

A FRONT-TRACKING ALTERNATIVE TO THE RANDOM CHOICE METHOD

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ABSTRACT. An alternative to Glimm's proof of the existence of solutions to systems of hyperbolic conservation laws is presented. The proof is based on an idea by Dafermos for the single conservation law and in some respects simplifies Glimm's original argument. The proof is based on construction of approximate solutions of which a subsequence converges. It is shown that the constructed solution satisfies Lax's entropy inequalities. The construction also gives a numerical method for solving such systems.

1. INTRODUCTION

We study the initial value problem for the general system of hyperbolic conservation laws

$$u_t + f(u)_x = 0.$$

Our analysis is based on Lax's [1] solution of the Riemann problem. We give here an alternative proof of Glimm's fundamental result [2] not based on a random sequence. Since Glimm's paper, there have been few generalizations of his result, but Liu [3] showed that Glimm's proof did not actually depend on the random sequence and that it converged for any equidistributed sequence. Chorin [4] developed Glimm's construction into a numerical method. Using Glimm's construction, Lax [5] showed that the constructed solution satisfied the entropy inequalities provided the system admitted an additional conservation law. This system of equations models a diverse range of physical phenomena, e.g., traffic flow [6], gas dynamics [7], and multiphase flow in porous media [8].

Our proof is based on ideas from the study of the single conservation law. Dafermos [9] used a piecewise linear continuous approximation to the flux function f to obtain approximate piecewise constant solutions. This idea was further developed into a numerical method by LeVeque [10] and by Holden et al. [11] and was generalized to several space dimensions by Høegh-Krohn and Risebro [12]. The idea of approximating rarefaction waves by piecewise constant states was also investigated by Swartz and Wendroff [13] for the system of gas dynamics.

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We construct our solutions by starting with an approximation to the solution of the Riemann problem where the rarefaction part of the solution is replaced by an approximating step function. The initial value function is also approximated by a step function, which defines a series of Riemann problems. Each discontinuity in the approximate solution is then tracked until it interacts with other discontinuities. For such interactions we can use some of the estimates in [2] directly, and here we only give the differences from Glimm's proof. Our main result is that if the total variation of the initial data is small, then a weak solution of the initial value problem exists. Without assuming the existence of an additional conservation law, we show that our constructed solution satisfies Lax's entropy inequalities, and therefore is not of what Glimm [2] called "extraneous" type. The construction in a natural way defines a numerical method for solving hyperbolic conservation laws. For general background we refer the reader to [14, part 3] and the references therein.

2. METHOD AND NOTATION

We will consider the Cauchy problem

$$(2.1) \quad \begin{aligned} u_t + f(u)_x &= 0, \\ u(x, 0) &= u_0(x) \end{aligned}$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth function. We assume that the system is strictly hyperbolic, that is, the Jacobian df has real eigenvalues $\lambda_1(u), \dots, \lambda_n(u)$ such that $\lambda_1(u) < \lambda_2(u) < \dots < \lambda_n(u)$. We want to construct a *weak solution* to (2.1), that is, a function $u: \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$

$$(2.2) \quad \int_0^\infty \int_{-\infty}^\infty (\phi_t u + \phi_x f(u)) dx dt + \int_{-\infty}^\infty \phi(x, 0) u_0(x) dx = 0$$

for all smooth ϕ with compact support in (x, t) .

The Riemann problem for (2.1) is the initial value problem where

$$(2.3) \quad u_0(x) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases}$$

The solution of this Riemann problem consists of three ingredients: *shocks*, *rarefaction waves*, and *contact discontinuities*. For an explanation of these see [14, Chapter 17].

Theorem 1 (Lax). *Let $u_l \in N \subset \mathbb{R}^n$ and suppose (2.1) is strictly hyperbolic and that each characteristic field is either genuinely nonlinear or linearly degenerate in N . Then there is a neighbourhood $M \subset N$ such that for $u_r \in M$, (2.3) has a solution. This solution consists of at most $n + 1$ constant states separated by shocks, rarefaction waves, and contact discontinuities. There is only one such solution in M , and discontinuities in this solution will satisfy the Lax inequalities (3.13).*

Proof. See [14, Theorem 17.18]. \square

We will construct an approximation to this solution. Through each point u in M we have n one-parameter families of curves $U_k(u, \varepsilon)$, $k = 1, \dots, n$. These have continuous derivatives of order two at $\varepsilon = 0$ and have the property that if u_r is on $U_k(u_l, \varepsilon)$ and the k th field is genuinely nonlinear, u_r can

be connected to u_l by a rarefaction wave iff $\varepsilon > 0$ and by a shock iff $\varepsilon < 0$, $U_k(u_l, 0) = u_l$. We call $|\varepsilon|$ the *strength* of the wave. If the k th field is linearly degenerate $U_k(u, \varepsilon)$ consists of the states that can be connected to u by a contact discontinuity. For a more detailed description of these concepts we again refer to [14].

