with the fact that the Hermite functions have L^2 norms equal to 1 ensure that

$$
(3.4) \quad \|H_{n,\lambda_1} - H_{n,\lambda_2}\|_{L^2} \leqslant \frac{\left| |\lambda_1|^{\frac{1}{4}} - |\lambda_2|^{\frac{1}{4}} \right|}{|\lambda_1|^{\frac{1}{4}}} + 4 \frac{\left| |\lambda_1|^{\frac{1}{2}} - |\lambda_2|^{\frac{1}{2}} \right|}{|\lambda_2|^{\frac{1}{2}}} \|M H_n'\|_{L^2}.
$$

Using (A.4), we get that

$$
M H_n' = \frac{1}{2} \left(\sqrt{n(n-1)} H_{n-2} - H_n - \sqrt{(n+1)(n+2)} H_{n+2} \right).
$$

As the family of Hermite functions is an orthonormal basis of L^2, one can write that

$$
4\|M H_n'\|_{L^2}^2 = n(n-1) + 1 + (n+1)(n+2) = 2n^2 + 2n + 3.
$$

Hence, one can conclude that if $\big||\lambda_1| - |\lambda_2|\big| \leqslant \frac{1}{2}|\lambda_2|^{\frac{1}{2}}\big(|\lambda_1|^{\frac{1}{2}} + |\lambda_2|^{\frac{1}{2}}\big)$ and $(|\lambda_1| + |\lambda_2|)|n + m| + |\lambda_1| + |\lambda_2| + |Y| \leqslant R_0$ then

$$(3.5) \qquad \big|\widetilde{\mathcal{W}}(\widehat{w}_1, Y) - \widetilde{\mathcal{W}}(\widehat{w}_2, Y)\big| \leqslant C(R_0)|\lambda_1 - \lambda_2|\left(\frac{1}{|\lambda_1|} + \frac{1}{\lambda_1^2}\right).$$

That estimate fails if the above condition on $\big||\lambda_1| - |\lambda_2|\big|$ is not satisfied. To overcome that difficulty, we need the following lemma.

LEMMA 3.2. — *The series*

$$\sum_{\ell_1, \ell_2} (\operatorname{sgn}\lambda)^{\ell_1} |\lambda|^{\frac{\ell_1 + \ell_2}{2}} \frac{(2i\eta)^{\ell_1}(2y)^{\ell_2}}{\ell_1! \ell_2!} \big(M^{\ell_1} H_m | \partial^{\ell_2} H_n\big)_{L^2(\mathbb{R}^d)}$$

converges normally towards $\widetilde{\mathcal{W}}$ *on* $\mathcal{B}(R_0)$.

Proof. — Again, as Hermite functions in dimension d are tensor products of one-dimensional Hermite functions, it is enough to prove the above lemma in dimension 1. Now, using the expansion of the exponential function and Lebesgue theorem, we get that for any fixed $(\widehat{w}, Y)$ in $\widetilde{\mathbb{H}}^1 \times T^\star\mathbb{R}$,

$$\widetilde{\mathcal{W}}(\widehat{w}, Y) = \sum_{\ell_1=0}^{\infty} \frac{1}{\ell_1!} (2i\lambda\eta)^{\ell_1} \int_{\mathbb{R}} H_{n,\lambda}(2y+z) z^{\ell_1} H_{m,\lambda}(z) \mathrm{d}z$$

$$(3.6) \qquad = \sum_{\ell_1=0}^{\infty} (\operatorname{sgn}\lambda)^{\ell_1} \frac{(2i\eta)^{\ell_1}}{\ell_1!} |\lambda|^{\frac{\ell_1}{2}} \int_{\mathbb{R}} H_{n,\lambda}(2y+z)(M^{\ell_1}H_m)_\lambda(z) \mathrm{d}z.$$

Let us prove that the series converges for the supremum norm on $\mathcal{B}(R_0)$. Clearly, (A.4) implies that for all integers $\ell \geqslant 1$ and m in $\mathbb{N}$,

$$\|(\sqrt{2}M)^\ell H_m\|_{L^2(\mathbb{R})} \leqslant \sqrt{m}\,\|(\sqrt{2}M)^{\ell-1} H_{m-1}\|_{L^2(\mathbb{R})}$$
$$+ \sqrt{m+1}\,\|(\sqrt{2}M)^{\ell-1} H_{m+1}\|_{L^2(\mathbb{R})},$$

which, by an obvious induction yields for all (ℓ_1, m) in $\mathbb{N}^2$,

$$(3.7) \qquad \|M^{\ell_1} H_m\|_{L^2(\mathbb{R})} \leqslant (2m + 2\ell_1)^{\frac{\ell_1}{2}}.$$

Hence the generic term of the series of (3.6) can be bounded by:

$$W_{\ell_1}(\widehat{w}) \stackrel{\text{def}}{=} \frac{(2\sqrt{2}R_0)^{\ell_1}}{\ell_1!} |\lambda|^{\frac{\ell_1}{2}} (m + \ell_1)^{\frac{\ell_1}{2}}.$$

Let us observe that, because $|\lambda|m$ and $|\lambda|$ are less than R_0, we have

$$\frac{W_{\ell_1+1}(\widehat{w})}{W_{\ell_1}(\widehat{w})} = \frac{2\sqrt{2}R_0}{\ell_1 + 1} \sqrt{|\lambda|(m + \ell_1 + 1)} \left(1 + \frac{1}{m + \ell_1}\right)^{\frac{\ell_1}{2}}$$

$$\leqslant \frac{2\sqrt{2e}\,R_0}{\ell_1 + 1} \sqrt{R_0}\big(1 + \sqrt{\ell_1 + 1}\big).$$

This implies that the series converges with respect to the supremum norm on $\mathcal{B}(R_0)$.

Next, for fixed ℓ_1, we want to expand

$$|\lambda|^{\frac{\ell_1}{2}} \int_{\mathbb{R}} H_{n,\lambda}(2y+z)(M^{\ell_1}H_m)_\lambda(z)\mathrm{d}z$$

as a series with respect to the variable y. To this end, we just have to expand the real analytic Hermite functions as follows:

$$H_{n,\lambda}(z+2y) = \sum_{\ell_2=0}^{\infty} \frac{(2y)^{\ell_2}}{\ell_2!}|\lambda|^{\frac{\ell_2}{2}}(H_n^{(\ell_2)})_\lambda(z).$$

Then we have to study (for fixed ℓ_1) the convergence of the series with general term,

$$W_{\ell_1,\ell_2}(\widehat{w},Y) \stackrel{\mathrm{def}}{=} \frac{(2y)^{\ell_2}}{\ell_2!}|\lambda|^{\frac{\ell_2}{2}}\left(H_n^{(\ell_2)}|M^{\ell_1}H_m\right)_{L^2}.$$

Using again (A.4), we see that

$$\|H_n^{(\ell_2)}\|_{L^2(\mathbb{R})} \leqslant (2n+2\ell_2)^{\frac{\ell_2}{2}}.$$

Hence, arguing as above, we get for any $(\widehat{w},Y)$ in $\mathcal{B}(R_0)$,

$$W_{\ell_1,\ell_2}(\widehat{w},Y) \leqslant 2^{\frac{\ell_1}{2}}(m+\ell_1)^{\frac{\ell_1}{2}}\widetilde{W}_{\ell_2}(\widehat{w},Y)$$

$$\text{with}\quad \widetilde{W}_{\ell_2}(\widehat{w},Y) \stackrel{\mathrm{def}}{=} \frac{(2\sqrt{2}R_0)^{\ell_2}}{\ell_2!}|\lambda|^{\frac{\ell_2}{2}}(n+\ell_2)^{\frac{\ell_2}{2}},$$

and it is now easy to complete the proof of the lemma. $\qquad\square$

Reverting to the proof of the continuity of $\widetilde{\mathcal{W}}$ in the neighborhood of $\widehat{\mathbb{H}}_0^d$, the problem now consists in investigating the behavior of the function

$$\mathcal{H}_{\ell_1,\ell_2}: \begin{cases} \widetilde{\mathbb{H}}^1 \longrightarrow \mathbb{R} \\ \widehat{w} = (n,m,\lambda) \longmapsto |\lambda|^{\frac{\ell_1+\ell_2}{2}}\left(M^{\ell_1}H_m|H_n^{(\ell_2)}\right)_{L^2(\mathbb{R})} \end{cases}$$

when λ tends to 0 and $\lambda(n+m) \to \dot{x}$ for fixed $k \stackrel{\mathrm{def}}{=} m-n$.

From Relations (A.3), we infer that

$$\mathcal{H}_{\ell_1,\ell_2}(\widehat{w}) = 2^{-(\ell_1+\ell_2)}|\lambda|^{\frac{\ell_1+\ell_2}{2}}\left((A+C)^{\ell_1}H_m|(A-C)^{\ell_2}H_n\right)_{L^2(\mathbb{R})}.$$

The explicit computation of $(A\pm C)^\ell$ is doable but tedious and fortunately turns out to be useless when λ tends to 0. Indeed, we have the following lemma:

LEMMA 3.3. — *A constant $C_\ell(R_0)$ (depending only on R_0 and ℓ) exists such that, for any (n, λ) with $\lambda > 0$ and $\lambda n \leqslant R_0$, we have (with the convention that $H_p = 0$ if $p < 0$):*

$$\left\| \lambda^{\frac{\ell}{2}} \left(\frac{A \pm C}{2} \right)^\ell H_n - \left(\frac{\lambda n}{2} \right)^{\frac{\ell}{2}} \sum_{\ell'=0}^{\ell} (\pm 1)^{\ell - \ell'} \binom{\ell}{\ell'} H_{n+\ell-2\ell'} \right\|_{L^2(\mathbb{R})} \leqslant C_\ell(R_0) \lambda^{\frac{1}{2}}.$$

Proof. — Let $\mathcal{V}_{n,\ell}$ be the vector space generated by $(H_{n+\ell'})_{-\ell \leqslant \ell' \leqslant \ell}$, equipped with the $L^2(\mathbb{R})$-norm. Let

$$R_{n,\ell} \stackrel{\text{def}}{=} \lambda^{\frac{\ell}{2}} (A \pm C)^\ell H_n - (2\lambda n)^{\frac{\ell}{2}} \sum_{\ell'=0}^{\ell} (\pm 1)^{\ell - \ell'} \binom{\ell}{\ell'} H_{n+\ell-2\ell'}.$$

Formulae (A.2) guarantee that $R_{n,\ell}$ is in $\mathcal{V}_{n,\ell}$. Let us now prove by induction on ℓ that

$$(3.8) \qquad \|R_{n,\ell}\|_{\mathcal{V}_{n,\ell}} \leqslant C_\ell(R_0) \lambda^{\frac{1}{2}}.$$

In the case when ℓ equals 1, by definition of A and C, we have

$$\lambda^{\frac{1}{2}} (A \pm C) H_n = \lambda^{\frac{1}{2}} \left(\sqrt{2n} H_{n-1} \pm \sqrt{2n+2} H_{n+1} \right)$$

$$= \sqrt{2\lambda n} (H_{n-1} \pm H_{n+1}) \pm \frac{2\sqrt{\lambda}}{\sqrt{2n+2} + \sqrt{2n}} H_{n+1}$$

and (3.8) is thus obvious.

Let us now observe that, for any ℓ' in $\{-\ell, \ldots, \ell\}$, we have

$$\lambda^{\frac{1}{2}} \|A H_{n+\ell'}\|_{L^2(\mathbb{R})} = \sqrt{2\lambda(n+\ell')} \, \|H_{n+\ell'-1}\|_{L^2(\mathbb{R})} \quad \text{and}$$

$$\lambda^{\frac{1}{2}} \|C H_{n+\ell'}\|_{L^2(\mathbb{R})} = \sqrt{2\lambda(n+\ell'+1)} \, \|H_{n+\ell'+1}\|_{L^2(\mathbb{R})}.$$

This gives that for all $\lambda(n+1) \leqslant R_0$,

$$(3.9) \qquad \left\| \lambda^{\frac{1}{2}} (A \pm C) \right\|_{\mathcal{L}(\mathcal{V}_{n,\ell}; \mathcal{V}_{n,\ell+1})} \leqslant C_\ell(R_0).$$

Let us assume that (3.8) holds for some ℓ. Inequality (3.9) implies that

$$(3.10) \qquad \left\| \lambda^{\frac{1}{2}} (A \pm C) R_{n,\ell} \right\|_{\mathcal{V}_{n,\ell+1}} \leqslant \lambda^{\frac{1}{2}} C_\ell(R_0).$$

Then, for any ℓ' in $\{0, \ldots, \ell\}$, we have

$$
\begin{aligned}
\lambda^{\frac{1}{2}}(A \pm C) H_{n+\ell-2\ell'} &= \lambda^{\frac{1}{2}}\left(\sqrt{2n + 2\ell - 4\ell'}\, H_{n+\ell-2\ell'-1}\right. \\
&\quad \left. \pm \sqrt{2n + 2\ell - 4\ell' + 2}\, H_{n+\ell-2\ell'+1}\right) \\
&= \sqrt{2\lambda n}\left(H_{n+\ell+1-2(\ell'+1)} \pm H_{n+\ell+1-2\ell'}\right) \\
&\quad + \frac{2\lambda^{\frac{1}{2}}(\ell - 2\ell')}{\sqrt{2n + 2\ell - 4\ell'} + \sqrt{2n}} H_{n+\ell-2\ell'-1} \\
&\quad \pm \frac{2\lambda^{\frac{1}{2}}(\ell - 2\ell' + 1)}{\sqrt{2n + 2\ell - 4\ell' + 2} + \sqrt{2n}} H_{n+\ell-2\ell'+1}.
\end{aligned}
$$

