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A FRICTIONAL CONTACT PROBLEM WITH WEAR AND DAMAGE FOR ELECTRO-VISCOELASTIC MATERIALS

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Abstract. We consider a quasistatic contact problem for an electro-viscoelastic body. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. The damage of the material caused by elastic deformation is taken into account, its evolution is described by an inclusion of parabolic type. We present a weak formulation for the model and establish existence and uniqueness results. The proofs are based on classical results for elliptic variational inequalities, parabolic inequalities and fixed point arguments.

Keywords: quasistatic process, electro-viscoelastic materials, bilateral contact, friction, damage, existence and uniqueness, monotone operator, fixed point, weak solution

MSC 2010: 74M15, 74M10, 74R99

1. Introduction

The piezoelectric effect is characterized by the coupling between the mechanical and electrical behavior of the materials. Indeed, the appearance of electric charges on some crystals submitted to the action of body forces and surface tractions was observed and their dependence on the deformation process was underlined. Conversely, it was proved experimentally that the action of electric field on the crystals may generate strain and stress. A deformable material which exhibits such a behavior is called a piezoelectric material. Piezoelectric materials are used extensively as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and measuring equipment. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore, there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [2], [10]. A static frictional contact problem for electric-elastic materials was considered in [3], [12]. A slip-dependent frictional contact problem for electro-elastic materials was studied in [16]. Contact problems with friction or adhesion for electro-viscoelastic materials were studied recently in [5], [11], [14], [15].

The goal of this paper is to make the coupling of an electro-viscoelastic problem with damage and a frictional contact problem with wear. We study a quasistatic problem of frictional bilateral contact with wear. We model the material behavior with an electro-viscoelastic constitutive law with damage and the contact is frictional and bilateral with a moving rigid foundation. We derive a variational formulation and prove the existence and uniqueness of the weak solution.

The paper is structured as follows. In Section 2 we present notation and some preliminaries. The model is described in Section 3, where the variational formulation is given. In Section 4, we present our main result stated in Theorem 4.1 and its proof which is based on arguments for elliptic variational inequalities, parabolic inequalities and fixed point.

2. Notation and preliminaries

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [4], [6], [13]. We denote by S^d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d=2,3)$, while "$\cdot$" and $|\cdot|$ represent the inner product and the Euclidean norm on S^d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with outer Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on $\partial\Omega = \Gamma$. We shall use the notation

$$\begin{gathered} H = L^2(\Omega)^d = \{\mathbf{u} = (u_i)\colon\ u_i \in L^2(\Omega)\}, \\ H^1(\Omega)^d = \{\mathbf{u} = (u_i)\colon\ u_i \in H^1(\Omega)\}, \\ \mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij})\colon\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \\ \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H}\colon\ \operatorname{Div} \boldsymbol{\sigma} \in H\}, \end{gathered}$$

where $\varepsilon\colon H^1(\Omega)^d \to \mathcal{H}$ and $\operatorname{Div}\colon \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

The spaces H, $H^1(\Omega)^d$, $\mathcal{H}$, and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x \quad \forall\,\mathbf{u},\mathbf{v}\in H,$$
$$(\mathbf{u},\mathbf{v})_{H^1(\Omega)^d} = \int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x + \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,\mathrm{d}x \quad \forall\,\mathbf{u},\mathbf{v}\in H^1(\Omega)^d,$$

where

$$\nabla\mathbf{v} = (v_{i,j}) \quad \forall\,\mathbf{v}\in H^1(\Omega)^d,$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,\mathrm{d}x \quad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{H},$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div}\boldsymbol{\sigma},\operatorname{Div}\boldsymbol{\tau})_H \quad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{H}_1.$$

The associated norms on the spaces H, $H^1(\Omega)^d$, $\mathcal{H}$, and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{H^1(\Omega)^d}$, $|\cdot|_{\mathcal{H}}$, and $|\cdot|_{\mathcal{H}_1}$, respectively. Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma\colon\ H^1(\Omega)^d \to H_\Gamma$ be the trace map. For every element $\mathbf{v}\in H^1(\Omega)^d$, we also use the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on the boundary Γ given by

$$v_\nu = \mathbf{v}\cdot\boldsymbol{\nu},\quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu}. \tag{2.1}$$

Similarly, for a regular (say C^1) tensor field $\boldsymbol{\sigma}\colon\ \Omega\to S^d$ we define its normal and tangential components by

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu},\quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}, \tag{2.2}$$

and for all $\boldsymbol{\sigma}\in\mathcal{H}_1$ the following Green's formula holds:

$$(\boldsymbol{\sigma},\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\operatorname{Div}\boldsymbol{\sigma},\mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\mathbf{v}\,\mathrm{d}a \quad \forall\,\mathbf{v}\in H^1(\Omega)^d. \tag{2.3}$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1\leqslant p\leqslant\infty$ and $k\geqslant 1$. For $T>0$ we denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$|\mathbf{f}|_{C(0,T;X)} = \max_{t\in[0,T]} |\mathbf{f}(t)|_X,$$
$$|\mathbf{f}|_{C^1(0,T;X)} = \max_{t\in[0,T]} |\mathbf{f}(t)|_X + \max_{t\in[0,T]} |\dot{\mathbf{f}}(t)|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and if X_1 and X_2 are real Hilbert spaces then $X_1\times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$.

3. Mechanical and variational formulations

We describe the model for the process and present its variational formulation. The physical setting is the following. An electro-viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. The body is submitted to the action of body forces of density $\mathbf{f}_0$ and volume electric charges of density q_0. It is also submitted to mechanical and electric constraint on the boundary. We consider a partition of Γ into three disjoint measurable parts Γ_1, Γ_2, and Γ_3, on the one hand, and into two measurable parts Γ_a and Γ_b, on the other hand, such that $\mathrm{meas}(\Gamma_1) > 0$, $\mathrm{meas}(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface traction of density $\mathbf{f}_2$ acts on $\Gamma_2 \times (0, T)$ and a body force of density $\mathbf{f}_0$ acts in $\Omega \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach to the process. We denote by $\mathbf{u}$ the displacement field, by $\boldsymbol{\sigma}$ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor. We use an electro-viscoelastic constitutive law with damage given by

$$\boldsymbol{\sigma} = \mathcal{A}(\varepsilon(\dot{\mathbf{u}})) + \mathcal{G}(\varepsilon(\mathbf{u}), \beta) - \mathcal{E}^*\mathbf{E}(\varphi),$$
$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) + B\mathbf{E}(\varphi),$$

where $\mathcal{A}$ is a given nonlinear function, $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transpose and B denotes the electric permittivity tensor. The symbol $\mathcal{G}$ represents the elasticity operator, where β is an internal variable describing the damage of the material caused by elastic deformations. The inclusion used for the evolution of the damage field is

$$\dot{\beta} - k\triangle\beta + \partial\varphi_K(\beta) \ni S(\varepsilon(\mathbf{u}), \beta),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in H^1(\Omega)\colon\ 0 \leqslant \xi \leqslant 1 \text{ a.e. in } \Omega\},$$

k is a positive coefficient, $\partial\varphi_K$ denotes the subdifferential of the indicator function φ_K and S is a given constitutive function which describes the sources of the damage in the system. When $\beta = 1$ the material is undamaged, when $\beta = 0$ the material is completely damaged, and for $0 < \beta < 1$ there is partial damage. General models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [7] and [8] and references therein. The models describe the evolution of the material damage which results from the excess tension or compression in the body as a result of applied forces and tractions. Mathematical analysis of one-dimensional damage models can be found in [9].