We will take this correct solution to the Riemann problem and approximate it in the following way: We start with the correct solution to (2.3). Leave each shock or contact discontinuity as it is. Along the rarefaction curves, we fix an initial $\delta > 0$, approximate the rarefaction fan by constant states

$$u_i^{(k)} = U_k(u^k, i\delta) = U_k(u_{i-1}^{(k)}, \delta)$$

for $i = 1, \dots, m$, where m is chosen such that $u_{m+2}^{(k)}$ is “past” the next constant state in the solution: u^{k+1} . The states $u_i^{(k)}$ and $u_{i+1}^{(k)}$ will be separated by a discontinuity moving with speed $\lambda_k(u_{i+1}^{(k)})$. This approximation corresponds to making a step function approximation of $u(x, t)$ at each fixed t . We call our approximation $u_\delta(x, t)$. We have that

$$\lim_{\delta \rightarrow 0} u_\delta = u \quad \text{for all } t.$$

The limit is in $L_1^{loc}(\mathbb{R}, dx)$ for each t . Furthermore u will satisfy (2.2), and since $\text{supp } \phi$ is confined to $t < T < \infty$, we have that

$$\int_0^\infty \int_{-\infty}^\infty (\phi_t u_\delta + \phi_x f(u_\delta)) dx dt + \int_{-\infty}^\infty \phi(x, 0) u_0(x) dx \rightarrow 0$$

as $\delta \rightarrow 0$, since, by the bounded convergence theorem, $f(u_\delta) \rightarrow f(u)$ in L_1 . Therefore u_δ is an approximate solution to (2.3), and we will call this a δ -approximation to the solution of (2.3).

Our strategy will now be to construct an approximation to a more general initial value problem, by using this δ -approximation on a series of Riemann problems. Assume that $u_0(x)$ is in $L_1^{loc} \cap B.V.$, then we define the sequence $\{x_i\}_1^M$ by

$$x_1 = \inf \left\{ x : |u_0(x) - \lim_{x \rightarrow -\infty} u_0(x)| > \delta \right\}$$

$$x_{n+1} = \inf \left\{ x : x > x_n \text{ and } |u_0(x) - \lim_{x \rightarrow x_n^+} u_0(x)| > \delta \right\}.$$

Now we define the approximated initial function u_0^δ by

$$u_0^\delta(x) = \begin{cases} \lim_{x \rightarrow -\infty} u_0(x), & x < x_1, \\ u_0(x_n), & x \in [x_n, x_{n+1}), \\ \lim_{x \rightarrow \infty} u_0(x), & x > x_M. \end{cases}$$

We have that $\|u_0^\delta - u_0\|_{L_1} \rightarrow 0$ as $\delta \rightarrow 0$. At each discontinuity in u_0^δ we construct the δ -approximation to the solution of the Riemann problem defined by $(u_0(x_{n-1}), u_0(x_n))$. When two δ -approximations interact at some $t > 0$, we are still in the class of step functions with compact support and a finite number of steps. Therefore the process can be repeated. With a slight abuse of notation we will call this “solution” u_δ .

It is clear that this process can be repeated an arbitrary number of times. If we define the t_i as the i th time discontinuities collide, we can continue our approximation up to a time $T = \lim_{i \rightarrow \infty} t_i$. In order to remove this restriction we will make a modification of our method: We construct our solution up to some time t_N . At t_N we will not use δ -approximations to "solve" the Riemann problems, but we will use an approximation where some of the small waves originating from these are ignored. The next time discontinuities collide we will again use δ -approximations until some new $t_{N'}$, and the process is repeated. We will show that it is sufficient to do this a finite number of times depending on δ in order to carry our approximation up to infinite time. In order to show that this is a well-defined construct we need some lemmas.

We follow the notation in [14, p. 370]. By

$$(2.4) \quad (u_l, u_r) = [(u_0, \dots, u_n)/(\varepsilon_1, \dots, \varepsilon_n)]$$

we mean that u_k is connected to u_{k-1} by a k -shock or a k -rarefaction wave with strength $|\varepsilon_k|$, i.e., $u_k = U^{(k)}(u_{k-1}, \varepsilon_k)$, and $u_l = u_0$ and $u_n = u_r$. Now let u_l, u_m, u_r be given states near a given state u , and let

$$(2.5) \quad (u_l, u_m) = [(u_0, \dots, u_n)/(\alpha_1, \dots, \alpha_n)],$$

$$(2.6) \quad (u_m, u_r) = [(u_0, \dots, u_n)/(\beta_1, \dots, \beta_n)].$$

With these definitions in hand we can prove the following slight modification of [14, Theorem 19.2] or [2, Theorem 2.1].