We deduce that for any ℓ' in $\{0, \ldots, \ell\}$,

$$
\left\| \lambda^{\frac{1}{2}}(A \pm C) H_{n+\ell-2\ell'} - \sqrt{2\lambda n}\left(H_{n+\ell+1-2(\ell'+1)} \pm H_{n+\ell+1-2\ell'}\right) \right\|_{\mathcal{V}_{n,\ell+1}} \\
\leqslant C_{\ell+1}(R_0)\lambda^{\frac{1}{2}}.
$$

Using (3.10) gives

$$
\left\| \lambda^{\frac{\ell+1}{2}}(A \pm C)^{\ell+1} H_n - (2\lambda n)^{\frac{\ell+1}{2}} \Sigma_{n,\ell} \right\|_{L^2(\mathbb{R})} \leqslant C_{\ell+1}(R_0)\lambda^{\frac{1}{2}}
$$

$$
\text{with} \quad \Sigma_{n,\ell} \overset{\text{def}}{=} \sum_{\ell'=0}^{\ell} (\pm 1)^{\ell-\ell'} \binom{\ell}{\ell'} \left(H_{n+\ell+1-2(\ell'+1)} \pm H_{n+\ell+1-2\ell'}\right).
$$

Now, Pascal's rule ensures that

$$
\begin{aligned}
\Sigma_{n,\ell} &= \sum_{\ell'=1}^{\ell+1} (\pm 1)^{\ell+1-\ell'} \binom{\ell}{\ell'-1} H_{n+\ell+1-2\ell'} + \sum_{\ell'=0}^{\ell} (\pm 1)^{\ell+1-\ell'} \binom{\ell}{\ell'} H_{n+\ell+1-2\ell'} \\
&= \sum_{\ell'=0}^{\ell+1} (\pm 1)^{\ell+1-\ell'} \binom{\ell+1}{\ell'} H_{n+\ell+1-2\ell'}.
\end{aligned}
$$

The lemma is proved. $\qquad\qquad\square$

From this lemma, we can deduce the following corollary.

COROLLARY 3.4. — For any (ℓ_1, ℓ_2) in $\mathbb{N}^2$ and $R_0 > 0$, there exists a constant $C_{\ell_1, \ell_2}(R_0)$ such that for all $(n, n+k, \lambda)$ in $\widetilde{\mathbb{H}}^1$ with $|\lambda n| + |k| + |\lambda| \leqslant R_0$, we have

$$
\left| \mathcal{H}_{\ell_1, \ell_2}(\widehat{w}) - F_{\ell_1, \ell_2}(k)\left(\frac{|\lambda| n}{2}\right)^{\frac{\ell_1 + \ell_2}{2}} \right| \leqslant C_{\ell_1, \ell_2}(R_0)|\lambda|^{\frac{1}{2}}
$$

$$
\text{with} \quad F_{\ell_1, \ell_2}(k) \overset{\text{def}}{=} \sum_{\substack{\ell_1' \leqslant \ell_1, \ell_2' \leqslant \ell_2 \\ k+\ell_1-2\ell_1' = \ell_2 - 2\ell_2'}} (-1)^{\ell_2 - \ell_2'} \binom{\ell_1}{\ell_1'} \binom{\ell_2}{\ell_2'}.
$$

Proof. — Lemma 3.3 implies that

$$\left| \mathcal{H}_{\ell_1,\ell_2}(\widehat{w}) - \left(\frac{|\lambda|n}{2}\right)^{\frac{\ell_2}{2}} \left(\frac{|\lambda|(n+k)}{2}\right)^{\frac{\ell_1}{2}} \right.$$

$$\left. \times \sum_{\substack{\ell_1' \leqslant \ell_1 \\ \ell_2' \leqslant \ell_2}} (-1)^{\ell_2-\ell_2'} \binom{\ell_1}{\ell_1'} \binom{\ell_2}{\ell_2'} \left(H_{n+k+\ell_1-2\ell_1'} | H_{n+\ell_2-2\ell_2'}\right)_{L^2} \right|$$

$$\leqslant C_{\ell_1,\ell_2}(R_0)|\lambda|^{\frac{1}{2}}.$$

Now, let us notice that

$$(|\lambda|(n+k))^{\frac{\ell_1}{2}} - (|\lambda|n)^{\frac{\ell_1}{2}}$$

$$= \frac{|\lambda|k}{\sqrt{|\lambda|n} + \sqrt{|\lambda|(n+k)}} \sum_{\ell_1'=0}^{\ell_1-1} \sqrt{|\lambda|n}^{\ell_1'} \sqrt{|\lambda|(n+k)}^{\ell_1-1-\ell_1'}.$$

Hence it is clear that for fixed k in $\mathbb{Z}$ such that $|k| \leqslant R_0$, we have, for $|\lambda| \leqslant R_0$ and $|n\lambda| \leqslant R_0$,

$$\left| (|\lambda|n)^{\frac{\ell_2}{2}} (|\lambda|(n+k))^{\frac{\ell_1}{2}} - |\lambda n|^{\frac{\ell_1+\ell_2}{2}} \right| \leqslant C_{\ell_1,\ell_2}(R_0)|\lambda|^{\frac{1}{2}}.$$

Thanks to (2.7), we conclude the proof. $\qquad\square$

Conclusion of the proof of Proposition 3.1. — Consider a positive real number ε. Recall that

$$\widetilde{\mathcal{W}}(\widehat{w}, Y) = \sum_{\ell_1,\ell_2} (\operatorname{sgn}\lambda)^{\ell_1} \frac{(2i\eta)^{\ell_1}(2y)^{\ell_2}}{\ell_1!\ell_2!} \mathcal{H}_{\ell_1,\ell_2}(\widehat{w}).$$

Clearly, it suffices to prove the uniform continuity of $\widetilde{\mathcal{W}}$ for all subset of $\widehat{\mathbb{H}}^d$ corresponding to some *fixed* value k of $m - n$. Now, considering $\widehat{w}_1 = (n_1, n_1 + k, \lambda_1)$ and $\widehat{w}_2 = (n_2, n_2 + k, \lambda_2)$, Lemma 3.2 implies that for all $\varepsilon > 0$, there exist two integers $L_{1,\varepsilon}$ and $L_{2,\varepsilon}$ such that

$$\left| \widetilde{\mathcal{W}}(\widehat{w}_1, Y) - \widetilde{\mathcal{W}}(\widehat{w}_2, Y) \right| \leqslant \frac{\varepsilon}{4} + \sum_{\substack{\ell_1 \leqslant L_{1,\varepsilon} \\ \ell_2 \leqslant L_{2,\varepsilon}}} \frac{(2|\eta|)^{\ell_1}(2|y|)^{\ell_2}}{\ell_1!\ell_2!}$$

$$\times \left| (\operatorname{sgn}\lambda_1)^{\ell_1} \mathcal{H}_{\ell_1,\ell_2}(n_1, n_1 + k, \lambda_1) - (\operatorname{sgn}\lambda_2)^{\ell_1} \mathcal{H}_{\ell_1,\ell_2}(n_2, n_2 + k, \lambda_2) \right|.$$

Let $C_\varepsilon(R_0)$ be the supremum for $\ell_1 \leqslant L_{1,\varepsilon}$ and $\ell_2 \leqslant L_{2,\varepsilon}$ of all constants $C_{\ell_1,\ell_2}(R_0)$ which appear in Corollary 3.4. Then $|\lambda_1| + |\lambda_2| \leqslant A(\varepsilon, R_0)$

implies that

$$(3.11) \quad \left|\widetilde{\mathcal{W}}(\widehat{w}_1, Y) - \widetilde{\mathcal{W}}(\widehat{w}_2, Y)\right| \leqslant \frac{\varepsilon}{2} + \sum_{\substack{\ell_1 \leqslant L_{1,\varepsilon} \\ \ell_2 \leqslant L_{2,\varepsilon}}} \frac{(2\,R_0)^{\ell_1+\ell_2}}{\ell_1! \ell_2!} |F_{\ell_1,\ell_2}(k)|$$

$$\times \left| (\operatorname{sgn} \lambda_1)^{\ell_1} \left|\frac{\lambda_1 n_1}{2}\right|^{\frac{\ell_1+\ell_2}{2}} - (\operatorname{sgn} \lambda_2)^{\ell_1} \left|\frac{\lambda_2 n_2}{2}\right|^{\frac{\ell_1+\ell_2}{2}} \right|$$

$$\text{with} \quad A(\varepsilon, R_0) \stackrel{\text{def}}{=} \frac{\mathrm{e}^{-8R_0}\,\varepsilon^2}{32 C_\varepsilon^2(R_0)}.$$

If $\ell_1 + \ell_2 = 0$ then the last term of the above inequality is 0. If $\ell_1 + \ell_2$ is positive, as $|F_{\ell_1,\ell_2}(k)|$ is less than $2^{\ell_1+\ell_2}$, we have, using (3.11),

$$(3.12) \quad |\lambda_1| + |\lambda_2| \leqslant A(\varepsilon, R_0) \quad \text{and} \quad |\lambda_1 n_1| + |\lambda_2 n_2| \leqslant \frac{1}{16}\varepsilon^2\,\mathrm{e}^{-8R_0}$$

$$\Longrightarrow \left|\widetilde{\mathcal{W}}(\widehat{w}_1, Y) - \widetilde{\mathcal{W}}(\widehat{w}_2, Y)\right| \leqslant \varepsilon.$$

In the case when $|\lambda_1 n_1| + |\lambda_2 n_2|$ is greater than $\frac{1}{16}\varepsilon^2\,\mathrm{e}^{-8R_0}$ then if

$$\left|\lambda_1 n_1 - \lambda_2 n_2\right| \leqslant \frac{1}{32}\varepsilon^2\,\mathrm{e}^{-8R_0}$$

then λ_1 and λ_2 have the same sign. The sum in the right-hand side term is finite, and it is clear that each term converges uniformly to 0 if $\lambda_2 n_2$ tends to $\lambda_1 n_1$. Thus a positive real number η_ε exists such that

$$(3.13) \quad |\lambda_1| + |\lambda_2| \leqslant A(\varepsilon, R_0) \quad \text{and} \quad |\lambda_1 n_1 - \lambda_2 n_2| \leqslant \eta_\varepsilon$$

$$\Longrightarrow \left|\widetilde{\mathcal{W}}(\widehat{w}_1, Y) - \widetilde{\mathcal{W}}(\widehat{w}_2, Y)\right| \leqslant \varepsilon.$$

Finally, we have to consider the case where $|\lambda_1| + |\lambda_2| \geqslant A(\varepsilon, R_0)$. With no loss of generality, one can assume that $\lambda_2 \geqslant \frac{1}{2}A(\varepsilon, R_0)$. Thus, if $|\lambda_1 - \lambda_2|$ is less than $\frac{1}{4}A(\varepsilon, R_0)$ we have $\lambda_1 \geqslant \frac{1}{4}A(\varepsilon, R_0)$ and we can apply Inequality (3.5) which gives (supposing that $A(\varepsilon, R_0) \leqslant 1$):

$$\left|\widetilde{\mathcal{W}}(\widehat{w}_1, Y) - \widetilde{\mathcal{W}}(\widehat{w}_2, Y)\right| \leqslant 2C(R_0) \left(\frac{1}{4}A(\varepsilon, R_0)\right)^{-2} |\lambda_1 - \lambda_2|.$$

Together with (3.12), this gives, if (n_j, m_j, λ_j) are in $\mathcal{B}(R_0)$,

$$|\lambda_1 - \lambda_2| \leqslant \frac{\varepsilon A^2(\varepsilon, R_0)}{32 C(R_0)} \quad \text{and} \quad |\lambda_1 n_1 - \lambda_2 n_2| < \eta_\varepsilon$$

$$\Longrightarrow \left|\widetilde{\mathcal{W}}(\widehat{w}_1, Y) - \widetilde{\mathcal{W}}(\widehat{w}_2, Y)\right| < \varepsilon.$$

The proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

End of the proof of the first part of Theorem 2.7. — Because of the integrability of f, Proposition 3.1 implies that $\widehat{f}_{\mathbb{H}}$ is uniformly continuous on $\widetilde{\mathbb{H}}^d$, and can thus be extended to a uniformly continuous function on the complete metric space $\widehat{\mathbb{H}}^d$.