We now briefly describe the boundary conditions on the contact surface Γ_3, based on the model derived in [18], [19]. We introduce the wear function $w\colon \Gamma_3 \times [0,T] \to \mathbb{R}_+$ which measures the wear of the surface. The wear is identified as the normal depth of the material that is lost. Since the body is in bilateral contact with the foundation, it follows that

$$u_\nu = -w \quad \text{on } \Gamma_3. \tag{3.1}$$

Thus the location of the contact evolves with the wear. We point out that the effect of the wear is the recession on Γ_3 and therefore, it is natural to expect that $u_\nu \leqslant 0$ on Γ_3, which implies $w \geqslant 0$ on Γ_3.

The evolution of the wear of the contacting surface is governed by a simplified version of Archard's law (see [18], [19]) which we now describe. The rate form of Archard's law is

$$\dot{w} = -k_1\sigma_\nu|\dot{\mathbf{u}}_\tau - \mathbf{v}^*|,$$

where $k_1 > 0$ is a wear coefficient, $\mathbf{v}^*$ is the tangential velocity of the foundation and $|\dot{\mathbf{u}}_\tau - \mathbf{v}^*|$ represents the slip speed between the contact surface and the foundation. We see that the rate of wear is assumed to be proportional to the contact stress and the slip speed. For the sake of simplicity we assume in the rest of the section that the motion of the foundation is uniform, i.e., $\mathbf{v}^*$ does not vary in time. Denote $v^* = |\mathbf{v}^*| > 0$. We assume that v^* is large so that we can neglect in the sequel $\dot{\mathbf{u}}_\tau$ as compared with $\mathbf{v}^*$ to obtain the following version of Archard's law

$$\dot{w} = -k_1 v^*\sigma_\nu. \tag{3.2}$$

The use of the simplified law (3.2) for the evolution of the wear avoids some mathematical difficulties in the study of the quasistatic electro-viscoelastic contact problem.

We can now eliminate the unknown function w from the problem. Let $\zeta = k_1 v^*$ and $\alpha = 1/\zeta$. Using (3.1) and (3.2) we have

$$\sigma_\nu = \alpha\dot{u}_\nu. \tag{3.3}$$

We model the frictional contact between the electro-viscoelastic body and the foundation with Coulomb's law of dry friction. Since there is only sliding contact, it follows that

$$|\boldsymbol{\sigma}_\tau| = \mu|\sigma_\nu|, \quad \boldsymbol{\sigma}_\tau = -\lambda(\dot{\mathbf{u}}_\tau - \mathbf{v}^*), \quad \lambda \geqslant 0, \tag{3.4}$$

where $\mu > 0$ is the coefficient of friction. These relations set constraints on the evolution of the tangential stress; in particular, the tangential stress is in the direction opposite to the relative sliding velocity $\dot{\mathbf{u}}_\tau - \mathbf{v}^*$.

Naturally, the wear increases in time, i.e. $\dot{w} \geqslant 0$. Hence, it follows from (3.1) and (3.2) that $\dot{u}_\nu \leqslant 0$ and $\sigma_\nu \leqslant 0$ on Γ_3. Thus, the conditions (3.3) and (3.4) imply

$$-\sigma_\nu = \alpha|\dot{u}_\nu|, \quad |\boldsymbol{\sigma}_\tau| = -\mu\sigma_\nu, \quad \boldsymbol{\sigma}_\tau = -\lambda(\dot{\mathbf{u}}_\tau - \mathbf{v}^*), \quad \lambda \geqslant 0. \tag{3.5}$$

To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$. Then, the classical formulation of the mechanical problem of a frictional bilateral contact with wear may be stated as follows.

Problem P. Find a displacement field $\mathbf{u}\colon \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon \Omega \times [0,T] \to S^d$, an electric potential field $\varphi\colon \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $\mathbf{D}\colon \Omega \times [0,T] \to \mathbb{R}^d$ and a damage field $\beta\colon \Omega \times [0,T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma} = \mathcal{A}(\varepsilon(\dot{\mathbf{u}})) + \mathcal{G}(\varepsilon(\mathbf{u}), \beta) + \mathcal{E}^*\nabla\varphi \quad \text{in } \Omega \times (0,T), \tag{3.6}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) - B\nabla\varphi \quad \text{in } \Omega \times (0,T), \tag{3.7}$$

$$\dot{\beta} - k\triangle\beta + \partial\varphi_K(\beta) \ni S(\varepsilon(\mathbf{u}), \beta) \quad \text{in } \Omega \times (0,T), \tag{3.8}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{3.9}$$

$$\operatorname{div}\mathbf{D} = q_0 \quad \text{in } \Omega \times (0,T), \tag{3.10}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{3.11}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.12}$$

$$\begin{cases} \sigma_\nu = -\alpha|\dot{u}_\nu|, \ |\boldsymbol{\sigma}_\tau| = -\mu\sigma_\nu, \\ \boldsymbol{\sigma}_\tau = -\lambda(\dot{\mathbf{u}}_\tau - \mathbf{v}^*), \ \lambda \geqslant 0 \ \text{on } \Gamma_3 \times (0,T), \end{cases} \tag{3.13}$$

$$\frac{\partial\beta}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{3.14}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{3.15}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{3.16}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \beta(0) = \beta_0 \ \text{in } \Omega. \tag{3.17}$$

First, (3.6) and (3.7) represent the electro-viscoelastic constitutive law with damage, the evolution of the damage field is governed by the inclusion of parabolic type given by relation (3.8), where S is the mechanical source of the damage growth, assumed to be rather general function of the strains and damage itself, and $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K. Equations (3.9) and (3.10) represent the equilibrium equations for the stress and electric-displacement fields while (3.11) and (3.12) are the displacement and traction boundary condition, respectively, (3.13) represents the frictional bilateral contact with wear described above. The relation (3.14) represents a homogeneous Neumann boundary condition, where $\partial\beta/\partial\nu$ represents the normal derivative of β, (3.15) and (3.16) represent the electric boundary conditions, and (3.17) represents the initial displacement field and the initial damage field. To obtain the variational formulation of the problem (3.6)–(3.17), we introduce the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\mathbf{v} \in H^1(\Omega)^d\colon\ \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Since $\mathrm{meas}(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$, depending only on Ω and Γ_1, such that

$$|\varepsilon(\mathbf{v})|_{\mathcal{H}} \geqslant C_k |\mathbf{v}|_{H^1(\Omega)^d} \quad \forall\, \mathbf{v} \in V.$$

A proof of Korn's inequality may be found in [13, p. 79]. On the space V we consider the inner product and the associated norm given by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad |\mathbf{v}|_V = |\varepsilon(\mathbf{v})|_{\mathcal{H}} \quad \forall\, \mathbf{u}, \mathbf{v} \in V. \tag{3.18}$$