Lemma 1. *Assume that a discontinuity α (in our scheme) of family j separating (u_l, u_m) and a discontinuity β of family k separating (u_m, u_r) collide and that (2.4), (2.5), and (2.6) hold. Then*

$$\varepsilon_i = \delta_{ij}\alpha + \delta_{ik}\beta + O(1)|\alpha||\beta|.$$

Proof. The proof of this is the same as the proof of Theorem 2.1 in [2]. \square

Let t_i be the i th time discontinuities in u_δ collide, and let $|\varepsilon_i^j|$ be the strength of the j th discontinuity from the left in the strip $t_i < t < t_{i+1}$. We say two discontinuities in u_δ are approaching if the speed of the one on the left is larger than the speed of the one on the right. We define

$$T_k = \sum_j |\varepsilon_k^j|,$$

$$Q_k = \sum |\alpha_k||\beta_k|$$

where the sum in Q_k is taken over all approaching pairs in $t_k < t < t_{k+1}$.

Lemma 2. *Let $m, n > 0$. If $n > m$ and T_m is sufficiently small, then*

$$Q_n \leq Q_m, \quad T_n + KQ_n \leq T_m + KQ_m$$

for some $K > 0$.

Proof. The proof is similar in spirit to the proof of the corresponding theorem for Glimm's construction, see [2] or [14]. We first assume that $n = m + 1$. Let the collision at t_{m+1} take place at x (if there are several collisions we can use the same argument at each collision). Since discontinuities propagate at finite speed, we can find $\varepsilon > 0$ and an interval J such that $x \in J$, and at $t_{m+1} - \varepsilon$ all

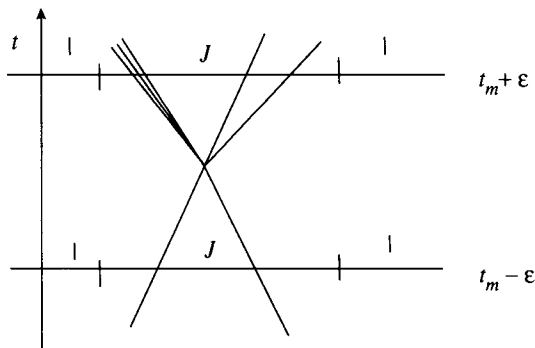


FIGURE 1

colliding discontinuities are in J , and at $t_{m+1} + \epsilon$ all discontinuities emanating from (x, t_{m+1}) are in J . These are also the only discontinuities in J in the time interval $[t_{m+1} - \epsilon, t_{m+1} + \epsilon]$. Let $I = \mathbb{R} \setminus J$, cf. Figure 1. Let $T_k(I)$ be T_k with the summation restricted to I , similarly for $Q_k(I)$. From Lemma 1 we have

$$T_n \leq T_m + K_0 Q_m(J), \quad Q_n = Q_n(I) + Q_n(I, J)$$

where $Q_k(I, J)$ is the sum with one wave from I and the other from J .

$$\begin{aligned} Q_n(I, J) &= \sum_{\delta \text{ appr } \epsilon_n} |\epsilon_n| |\delta| \leq \sum_{\delta \text{ appr } \alpha \text{ or } \beta} (|\alpha| + |\beta|) |\delta| + K_0 Q_m(J) T_m(I) \\ &\leq Q_m(I, J) + K_0 Q_m(J) T_m \leq Q_m(I, J) + \frac{1}{2} Q_m(J) \quad \text{if } K_0 T_m \leq \frac{1}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} (2.7) \quad Q_n - Q_m &= [Q_n(I) + Q_n(I, J)] - [Q_m(I) + Q_m(J) + Q_m(I, J)] \\ &\leq Q_m(I, J) + \frac{1}{2} Q_m(J) - Q_m(J) - Q_m(I, J) = -\frac{1}{2} Q_m(J) \leq 0. \end{aligned}$$

Now

$$\begin{aligned} T_n + K Q_n &\leq T_m + K_0 Q_m(J) + K Q_m - (K/2) Q_m(J) \\ &\leq T_m + K Q_m \quad \text{if } K_0 - K/2 \leq 0. \end{aligned}$$

Summing we have that the inequalities hold for any $n > m$. \square

Corollary 1. *If $T.V.(u_0)$ is sufficiently small then*

$$\text{osc } u_\delta \leq T.V.(u_\delta) \leq c T_n \leq \bar{c} T_0 \leq \bar{\bar{c}} T.V.(u_0)$$

where all constants are independent of t and δ .