Let us finally establish that $\widehat{f}_{\mathbb{H}}(\widehat{w}) \to 0$ when $\widehat{w}$ goes to infinity. In the case where f is in $\mathcal{S}(\mathbb{H}^d)$, this in an obvious consequence of Lemma 2.3. The general case of an integrable function on $\mathbb{H}^d$ follows by density as, obviously, Formula (2.10) implies that the map $f \to \widehat{f}_{\mathbb{H}}$ is continuous from $L^1(\mathbb{H}^d)$ to $L^\infty(\widehat{\mathbb{H}}^d)$. $\qquad\square$

We are now ready to establish Formula (2.21) for any integrable function f on $\mathbb{H}^d$. So let us fix some $(\dot{x}, k)$ in $\widehat{\mathbb{H}}_0^d$, and consider a sequence $(\widehat{w}_p)_{p \in \mathbb{N}} = (n_p, n_p + k, \lambda_p)_{p \in \mathbb{N}}$ such that

$$\lim_{p \to \infty} \widehat{w}_p = (\dot{x}, k) \text{ in the sense of } \widehat{d}.$$

According to Proposition 3.1, if we set

$$\mathcal{K}_d(\dot{x}, k, Y) \stackrel{\text{def}}{=} \lim_{p \to \infty} \mathcal{W}(\widehat{w}_p, Y)$$

then the definition of $\widehat{f}_{\mathbb{H}}$ on $\widetilde{\mathbb{H}}^d$ and the Lebesgue dominated convergence theorem imply that

$$\widehat{f}_{\mathbb{H}}(\dot{x}, k) = \lim_{p \to \infty} \widehat{f}_{\mathbb{H}}(\widehat{w}_p) = \int_{\mathbb{H}^d} \overline{\mathcal{K}}_d(\dot{x}, k, Y) f(Y, s) \mathrm{d}Y \mathrm{d}s.$$

Now, Lemma 3.2 gives

$$\mathcal{K}_d(\dot{x}, k, Y) = \sum_{\ell_1, \ell_2} \frac{(2i\eta)^{\ell_1}}{\ell_1!} \frac{(2y)^{\ell_2}}{\ell_2!} \lim_{p \to \infty} \mathcal{H}_{\ell_1, \ell_2}(\widehat{w}_p)(\operatorname{sgn} \lambda_p)^{\ell_1}$$

$$\text{with} \quad \mathcal{H}_{\ell_1, \ell_2}(\widehat{w}) \stackrel{\text{def}}{=} |\lambda|^{\frac{|\ell_1 + \ell_2|}{2}} \left(M^{\ell_1} H_m | \partial^{\ell_2} H_n \right)_{L^2(\mathbb{R}^d)}.$$

If $d = 1$ then Corollary 3.4 implies that

$$\lim_{p \to \infty} \mathcal{H}_{\ell_1, \ell_2}(\widehat{w}_p) = F_{\ell_1, \ell_2}(k) \left(\frac{|\dot{x}|}{4} \right)^{\frac{\ell_1 + \ell_2}{2}}$$

and, because $\operatorname{sgn}(\lambda_p) = \operatorname{sgn} \dot{x}$ for large enough p, this guarantees (2.21) and Formula (3.1).

Once again, as in general dimension $d \geqslant 1$ the term $\mathcal{H}_{\ell_1, \ell_2}$ may be written as the product of d terms involving only one-dimensional Hermite functions, the above formula still holds true (with the notation convention given in Proposition 3.1 of course).

This concludes the proof of the first part of Theorem 2.7 and of Identity (2.21). $\qquad\square$

Remark 3.5. — Computing $\mathcal{K}_d$ will be carried out later on, in Section 5. For the time being, let us just underline that the expression of $F_{\ell_1,\ell_2}(k)$ which appears in (3.1) ensures that $F_{0,0}(k) = \delta_{0,k}$. We thus have

$$(3.14) \qquad \mathcal{K}_d(\dot{x}, k, 0) = \mathcal{K}_d(0, k, Y) = F_{0,0}(k) = \delta_{0,k}.$$

Let us also notice that, denoting by $\widehat{0}$ the point $(0,0)$ of $\widehat{\mathbb{H}}_0^d$, we recover the following property:

$$(3.15) \qquad \widehat{f_{\mathbb{H}}}(\widehat{0}) = \int_{\mathbb{H}^d} f(w)\mathrm{d}w.$$

4. The case of functions that do not depend on the vertical variable

The purpose of this section is to prove Theorem 2.8. As already pointed out in the introduction, a key issue is to study the limit (in the sense of weak convergence of measures) of functions which concentrate near the set $\widehat{\mathbb{H}}_0^d$. This is the aim of the following lemma.

LEMMA 4.1. — *Let $\widehat{\chi} : \mathbb{R} \to \mathbb{R}$ be integrable, compactly supported and with integral 1. Then for any continuous function θ from $\widetilde{\mathbb{H}}^d$ to $\mathbb{C}$ satisfying*

$$(4.1) \qquad \sup_{(n,m,\lambda)\in\widetilde{\mathbb{H}}^d} \left(1 + |\lambda|(|n+m| + d) + |n - m|\right)^{2d+1} |\theta(n, m, \lambda)| < \infty,$$

we have

$$\lim_{\varepsilon\to 0} \int_{\widehat{\mathbb{H}}^d} \varepsilon^{-1} \widehat{\chi}(\varepsilon^{-1}\lambda)\theta(n, m, \lambda)\mathrm{d}\widehat{w} = \langle \mu_{\widehat{\mathbb{H}}_0^d}, \theta \rangle$$

where the measure in the right-hand side has been defined in (2.24).

Proof. — Let us first prove the result if the function θ is supported in the closure of

$$\mathcal{B}_K \overset{\text{def}}{=} \left\{(n, m, \lambda) \in \widetilde{\mathbb{H}}^d \ : \ |\lambda|(2|n| + d) \leqslant K \ \text{ and } \ |m - n| \leqslant K\right\}$$

for some positive K. Then we have

$$\mathcal{I}_\varepsilon \overset{\text{def}}{=} \int_{\widehat{\mathbb{H}}^d} \varepsilon^{-1} \widehat{\chi}(\varepsilon^{-1}\lambda)\theta(n, m, \lambda)\mathrm{d}\widehat{w} = \sum_{|k|\leqslant K} \left(\mathcal{I}_\varepsilon^-(k) + \mathcal{I}_\varepsilon^+(k)\right) \quad \text{with}$$

$$\mathcal{I}_\varepsilon^\pm(k) \overset{\text{def}}{=} \int_{\mathbb{R}_\pm} \varepsilon^{-1}\widehat{\chi}(\varepsilon^{-1}\lambda)\left(\sum_{n\in\mathbb{N}^d} \theta(n, n + k, \lambda)\right)|\lambda|^d\mathrm{d}\lambda.$$

Above, we agreed that $\theta(n, n + k, \lambda) = 0$ whenever at least one component of $n + k$ is negative. Then the idea is to use Riemann type sums. More

concretely, for all n in $\mathbb{N}^d$ and λ in $\mathbb{R} \setminus \{0\}$, let us define the family of cubes $Q_{n,\lambda} \stackrel{\text{def}}{=} 2\lambda n + 2\lambda[0,1[^d$. It is obvious that

$$(4.2) \qquad \text{Vol}(Q_{n,\lambda}) = (2|\lambda|)^d \quad \text{and} \quad \sum_{n \in \mathbb{N}^d} \mathbf{1}_{Q_{n,\lambda}} = 1 \quad \text{on} \ (\mathbb{R}_{\text{sgn}\,\lambda})^d.$$

From the volume property and the definition of $\mathcal{I}_\varepsilon^+(k)$, we readily get

$$\mathcal{I}_\varepsilon^+(k) = 2^{-d} \int_{\mathbb{R}} \int_{(\mathbb{R}_+)^d} \sum_{n \in \mathbb{N}^d} \varepsilon^{-1} \widehat{\chi}(\varepsilon^{-1}\lambda) \theta(n, n+k, \lambda) \mathbf{1}_{Q_{n,\lambda}}(\dot{x}) \mathrm{d}\dot{x} \mathrm{d}\lambda.$$

Let us write that

$$2^d \mathcal{I}_\varepsilon^+(k) = \int_{\mathbb{R}} \int_{(\mathbb{R}_+)^d} \sum_{n \in \mathbb{N}^d} \varepsilon^{-1} \widehat{\chi}(\varepsilon^{-1}\lambda) \theta(\dot{x}, k) \mathbf{1}_{Q_{n,\lambda}}(\dot{x}) \mathrm{d}\dot{x} \mathrm{d}\lambda$$

$$+ \int_{\mathbb{R}} \int_{(\mathbb{R}_+)^d} \varepsilon^{-1} \widehat{\chi}(\varepsilon^{-1}\lambda) \sum_{n \in \mathbb{N}^d} \big(\theta(n, n+k, \lambda) - \theta(\dot{x}, k)\big) \mathbf{1}_{Q_{n,\lambda}}(\dot{x}) \mathrm{d}\dot{x} \mathrm{d}\lambda.$$

Using the second property of (4.2), the fact that $\widehat{\chi}$ is of integral 1, and that the summation may be restricted to those indices n in $\mathbb{N}^d$ such that $|\lambda n| \leqslant K$ (because θ is supported in $\mathcal{B}_K$), we end up with

$$2^d \mathcal{I}_\varepsilon^+(k) - \int_{(\mathbb{R}_+)^d} \theta(\dot{x}, k) \mathrm{d}\dot{x}$$

$$= \int_{\mathbb{R}} \int_{(\mathbb{R}_+)^d} \varepsilon^{-1} \widehat{\chi}(\varepsilon^{-1}\lambda) \sum_{|n\lambda| \leqslant K} \big(\theta(n, n+k, \lambda) - \theta(\dot{x}, k)\big) \mathbf{1}_{Q_{n,\lambda}}(\dot{x}) \mathrm{d}\dot{x} \mathrm{d}\lambda.$$

As θ is uniformly continuous on $\widehat{\mathbb{H}}^d$ (being compactly supported), we have

$$\forall\, \eta > 0, \ \exists\, \varepsilon > 0, \ |2\lambda n - \dot{x}| + |\lambda| < \varepsilon \Longrightarrow \big|\theta(n, n+k, \lambda) - \theta(\dot{x}, k)\big| < \eta.$$

One can thus conclude that for any $\eta > 0$, if ε is small enough then we have

$$\left| 2^d \mathcal{I}_\varepsilon^+(k) - \int_{(\mathbb{R}_+)^d} \theta(\dot{x}, k) \mathrm{d}\dot{x} \right|$$

$$\leqslant \eta \int_{\mathbb{R}} \varepsilon^{-1} \widehat{\chi}(\varepsilon^{-1}\lambda) \bigg(\sum_{|n\lambda| \leqslant K} \int_{(\mathbb{R}_+)^d} \mathbf{1}_{Q_{n,\lambda}}(\dot{x}) \mathrm{d}\dot{x} \bigg) \mathrm{d}\lambda.$$

Using once again that the measure of $Q_{n,\lambda}$ is $(2|\lambda|)^d$ and noting that the set of indices n in $\mathbb{N}^d$ for which $|n\lambda| \leqslant K$ is bounded by $C_d K^d |\lambda|^{-d}$ for some constant C_d depending only on d, we conclude that for small enough ε, we have

$$(4.3) \qquad \left| \mathcal{I}_\varepsilon^+(k) - 2^{-d} \int_{(\mathbb{R}_+)^d} \theta(\dot{x}, k) \mathrm{d}\dot{x} \right| \leqslant C_d \eta K^d.$$

Of course, handling $\mathcal{I}_\varepsilon^-(k)$ is strictly similar. Because the set of k in $\mathbb{Z}^d$ with $|k| \leqslant K$ is finite (and independent of ε), this proves the lemma in the case where θ is compactly supported.

To handle the general case, one may fix some cut-off function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with value 1 on $[0,1]$ and supported in $[0,2]$, and, for all $K > 0$, decompose θ into

$$\theta = \theta_K + \theta^K \quad \text{with} \quad \theta_K(\widehat{w}) \overset{\text{def}}{=} \psi(K^{-1}(|\lambda|(2|n| + d) + |m - n|))\theta(\widehat{w}).$$

The first part of the proof applies to θ_K and for all positive real number η, one may thus find some $\varepsilon_{K,\eta}$ so that

$$(4.4) \qquad \left| \int_{\widehat{\mathbb{H}}^d} \varepsilon^{-1} \widehat{\chi}(\varepsilon^{-1}\lambda) \theta_K(\widehat{w}) \mathrm{d}\widehat{w} - \langle \mu_{\widehat{\mathbb{H}}_0^d}, \theta_K \rangle \right| \leqslant \eta \ \text{ for } \ \varepsilon < \varepsilon_{K,\eta}.$$

To bound the term corresponding to θ^K, we shall use the fact that Condition (4.1) ensures that there exists some constant C so that

$$(4.5) \qquad \forall \, (\dot{x}, k) \in \widehat{\mathbb{H}}_0^d, \ |\theta(\dot{x}, k)| \leqslant C(1 + |\dot{x}| + |k|)^{-2d-1}.$$

Now, we have, denoting $\mathbb{R}_{\mp}^d \overset{\text{def}}{=} (\mathbb{R}_-)^d \cup (\mathbb{R}_+)^d$,

$$\int_{\widehat{\mathbb{H}}_0^d} |\theta^K(\dot{x}, k)| \mathrm{d}\mu_{\widehat{\mathbb{H}}_0^d}(\dot{x}, k)$$

$$\leqslant 2^{-d} \left(\sum_{|k| \geqslant K} \int_{\mathbb{R}_{\mp}^d} |\theta(\dot{x}, k)| \mathrm{d}\dot{x} + \sum_{k \in \mathbb{Z}^d} \int_{|\dot{x}| \geqslant K} |\theta(\dot{x}, k)| \mathrm{d}\dot{x} \right).$$