It follows that $|\cdot|_{H^1(\Omega)^d}$ and $|\cdot|_V$ are equivalent norms on V and therefore, $(V, |\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev Trace Theorem and (3.18), there exists a constant $C_0 > 0$, depending only on Ω, Γ_1, and Γ_3, such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \leqslant C_0 |\mathbf{v}|_V \quad \forall\, \mathbf{v} \in V. \tag{3.19}$$

We also introduce the spaces

$$W = \{\phi \in H^1(\Omega)\colon\ \phi = 0 \text{ on } \Gamma_a\},$$
$$\mathcal{W} = \{\mathbf{D} = (D_i)\colon\ D_i \in L^2(\Omega),\ \operatorname{div}\mathbf{D} \in L^2(\Omega)\},$$

where $\operatorname{div}\mathbf{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi, \phi)_W = \int_\Omega \nabla\varphi \cdot \nabla\phi \,\mathrm{d}x,$$
$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_\Omega \mathbf{D} \cdot \mathbf{E} \,\mathrm{d}x + \int_\Omega \operatorname{div}\mathbf{D} \cdot \operatorname{div}\mathbf{E} \,\mathrm{d}x.$$

The associated norms will be denoted by $|\cdot|_W$ and $|\cdot|_{\mathcal{W}}$, respectively. Moreover, when $\mathbf{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds:

$$(\mathbf{D}, \nabla\phi)_H + (\operatorname{div}\mathbf{D}, \phi)_{L^2(\Omega)} = \int_\Gamma \mathbf{D}\cdot\boldsymbol{\nu}\phi\,\mathrm{d}a \quad \forall\,\phi \in H^1(\Omega).$$

Notice also that, since $\mathrm{meas}(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$|\nabla\phi|_H \geqslant C_F|\phi|_{H^1(\Omega)} \quad \forall\,\phi \in W, \tag{3.20}$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a. In the study of the mechanical problem (3.6)–(3.17), we assume that the viscosity function $\mathcal{A}\colon\ \Omega\times S^d \to S^d$ satisfies

$$\left\{\begin{array}{l} \text{(a) There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad |\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)| \leqslant L_{\mathcal{A}}|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|\ \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in S^d,\ \text{a.e. } \mathbf{x}\in\Omega. \\ \text{(b) There exists a constant } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m_{\mathcal{A}}|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \\ \quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in S^d,\ \text{a.e. } \mathbf{x}\in\Omega. \\ \text{(c) The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \\ \quad \text{for any } \boldsymbol{\varepsilon} \in S^d. \\ \text{(d) The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}. \end{array}\right. \tag{3.21}$$

The elasticity operator $\mathcal{G}\colon\ \Omega\times S^d\times\mathbb{R} \to S^d$ satisfies

$$\left\{\begin{array}{l} \text{(a) There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad |\mathcal{G}(\mathbf{x},\boldsymbol{\varepsilon}_1,\alpha_1) - \mathcal{G}(\mathbf{x},\boldsymbol{\varepsilon}_2,\alpha_2)| \leqslant L_{\mathcal{G}}(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\alpha_1 - \alpha_2|) \\ \quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in S^d,\ \forall\,\alpha_1,\alpha_2 \in \mathbb{R} \text{ a.e. } \mathbf{x}\in\Omega. \\ \text{(b) The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x},\boldsymbol{\varepsilon},\alpha) \text{ is Lebesgue measurable on } \Omega \\ \quad \text{for any } \boldsymbol{\varepsilon} \in S^d \text{ and } \alpha\in\mathbb{R}. \\ \text{(c) The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x},\mathbf{0},0) \text{ belongs to } \mathcal{H}. \end{array}\right. \tag{3.22}$$

The damage source function $S\colon\ \Omega\times S^d\times\mathbb{R} \to \mathbb{R}$ satisfies

$$\left\{\begin{array}{l} \text{(a) There exists a constant } L_S > 0 \text{ such that} \\ \quad |S(\mathbf{x},\boldsymbol{\varepsilon}_1,\alpha_1) - S(\mathbf{x},\boldsymbol{\varepsilon}_2,\alpha_2)| \leqslant L_S(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\alpha_1 - \alpha_2|) \\ \quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in S^d,\ \forall\,\alpha_1,\alpha_2 \in \mathbb{R} \text{ a.e. } \mathbf{x}\in\Omega. \\ \text{(b) For any } \boldsymbol{\varepsilon} \in S^d \text{ and } \alpha\in\mathbb{R},\ \mathbf{x} \to S(\mathbf{x},\boldsymbol{\varepsilon},\alpha) \\ \quad \text{is Lebesgue measurable on } \Omega. \\ \text{(c) The mapping } \mathbf{x} \to S(\mathbf{x},\mathbf{0},0) \text{ belongs to } L^2(\Omega). \end{array}\right. \tag{3.23}$$

The electric permittivity operator $B = (b_{ij})\colon\ \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$(3.24)\qquad \begin{cases} \text{(a) } B(\mathbf{x})\mathbf{E} = (b_{ij}(\mathbf{x})E_j)\ \forall\, \mathbf{E} = (E_i) \in \mathbb{R}^d, \ \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) } b_{ij} = b_{ji},\ b_{ij} \in L^\infty(\Omega). \\ \text{(c) There exists a constant } m_B > 0 \text{ such that} \\ \qquad B\mathbf{E}\cdot\mathbf{E} \geqslant m_B|\mathbf{E}|^2\ \ \forall\, \mathbf{E} = (E_i) \in \mathbb{R}^d,\ \text{a.e. } \mathbf{x} \in \Omega. \end{cases}$$

The piezoelectric operator $\mathcal{E}\colon\ \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$(3.25)\qquad \begin{cases} \text{(a) } \mathcal{E}(\mathbf{x})\boldsymbol{\tau} = (e_{ijk}(\mathbf{x})\tau_{jk})\ \ \forall\, \boldsymbol{\tau} = (\tau_{ij}) \in S^d, \ \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b) } e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{cases}$$

We also suppose that the body forces and surface tractions have the regularity

$$(3.26)\qquad \mathbf{f}_0 \in C(0,T;H), \quad \mathbf{f}_2 \in C(0,T;L^2(\Gamma_2)^d),$$

$$(3.27)\qquad q_0 \in C(0,T;L^2(\Omega)), \quad q_2 \in C(0,T;L^2(\Gamma_b)),$$

$$(3.28)\qquad q_2(t) = 0 \text{ on } \Gamma_3 \quad \forall\, t \in [0,T].$$

The functions α and μ have the following properties:

$$(3.29)\qquad \alpha \in L^\infty(\Gamma_3), \quad \alpha(\mathbf{x}) \geqslant \alpha^* > 0 \text{ a.e. on } \Gamma_3,$$

$$(3.30)\qquad \mu \in L^\infty(\Gamma_3), \quad \mu \geqslant 0 \text{ a.e. on } \Gamma_3.$$