Proof. $\text{osc} \leq T.V.$ is always true. $T.V. \leq cT$ since they are equivalent norms. $T_n \leq T_n + K Q_n \leq T_0 + K Q_0 \leq T_0 + K T_0^2 \leq 2T_0$ if $K T_0 \leq 1$. \square

Now define

$$P_k = \sum |\epsilon_k^{j-1}| |\epsilon_k^j|$$

where the sum is taken over those discontinuities that collide at t_k (typically $P_k = |\epsilon_k^{j-1}| |\epsilon_k^j|$).

Lemma 3. $\sum_i P_i < \infty$.

Proof. If $\Delta Q_k = Q_k - Q_{k+1}$, then by (2.7) $\Delta Q_k \geq \frac{1}{2} P_k$. Therefore

$$Q_0 \geq Q_0 - Q_k = \sum_{i=0}^k \Delta Q_i \geq \frac{1}{2} \sum_{i=0}^k P_i. \quad \square$$

Now we can define t_{N_1} , where $N_1 < \infty$ is the smallest number such that $P_k < \delta$ for all $k \geq N_1$, and we relabel the collision times up to t_{N_1} : $t_0^1, t_1^1, \dots, t_{N_1}^1$. At the collision(s) at $t_{N_1}^1$, we do not construct waves of families different from the ones that are colliding, although we allow these to change their speeds. What this means is perhaps best illustrated by an example: Assume that the collision at $t_{N_1}^1$ is between a discontinuity of family l and one of family m . Assume also that the solution of the Riemann problem defined by the states to the left and right of this collision has a solution that contains waves of families k, m, n , and l . Note that Lemma 1 says that the waves of families k and n will be small. In making the step function approximation to the solution of the Riemann problem the small waves of family n and k are ignored and the solution is regarded as constant over these waves.

This constant is chosen as follows: If the small wave is of a family smaller than the family of both the colliding waves, i.e., its speed is strictly smaller than the speeds of the colliding waves, then the state to the right of this wave is set equal to the state to the left of it. Similarly if the family of the wave to be ignored is larger than the families of the colliding waves, then the state to the left of the small wave is set equal to the state to the right of it. If the family of the small wave is between the families of the colliding waves, we may choose either to set the state to the left of it equal to the state on the right, or vice versa.

This of course introduces an additional error into our approximation, but it is necessary to remove some discontinuities in order to limit the number of fronts to track. In Lemmas 5 and 6 we show that this error is so small that the approximation remains an approximate weak solution.

The next collision time after $t_{N_1}^1$ we label t_0^2 . At this collision we again use the original approximation technique where all waves in the solution of a Riemann problem are approximated. We continue using this approximation up to a collision time $t_{N_2}^2$, where N_2 is defined like N_1 . When solving the Riemann problems at $t_{N_2}^2$ we again ignore waves of new families. Continuing in this way we get collision times for the u_δ

$$t_0^1, \dots, t_{N_1}^1, t_0^2, \dots, t_{N_2}^2, t_0^3, \dots, t_{N_3}^3, \dots, t_0^i, \dots, t_{N_i}^i, \dots$$

Lemma 2 was shown only for collisions at t_j^i where $j < N_i$, but it is easily seen that Q_n and $T_n + KQ_n$ are decreasing also for the collisions at $t_{N_i}^i$. Hence Lemma 3 will hold when the sum is taken over all collisions. We restate this as

$$\sum_i \sum_{k=0}^{N_i} P_k^i < \infty,$$

where P_k^i refers to the collision(s) taking place at t_k^i . Therefore there is an integer M such that $\sum_{k=0}^{N_i} P_k^i \leq \delta$ for all $i \geq M$. For such i we have that

$P_k^i \leq \delta$ for all k . Hence, after $t_{N_{M-1}-1}^{M-1}$ we do not create waves of new families at collision points. Since we have a strictly hyperbolic system where the speeds of waves of different families are different, we see that after a certain time all discontinuities will have passed through each other and there will be no more interactions. Thus the approximation u_δ can be defined at any (x, t) in the upper halfplane.

We remark that the above reasoning could also be used on the following approximation strategy: If P_k is less than δ for some k , we do not construct waves of new families at t_k , and we define t_0^2 to be the first time after t_k that discontinuities existing already before t_k collide. This strategy may be more practical and one can also show (as we will do) that it gives an approximate weak solution.

3. RESULTS

Corollary 2. *If $T.V.(u_0)$ is small then*

$$T.V.x(u_\delta) + \sup_x |u_\delta| \leq cT.V.(u_0)$$

where c is independent of t and δ .

This corollary is a consequence of Corollary 1, and its proof may be found in [14, p. 384].