In light of (4.5) and making an obvious change of variables, we get

$$\sum_{|k| \geqslant K} \int_{\mathbb{R}_{\mp}^d} |\theta(\dot{x}, k)| \mathrm{d}\dot{x} \leqslant C \sum_{|k| \geqslant K} \int_{\mathbb{R}_+^d} (1 + |\dot{x}| + |k|)^{-2d-1} \mathrm{d}\dot{x}$$

$$\leqslant C \sum_{|k| \geqslant K} (1 + |k|)^{-d-1} \int_{\mathbb{R}_+^d} (1 + |\dot{y}|)^{-2d-1} \mathrm{d}\dot{y}$$

$$\leqslant CK^{-1}.$$

Likewise,

$$\sum_{k\in\mathbb{Z}^d}\int_{|\dot{x}|\geqslant K}|\theta(\dot{x},k)|\mathrm{d}\dot{x} \leqslant C\sum_{k\in\mathbb{Z}^d}\int_{|\dot{x}|\geqslant K}(1+|\dot{x}|+|k|)^{-2d-1}\mathrm{d}\dot{x}$$

$$\leqslant C\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|k|)^{d+1}}\int_{|\dot{y}|>\frac{K}{1+|k|}}\frac{\mathrm{d}\dot{y}}{(1+|\dot{y}|)^{2d+1}}$$

$$\leqslant C\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|k|)^{d+1}}\frac{1}{(1+K/(1+|k|))^{d+1}}$$

$$\leqslant CK^{-1}.$$

Therefore, if we take K large enough then one may ensure that

(4.6) $$\left|\langle\mu_{\widehat{\mathbb{H}}^d_0},\theta^K\rangle\right| \leqslant \eta.$$

Finally,

$$\left|\int_{\widehat{\mathbb{H}}^d}\varepsilon^{-1}\widehat{\chi}(\varepsilon^{-1}\lambda)\theta^K(\widehat{w})\mathrm{d}\widehat{w}\right| \leqslant \mathcal{J}^1_K(\varepsilon)+\mathcal{J}^2_K(\varepsilon) \quad\text{with}$$

$$\mathcal{J}^1_K(\varepsilon)\overset{\text{def}}{=}\varepsilon^{-1}\int_{\mathbb{R}}\sum_{|k|\geqslant K}\sum_{n\in\mathbb{N}^d}\widehat{\chi}(\varepsilon^{-1}\lambda)|\theta(\widehat{w})|\,|\lambda|^d\mathrm{d}\lambda$$

$$\text{and}\quad \mathcal{J}^2_K(\varepsilon)\overset{\text{def}}{=}\varepsilon^{-1}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}^d}\sum_{|n\lambda|\geqslant K}\widehat{\chi}(\varepsilon^{-1}\lambda)|\theta(\widehat{w})|\,|\lambda|^d\mathrm{d}\lambda.$$

Because θ satisfies (4.1), we have

$$\mathcal{J}^1_K(\varepsilon)\leqslant C\sum_{|k|\geqslant K}\sum_{n\in\mathbb{N}^d}\int_{\mathbb{R}}\varepsilon^{-1}\widehat{\chi}(\varepsilon^{-1}\lambda)(1+|k|+|\lambda n|)^{-2d-1}\,|\lambda|^d\mathrm{d}\lambda.$$

Clearly, because the sum below has $\mathcal{O}(|k|/|\lambda|)^d$ terms, we may write

$$\int_{\mathbb{R}}\varepsilon^{-1}\widehat{\chi}(\varepsilon^{-1}\lambda)\sum_{|n\lambda|\leqslant|k|}(1+|k|+|\lambda n|)^{-2d-1}\,|\lambda|^d\mathrm{d}\lambda \leqslant C(1+|k|)^{-d-1}$$

and, similarly, because

$$\sum_{|n\lambda|\geqslant|k|}|\lambda|^d(1+|k|+|\lambda n|)^{-2d-1} \leqslant C\sum_{|n\lambda|\geqslant|k|}|\lambda|^d(1+|\lambda n|)^{-2d-1}$$

$$\leqslant C(1+|k|)^{-d-1},$$

we get

$$\int_{\mathbb{R}}\varepsilon^{-1}\widehat{\chi}(\varepsilon^{-1}\lambda)\sum_{|n\lambda|\geqslant|k|}(1+|k|+|\lambda n|)^{-2d-1}\,|\lambda|^d\mathrm{d}\lambda \leqslant C(1+|k|)^{-d-1}.$$

Therefore
$$\mathcal{J}_K^1(\varepsilon) \leqslant CK^{-1}.$$
Proving that $\mathcal{J}_K^2(\varepsilon) \leqslant CK^{-1}$ relies on similar arguments. Putting together with (4.4) and (4.6), it is now easy to conclude the proof of the lemma. □

Proof of Theorem 2.8. — Let χ in $\mathcal{S}(\mathbb{R})$ have a compactly supported Fourier transform, and value 1 at 0 (hence the integral of $\widehat{\chi}$ is 2π). Let $\theta :$ $\widehat{\mathbb{H}}^d \to \mathbb{C}$ be continuous and compactly supported, and set

$$\mathcal{I}_\varepsilon(g, \theta) \overset{\text{def}}{=} \langle \mathcal{F}_\mathbb{H}(g \otimes \chi(\varepsilon \cdot)), \theta \rangle.$$

By definition of the Fourier transform of L^1 functions, one may write:

$$\mathcal{I}_\varepsilon(g, \theta) = \int_{\mathbb{H}^d \times \widehat{\mathbb{H}}^d} \mathrm{e}^{-\mathrm{i}s\lambda} \chi(\varepsilon s) \overline{\mathcal{W}}(\widehat{w}, Y) g(Y) \theta(\widehat{w}) \mathrm{d}Y \mathrm{d}s \mathrm{d}\widehat{w}$$

$$= \int_{\widehat{\mathbb{H}}^d} \frac{1}{\varepsilon} \widehat{\chi}\left(\frac{\lambda}{\varepsilon}\right) G(\widehat{w}) \theta(\widehat{w}) \mathrm{d}\widehat{w}$$

$$\text{with} \quad G(\widehat{w}) \overset{\text{def}}{=} \int_{T^\star \mathbb{R}^d} \overline{\mathcal{W}}(\widehat{w}, Y) g(Y) \mathrm{d}Y.$$

As the function g is integrable on $T^\star \mathbb{R}^d$, Proposition 3.1 implies that the (numerical) product $G\theta$ is a continuous compactly supported function on $\widehat{\mathbb{H}}^d$. Lemma 4.1 applied to this function $G\theta$ implies that

$$\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon(g, \theta) = 2\pi \int_{\widehat{\mathbb{H}}_0^d} \mathcal{G}_\mathbb{H} g(\dot{x}, k) \theta(\dot{x}, k) \mathrm{d}\mu_{\widehat{\mathbb{H}}_0^d}(\dot{x}, k).$$

In other words, the measure $\mathcal{F}_\mathbb{H}(g \otimes \chi(\varepsilon \cdot)) \mathrm{d}\widehat{w}$ converges weakly to the measure $2\pi(\mathcal{G}_\mathbb{H} g) \mathrm{d}\mu_{\widehat{\mathbb{H}}_0^d}$ which is exactly Theorem 2.8. □

5. Computing the kernel $\mathcal{K}$

We have already seen in Remark 3.5 that $\mathcal{K}_d(0, k, Y) = \delta_{0,k}$ for all Y in $T^\star \mathbb{R}^d$, so let us now prove the symmetry identities pointed out in the introduction. The first relation in (2.26) stems from the observation that for all (n, m, λ) in $\widetilde{\mathbb{H}}^d$ and Y in $T^\star \mathbb{R}^d$, we have

$$\mathcal{W}(m, n, \lambda, -Y) = \overline{\mathcal{W}(n, m, \lambda, Y)}.$$

Therefore, for any $(\dot{x}, k)$ in $\widehat{\mathbb{H}}_0^d$ passing to the limit $(n, m, \lambda) \to (\dot{x}, k)$ yields

(5.1) $$\mathcal{K}_d(\dot{x}, -k, -Y) = \overline{\mathcal{K}_d(\dot{x}, k, Y)}.$$

In order to establish the second symmetry relation for $\mathcal{K}_d$, it suffices to notice that

$$\forall (n, m, \lambda, Y) \in \widetilde{\mathbb{H}}^d \times T^\star \mathbb{R}^d, \ \mathcal{W}(n, m, \lambda, Y) = (-1)^{|n+m|} \mathcal{W}(m, n, -\lambda, Y).$$

and to pass to the limit $(n, m, \lambda) \to (\dot{x}, k)$.

The last relation in (2.26) follows from having (n, m, λ) tend to $(\dot{x}, k)$ in

$$(5.2) \qquad \mathcal{W}(n, m, -\lambda, Y) = \overline{\mathcal{W}}(n, m, \lambda, Y).$$

Identity (2.27) is a consequence of Relation (2.13). Indeed, observe that for any smooth function $f : T^\star \mathbb{R}^d \to \mathbb{C}$, we have

$$e^{-is\lambda} \Delta_{\mathbb{H}}\left(e^{is\lambda} f(Y)\right) = \Delta_Y f(Y) + 4i\lambda \sum_{j=1}^{d} \mathcal{T}_j f(Y) - 4\lambda^2 |Y|^2 f(Y)$$

$$\text{with} \quad \mathcal{T}_j \overset{\text{def}}{=} \eta_j \partial_{y_j} - y_j \partial_{\eta_j}.$$

Taking $f(Y) = \mathcal{W}(\widehat{w}, Y)$, using (2.13) and having (n, m, λ) tend to $(\dot{x}, k)$ yields

$$(5.3) \qquad \Delta_Y \mathcal{K}_d(\dot{x}, k, Y) = -4|\dot{x}|\mathcal{K}_d(\dot{x}, k, Y).$$

Relation (2.28) is a consequence of (A.10) which implies in particular that

$$|\lambda|(n_j - m_j)\mathcal{W}(\widehat{w}, Y) = i\lambda \mathcal{T}_j \mathcal{W}(\widehat{w}, Y).$$

Passing to the limit when (n, m, λ) tends to $(\dot{x}, k)$ ensures

$$(5.4) \qquad ik_j \mathcal{K}_d(\dot{x}, k, Y) = \text{sgn}(\dot{x})\mathcal{T}_j \mathcal{K}_d(\dot{x}, k, Y)$$

which is exactly (2.28).

Proving Identity (2.29) is bit more involved. To achieve it, let us fix some function α of $\mathcal{S}(\mathbb{R})$ and two functions g_1 and g_2 of $\mathcal{S}(T^\star \mathbb{R}^d)$. By definition of convolution and Fourier transform, we have

$$\mathcal{F}_{\mathbb{H}}\left((g_1 \otimes \alpha) \star (g_2 \otimes \alpha)\right)(\widehat{w})$$

$$= \int_{\mathbb{H}^d \times \mathbb{H}^d} e^{-is\lambda} \overline{\mathcal{W}}(\widehat{w}, Y)g_1(Y - Y')\alpha\left(s - s' - 2\sigma(Y', Y)\right)g_2(Y')\alpha(s')\mathrm{d}w\mathrm{d}w'.$$

Integrating first with respect to s and next with respect to s' yields

$$\mathcal{F}_{\mathbb{H}}\left((g_1 \otimes \alpha) \star (g_2 \otimes \alpha)\right)(\widehat{w})$$

$$= \widehat{\alpha}^2(\lambda) \int_{(T^\star \mathbb{R}^d)^2} e^{2i\lambda\sigma(Y, Y')} \overline{\mathcal{W}}(\widehat{w}, Y)g_1(Y - Y')g_2(Y')\mathrm{d}Y\mathrm{d}Y'.$$

From the fact that σ is symplectic, we infer that

$$(5.5) \quad \mathcal{F}_{\mathbb{H}}\left((g_1 \otimes \alpha) \star (g_2 \otimes \alpha)\right)(\widehat{w})$$

$$= \widehat{\alpha}^2(\lambda) \int_{(T^\star \mathbb{R}^d)^2} e^{2i\lambda\sigma(Y_1, Y_2)} \overline{\mathcal{W}}(\widehat{w}, Y_1 + Y_2)g_1(Y_1)g_2(Y_2)\mathrm{d}Y_1\mathrm{d}Y_2.$$

Of course, because both $g_1 \otimes \alpha$ and $g_2 \otimes \alpha$ are in $\mathcal{S}(\mathbb{H}^d)$, we are guaranteed, thanks to the convolution formula (2.19), that

$$\mathcal{F}_{\mathbb{H}}\big((g_1 \otimes \alpha) \star (g_2 \otimes \alpha)\big)(n, n+k, \lambda) = G_{12}$$

$$\text{with} \quad G_{12} \stackrel{\text{def}}{=} \big(\mathcal{F}_{\mathbb{H}}(g_1 \otimes \alpha) \cdot \mathcal{F}_{\mathbb{H}}(g_2 \otimes \alpha)\big)(n, n+k, \lambda).$$

Now, we have, setting $k' = n + k - \ell$ in the second line,

$$G_{12} = \sum_{\ell \in \mathbb{N}^d} \mathcal{F}_{\mathbb{H}}(g_1 \otimes \alpha)(n, \ell, \lambda) \mathcal{F}_{\mathbb{H}}(g_2 \otimes \alpha)(\ell, n+k, \lambda)$$

$$= \widehat{\alpha}^2(\lambda) \int_{(T^\star \mathbb{R}^d)^2} \sum_{k' \leqslant n+k} \overline{\mathcal{W}}(n, n+k-k', \lambda, Y_1)$$

$$\times \overline{\mathcal{W}}(n+k-k', n+k, \lambda, Y_2) g_1(Y_1) g_2(Y_2) \mathrm{d}Y_1 \mathrm{d}Y_2.$$

Hence, reverting to Relation (5.5) and keeping in mind that the above computations hold true for any functions α, g_1 and g_2 in the Schwartz class, one may conclude that

$$\mathrm{e}^{-2\mathrm{i}\lambda\sigma(Y_1, Y_2)} \mathcal{W}(n, n+k, \lambda, Y_1 + Y_2)$$

$$= \sum_{k' \in \mathbb{Z}^d} \mathcal{W}(n, n+k-k', \lambda, Y_1) \mathcal{W}(n+k-k', n+k, \lambda, Y_2).$$

Taking advantage of the decay of $\mathcal{W}$ with respect to the variable k given by Proposition A.1, we can pass to the limit for $2\lambda n$ tending to $\dot{x}$ and λ tending to 0. This gives

$$(5.6) \qquad \mathcal{K}_d(\dot{x}, k, Y_1 + Y_2) = \sum_{k' \in \mathbb{Z}^d} \mathcal{K}_d(\dot{x}, k - k', Y_1) \mathcal{K}_d(\dot{x}, k', Y_2)$$

which is the generalization of Formula (2.29) in any dimension.