Note that we need to impose the assumption (3.28) for physical reasons, indeed the foundation is assumed to be an insulator and therefore, the electric charges (which are prescribed on $\Gamma_b \supset \Gamma_3$) have to vanish on the potential contact surface. The initial displacement field satisfies

$$(3.31)\qquad \mathbf{u}_0 \in V,$$

and the initial damage field satisfies

$$(3.32)\qquad \beta_0 \in K.$$

We define the bilinear form $a\colon\ H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$(3.33)\qquad a(\xi,\vartheta) = k\int_\Omega \nabla\xi\cdot\nabla\vartheta\,\mathrm{d}x.$$

Next, we denote by $\mathbf{f}\colon\ [0,T] \to V$ the function defined by

$$(3.34)\qquad (\mathbf{f}(t),\mathbf{v})_V = \int_\Omega \mathbf{f}_0(t)\cdot\mathbf{v}\,\mathrm{d}x + \int_{\Gamma_2} \mathbf{f}_2(t)\cdot\mathbf{v}\,\mathrm{d}a \quad \forall\, \mathbf{v} \in V,\ t \in [0,T],$$

and we denote by $q\colon\ [0,T]\to W$ the function defined by

$$(q(t),\phi)_W=\int_\Omega q_0(t)\cdot\phi\,\mathrm{d}x-\int_{\Gamma_b} q_2(t)\cdot\phi\,\mathrm{d}a\quad\forall\,\phi\in W,\ t\in[0,T]. \tag{3.35}$$

Next, we denote by $j\colon\ V\times V\to\mathbb{R}$ the functional defined by

$$j(\mathbf{u},\mathbf{v})=\int_{\Gamma_3}\alpha|u_\nu|(\mu|\mathbf{v}_\tau-\mathbf{v}^*|+v_\nu)\,\mathrm{d}a\quad\forall\,\mathbf{u},\mathbf{v}\in V. \tag{3.36}$$

We note that conditions (3.26) and (3.27) imply

$$\mathbf{f}\in C(0,T;V),\quad q\in C(0,T;W). \tag{3.37}$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.6)–(3.17).

Problem PV. Find a displacement field $\mathbf{u}\colon\ [0,T]\to V$, a stress field $\boldsymbol{\sigma}\colon$ $[0,T]\to\mathcal{H}_1$, an electric potential field $\varphi\colon\ [0,T]\to W$, an electric displacement field $\mathbf{D}\colon\ [0,T]\to H$ and a damage field $\beta\colon\ [0,T]\to H^1(\Omega)$ such that

$$\boldsymbol{\sigma}(t)=\mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t)))+\mathcal{G}(\varepsilon(\mathbf{u}(t)),\beta(t))+\mathcal{E}^*\nabla\varphi(t),\quad t\in(0,T), \tag{3.38}$$

$$\begin{gathered}(\boldsymbol{\sigma}(t),\varepsilon(\mathbf{v}-\dot{\mathbf{u}}(t)))_{\mathcal{H}}+j(\dot{\mathbf{u}}(t),\mathbf{v})-j(\dot{\mathbf{u}}(t),\dot{\mathbf{u}}(t))\\ \geqslant(\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}(t))_V\quad\forall\,\mathbf{v}\in V,\ t\in(0,T),\end{gathered} \tag{3.39}$$

$$\begin{gathered}\beta(t)\in K\ \text{ for all } t\in[0,T],\\ (\dot\beta(t),\xi-\beta(t))_{L^2(\Omega)}+a(\beta(t),\xi-\beta(t))\geqslant(S(\varepsilon(\mathbf{u}(t)),\beta(t)),\xi-\beta(t))_{L^2(\Omega)}\\ \forall\,\xi\in K,\ \text{a.e. } t\in(0,T),\end{gathered} \tag{3.40}$$

$$\mathbf{D}(t)=\mathcal{E}\varepsilon(\mathbf{u}(t))-B\nabla\varphi(t),\quad t\in(0,T), \tag{3.41}$$

$$(\mathbf{D}(t),\nabla\phi)_H=-(q(t),\phi)_W\quad\forall\,\phi\in W,\ t\in(0,T), \tag{3.42}$$

$$\mathbf{u}(0)=\mathbf{u}_0,\quad\beta(0)=\beta_0. \tag{3.43}$$

We notice that the variational problem PV is formulated in terms of displacement field, stress field, electrical potential field, electric displacement field and damage field and the functions $\mathbf{u}$, $\boldsymbol{\sigma}$, φ, $\mathbf{D}$, and β which satisfy (3.38)–(3.43) are called a weak solution of the contact problem P. The existence of the unique solution of Problem PV is stated and proved in the next section.

4. An existence and uniqueness result

Now, we propose our existence and uniqueness result.

Theorem 4.1. *Assume that* (3.21)–(3.32) *hold. Then there exists a constant* α_0 *which depends only on* Ω, Γ_1, Γ_3 *and* $\mathcal{A}$ *such that if*

$$|\alpha|_{L^\infty(\Gamma_3)}(|\mu|_{L^\infty(\Gamma_3)}+1)<\alpha_0, \tag{4.1}$$

then there exists a unique solution $\{\mathbf{u},\boldsymbol{\sigma},\varphi,\mathbf{D},\beta\}$ *to Problem* PV. *Moreover, the solution satisfies*

$$\mathbf{u}\in C^1(0,T;V), \tag{4.2}$$
$$\boldsymbol{\sigma}\in C(0,T;\mathcal{H}_1), \tag{4.3}$$
$$\varphi\in C(0,T;W), \tag{4.4}$$
$$\mathbf{D}\in C(0,T;\mathcal{W}), \tag{4.5}$$
$$\beta\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)). \tag{4.6}$$

The functions $\mathbf{u}$, $\boldsymbol{\sigma}$, φ, $\mathbf{D}$, and β which satisfy (3.38)–(3.43) are called a weak solution of the contact problem P. We conclude that, under the assumptions (3.21)–(3.32) and if (4.1) is satisfied, the mechanical problem (3.6)–(3.17) has a unique weak solution satisfying (4.2)–(4.6). The proof of Theorem 4.1 is carried out in several steps that we prove in what follows. Everywhere in this section we suppose that the assumptions of Theorem 4.1 hold, and we denote by C a generic positive constant which depends on Ω, Γ_1, and Γ_3 and may change from place to place.

R e m a r k 4.1. We remark that if v^* is large enough then $\alpha=1/(k_1v^*)$ is sufficiently small and therefore, the condition (4.1) for the unique solvability of Problem PV is satisfied. We conclude that the mechanical problem (3.6)–(3.17) has a unique weak solution if the tangential velocity of the foundation is large enough. Moreover, having solved the problem (3.6)–(3.17), we can find the wear function by integrating (3.2) and using the initial condition $w(0)=0$ which means that at the initial moment the body is not subject to any prior wear.

Let $\boldsymbol{\eta}\in C(0,T;\mathcal{H})$ and $\mathbf{g}\in C(0,T;V)$ be given. In the first step we consider the following variational problem.