Corollary 3.

$$\|u_\delta(\cdot, t_1) - u_\delta(\cdot, t_2)\|_{L_1} \leq c|t_2 - t_1|$$

where c is independent of δ, t_1 , and t_2 .

Proof. Let $M < \infty$ be the maximum speed at which a wave may propagate. Thus, if $t_1 < t_2$ then $|u_\delta(x, t_2) - u_\delta(x, t_1)|$ is bounded by the spatial variation of $u_\delta(y, t_1)$ over the interval $(x - M|t_2 - t_1|) < y < (x + M|t_2 - t_1|)$. However, $u_\delta(\cdot, t)$ is of bounded variation, so that we may write

$$\int_{-\infty}^{\infty} |u_\delta(x, t_2) - u_\delta(x, t_1)| dx = O(1) \int_{-\infty}^{\infty} \int_{x-M|t_2-t_1|}^{x+M|t_2-t_1|} \left| \frac{du_\delta}{dy} \right| dx dy.$$

Here, $|du_\delta/dy| dx dy$ is a measure of mass $T.V.u_\delta(x, t)$, and by changing the order of integration we have

$$\begin{aligned} \int_{-\infty}^{\infty} |u_\delta(x, t_2) - u_\delta(x, t_1)| dx &= O(1)M|t_2 - t_1|T.V. u_\delta(x, t) \\ &\leq O(1)M|t_2 - t_1|T.V. u_{0,\delta}(x), \end{aligned}$$

where the last inequality holds by Corollary 1. \square

Now we have that the u_δ functions satisfy

(3.1) $\|u_\delta(\cdot, \cdot)\|_\infty \leq M_1,$

(3.2) $T.V.x(u_\delta(\cdot, t)) \leq M_2,$

(3.3) $\|u_\delta(\cdot, t_1) - u_\delta(\cdot, t_2)\|_{L_1} \leq M_3|t_2 - t_1|.$

The constants M_i are independent of the δ and the times t_1 and t_2 . Using Helly's theorem as in [14] one can show that (3.1) to (3.3) imply the following

Theorem 3. *If (3.1)–(3.3) hold then a subsequence of the family $\{u_\delta\}$ converges in L_1^{loc} . For this subsequence $f(u_\delta) \rightarrow f(u)$ in L_1^{loc} , where u is the limit function.*

We use this theorem for the sequence of functions $\{u_\delta\}$ as $\delta \rightarrow 0$. Now we have to check whether the limit is a weak solution to our problem. To this end we define

$$(3.4) \quad \mathcal{J}_\phi(u, f) = \int_0^\infty \int_{-\infty}^\infty (\phi_t u + \phi_x f(u)) \, dx \, dt + \int_{-\infty}^\infty \phi(x, 0) u_0(x) \, dx$$

and

$$(3.5) \quad \begin{aligned} \mathcal{J}_\phi^{t_1, t_2}(u, f) &= \int_{t_1}^{t_2} \int_{-\infty}^\infty (\phi_t u + \phi_x f(u)) \, dx \, dt \\ &\quad + \int_{-\infty}^\infty \phi(x, t_1) u(x, t_1) \, dx - \int_{-\infty}^\infty \phi(x, t_2) u(x, t_2) \, dx \end{aligned}$$

for all smooth ϕ with compact support in $\mathbb{R} \times \mathbb{R}_0^+$. We now fix δ and ϕ . Let s, t be consecutive times when discontinuities of u_δ collide. If we had tracked all waves from the Riemann problems in $u_\delta(x, s)$, we could compare it with an exact solution in the strip $[s, t]$, since the exact solution here would be a series of noninteracting solutions to Riemann problems. Comparing u_δ with this exact solution we would get an “error” estimate, telling us how far u_δ is from being a weak solution. But we cannot do this directly, since we do not necessarily know the weak solution in the whole strip $\langle t, s \rangle$. Therefore we define v_δ to be a δ -approximation to the initial value problem

$$(3.6) \quad v_t + f(v)_x = 0, \quad v(x, s) = u_\delta(x, s).$$

Let $t_0 = s$. Since discontinuities of v_δ propagate with finite speed, they will either collide at some $t_1 < t$, or else not collide until t , in which case we define $t_1 = t$. If $t_1 < t$ then from t_1 we define v_δ to be the δ -approximation to the same problem with initial values $u_\delta(x, t_1)$. Now either the discontinuities of v_δ will either not collide until t , in which case we set $t_2 = t$, or else collide at some $t_2 < t$. We can continue in this fashion to obtain a sequence $\{t_i\}$. We have that either this sequence is finite and $t_k = t$ for some k , or else $\lim_i t_i = t$. To see this, let d_i denote the smallest distance between discontinuities of $u_\delta(\cdot, t_i)$. Note that $d_i > 0$ if $t_i \neq t$. But since discontinuities of v_δ have finite speed we have that $d_i \leq M(t_{i+1} - t_i)$, where M is a bound on the speed of discontinuities. Thus we see that $\lim t_i \geq t$. Thus we have filled the interval $[t, s]$ with at most countably many intervals $[t_i, t_{i+1}]$ such that $t - s = \sum(t_{i+1} - t_i)$, cf. Figure 2.