In order to fully benefit from Relations (2.27), (2.28) and (2.29) so as to eventually compute the function $\mathcal{K}$, it is wise to introduce the following function $\widetilde{\mathcal{K}}$ on $\mathbb{R} \times \mathbb{T} \times T^\star \mathbb{R}$, where $\mathbb{T}$ denotes the one-dimensional torus:

$$(5.7) \qquad \widetilde{\mathcal{K}}(\dot{x}, z, Y) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} \mathcal{K}(\dot{x}, k, Y) \mathrm{e}^{\mathrm{i}kz}.$$

From Proposition A.1, after having (n, m, λ) tend to $(\dot{x}, k)$, we infer that for all $(\dot{x}, k, Y)$ in $\mathbb{R} \times \mathbb{Z} \times T^\star \mathbb{R}$, we have, denoting $\langle Y \rangle \stackrel{\text{def}}{=} \sqrt{1 + |Y|^2}$,

$$(5.8) \qquad \forall N \in \mathbb{N}, \ |\mathcal{K}(\dot{x}, k, Y)| \leqslant C_N \left(\frac{\langle Y \rangle (1 + |\dot{x}|)^{\frac{1}{2}}}{1 + |k|} \right)^N.$$

Thus the series (5.7) defines a function $\widetilde{\mathcal{K}}$ on $\mathbb{R} \times \mathbb{T} \times T^\star \mathbb{R}$.

Furthermore, from (2.29) we infer immediately that

$$\widetilde{\mathcal{K}}(\dot{x}, z, Y_1 + Y_2) = \widetilde{\mathcal{K}}(\dot{x}, z, Y_1)\, \widetilde{\mathcal{K}}(\dot{x}, z, Y_2), \tag{5.9}$$

and, in light of (5.1), we discover that for any $(\dot{x}, z, Y)$ in $\mathbb{R} \times \mathbb{T} \times T^\star\mathbb{R}$,

$$\widetilde{\mathcal{K}}(\dot{x}, z, -Y) = \overline{\widetilde{\mathcal{K}}(\dot{x}, z, Y)}. \tag{5.10}$$

Combined with (3.14) and (5.9) this implies that for any couple $(\dot{x}, z)$ in $\mathbb{R} \times \mathbb{T}$, the function $Y \mapsto \widetilde{\mathcal{K}}(\dot{x}, z, Y)$ is a character of $\mathbb{R}^2$. Identifying $T^\star\mathbb{R}$ with $\mathbb{R}^2$, we thus conclude that there exists a function $\Phi = (\Phi_y, \Phi_\eta)$ from $\mathbb{R} \times \mathbb{T}$ to $\mathbb{R}^2$ such that

$$\widetilde{\mathcal{K}}(\dot{x}, z, Y) = \mathrm{e}^{\mathrm{i}Y \cdot \Phi(\dot{x}, z)} = \mathrm{e}^{\mathrm{i}(y\Phi_y(\dot{x}, z) + \eta\Phi_\eta(\dot{x}, z))}.$$

Taking advantage of (5.3) which implies that $\mathcal{K}$ is a smooth function of Y, and arguing as for proving (5.8), we discover that for all multi-index $\alpha = (\alpha_1, \alpha_2)$ in $\mathbb{N}^2$ and all $(\dot{x}, Y)$ in some bounded set $\mathcal{B}$, we have

$$\forall\, N \in \mathbb{N}, \quad \sup_{\substack{(\dot{x}, Y) \in \mathcal{B} \\ k \in \mathbb{Z}^d}} (1 + |k|)^N |\partial_{\dot{x}, Y}^\alpha \mathcal{K}(\dot{x}, k, Y)| < \infty.$$

Therefore invoking Relation (2.28), we deduce that for any positive $\dot{x}$

$$\partial_z \widetilde{\mathcal{K}}(\dot{x}, z, Y) = \eta\partial_y \widetilde{\mathcal{K}}(\dot{x}, z, Y) - y\partial_\eta \widetilde{\mathcal{K}}(\dot{x}, z, Y)$$

which entails that $\partial_z \Phi(\dot{x}, z) = R\Phi(\dot{x}, z)$ where R denotes the rotation of angle $\pi/2$. Hence

$$\Phi(\dot{x}, z) = R(z)\widetilde{\Phi}(\dot{x})$$

where $R(z)$ denotes the rotation of angle z. Now, Relation (5.3) ensures that $|\widetilde{\Phi}(\dot{x})| = 2|\dot{x}|^{\frac{1}{2}}$, and thus there exists a function ϕ from $\mathbb{R}$ to the unit circle of $\mathbb{R}^2$ so that for positive $\dot{x}$

$$\widetilde{\mathcal{K}}(\dot{x}, z, Y) = \mathrm{e}^{2\mathrm{i}|\dot{x}|^{\frac{1}{2}} Y \cdot (R(z)\phi(\dot{x}))}. \tag{5.11}$$

Let us finally establish Identity (2.30). It relies on the study of the action of the Fourier transform on the *weight function M^2* defined by

$$(M^2 f)(Y, s) \stackrel{\mathrm{def}}{=} |Y|^2 f(Y, s).$$

For any functions g in $\mathcal{S}(T^\star\mathbb{R})$ and $\psi : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R}$, smooth and compactly supported in $[r_0, \infty[\times\mathbb{Z}$ for some positive real number r_0, let us define

$$\Theta_\psi(\widehat{w}) \stackrel{\mathrm{def}}{=} \psi\big(|\lambda|(n + m + 1), m - n\big) \quad \text{and}$$

$$\mathcal{B}(g, \psi) \stackrel{\mathrm{def}}{=} \int_{T^\star\mathbb{R} \times \widehat{\mathbb{H}}_0^1} |Y|^2 \mathcal{K}(\dot{x}, k, Y) g(Y) \psi(\dot{x}, k)\, \mathrm{d}Y\, \mathrm{d}\mu_{\widehat{\mathbb{H}}_0^1}(\dot{x}, k).$$

Lemma 4.1 implies that if $\widehat{\chi} : \mathbb{R} \to \mathbb{R}$ is integrable, supported in $[-1, 1]$ and with integral 1, then $\mathcal{B}(g, \psi) = \lim_{\varepsilon \to 0} \mathcal{B}_\varepsilon(g, \psi)$ with

$$\mathcal{B}_\varepsilon(g, \psi)$$
$$\stackrel{\text{def}}{=} \int_{T^\star\mathbb{R}} g(Y) \int_{\mathbb{R}} \sum_{(n,m)\in\mathbb{N}^2} |Y|^2 \mathcal{W}(n, m, \lambda, Y) \Theta_\psi(n, m, \lambda) \frac{1}{\varepsilon} \widehat{\chi}\left(\frac{\lambda}{\varepsilon}\right) |\lambda| \mathrm{d}\lambda \mathrm{d}Y.$$

The following lemma gives a formula for $|Y|^2 \mathcal{W}(\widehat{w}, Y)$.

LEMMA 5.1. — For all $\widehat{w}$ in $\widetilde{\mathbb{H}}^d$ and Y in $T^\star\mathbb{R}^d$, we have

$$|Y|^2 \mathcal{W}(\widehat{w}, Y) = -\widehat{\Delta}\mathcal{W}(\,\cdot\,, Y)(\widehat{w})$$

with

$$(5.12) \quad \widehat{\Delta}\theta(\widehat{w}) \stackrel{\text{def}}{=} -\frac{1}{2|\lambda|}(|n+m|+d)\theta(\widehat{w})$$
$$+ \frac{1}{2|\lambda|} \sum_{j=1}^d \left\{ \sqrt{(n_j+1)(m_j+1)}\,\theta(\widehat{w}_j^+) + \sqrt{n_j m_j}\,\theta(\widehat{w}_j^-) \right\}$$

where $\widehat{w}_j^\pm \stackrel{\text{def}}{=} (n \pm \delta_j, m \pm \delta_j, \lambda)$.

Proof. — From the definition of $\mathcal{W}$ and integrations by parts, we get

$$|Y|^2 \mathcal{W}(\widehat{w}, Y) = \int_{\mathbb{R}^d} \left(|y|^2 - \frac{1}{4\lambda^2}\Delta_z\right)(\mathrm{e}^{2\mathrm{i}\lambda\langle\eta,z\rangle}) H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z)\mathrm{d}z$$
$$= \int_{\mathbb{R}^d} \mathrm{e}^{2\mathrm{i}\lambda\langle\eta,z\rangle} |\lambda|^{\frac{d}{2}} \mathcal{I}(\widehat{w}, y, z)\mathrm{d}z$$

with $\mathcal{I}(\widehat{w}, y, z) \stackrel{\text{def}}{=} \left(|y|^2 - \frac{1}{4\lambda^2}\Delta_z\right)\left(H_n(|\lambda|^{\frac{1}{2}}(y+z))H_m(|\lambda|^{\frac{1}{2}}(-y+z))\right)$.

Using Leibniz formula, the chain rule and $4|y|^2 = |y+z|^2 + |y-z|^2 + 2(y+z)\cdot(y-z)$, we get

$$\mathcal{I}(\widehat{w}, y, z) = -\frac{1}{4\lambda^2}\left((\Delta_z - \lambda^2|y+z|^2)H_n(|\lambda|^{\frac{1}{2}}(y+z))\right)H_m(|\lambda|^{\frac{1}{2}}(-y+z))$$
$$- \frac{1}{4\lambda^2}\left((\Delta_z - \lambda^2|y-z|^2)H_m(|\lambda|^{\frac{1}{2}}(-y+z))\right)H_n(|\lambda|^{\frac{1}{2}}(y+z))$$
$$- \frac{1}{2|\lambda|}\sum_{j=1}^d (\partial_j H_n)(|\lambda|^{\frac{1}{2}}(y+z))(\partial_j H_m)(|\lambda|^{\frac{1}{2}}(-y+z))$$
$$- \frac{1}{2}(z+y)\cdot(z-y)H_n(|\lambda|^{\frac{1}{2}}(y+z))H_m(|\lambda|^{\frac{1}{2}}(-y+z)).$$

Using (2.9), we end up with

$$\mathcal{I}(\widehat{w}, y, z) = \frac{1}{2|\lambda|}(|n + m| + d)H_n(|\lambda|^{\frac{1}{2}}(y + z))H_m(|\lambda|^{\frac{1}{2}}(-y + z))$$

$$- \frac{1}{2|\lambda|}\sum_{j=1}^{d}\left\{(\partial_j H_n)(|\lambda|^{\frac{1}{2}}(y + z))(\partial_j H_m)(|\lambda|^{\frac{1}{2}}(-y + z))\right.$$

$$\left. + (M_j H_n)(|\lambda|^{\frac{1}{2}}(y + z))(M_j H_m)(|\lambda|^{\frac{1}{2}}(-y + z))\right\}.$$

Then, taking advantage of (A.4), we get Identity (5.12).   □

Let us resume to the proof of Identity (2.30). Using the above lemma for $d = 1$ and performing obvious changes of variable in the sum give

$$\mathcal{B}_\varepsilon(g, \psi)$$

$$= -\int_{T^\star\mathbb{R}} g(Y)\int_{\mathbb{R}}\sum_{(n,m)\in\mathbb{N}^2}(\widehat{\Delta}\mathcal{W}(\cdot, Y))(n, m, \lambda)\Theta_\psi(n, m, \lambda)\frac{1}{\varepsilon}\widehat{\chi}\left(\frac{\lambda}{\varepsilon}\right)|\lambda|\mathrm{d}\lambda\mathrm{d}Y$$

$$= -\int_{T^\star\mathbb{R}} g(Y)\int_{\mathbb{R}}\sum_{(n,m)\in\mathbb{N}^2}\mathcal{W}(n, m, \lambda, Y)(\widehat{\Delta}\Theta_\psi)(n, m, \lambda)\frac{1}{\varepsilon}\widehat{\chi}\left(\frac{\lambda}{\varepsilon}\right)|\lambda|\mathrm{d}\lambda\mathrm{d}Y.$$

The key to proving the convergence of $\mathcal{B}_\varepsilon$ for $\varepsilon \to 0$ is the asymptotic description of the operator $\widehat{\Delta}$ when λ tends to 0, given in the following lemma:

LEMMA 5.2. — Let ψ be a smooth function compactly supported in the set $[r_0, \infty[\times\mathbb{Z}$ for some positive real number r_0. Then

$$\widehat{\Delta}\Theta_\psi(n, m, \lambda) \overset{1}{\sim} \Theta_{L\psi}(n, m, \lambda)$$

$$\text{with} \quad (L\psi)(\dot{x}, k) \overset{\text{def}}{=} \dot{x}\psi''(\dot{x}, k) + \psi'(\dot{x}, k) - \frac{k^2}{4\dot{x}}\psi(\dot{x}, k)$$

where the notation $\Theta_1 \overset{p}{\sim} \Theta_2$ means that for any positive integer N, there is a constant $C_{N,p}$ such that for all (n, m, λ) in $\mathbb{N}^2\times]0, \infty[$ satisfying

$$\lambda(n + m) \geqslant \frac{r_0}{2} \quad \text{and} \quad \lambda \leqslant \lambda_0/(1 + |n - m|),$$

with a sufficiently small positive real number λ_0 depending only on r_0, we have

$$\left|\Theta_1(n, m, \lambda) - \Theta_2(n, m, \lambda)\right| \leqslant C_{N,p}\,\lambda^p\left(1 + |\lambda|(|n + m| + 1) + |m - n|\right)^{-N}.$$

Proof. — By definition of the operator $\widehat{\Delta}$, and for $\lambda > 0$, we have, denoting $k \overset{\text{def}}{=} m - n$ and $y \overset{\text{def}}{=} \lambda(n + m)$,

$$-2\lambda^2 \widehat{\Delta}\Theta_\psi(\widehat{w}) = (y + \lambda)\psi(y + \lambda, k)$$
$$- \lambda\sqrt{(n+1)(m+1)}\,\psi(y + 3\lambda, k) - \lambda\sqrt{nm}\,\psi(y - \lambda, k).$$

Using that

$$\lambda^2 nm = \frac{\lambda^2}{4}(n + m)^2 - \frac{\lambda^2}{4}(m - n)^2 = \frac{y^2}{4} - \frac{\lambda^2}{4}k^2,$$

we get that

$$\lambda\sqrt{(n+1)(m+1)} = \frac{y}{2}\sqrt{1 + \frac{4\lambda}{y} + \left(\frac{4 - k^2}{y^2}\right)\lambda^2} \overset{3}{\sim} \frac{y}{2} + \lambda - \frac{k^2}{4y}\lambda^2$$

$$\text{and} \quad \lambda\sqrt{nm} \overset{3}{\sim} \frac{y}{2} - \frac{k^2}{4y}\lambda^2.$$

Writing the Taylor expansion for ψ gives (omitting the dependency with respect to k for notational simplicity),

$$(y + \lambda)\psi(y + \lambda) \overset{3}{\sim} y\psi(y) + (\psi(y) + y\psi'(y))\lambda + \left(\psi'(y) + \frac{y}{2}\psi''(y)\right)\lambda^2,$$

$$- \lambda\sqrt{(n+1)(m+1)}\,\psi(y + 3\lambda) \overset{3}{\sim} -\frac{y}{2}\psi(y) - \left(\psi(y) + \frac{3}{2}y\psi'(y)\right)\lambda$$
$$- \left(\frac{9}{4}y\psi''(y) + 3\psi'(y) - \frac{k^2}{4y}\psi(y)\right)\lambda^2,$$

$$- \lambda\sqrt{nm}\,\psi(y - \lambda) \overset{3}{\sim} -\frac{y}{2}\psi(y) + \frac{1}{2}y\psi'(y)\lambda - \left(\frac{y}{4}\psi''(y) - \frac{k^2}{4y}\psi(y)\right)\lambda^2.$$

By summation of these three identities, we get

$$-2\lambda^2 \widehat{\Delta}\Theta_\psi(\widehat{w}) \overset{3}{\sim} -\left(2y\psi''(y) + 2\psi'(y) - \frac{k^2}{2y}\psi(y)\right)\lambda^2,$$

whence the lemma. $\square$

From the above lemma, it is easy to complete the proof of Identity (2.30). Indeed, we get

$$\mathcal{B}_\varepsilon(g, \psi) = -\int_{T^\star\mathbb{R}} g(Y) \int_{\mathbb{R}} \sum_{(n,m)\in\mathbb{N}^2} \mathcal{W}(n, m, \lambda, Y)$$
$$\times L\psi\big(|\lambda|(n+m), m-n\big)\frac{1}{\varepsilon}\widehat{\chi}\left(\frac{\lambda}{\varepsilon}\right)|\lambda|\mathrm{d}\lambda\mathrm{d}Y + \mathcal{R}_\varepsilon(g, \psi),$$

where the remainder $\mathcal{R}_\varepsilon$ is such that for all $N \in \mathbb{N}$ there exists C_N so that

$$\left|\mathcal{R}_\varepsilon(g,\psi)\right| \leqslant C_N \|g\|_{L^1(T^\star\mathbb{R}^d)}$$
$$\times \sum_{(n,m)\in\mathbb{N}^2} \int_\mathbb{R} |\lambda|\big(1+|\lambda|(|n+m|+1)+|m-n|\big)^{-N}\frac{1}{\varepsilon}|\widehat{\chi}|\left(\frac{\lambda}{\varepsilon}\right)|\lambda|\mathrm{d}\lambda.$$

Taking N large enough, we find out that

$$\sum_{(n,m)\in\mathbb{N}^2} \int_\mathbb{R} |\lambda|\big(1+|\lambda|(|n+m|+1)+|m-n|\big)^{-N}\frac{1}{\varepsilon}|\widehat{\chi}|\left(\frac{\lambda}{\varepsilon}\right)|\lambda|\mathrm{d}\lambda$$
$$\leqslant C_N \int_\mathbb{R} \frac{|\lambda|}{\varepsilon}|\widehat{\chi}|\left(\frac{\lambda}{\varepsilon}\right)\mathrm{d}\lambda \leqslant C_N\varepsilon.$$

Then Lemma 4.1 ensures

$$\mathcal{B}(g,\psi) = -\int_{T^\star\mathbb{R}\times\widehat{\mathbb{H}}^1_0} \mathcal{K}(\dot{x},k,Y)g(Y)$$
$$\left(\dot{x}\psi''(\dot{x},k)+\psi'(\dot{x},k)-\frac{k^2}{4\dot{x}}\psi(\dot{x},k)\right)\mathrm{d}Y\mathrm{d}\mu_{\widehat{\mathbb{H}}^1_0}(\dot{x},k).$$

Integration by parts yields

$$\mathcal{B}(g,\psi) = \int_{T^\star\mathbb{R}\times\widehat{\mathbb{H}}^1_0} g(Y)\left(\frac{k^2}{4\dot{x}}\mathcal{K}(\dot{x},k,Y)-\partial_{\dot{x}}\mathcal{K}(\dot{x},k,Y)\right.$$
$$\left. -\dot{x}\partial^2_{\dot{x}}\mathcal{K}(\dot{x},k,Y)\right)\psi(\dot{x},k)\mathrm{d}Y\mathrm{d}\mu_{\widehat{\mathbb{H}}^1_0}(\dot{x},k).$$

Using the fact that the above equality holds true for all g in $\mathcal{S}(T^\star\mathbb{R})$ and for functions ψ smooth and compactly supported in $[r_0,\infty[\times\mathbb{Z}$ for some $r_0 > 0$, and combining with a density argument, one can conclude to Identity (2.30) for all positive $\dot{x}$ and k in $\mathbb{Z}$.

In order to complete the proof of (2.22), let us translate (2.30) in terms of $\widetilde{\mathcal{K}}$. We have

$$\frac{1}{4\dot{x}}\partial^2_z\widetilde{\mathcal{K}} + \partial_{\dot{x}}(\dot{x}\partial_{\dot{x}}\widetilde{\mathcal{K}}) + |Y|^2\widetilde{\mathcal{K}} = 0.$$

Now, plugging the ansatz (5.11) into the above relation yields for any positive $\dot{x}$, any k in $\mathbb{Z}$ and any Y in $T^\star\mathbb{R}$,

$$|Y|^2 = \big(Y \cdot (R'(z)\phi(\dot{x}))\big)^2 + \big(Y \cdot (R(z)\phi(\dot{x}))+2\dot{x}Y\cdot(R(z)\phi'(\dot{x}))\big)^2$$
$$- 4i\sqrt{\dot{x}}Y \cdot (R(z)\phi'(\dot{x})) - 2i\dot{x}^{3/2}Y \cdot (R(z)\phi''(\dot{x})).$$

Taking the imaginary part implies that ϕ satisfies

$$\dot{x}\phi''(\dot{x}) + 2\phi'(\dot{x}) = 0 \quad \text{for} \ \ \dot{x} > 0.$$

Now, as ϕ is valued in the unit circle, this implies that ϕ is a constant. Therefore there exists some number z_0 in $(-\pi, \pi]$ so that for any positive $\dot{x}$, any z in $\mathbb{R}$ and any Y in $T^{\star}\mathbb{R}$, we have

$$\widetilde{\mathcal{K}}(\dot{x}, z, Y) = e^{2i|\dot{x}|^{\frac{1}{2}}(y\cos(z+z_0)+\eta\sin(z+z_0))}.$$

Inverse Fourier theorem for periodic functions implies that

$$\mathcal{K}(\dot{x}, k, Y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2i|\dot{x}|^{\frac{1}{2}}(y\cos(z+z_0)+\eta\sin(z+z_0))} e^{-ikz} \, dz.$$

In order to compute the value of z_0, one may take advantage of the symmetry relations in (2.26) that imply

(5.13) $$\mathcal{K}(\dot{x}, -k, y, -\eta) = (-1)^k \mathcal{K}(\dot{x}, k, y, \eta).$$

Now, the above formula for $\mathcal{K}$ and an obvious change of variable give

$$2\pi\mathcal{K}(\dot{x}, -k, y, -\eta) = \int_{-\pi}^{\pi} e^{ikz} e^{2i|\dot{x}|^{\frac{1}{2}}(y\cos(z+z_0)-\eta\sin(z+z_0))} \, dz$$

$$= \int_{-\pi}^{\pi} e^{ik(\pi-z)} e^{2i|\dot{x}|^{\frac{1}{2}}(y\cos(\pi-z+z_0)-\eta\sin(\pi-z+z_0))} \, dz$$

$$= (-1)^k \int_{-\pi}^{\pi} e^{-ikz} e^{-2i|\dot{x}|^{\frac{1}{2}}(y\cos(z-z_0)+\eta\sin(z-z_0))} \, dz.$$

Hence (5.13) is fulfilled for all positive $\dot{x}$, k in $\mathbb{Z}$ and (y, η) in $T^{\star}\mathbb{R}$ if and only if

$$\forall z \in (-\pi, \pi), \quad \cos(z+z_0) = -\cos(z-z_0) \quad \text{and} \quad \sin(z+z_0) = -\sin(z-z_0)$$

which is equivalent to $z_0 \equiv \frac{\pi}{2}[\pi]$. Hence there exists $\varepsilon \in \{-1, 1\}$ so that

$$\widetilde{\mathcal{K}}(\dot{x}, z, Y) = e^{2i\varepsilon\sqrt{\dot{x}}(y\sin z - \eta\cos z)}.$$

To determine the value of ε, one may use the fact that for all positive $\dot{x}$ and η in $\mathbb{R}$, the above formula implies that

$$\sum_{k\in\mathbb{Z}} \mathcal{K}(\dot{x}, k, (0, \eta)) = \widetilde{\mathcal{K}}(\dot{x}, 0, (0, \eta)) = e^{-2i\varepsilon\sqrt{\dot{x}}\eta} = \cos(2\sqrt{\dot{x}}\,\eta) - i\varepsilon\sin(2\sqrt{\dot{x}}\,\eta).$$

Now, from the expansion of $\mathcal{K}$ given in (3.1), we infer that for all $\eta \in \mathbb{R}$ and $\dot{x} > 0$,

$$\widetilde{\mathcal{K}}(\dot{x}, 0, (0, \eta)) = \sum_{\ell_1 \in \mathbb{N}} \sum_{|k| \leqslant \ell_1} \frac{i^{\ell_1}}{\ell_1!} F_{\ell_1,0}(k)\eta^{\ell_1}\dot{x}^{\frac{\ell_1}{2}}.$$

Note that the imaginary part of the term corresponding to ℓ_1 is positive (indeed $F_{1,0}(k)$ is positive), which implies that $\varepsilon = -1$. This completes the proof of Identity (2.22) in the case where $\dot{x}$ is non negative. The negative case just follows from (2.26). Thus the whole Theorem 2.7 is proved.

6. Some properties of operator $\mathcal{G}_{\mathbb{H}}$

We end this paper with a short presentation of basic properties of the transformation $\mathcal{G}_{\mathbb{H}}$, that highlight some analogy (but also some difference) with the classical Fourier transform on $T^\star\mathbb{R}^d$. The main result of this section reads as follows.

THEOREM 6.1. — The operator $\mathcal{G}_{\mathbb{H}}$ maps continuously $L^1(T^\star\mathbb{R}^d)$ to the space $\mathcal{C}_0(\widehat{\mathbb{H}}_0^d)$ of continuous functions on $\widehat{\mathbb{H}}_0^d$ going to 0 at infinity and, for any couple (f,g) of functions in $L^1(T^\star\mathbb{R}^d)$, we have the convolution identity:

(6.1) $\mathcal{G}_{\mathbb{H}}(f \star g)(\dot{x},k) = \sum_{k'\in\mathbb{Z}^d} \mathcal{G}_{\mathbb{H}}f(\dot{x},k-k')\,\mathcal{G}_{\mathbb{H}}(\dot{x},k')$ for all $(\dot{x},k)\in\widehat{\mathbb{H}}_0^d$.