P r o b l e m $\mathrm{PV}_{\eta g}$. Find a displacement field $\mathbf{v}_{\eta g}\colon [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_{\eta g}\colon [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_{\eta g}(t) = \mathcal{A}(\varepsilon(\mathbf{v}_{\eta g}(t))) + \boldsymbol{\eta}(t), \quad t \in [0,T], \tag{4.7}$$

$$(\boldsymbol{\sigma}_{\eta g}(t), \varepsilon(\mathbf{v} - \mathbf{v}_{\eta g}(t)))_{\mathcal{H}} + j(\mathbf{g}(t), \mathbf{v}) - j(\mathbf{g}(t), \mathbf{v}_{\eta g}(t)) \geqslant (\mathbf{f}(t), \mathbf{v} - \mathbf{v}_{\eta g}(t))_V \quad \forall\, \mathbf{v} \in V,\ \ t \in [0,T]. \tag{4.8}$$

In the study of Problem $\mathrm{PV}_{\eta g}$ we have the following result.

Lemma 4.1. $\mathrm{PV}_{\eta g}$ *has a unique weak solution such that*

$$\mathbf{v}_{\eta g} \in C(0,T;V), \quad \boldsymbol{\sigma}_{\eta g} \in C(0,T;\mathcal{H}_1). \tag{4.9}$$

P r o o f. We define the operator $A\colon V \to V$ such that

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}(\varepsilon(\mathbf{u})), \varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall\, \mathbf{u}, \mathbf{v} \in V. \tag{4.10}$$

It follows from (4.10) and (3.21) (a) that

$$|A\mathbf{u} - A\mathbf{v}|_V \leqslant L_{\mathcal{A}} |\mathbf{u} - \mathbf{v}|_V \quad \forall\, \mathbf{u}, \mathbf{v} \in V, \tag{4.11}$$

which shows that $A\colon V \to V$ is Lipschitz continuous. Now, by (4.10) and (3.21) (b) we find

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_V \geqslant m_{\mathcal{A}} |\mathbf{u} - \mathbf{v}|_V^2 \quad \forall\, \mathbf{u}, \mathbf{v} \in V, \tag{4.12}$$

i.e., that $A\colon V \to V$ is a strongly monotone operator on V. Moreover, using Riesz Representation Theorem, we may define an element $\mathbf{F} \in C(0,T;V)$ by

$$(\mathbf{F}(t), \mathbf{v})_V = (\mathbf{f}(t), \mathbf{v})_V - (\boldsymbol{\eta}(t), \varepsilon(\mathbf{v}))_{\mathcal{H}}.$$

Since A is a strongly monotone and Lipschitz continuous operator on V and since $\mathbf{v} \to j(g(t), \mathbf{v})$ is a proper convex lower semicontinuous functional, it follows from classical result on elliptic inequalities (see for example [4]) that there exists a unique function $\mathbf{v}_{\eta g}(t) \in V$ which satisfies

$$(A\mathbf{v}_{\eta g}(t), \mathbf{v} - \mathbf{v}_{\eta g}(t))_V + j(\mathbf{g}(t), \mathbf{v}) - j(\mathbf{g}(t), \mathbf{v}_{\eta g}(t)) \geqslant (\mathbf{F}(t), \mathbf{v} - \mathbf{v}_{\eta g}(t))_V \quad \forall\, \mathbf{v} \in V. \tag{4.13}$$

We use the relation (4.7), the assumption (3.21), and the properties of the deformation tensor to obtain that $\boldsymbol{\sigma}_{\eta g}(t) \in \mathcal{H}$. Since $\mathbf{v} = \mathbf{v}_{\eta g}(t) \pm \psi$ satisfies (4.8), where $\psi \in \mathcal{D}(\Omega)^d$ is arbitrary, using the definition (3.34) for $\mathbf{f}(t)$, we find

$$\operatorname{Div} \boldsymbol{\sigma}_{\eta g}(t) + \mathbf{f}_0(t) = 0, \quad t \in (0,T). \tag{4.14}$$

With the regularity assumption (3.26) on $\mathbf{f}_0$ we see that $\operatorname{Div} \boldsymbol{\sigma}_{\eta g}(t) \in H$. Therefore, $\boldsymbol{\sigma}_{\eta g}(t) \in \mathcal{H}_1$. Let $t_1, t_2 \in [0,T]$ and denote $\boldsymbol{\eta}(t_i) = \boldsymbol{\eta}_i$, $\mathbf{f}(t_i) = \mathbf{f}_i$, $\mathbf{g}(t_i) = \mathbf{g}_i$, $\mathbf{v}_{\eta g}(t_i) = \mathbf{v}_i$, $\boldsymbol{\sigma}_{\eta g}(t_i) = \boldsymbol{\sigma}_i$ for $i = 1,2$. Using the relation (4.8), we find that

$$\begin{aligned}(\mathcal{A}(\varepsilon(\mathbf{v}_1)) - \mathcal{A}(\varepsilon(\mathbf{v}_2)), \varepsilon(\mathbf{v}_1 - \mathbf{v}_2))_{\mathcal{H}} &\\ \leqslant (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{v}_1 - \mathbf{v}_2)_V + (\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \varepsilon(\mathbf{v}_1 - \mathbf{v}_2))_{\mathcal{H}} &\\ + j(\mathbf{g}_1, \mathbf{v}_2) - j(\mathbf{g}_1, \mathbf{v}_1) + j(\mathbf{g}_2, \mathbf{v}_1) - j(\mathbf{g}_2, \mathbf{v}_2).&\end{aligned} \tag{4.15}$$

From the definition of the functional j given by (3.36) we have

$$\begin{aligned}&j(\mathbf{g}_1, \mathbf{v}_2) - j(\mathbf{g}_1, \mathbf{v}_1) + j(\mathbf{g}_2, \mathbf{v}_1) - j(\mathbf{g}_2, \mathbf{v}_2)\\ &\quad = \int_{\Gamma_3} (\alpha|g_{1\nu}| - \alpha|g_{2\nu}|)(\mu|\mathbf{v}_{2\tau} - \mathbf{v}^*| - \mu|\mathbf{v}_{1\tau} - \mathbf{v}^*| + v_{2\nu} - v_{1\nu})\, da.\end{aligned}$$

The relation (3.19) and the assumptions (3.29) and (3.30) imply

$$\begin{aligned}&|j(\mathbf{g}_1, \mathbf{v}_2) - j(\mathbf{g}_1, \mathbf{v}_1) + j(\mathbf{g}_2, \mathbf{v}_1) - j(\mathbf{g}_2, \mathbf{v}_2)|\\ &\qquad \leqslant C_0^2 |\alpha|_{L^\infty(\Gamma_3)}(|\mu|_{L^\infty(\Gamma_3)} + 1)|\mathbf{g}_1 - \mathbf{g}_2|_V |\mathbf{v}_1 - \mathbf{v}_2|_V.\end{aligned} \tag{4.16}$$

The relation (3.18), the assumption (3.21), and the inequality (4.16) combined with (4.15) give us

$$\begin{aligned}&m_{\mathcal{A}}|\mathbf{v}_1 - \mathbf{v}_2|_V\\ &\qquad \leqslant C_0^2 |\alpha|_{L^\infty(\Gamma_3)}(|\mu|_{L^\infty(\Gamma_3)} + 1)|\mathbf{g}_1 - \mathbf{g}_2|_V + |\mathbf{f}_1 - \mathbf{f}_2|_V + |\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2|_{\mathcal{H}}.\end{aligned} \tag{4.17}$$