In the strip $[t_i, t_{i+1}]$ we define $v(x, t)$ to be the weak solution of

$$\begin{aligned} v_t + f(v)_x &= 0, \\ v(x, t_i) &= v_\delta(x, t_i) = u_\delta(x, t_i). \end{aligned}$$

The function $v(x, t)$ can be defined in this strip since the different Riemann problems only will interact at t_{i+1} . With these definitions in mind we can state the following

Lemma 4. *For $t \in [t_i, t_{i+1}]$ we have*

$$(3.7) \quad \int_{x_1}^{x_2} |v - u_\delta| \, dx = O(\delta)(t - t_i).$$

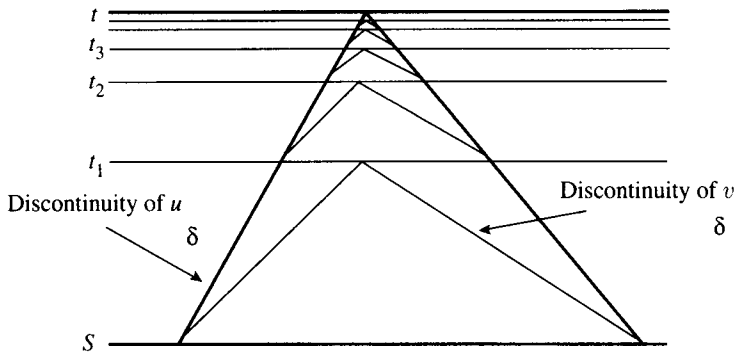


FIGURE 2

Proof.

$$(3.8) \quad \int_{x_1}^{x_2} |v - u_\delta| dx \leq \int_{x_1}^{x_2} |v - v_\delta| dx + \int_{x_1}^{x_2} |v_\delta - u_\delta| dx.$$

For the first term note that $v = v_\delta$ except when (x, t) is in a rarefaction fan. In a rarefaction fan the difference $|v_\delta - v|$ is always less than or equal to $O(\delta)$. If \hat{u}_r, \hat{u}_l are the states to the right and left of such a fan respectively, then the integral across the fan will be a sum of integrals across each step of v_δ . The number of such steps is $O(|\hat{u}_r - \hat{u}_l|/\delta)$, and the width of each region where v_δ differs from v is $(t - t_i)O(\Delta\lambda)$, where $\Delta\lambda = |\lambda(v_j^{(k)}) - \lambda(v_{j+1}^{(k)})| = O(\delta)$. Therefore the first term in (3.8) is a sum over all rarefaction fans of v_δ

$$\sum_j O\left(\frac{|\hat{u}_{r_j} - \hat{u}_{l_j}|}{\delta}\right) O(\delta)(t - t_i)O(\delta).$$

But this is less than

$$T.V.(u_\delta(x, t_i))(t - t_i)O(\delta).$$

We have that v_δ and u_δ both are step functions with a finite number of steps. Furthermore they are equal except possibly in a fan emanating from each discontinuity in $v_\delta(x, t_i)$. Let x_j be in the j th interval from the left where $v_\delta(x, t)$ differs from $u_\delta(x, t)$. We label the discontinuities not tracked in u_δ , but tracked in v_δ , by $\{\eta_j\}$, where $|\eta_j|$ is the strength of this discontinuity.

$$\int_{x_1}^{x_2} |v_\delta(x, t) - u_\delta(x, t)| dx \leq \sum_j M(t - t_i)|v_\delta(x_j, t) - u_\delta(x_j, t)|$$

(here M is a constant such that $M > \sup_{u \in C} |\lambda_n(u) - \lambda_1(u)|$, $C = \text{convex hull of } \{\text{Ran}\{u_\delta\}\}$)

$$\begin{aligned} &= M(t - t_i) \sum_j |v_\delta(x_j, t) - u_\delta(x_j, t)| \\ &= M(t - t_i) O\left(\sum_j |\eta_j|\right). \end{aligned}$$

But by construction of u_δ we have that $\sum_j |\eta_j| \leq O(\delta)$ and the result follows. \square

Lemma 5. *If we let t_i, t_{i+1} be as before we have*

$$\mathcal{I}_\phi^{t_i, t_{i+1}}(u_\delta, f) = O(\delta)((t_{i+1} - t_i) + (t_{i+1} - t_i)^2).$$