Moreover, for any g in $\mathcal{S}(T^\star\mathbb{R}^d)$, we have the following inversion formula:

$$g(Y) = \left(\frac{2}{\pi}\right)^d \int_{\widehat{\mathbb{H}}_0^d} \mathcal{K}_d(\dot{x},k,Y)\mathcal{G}_{\mathbb{H}}g(\dot{x},k)\mathrm{d}\mu_{\widehat{\mathbb{H}}_0^d}(\dot{x},k).$$

Finally, the following Fourier–Plancherel identity holds true:

$$\forall\, g \in \mathcal{S}(T^\star\mathbb{R}^d)\,,\ \|g\|^2_{L^2(T^\star\mathbb{R}^d)} = \left(\frac{2}{\pi}\right)^d \|\mathcal{G}_{\mathbb{H}}g\|^2_{L^2(\widehat{\mathbb{H}}_0^d)}.$$

Proof. — The first property stems from the fact that, because $|\mathcal{K}_d| \leqslant 1$, we have

$$\|\mathcal{G}_{\mathbb{H}}g\|_{L^\infty(\widehat{\mathbb{H}}_0^d)} \leqslant \|g\|_{L^1(T^\star\mathbb{R}^d)}.$$

Furthermore, as the kernel $\mathcal{K}_d$ is continuous with respect to $(\dot{x},k)$, we get from the explicit expression of $\mathcal{G}_{\mathbb{H}}$ that the range of $L^1(T^\star\mathbb{R}^d)$ by $\mathcal{G}_{\mathbb{H}}$ is included in the set of continuous functions on $\widehat{\mathbb{H}}_0^d$.

Proving that $(\mathcal{G}_{\mathbb{H}}g)(\dot{x},k)$ tends to 0 when $(\dot{x},k)$ goes to infinity is based on the regularity and decay properties of the kernel $\mathcal{K}_d$. More specifically, Identity (2.27) implies that

$$\forall\, p \in \mathbb{N}\,,\ 4^p|\dot{x}|^p\mathcal{K}_d(\dot{x},k,Y) = \big((-\Delta_Y)^p\mathcal{K}_d\big)(\dot{x},k,Y),$$

while Relation (2.28) gives for all multi-index α in $\mathbb{N}^d$,

$$(ik\,\mathrm{sgn}\,\dot{x})^\alpha\mathcal{K}_d(\dot{x},k,Y) = (\mathcal{T}^\alpha\mathcal{K}_d)(\dot{x},k,Y))$$

$$\text{with}\quad \mathcal{T}^\alpha \overset{\text{def}}{=} \prod_{j=1}^d (\eta_j\partial_{y_j} - y_j\partial_{\eta_j})^{\alpha_j}.$$

Hence, if $g \in \mathcal{S}(T^\star \mathbb{R}^d)$ then performing suitable integration by parts in the integral defining $\mathcal{G}_\mathbb{H} g$ yields

$$4^p |\dot{x}|^p (\mathcal{G}_\mathbb{H} g)(\dot{x}, k) = \mathcal{G}_\mathbb{H}((-\Delta_Y)^p g)(\dot{x}, k)$$

$$\text{and} \quad (-ik \operatorname{sgn} \dot{x})^\alpha (\mathcal{G}_\mathbb{H} g)(\dot{x}, k) = (\mathcal{G}_\mathbb{H} \mathcal{T}^\alpha g)(\dot{x}, k).$$

This implies that, for any positive integer p, a constant C_p and an integer N_p exist such that

(6.2) $$\qquad (1 + |\dot{x}| + |k|)^p |\mathcal{G}_\mathbb{H}(g)(\dot{x}, k)| \leqslant C_p \, \|g\|_{N_p, \mathcal{S}(T^\star \mathbb{R}^d)}.$$

This proves that $(\mathcal{G}_\mathbb{H} g)(\dot{x}, k)$ tends to 0 when $(\dot{x}, k)$ goes to infinity for any g in $\mathcal{S}(T^\star \mathbb{R}^d)$. Now, because $L^1(T^\star \mathbb{R}^d)$ is dense in $\mathcal{S}(T^\star \mathbb{R}^d)$ and $\mathcal{G}_\mathbb{H}$ is continuous from $L^1(T^\star \mathbb{R}^d)$ to the set $\mathcal{C}_b(\widehat{\mathbb{H}}_0^d)$ of bounded continuous functions on $\widehat{\mathbb{H}}_0^d$, one can conclude that the range of $L^1(T^\star \mathbb{R}^d)$ by $\mathcal{G}_\mathbb{H}$ is included in $\mathcal{C}_0(\widehat{\mathbb{H}}_0^d)$.

In order to establish (6.1), it suffices to see that, by virtue of the definition of $\mathcal{G}_\mathbb{H}$, of Identity (5.6) and of Fubini theorem (here the decay inequality (5.8) comes into play), one may write that for any couple (f, g) of integrable functions on $T^\star \mathbb{R}^d$, we have

$$\mathcal{G}_\mathbb{H}(f \star g)(\dot{x}, k) = \int_{(T^\star \mathbb{R}^d)^2} \overline{\mathcal{K}}_d(\dot{x}, k, Y) \, f(Y - Y') \, g(Y') \mathrm{d}Y \mathrm{d}Y'$$

$$= \sum_{k' \in \mathbb{Z}^d} \int_{(T^\star \mathbb{R}^d)^2} \overline{\mathcal{K}}_d(\dot{x}, k', Y') g(Y')$$

$$\times \overline{\mathcal{K}}_d(\dot{x}, k - k', Y - Y') f(Y - Y') \mathrm{d}Y \mathrm{d}Y'.$$

Then performing an obvious change of variable, and using again Fubini theorem and the definition of $\mathcal{G}_\mathbb{H}$ gives (6.1).

In order to prove the *inversion Fourier formula for* $\mathcal{G}_\mathbb{H}$, let us consider g in $\mathcal{S}(T^\star \mathbb{R}^d)$ and χ in $\mathcal{S}(\mathbb{R})$ with value 1 near 0. For any sequence $(\varepsilon_p)_{p \in \mathbb{N}}$ of positive real numbers which tends to 0, we have according to the inverse Fourier formula (2.17),

$$g(Y)\chi(\varepsilon_p s) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} \mathrm{e}^{is\lambda} \, \mathcal{W}(\widehat{w}, Y) \mathcal{F}_\mathbb{H}(g \otimes \chi(\varepsilon_p \cdot))(\widehat{w}) \mathrm{d}\widehat{w}$$

$$= \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} \mathrm{e}^{is\lambda} \, \mathcal{W}(\widehat{w}, Y) \left(\int_{T^\star \mathbb{R}^d} \overline{\mathcal{W}}(\widehat{w}, Y') g(Y') \mathrm{d}Y' \right) \frac{1}{\varepsilon_p} \widehat{\chi}\left(\frac{\lambda}{\varepsilon_p}\right) \mathrm{d}\widehat{w}.$$

From the definition of $\Delta_\mathbb{H}$ in (2.5), we gather that for any integer p and positive real number ε, there exist some function f_ε^p on $\mathbb{H}^d$, and constant C_p (depending only on p) so that

(6.3) $$\qquad (-\Delta_\mathbb{H})^p \chi(\varepsilon s) g(Y) = \chi(\varepsilon s)(-\Delta_Y)^p g(Y) + \varepsilon f_\varepsilon^p(Y, s)$$

with $\|f^p_\varepsilon(\,\cdot\,,s)\|_{L^1(T^\star\mathbb{R}^d)} \leqslant C_p$.

Therefore, having ε tend to 0, we deduce that

$$|\lambda|^p(2|m|+d)^p\left|\int_{T^\star\mathbb{R}^d}\overline{\mathcal{W}}(\widehat{w},Y)g(Y)\mathrm{d}Y\right| \leqslant C_p\int_{T^\star\mathbb{R}^d}\left|(-\Delta_Y)^pg(Y)\right|\mathrm{d}Y\,.$$

Along the same lines, taking advantage of the *right-invariant* vector fields defined in (A.9), we get for any integer p

$$|\lambda|^p(2|n|+d)^p\left|\int_{T^\star\mathbb{R}^d}\overline{\mathcal{W}}(\widehat{w},Y)g(Y)\mathrm{d}Y\right| \leqslant C_p\int_{T^\star\mathbb{R}^d}\left|(-\Delta_Y)^pg(Y)\right|\mathrm{d}Y\,.$$

Identity (A.10) together with integrations by parts implies that for any multi-index α

$$(-i\,\mathrm{sgn}\,\lambda)^{|\alpha|}\prod_{j=1}^d(n_j-m_j)^{\alpha_j}\int_{T^\star\mathbb{R}^d}\overline{\mathcal{W}}(\widehat{w},Y)g(Y)\mathrm{d}Y$$
$$=\int_{T^\star\mathbb{R}^d}\overline{\mathcal{W}}(\widehat{w},Y)\mathcal{T}^\alpha g(Y)\mathrm{d}Y\,.$$

We deduce that the function

$$\widehat{w}\longmapsto \mathcal{W}(\widehat{w},Y)\left(\int_{T^\star\mathbb{R}^d}\overline{\mathcal{W}}(\widehat{w},Y')g(Y')\mathrm{d}Y'\right)$$

satisfies the hypothesis of Lemma 4.1. Thus combining with Proposition 3.1 gives

$$g(Y) = \frac{2^{d-1}}{\pi^{d+1}}\,2\pi\int_{\widehat{\mathbb{H}^d_0}}\mathcal{K}_d(\dot{x},k,Y)\left(\int_{T^\star\mathbb{R}^d}\overline{\mathcal{K}}_d(\dot{x},k,Y')g(Y')\mathrm{d}Y'\right)\mathrm{d}\mu_{\widehat{\mathbb{H}^d_0}}(\dot{x},k)$$
$$= \left(\frac{2}{\pi}\right)^d\int_{\widehat{\mathbb{H}^d_0}}\mathcal{K}_d(\dot{x},k,Y)\,\mathcal{G}_\mathbb{H}g(\dot{x},k)\mathrm{d}\mu_{\widehat{\mathbb{H}^d_0}}(\dot{x},k),$$

which completes the proof of the inversion formula.

Of course, as in the classical Fourier theory, having an inversion formula implies a Fourier–Plancherel type relation. Indeed we have for any

function g in $\mathcal{S}(T^\star\mathbb{R}^d)$, using Fubini theorem,

$$
\int_{T^\star\mathbb{R}^d} g(Y)\overline{g}(Y)\,\mathrm{d}Y
$$

$$
= \left(\frac{2}{\pi}\right)^d \int_{T^\star\mathbb{R}^d} \left(\int_{\widehat{\mathbb{H}}_0^d} \mathcal{K}_d(\dot{x},k,Y)\,\mathcal{G}_{\mathbb{H}}g(\dot{x},k)\mathrm{d}\mu_{\widehat{\mathbb{H}}_0^d}(\dot{x},k)\right)\overline{g}(Y)\mathrm{d}Y
$$

$$
= \left(\frac{2}{\pi}\right)^d \int_{\widehat{\mathbb{H}}_0^d} \mathcal{G}_{\mathbb{H}}g(\dot{x},k)\overline{\left(\int_{T^\star\mathbb{R}^d} \overline{\mathcal{K}}_d(\dot{x},k,Y)g(Y)\mathrm{d}Y\right)}\mathrm{d}\mu_{\widehat{\mathbb{H}}_0^d}(\dot{x},k)
$$

$$
= \left(\frac{2}{\pi}\right)^d \int_{\widehat{\mathbb{H}}_0^d} \mathcal{G}_{\mathbb{H}}g(\dot{x},k)\,\overline{\mathcal{G}_{\mathbb{H}}g(\dot{x},k)}\mathrm{d}\mu_{\widehat{\mathbb{H}}_0^d}(\dot{x},k).
$$

The whole Theorem 6.1 is proved. $\square$

Appendix A. Useful tools and results

Let us first recall standard properties of Hermite functions that have been used repeatedly in the paper, when establishing identities pertaining to the function $\mathcal{W}$.