The inequality (4.17) and the regularity of the functions $\mathbf{f}$, $\mathbf{g}$, and $\boldsymbol{\eta}$ show that

$$\mathbf{v}_{\eta g} \in C(0,T;V).$$

From the assumption (3.21) and the relation (4.7) we have

$$|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2|_{\mathcal{H}} \leqslant L_{\mathcal{A}}|\mathbf{v}_1 - \mathbf{v}_2|_V + |\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2|_{\mathcal{H}}, \tag{4.18}$$

and from (4.14) we have

$$\operatorname{Div} \boldsymbol{\sigma}(t_i) + \mathbf{f}_0(t_i) = 0, \quad i = 1,2. \tag{4.19}$$

The regularity of the function $\boldsymbol{\eta}$, $\mathbf{v}$, $\mathbf{f}_0$ and the relations (4.18)–(4.19) show that

$$\boldsymbol{\sigma}_{\eta g} \in C(0,T;\mathcal{H}_1).$$

□

Let $\mathbf{g} \in C(0,T;V)$ and let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$ be given. We consider the following operator

$$\Lambda_\eta \colon C(0,T;V) \to C(0,T;V)$$

defined by

$$\Lambda_\eta \mathbf{g} = \mathbf{v}_{\eta g} \quad \forall\, \mathbf{g} \in C(0,T;V). \tag{4.20}$$

Lemma 4.2. *Assume that* (3.21)–(3.32) *hold. Then there exists a real* $\alpha_0 > 0$ *which depends only on* Ω, Γ_1, Γ_3, *and* $\mathcal{A}$ *such that if* (4.1) *is satisfied then the operator* Λ_η *has a unique fixed point* $\mathbf{g}^*_\eta \in C(0,T;V)$.

Proof. Let $\mathbf{g}_1, \mathbf{g}_2 \in C(0,T;V)$ and let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$. We use the notation $\mathbf{v}_i = \mathbf{v}_{\eta g_i}$ and $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{\eta g_i}$ for $i = 1,2$. Using similar arguments as those in (4.17), we find

$$\begin{aligned} m_{\mathcal{A}} |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_V \\ \leqslant C_0^2 |\alpha|_{L^\infty(\Gamma_3)} (|\mu|_{L^\infty(\Gamma_3)} + 1) |\mathbf{g}_1(t) - \mathbf{g}_2(t)|_V \quad \forall\, t \in [0,T]. \end{aligned} \tag{4.21}$$

From (4.20) and (4.21) we find that

$$\begin{aligned} |\Lambda_\eta \mathbf{g}_1(t) - \Lambda_\eta \mathbf{g}_2(t)|_V \\ \leqslant \frac{C_0^2}{m_A} |\alpha|_{L^\infty(\Gamma_3)} (|\mu|_{L^\infty(\Gamma_3)} + 1) |\mathbf{g}_1(t) - \mathbf{g}_2(t)|_V \quad \forall\, t \in [0,T]. \end{aligned} \tag{4.22}$$

Let

$$\alpha_0 = \frac{m_A}{C_0^2},$$

where α_0 is a positive constant which depends on Ω, Γ_1, Γ_3 and on the operator $\mathcal{A}$. If (4.1) is satisfied we deduce from (4.22) that the operator Λ_η is a contraction. From Banach's Fixed Point Theorem we conclude that the operator Λ_η has a unique fixed point $\mathbf{g}^*_\eta \in C(0,T;V)$. □

Denote

$$\mathbf{v}_\eta = \mathbf{v}_{\eta g^*_\eta}, \quad \boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_{\eta g^*_\eta}, \tag{4.23}$$

and let $\mathbf{u}_\eta \colon [0,T] \to V$ be the function defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_\eta(s)\,\mathrm{d}s + \mathbf{u}_0 \quad \forall\, t \in [0,T]. \tag{4.24}$$

Using (4.9) we find that $\mathbf{u}_\eta$ satisfies the regularity expressed in (4.2). In the second step, let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$; we use the displacement field $\mathbf{u}_\eta$ obtained in (4.24) and consider the following variational problem.

P r o b l e m QV_η. Find the electric potential field $\varphi_\eta\colon [0,T]\to W$ such that

$$(B\nabla\varphi_\eta(t),\nabla\phi)_H-(\mathcal{E}\varepsilon(\mathbf{u}_\eta(t)),\nabla\phi)_H \\ =(q(t),\phi)_W \quad \forall\,\phi\in W,\ \ t\in(0,T). \tag{4.25}$$

We have the following result.

Lemma 4.3. QV_η *has a unique solution* φ_η *which satisfies the regularity* (4.4).

P r o o f. We define a bilinear form $b(\cdot,\cdot)\colon W\times W\to\mathbb{R}$ such that

$$b(\varphi,\phi)=(B\nabla\varphi,\nabla\phi)_H \quad \forall\,\varphi,\phi\in W. \tag{4.26}$$

We use (3.24) to show that the bilinear form b is continuous, symmetric, and coercive on W. Moreover using the Riesz Representation Theorem we may define an element $q_\eta\colon [0,T]\to W$ such that

$$(q_\eta(t),\phi)_W=(q(t),\phi)_W+(\mathcal{E}\varepsilon(\mathbf{u}_\eta(t)),\nabla\phi)_H \quad \forall\,\phi\in W,\ \ t\in(0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t)\in W$ such that

$$b(\varphi_\eta(t),\phi)=(q_\eta(t),\phi)_W \quad \forall\,\phi\in W. \tag{4.27}$$

We conclude that $\varphi_\eta(t)$ is a solution of QV_η. Let $t_1,t_2\in[0,T]$. It follows from (3.20), (3.24), (3.25), (4.26), and (4.27) that

$$|\varphi_\eta(t_1)-\varphi_\eta(t_2)|_W\leqslant C(|\mathbf{u}_\eta(t_1)-\mathbf{u}_\eta(t_2)|_V+|q(t_1)-q(t_2)|_W),$$

and the previous inequality and the regularity of $\mathbf{u}_\eta$ and q imply that $\varphi_\eta\in C(0,T;W)$. □

In the third step, we let $\theta\in C(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.

P r o b l e m PV_θ. Find a damage field $\beta_\theta\colon [0,T]\to H^1(\Omega)$ such that

$$\beta_\theta(t)\in K, \\ (\dot\beta_\theta(t),\xi-\beta_\theta(t))_{L^2(\Omega)}+a(\beta_\theta(t),\xi-\beta_\theta(t))\geqslant(\theta(t),\xi-\beta_\theta(t))_{L^2(\Omega)} \\ \forall\,\xi\in K,\ \ \text{a.e. } t\in(0,T), \tag{4.28}$$

$$\beta_\theta(0)=\beta_0. \tag{4.29}$$

To solve PV_θ, we recall the following standard result for parabolic variational inequalities (see [17, p. 47] or [1, p. 124]). Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual V' of V. We write

$$V \subset H \subset V'$$

and we say that the inclusions above define a Gelfand triple. We denote by $|\cdot|_V$, $|\cdot|_H$, and $|\cdot|_{V'}$, the norms on the spaces V, H and V' respectively, and we use $(\cdot,\cdot)_{V'\times V}$ for the duality pairing between V' and V.