Proof. Let $M \geq \sup\{|\phi_x|, |\phi_t|, |\phi|, |df|\}$, and let $v(x, t)$ be as before; then
(3.9)

$$\begin{aligned} \left| \mathcal{I}_\phi^{t_i, t_{i+1}}(u_\delta, f) \right| &= \left| \mathcal{I}_\phi^{t_i, t_{i+1}}(u_\delta, f) - \mathcal{I}_\phi^{t_i, t_{i+1}}(v, f) \right| \\ &= \left| \int_{t_i}^{t_{i+1}} \int ((u_\delta - v)\phi_t + (f(u_\delta) - f(v))\phi_x) dx dt - \int \phi(x, t_{i+1})(u_\delta - v) dx \right| \\ &\leq M \left(\int_{t_i}^{t_{i+1}} \int |u_\delta - v| dx dt + \int |u_\delta - v| dx + \int_{t_i}^{t_{i+1}} \int |f(u_\delta) - f(v)| dx dt \right). \end{aligned}$$

Now

$$f(u_\delta) - f(v) = df(u_\delta - v) + O^2(u_\delta - v)$$

and

$$(3.10) \quad \int |f(u_\delta) - f(v)| dx \leq M \int |u_\delta - v| dx + \int |O^2(u_\delta - v)| dx.$$

Using (3.10) in (3.9), Lemma 4 on (3.9), and integrating in t will give Lemma 5. \square

Lemma 6. *If t and s are as before we have*

$$\mathcal{I}_\phi^{t, s}(u_\delta, f) \leq O(\delta)((s - t) + (s - t)^2).$$

Proof. By Lemma 5

$$\mathcal{I}_\phi^{t, s}(u_\delta, f) = O(\delta) \sum ((t_{i+1} - t_i) + (t_{i+1} - t_i)^2)$$

where t_i is as before. We have $\sum (t_{i+1} - t_i) = (s - t)$, which means that the second term in the sum is less than $(s - t)^2$. \square

Lemma 7. $\lim_{\delta \rightarrow 0} \mathcal{I}_\phi(u_\delta, f) = 0$.

Proof. If we let s_i, s_{i+1} be consecutive times when discontinuities of u_δ collide we have

$$\mathcal{I}_\phi(u_\delta, f) = \sum_i \mathcal{I}_\phi^{s_i, s_{i+1}}(u_\delta, f);$$

therefore by Lemma 6

$$(3.11) \quad \mathcal{I}_\phi(u_\delta, f) = O(\delta) \sum ((s_{i+1} - s_i) + (s_{i+1} - s_i)^2).$$

We now have $\sum (s_{i+1} - s_i) \leq T$ where T is such that $\text{supp } \phi$ is contained in $\{t \leq T\}$. Therefore the sum in (3.11) is finite, and the lemma follows. \square

Thus u_δ converges to a weak solution, and we have proved the following theorem.

Theorem 4. Assume $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is strictly hyperbolic and genuinely nonlinear and $u_0: \mathbb{R} \rightarrow \mathbb{R}^n$ is such that $T.V.x(u_0)$ is sufficiently small. Then there exists a weak solution $u(x, t)$ to the initial value problem

$$(3.12) \quad u_t + f(u)_x = 0, \quad u(x, 0) = u_0(x).$$

For the solution of the Riemann problem we have that all discontinuities satisfy the Lax entropy conditions:

$$(3.13) \quad \lambda_k(u_r) < s_k \leq \lambda_{k+1}(u_r), \quad \lambda_{k-1}(u_l) \leq s_k < \lambda_k(u_l)$$

where s_k is the speed of a k -shock. Since discontinuities of u_δ , at least if they have a magnitude larger than δ , almost (in the sense of (3.14) and (3.15)) satisfy these conditions, one may ask whether discontinuities in the limit function satisfy (3.13). The next theorem gives a partial answer to this. If $u(x, t)$ has a discontinuity at (x, t) , and there is a neighbourhood N of (x, t) such that we can find a smooth curve $x(s)$ in N where $x(t) = x$, and in N we can define

$$u_+(t) = \lim_{x \rightarrow x(t)^+} u(x, t), \quad u_-(t) = \lim_{x \rightarrow x(t)^-} u(x, t),$$

and u_\pm are continuous in N , and $u_-(t) \neq u_+(t)$ for t in N , then we say that the discontinuity at (x, t) is isolated.

Theorem 5. An isolated discontinuity in $u = \lim_{\delta \rightarrow 0} u_\delta$ moving with a speed $s(t)$ satisfies (3.13) for some k .