In addition to the creation operator $C_j \stackrel{\text{def}}{=} -\partial_j + M_j$ already defined in the introduction, it is convenient to introduce the following *annihilation* operator:

$$
\text{(A.1)} \qquad\qquad\qquad A_j \stackrel{\text{def}}{=} \partial_j + M_j.
$$

It is very classical (see e.g. [11, 14]) that

$$
\text{(A.2)} \qquad A_j H_n = \sqrt{2n_j}\,H_{n-\delta_j} \quad \text{and} \quad C_j H_n = \sqrt{2n_j+2}\,H_{n+\delta_j},
$$

As Relations (A.1) imply that

$$
\text{(A.3)} \qquad\qquad 2M_j = C_j + A_j \quad \text{and} \quad 2\partial_j = A_j - C_j,
$$

we discover that

$$
\text{(A.4)} \qquad
\begin{aligned}
M_j H_n &= \frac{1}{2}\left(\sqrt{2n_j}\,H_{n-\delta_j} + \sqrt{2n_j+2}\,H_{n+\delta_j}\right) \quad \text{and} \\
\partial_j H_n &= \frac{1}{2}\left(\sqrt{2n_j}\,H_{n-\delta_j} - \sqrt{2n_j+2}\,H_{n+\delta_j}\right).
\end{aligned}
$$

Note also that

$$
\text{(A.5)} \qquad C_j A_j + \mathrm{Id} = -\partial_j^2 + M_j^2 \quad \text{and} \quad [C_j, A_j] = -2\,\mathrm{Id},
$$

and thus

$$
\Delta_{\mathrm{osc}}^1 = \sum_{j=1}^d C_j A_j + d\,\mathrm{Id}\,.
$$

Finally, we have

$$\tag{A.6} [-\partial_j^2 + M_j^2, C_j] = 2C_j.$$

Let us next prove Relation (2.12) and Lemma 2.3. To this end, we write that by definition of $\mathcal{X}_j$ and of $\mathcal{W}$, we have

$$\mathcal{X}_j\big(e^{is\lambda}\,\mathcal{W}(\widehat{w},Y)\big) = \int_{\mathbb{R}^d} \mathcal{X}_j\big(e^{is\lambda+2i\lambda\langle\eta,z\rangle}\,H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)\big)\mathrm{d}z$$

$$= \int_{\mathbb{R}^d} e^{is\lambda+2i\lambda\langle\eta,z\rangle}\big(2i\lambda\eta_j + \partial_{y_j}\big)\big(H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)\big)\mathrm{d}z.$$

As $2i\lambda\eta_j\,e^{2i\lambda\langle\eta,z\rangle} = \partial_{z_j}\big(e^{2i\lambda\langle\eta,z\rangle}\big)$, an integration by parts implies that

$$\tag{A.7}\mathcal{X}_j\big(e^{is\lambda}\,\mathcal{W}(\widehat{w},Y)\big)$$
$$= \int_{\mathbb{R}^d} e^{is\lambda+2i\lambda\langle\eta,z\rangle}\big(\partial_{y_j} - \partial_{z_j}\big)\big(H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)\big)\mathrm{d}z.$$

The action of Ξ_j is simply described by

$$\Xi_j\big(e^{is\lambda}\,\mathcal{W}(\widehat{w},Y)\big) = \int_{\mathbb{R}^d} \Xi_j\big(e^{is\lambda+2i\lambda\langle\eta,z\rangle}\big)H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)\mathrm{d}z$$

$$= \int_{\mathbb{R}^d} e^{is\lambda+2i\lambda\langle\eta,z\rangle}\,2i\lambda(z_j - y_j)H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)\mathrm{d}z.$$

Together with (A.7), this gives

$$\big(\mathcal{X}_j^2 + \Xi_j^2\big)\big(e^{is\lambda}\,\mathcal{W}(\widehat{w},Y)\big)$$
$$= 4\int_{\mathbb{R}^d} e^{is\lambda+2i\lambda\langle\eta,z\rangle}\,H_{n,\lambda}(y+z)\big((-\partial_j^2 + \lambda^2 M_j^2)H_{m,\lambda}\big)(-y+z)\mathrm{d}z.$$

Putting together with (2.9), we get Formula (2.12) and thus (2.13).

Now, if f belongs to the Schwartz space, then combining (2.13) with integrations by parts yields (2.14). Indeed, we have

$$-4|\lambda|(2|m| + d)\widehat{f_\mathbb{H}}(n,m,\lambda) = -4|\lambda|(2|m| + d)\int_{\mathbb{H}^d} e^{-is\lambda}\,\overline{\mathcal{W}}(\widehat{w},Y)f(Y,s)\mathrm{d}s$$

$$= \int_{\mathbb{H}^d} \Delta_\mathbb{H}\big(e^{-is\lambda}\,\overline{\mathcal{W}}(\widehat{w},Y)\big)f(Y,s)\mathrm{d}s$$

$$= (\mathcal{F}_\mathbb{H}\Delta_\mathbb{H}f)(n,m,\lambda).$$

This gives, after iteration

$$\tag{A.8} 4^p|\lambda|^p(2|m| + d)^p\big|\widehat{f_\mathbb{H}}(n,m,\lambda)\big| \leqslant \|\Delta_\mathbb{H}^p f\|_{L^1(\mathbb{H}^d)},$$

which is one part of the decay inequality (2.15).

To complete the proof of Lemma 2.3, it suffices to exhibit suitable decay properties with respect to k. To this end, we introduce the *right-invariant* vector fields $\widetilde{\mathcal{X}}_j$ and $\widetilde{\Xi}_j$ defined by

$$(A.9) \quad \widetilde{\mathcal{X}}_j \overset{\text{def}}{=} \partial_{y_j} - 2\eta_j \partial_s \quad \text{and} \quad \widetilde{\Xi}_j \overset{\text{def}}{=} \partial_{\eta_j} + 2y_j \partial_s \quad \text{with} \quad j \in \{1, \ldots, d\}.$$

Then arguing as above, we readily get

$$4|\lambda|(2n_j + 1)\,\mathrm{e}^{\mathrm{i}s\lambda}\,\mathcal{W}(\widehat{w}, Y)$$
$$= -\,\mathrm{e}^{\mathrm{i}s\lambda}(\partial_{y_j}^2 + \partial_{\eta_j}^2)\mathcal{W}(\widehat{w}, Y) + 4\mathrm{i}\lambda\,\mathrm{e}^{\mathrm{i}s\lambda}(\eta_j \partial_{y_j} - y_j \partial_{\eta_j})\mathcal{W}(\widehat{w}, Y)$$
$$- 4\,\mathrm{e}^{\mathrm{i}s\lambda}\,\lambda^2(y_j^2 + \eta_j^2)\mathcal{W}(\widehat{w}, Y).$$

As

$$-\widetilde{\mathcal{X}}_j^2 + \mathcal{X}_j^2 - \widetilde{\Xi}_j^2 + \Xi_j^2 = 8\partial_s \mathcal{T}_j\,,$$

we get by difference with (2.12),

$$(A.10) \qquad |\lambda|(n_j - m_j)\,\mathrm{e}^{\mathrm{i}s\lambda}\,\mathcal{W}(\widehat{w}, Y) = \partial_s \mathcal{T}_j\big(\mathrm{e}^{\mathrm{i}s\lambda}\,\mathcal{W}\big)(\widehat{w}, Y)$$
$$= \mathrm{i}\lambda\,\mathrm{e}^{\mathrm{i}s\lambda}\,\mathcal{T}_j\mathcal{W}(\widehat{w}, Y).$$

After an obvious iteration this gives

$$(A.11) \qquad \prod_{j=1}^{d}(n_j - m_j)^{\alpha_j}\mathcal{W}(\widehat{w}, Y) = (\mathrm{i}\,\mathrm{sgn}(\lambda))^{|\alpha|}\mathcal{T}^\alpha\mathcal{W}(\widehat{w}, Y),$$

whence, mimicking the proof of (A.8),

$$(A.12) \quad |n - m|^p|\widehat{f}_\mathbb{H}(n, m, \lambda)| \leqslant \sup_{|\alpha|=p} \|\mathcal{T}^\alpha f\|_{L^1(\mathbb{H}^d)} \quad \text{for all } f \text{ in } \mathcal{S}(\mathbb{H}^d).$$

This completes the proof of Lemma 2.3.

The following result was needed to prove the convergence of the series in (5.7).

PROPOSITION A.1. — *For any integer p, a constant C exists such that*

$$|\mathcal{W}(n, m, \lambda, Y)| \leqslant C_p\bigg(\frac{\langle Y\rangle\big(1 + \sqrt{|\lambda|(|n + m| + d)}\big)}{1 + |m - n|}\bigg)^p$$

with $\langle Y\rangle \overset{\text{def}}{=} \sqrt{1 + |Y|^2}$.

Proof. — Identity (A.11) implies that for all integer p,

$$(A.13) \qquad |m - n|^p|\mathcal{W}(n, m, \lambda, Y)| \leqslant C_k\langle Y\rangle^p \sup_{|\alpha|\leqslant p} |\partial_Y^\alpha \mathcal{W}(n, m, \lambda, Y)|.$$

Differentiating under the integral and using Leibniz formula, we get for all $(\alpha_1, \alpha_2) \in \mathbb{N}^d \times \mathbb{N}^d$,

$$\partial_y^{\alpha_1} \partial_\eta^{\alpha_2} \mathcal{W}(\widehat{w}, Y) = \int_{\mathbb{R}^d} e^{2i\lambda \langle \eta, z \rangle} (i \operatorname{sgn} \lambda)^{|\alpha_2|} |\lambda|^{\frac{|\alpha_2|}{2}}$$
$$\times |\lambda|^{\frac{|\alpha_2|}{2}} (y + z - y + z)^{\alpha_2} \partial_y^{\alpha_1} \big(H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z) \big) \mathrm{d}z,$$

whence

$$\partial_y^{\alpha_1} \partial_\eta^{\alpha_2} \mathcal{W}(\widehat{w}, Y) = (i \operatorname{sgn}(\lambda))^{|\alpha_2|} |\lambda|^{\frac{|\alpha_1| + |\alpha_2|}{2}} \sum_{\substack{\beta_1 \leqslant \alpha_1 \\ \beta_2 \leqslant \alpha_2}} (-1)^{\beta_1} C_{\alpha_1}^{\beta_1} C_{\alpha_2}^{\beta_2}$$
$$\times \int_{\mathbb{R}^d} e^{2i\lambda \langle \eta, z \rangle} (M^{\beta_2} \partial^{\beta_1} H_n)_\lambda (y+z) (M^{\beta_2 - \beta_2} \partial^{\alpha_1 - \beta_1} H_m)_\lambda (-y+z) \mathrm{d}z.$$

Stating $\alpha = (\alpha_1, \alpha_2)$, we infer that

$$\partial_y^{\alpha_1} \partial_\eta^{\alpha_2} \mathcal{W}(\widehat{w}, Y) = \big| \partial_Y^\alpha \mathcal{W}(\widehat{w}, Y) \big|$$
$$\leqslant |\lambda|^{\frac{|\alpha_1| + |\alpha_2|}{2}} \sum_{\substack{\beta_1 \leqslant \alpha_1 \\ \beta_2 \leqslant \alpha_2}} C_{\alpha_1}^{\beta_1} C_{\alpha_2}^{\beta_2} \| M^{\beta_2} \partial^{\beta_1} H_n \|_{L^2} \| M^{\alpha_2 - \beta_2} \partial^{\alpha_1 - \beta_1} H_m \|_{L^2}.$$

Relations (A.2)–(A.4) imply that

$$\| M^{\beta_2} \partial^{\beta_1} H_n \|_{L^2} \| M^{\alpha_2 - \beta_2} \partial^{\alpha_1 - \beta_1} H_m \|_{L^2} \leqslant C_{\alpha_1, \alpha_2} (|n + m| + d)^{\frac{\alpha_1 + \alpha_2}{2}}.$$

Plugging that inequality in (A.13) completes the proof of the proposition. $\qquad \square$

Let us finally prove the convolution identity (2.19). It is just based on the fact that for all (n, m, λ) in $\widetilde{\mathbb{H}}^d$ and any integrable function f on $\mathbb{H}^d$, we have

$$\sum_{m \in \mathbb{N}^d} |\widehat{f}_{\mathbb{H}}(n, m, \lambda)|^2 \leqslant \|f\|_{L^1(\mathbb{H}^d)}^2 \quad \text{and} \quad \sum_{n \in \mathbb{N}^d} |\widehat{f}_{\mathbb{H}}(n, m, \lambda)|^2 \leqslant \|f\|_{L^1(\mathbb{H}^d)}^2.$$

Indeed, if A and B are two bounded operators on a separable Hilbert space $\mathcal{H}$ endowed with an orthonormal basis $(e_n)_{n \in \mathbb{N}^d}$ then, denoting

$$A(n, m) \overset{\text{def}}{=} (Ae_m | e_n) \quad \text{and} \quad B(n, m) \overset{\text{def}}{=} (Be_m | e_n),$$

one may write

$$\sum_{\ell \in \mathbb{N}^d} |A(\ell, m)|^2 = \|Ae_m\|_{\mathcal{H}}^2 \leqslant \|A\|_{\mathcal{L}(\mathcal{H})}^2$$
$$\text{and} \quad \sum_{\ell \in \mathbb{N}^d} |A(n, \ell)|^2 = \|A^* e_n\|_{\mathcal{H}}^2 \leqslant \|A\|_{\mathcal{L}(\mathcal{H})}^2.$$

Therefore, from Inequality (2.1) and Definition 2.2, we readily infer that

$$
\begin{aligned}
\big(\mathfrak{F}_{\mathbb{H}}(f)(\lambda) \circ \mathfrak{F}_{\mathbb{H}}(g)(\lambda) H_{m,\lambda} \big| H_{n,\lambda}\big)_{L^2(\mathbb{R}^d)} & \\
= \lim_{(N,N') \to (\infty,\infty)} \sum_{\substack{|\ell| \leqslant N \\ |\ell'| \leqslant N'}} & \widehat{f}_{\mathbb{H}}(\ell',\ell,\lambda)\widehat{g}_{\mathbb{H}}(\ell,m,\lambda)(H_{\ell',\lambda}|H_{n,\lambda}) \\
= \lim_{N \to \infty} \sum_{|\ell| \leqslant N} & \widehat{f}_{\mathbb{H}}(n,\ell,\lambda)\,\widehat{g}_{\mathbb{H}}(\ell,m,\lambda) \\
= \sum_{\ell \in \mathbb{N}^d} & \widehat{f}_{\mathbb{H}}(n,\ell,\lambda)\,\widehat{g}_{\mathbb{H}}(\ell,m,\lambda).
\end{aligned}
$$

Then, remembering Relation (2.4) completes the proof of (2.19).

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