Note that if $f \in H$ then

$$(f,v)_{V'\times V} = (f,v)_H \quad \forall\, v \in H.$$

Theorem 4.2. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V. Assume that $a(\cdot,\cdot)\colon V\times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0,*

$$a(v,v) + c_0|v|_H^2 \geqslant \zeta |v|_V^2 \quad \forall\, v \in V.$$

Then, for every $u_0 \in K$ and $f \in L^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H)\cap L^2(0,T;V)$ such that $u(0) = u_0$, $u(t) \in K$ for all $t \in [0,T]$, and for almost all $t \in (0,T)$,

$$(\dot u(t), v - u(t))_{V'\times V} + a(u(t), v - u(t)) \geqslant (f(t), v - u(t))_H \quad \forall\, v \in K.$$

We apply this theorem to Problem PV_θ.

Lemma 4.4. *Problem PV_θ has a unique solution β_θ such that*

$$\beta_\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.30}$$

Proof. The inclusion mapping of $(H^1(\Omega), |\cdot|_{H^1(\Omega)})$ into $(L^2(\Omega), |\cdot|_{L^2(\Omega)})$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'.$$

We use the notation $(\cdot,\cdot)_{(H^1(\Omega))'\times H^1(\Omega)}$ to represent the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. We have

$$(\beta,\xi)_{(H^1(\Omega))'\times H^1(\Omega)} = (\beta,\xi)_{L^2(\Omega)} \quad \forall\, \beta \in L^2(\Omega),\ \xi \in H^1(\Omega),$$

and we note that K is a closed convex set in $H^1(\Omega)$. Then, using the definition (3.33) of the bilinear form a, and the fact that $\beta_0 \in K$ in (3.32), it is easy to see that Lemma 4.4 is a straightforward consequence of Theorem 4.2. □

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$, the operator $\mathcal{E}$ and the function S, for $t \in [0,T]$, we consider the element

$$\Lambda(\boldsymbol{\eta},\theta)(t) = (\Lambda_1(\boldsymbol{\eta},\theta)(t), \Lambda_2(\boldsymbol{\eta},\theta)(t)) \in \mathcal{H} \times L^2(\Omega), \tag{4.31}$$

defined by the equalities

$$\Lambda_1(\boldsymbol{\eta},\theta)(t) = \mathcal{G}(\varepsilon(\mathbf{u}_\eta(t)), \beta_\theta(t)) + \mathcal{E}^*\nabla\varphi_\eta(t), \quad t \in [0,T], \tag{4.32}$$

$$\Lambda_2(\boldsymbol{\eta},\theta)(t) = S(\varepsilon(\mathbf{u}_\eta(t)), \beta_\theta(t)), \quad t \in [0,T]. \tag{4.33}$$

We have the following result.

Lemma 4.5. *Let (4.1) be satisfied. Then for $(\boldsymbol{\eta},\theta) \in C(0,T;\mathcal{H} \times L^2(\Omega))$, the function $\Lambda(\boldsymbol{\eta},\theta)\colon [0,T] \to \mathcal{H} \times L^2(\Omega)$ is continuous, and there is a unique element $(\boldsymbol{\eta}^*,\theta^*) \in C(0,T;\mathcal{H} \times L^2(\Omega))$ such that $\Lambda(\boldsymbol{\eta}^*,\theta^*) = (\boldsymbol{\eta}^*,\theta^*)$.*

Proof. Let $(\boldsymbol{\eta},\theta) \in C(0,T;\mathcal{H}\times L^2(\Omega))$, and $t_1,t_2 \in [0,T]$. Using (3.18), (3.22), and (3.25), we have

$$\begin{aligned} |\Lambda_1(\boldsymbol{\eta},\theta)(t_1) - \Lambda_1(\boldsymbol{\eta},\theta)(t_2)|_{\mathcal{H}} &\leqslant |\mathcal{G}(\varepsilon(\mathbf{u}_\eta(t_1)), \beta_\theta(t_1)) - \mathcal{G}(\varepsilon(\mathbf{u}_\eta(t_2)), \beta_\theta(t_2))|_{\mathcal{H}} \\ &\quad + |\mathcal{E}^*\nabla\varphi_\eta(t_1) - \mathcal{E}^*\nabla\varphi_\eta(t_2)|_{\mathcal{H}} \\ &\leqslant C(|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)|_V + |\varphi_\eta(t_1) - \varphi_\eta(t_2)|_W \\ &\quad + |\beta_\theta(t_1) - \beta_\theta(t_2)|_{L^2(\Omega)}). \end{aligned} \tag{4.34}$$

Next, due to the regularities of $\mathbf{u}_\eta, \varphi_\eta$, and β_θ expressed in (4.2), (4.4), and (4.6), respectively, we deduce from (4.34) that $\Lambda_1(\boldsymbol{\eta},\theta) \in C(0,T;\mathcal{H})$. By similar arguments, from (4.33) and (3.23) it follows that

$$\begin{aligned} |\Lambda_2(\boldsymbol{\eta},\theta)(t_1) - \Lambda_2(\boldsymbol{\eta},\theta)(t_2)|_{L^2(\Omega)} \\ \leqslant C(|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)|_V + |\beta_\theta(t_1) - \beta_\theta(t_2)|_{L^2(\Omega)}). \end{aligned} \tag{4.35}$$

Therefore, $\Lambda_2(\boldsymbol{\eta},\theta) \in C(0,T;L^2(\Omega))$ and $\Lambda(\boldsymbol{\eta},\theta) \in C(0,T;\mathcal{H} \times L^2(\Omega))$. Let now $(\boldsymbol{\eta}_1,\theta_1),(\boldsymbol{\eta}_2,\theta_2) \in C(0,T;\mathcal{H} \times L^2(\Omega))$. We use the notation $\mathbf{g}^*_{\eta_i} = \mathbf{g}_i$, $\boldsymbol{\sigma}_{\eta_i g_i} = \boldsymbol{\sigma}_i$, $\mathbf{u}_{\eta_i g_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i g_i} = \mathbf{v}_{\eta_i g_i} = \mathbf{v}_i$, $\varphi_{\eta_i} = \varphi_i$ and $\beta_{\theta_i} = \beta_i$ for $i = 1,2$. From the notation used in (4.20) and (4.23), we deduce that $\mathbf{v}_i = \mathbf{g}_i$. Arguments similar to those used in the proof of (4.34) and (4.35) yield

$$\begin{aligned} |\Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t)|^2_{\mathcal{H}\times L^2(\Omega)} \\ \leqslant C(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V + |\varphi_1(t) - \varphi_2(t)|^2_W + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)}). \end{aligned} \tag{4.36}$$

Since

$$\mathbf{u}_i(t) = \int_0^t \mathbf{v}_i(s)\,\mathrm{d}s + \mathbf{u}_0, \quad t \in [0,T],$$

we have

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 \leqslant C \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V^2\,\mathrm{d}s \quad \forall\, t \in [0,T]. \tag{4.37}$$

It follows now from $\mathrm{PV}_{\eta g}$ for $\boldsymbol{\eta} = \boldsymbol{\eta}_i$, $i = 1,2$, that

$$\boldsymbol{\sigma}_i(t) = \mathcal{A}(\varepsilon(\mathbf{v}_i(t))) + \boldsymbol{\eta}_i(t) \quad \forall\, t \in [0,T], \tag{4.38}$$

$$(\boldsymbol{\sigma}_i(t), \varepsilon(\mathbf{v} - \mathbf{v}_i(t))_{\mathcal{H}} + j(\mathbf{g}_i(t), \mathbf{v}) - j(\mathbf{g}_i(t), \mathbf{v}_i(t)) \geqslant (\mathbf{f}(t), \mathbf{v} - \mathbf{v}_i(t)_V \quad \forall\, \mathbf{v} \in V,\ \forall\, t \in [0,T]. \tag{4.39}$$

Using the relations (4.39) we obtain that

$$(\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t), \varepsilon(\mathbf{v}_1(t) - \mathbf{v}_2(t)))_{\mathcal{H}} \leqslant j(\mathbf{g}_1(t), \mathbf{v}_2(t)) - j(\mathbf{g}_1(t), \mathbf{v}_1(t)) + j(\mathbf{g}_2(t), \mathbf{v}_1(t)) - j(\mathbf{g}_2(t), \mathbf{v}_2(t)) \quad \forall\, t \in [0,T],$$

and similar arguments to those used in (4.16) on the functional j yield

$$(\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t), \varepsilon(\mathbf{v}_1(t) - \mathbf{v}_2(t)))_{\mathcal{H}} \leqslant C_0^2 |\alpha|_{L^\infty(\Gamma_3)} (|\mu|_{L^\infty(\Gamma_3)} + 1) |\mathbf{g}_1(t) - \mathbf{g}_2(t)|_V |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_V.$$

Keeping in mind that $\mathbf{v}_i = \mathbf{g}_i$ for $i = 1,2$, it follows that

$$(\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t), \varepsilon(\mathbf{v}_1(t) - \mathbf{v}_2(t)))_{\mathcal{H}} \leqslant C_0^2 |\alpha|_{L^\infty(\Gamma_3)} (|\mu|_{L^\infty(\Gamma_3)} + 1) |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_V^2 \quad \forall\, t \in [0,T].$$

We substitute (4.38) into the previous inequality and use (3.18) and (3.21) to deduce that

$$(m_{\mathcal{A}} - C_0^2 |\alpha|_{L^\infty(\Gamma_3)} (|\mu|_{L^\infty(\Gamma_3)} + 1)) |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_V \leqslant |\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)|_{\mathcal{H}} \quad \forall\, t \in [0,T].$$

It follows from (4.1) that

$$|\mathbf{v}_1(t) - \mathbf{v}_2(t)|_V^2 \leqslant C |\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)|_{\mathcal{H}}^2. \tag{4.40}$$

For the electric potential field, we use (4.25), (3.20), (3.24), and (3.25) to obtain

$$|\varphi_1(t)-\varphi_2(t)|_W^2 \leqslant C|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2. \tag{4.41}$$

From (4.28) we deduce that

$$\begin{aligned}(\dot\beta_1-\dot\beta_2,\beta_1-\beta_2)_{L^2(\Omega)} &+ a(\beta_1-\beta_2,\beta_1-\beta_2)\\ &\leqslant (\theta_1-\theta_2,\beta_1-\beta_2)_{L^2(\Omega)} \quad \text{a.e. } t\in(0,T).\end{aligned}$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0)=\beta_2(0)=\beta_0$ and the inequality $a(\beta_1-\beta_2,\beta_1-\beta_2)\geqslant 0$, we find

$$\frac{1}{2}|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)} \leqslant \int_0^t (\theta_1(s)-\theta_2(s),\beta_1(s)-\beta_2(s))_{L^2(\Omega)}\,\mathrm{d}s,$$

which implies that

$$\begin{aligned}&|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)}\\ &\qquad\leqslant \int_0^t |\theta_1(s)-\theta_2(s)|^2_{L^2(\Omega)}\,\mathrm{d}s + \int_0^t |\beta_1(s)-\beta_2(s)|^2_{L^2(\Omega)}\,\mathrm{d}s.\end{aligned}$$

This inequality combined with Gronwall's inequality lead to

$$|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)} \leqslant C\int_0^t |\theta_1(s)-\theta_2(s)|^2_{L^2(\Omega)}\,\mathrm{d}s \quad \forall\, t\in[0,T]. \tag{4.42}$$

We substitute (4.41) into (4.36) and use (4.37) to obtain

$$\begin{aligned}&|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)|^2_{\mathcal{H}\times L^2(\Omega)}\\ &\qquad\leqslant C\Big(\int_0^t |\mathbf{v}_1(s)-\mathbf{v}_2(s)|_V^2\,\mathrm{d}s + |\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)}\Big).\end{aligned}$$

It follows now from the previous inequality, the estimates (4.40) and (4.42) that

$$\begin{aligned}&|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)|^2_{\mathcal{H}\times L^2(\Omega)}\\ &\qquad\leqslant C\int_0^t |(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)|^2_{\mathcal{H}\times L^2(\Omega)}\,\mathrm{d}s.\end{aligned}$$

Reiterating this inequality m times leads to

$$\begin{aligned}&|\Lambda^m(\boldsymbol{\eta}_1,\theta_1)-\Lambda^m(\boldsymbol{\eta}_2,\theta_2)|^2_{C(0,T;\mathcal{H}\times L^2(\Omega))}\\ &\qquad\leqslant \frac{C^mT^m}{m!}|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)|^2_{C(0,T;\mathcal{H}\times L^2(\Omega))}.\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathcal{H}\times L^2(\Omega))$, and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 4.1.

P r o o f of Theorem 4.1. *Existence.* Let $(\boldsymbol{\eta}^*, \theta^*) \in C(0,T;\mathcal{H} \times L^2(\Omega))$ be the fixed point of Λ defined by (4.31)–(4.33) and let $(\mathbf{v}, \boldsymbol{\sigma})$ be the solution of $\mathrm{PV}_{\eta g}$ for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and $\mathbf{g} = \mathbf{g}^*_{\eta^*}$ obtained in Lemma 4.1. Denote $\mathbf{u} = \mathbf{u}_{\eta^*}$ (see (4.24)). Let now $\varphi = \varphi_{\eta^*}$ and $\beta = \beta_{\theta^*}$ be the solutions of QV_η and PV_θ for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and $\theta = \theta^*$ obtained in Lemmas 4.3 and 4.4. The equalities $\Lambda_1(\boldsymbol{\eta}^*, \theta^*) = \boldsymbol{\eta}^*$ and $\Lambda_2(\boldsymbol{\eta}^*, \theta^*) = \theta^*$ combined with (4.32) and (4.33) show that (3.38)–(3.42) are satisfied. Next, (3.43) and the regularity (4.2)–(4.6) follow from Lemmas 4.1, 4.3, 4.4, and (3.42).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.31)–(4.33) and the unique solvability of Problems $\mathrm{PV}_{\eta g}$, QV_η, and PV_θ. □

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