Proof. Assume that for a fixed time t we have an isolated discontinuity in the limit function u at x moving with a speed s . Let

$$u_l = \lim_{x \rightarrow x^-} u(x, t), \quad u_r = \lim_{x \rightarrow x^+} u(x, t).$$

Since convergence in L_1^{loc} implies pointwise convergence almost everywhere, we can find sequences $\{x_{\delta,l,1}\}$, $\{x_{\delta,r,1}\}$, $\{x_{\delta,l,2}\}$, $\{x_{\delta,r,2}\}$, and $\{t_{\delta,1}\}$ and $\{t_{\delta,2}\}$ with $t_{\delta,1} < t_{\delta,2}$ such that

$$t_\delta, k \rightarrow t, \quad x_{\delta,l,k} \rightarrow x^-, \quad x_{\delta,r,k} \rightarrow x^+, \\ u_\delta(x_{\delta,l,k}, t_{\delta,k}) \rightarrow u_l, \quad u_\delta(x_{\delta,r,k}, t_{\delta,k}) \rightarrow u_r$$

as $\delta \rightarrow 0$ for $k = 1, 2$. Furthermore we may define a parallelepiped E_δ with corners $(x_{\delta,l,1}, t_\delta^1)$, $(x_{\delta,r,1}, t_\delta^1)$, $(x_{\delta,r,2}, t_\delta^2)$, and $(x_{\delta,l,2}, t_\delta^2)$ such that the diagonals have slope not too different from s , i.e.,

$$\frac{x_{\delta,r,1} - x_{\delta,l,2}}{t_\delta^1 - t_\delta^2} - \varepsilon(\delta) < s < \frac{x_{\delta,l,1} - x_{\delta,r,2}}{t_\delta^1 - t_\delta^2} + \varepsilon(\delta),$$

for some $\varepsilon(\delta)$ that vanishes as $\delta \rightarrow 0$.

We call a discontinuity line in u_δ an *approximate shock wave* if it represents a shock in the solution to the Riemann problem where it originated. Similarly we call a discontinuity line an *approximate rarefaction wave* if it represents part of a rarefaction fan. These are the only kind of discontinuity lines in u_δ . Let

$$M_\delta^k = \frac{\sum_i |\eta_i|}{x_{\delta,r,k} - x_{\delta,l,k}},$$

where the sum is taken over the approximate rarefaction waves crossing the line segment $t_\delta^k \times [x_{\delta,l,k}, x_{\delta,r,k}]$ for $k = 1, 2$. If this is unbounded as $\delta \rightarrow 0$ then

the limit function u must have a rarefaction wave centered at (x, t) and the discontinuity would not be isolated here. Thus M_δ^k is bounded as $\delta \rightarrow 0$. Now

$$\begin{aligned} \frac{|u_{\delta,r,k} - u_{\delta,l,k}|}{x_{\delta,r,k} - x_{\delta,l,k}} &\leq \frac{\sum |\text{appr. rarefaction}| + \sum |\text{appr. shock}|}{x_{\delta,r,k} - x_{\delta,l,k}} \\ &= M_\delta^k + \frac{\sum |\text{appr. shock}|}{x_{\delta,r,k} - x_{\delta,l,k}}. \end{aligned}$$

Since the fraction on the left is unbounded as $\delta \rightarrow 0$, there must be approximate shocks crossing the line segment $t_\delta^k \times [x_{\delta,l,k}, x_{\delta,r,k}]$ for such small δ .

Since the discontinuity is isolated, it follows that the total strength of all approximate waves of u_δ crossing the left and right sides of E_δ must vanish as $\delta \rightarrow 0$.

We now define a *shock line* to be a sequence of approximate shock waves of the same family in u_δ . Assuming that a shock line has been defined for $t < t_n$, where $\{t_n\}$ are the collision times of u_δ , and in the strip $t_n < t < t_{n+1}$ consists of the approximate shock η . If η does not collide at t_{n+1} then the shock line continues as η until t_{n+2} . If η collides at t_{n+1} the shock line stops if the approximate solution of the Riemann problem defined by the collision of η does not contain an approximate shock wave of the same family as η . If the approximated solution of this Riemann problem contains an approximate shock wave of the same family as η , it continues as that approximate shock wave. Note that to each shock line there corresponds one family.

It now follows that there must be shock lines in E_δ that do not intersect the left or right side of E_δ and that the state to the left of the leftmost of these must tend to u_l as δ tends to zero. To see this, label this state v_l , and let the position of this approximated shock line at t_δ^k be y_l , then $|u_l - v_l| \leq |u_l - u_{\delta,l,k}| + |v_l - u_{\delta,l,k}|$. The first of these two terms tends to zero. Between y_l and $x_{\delta,l,k}$, u_δ only varies over discontinuities that are either approximated rarefaction waves and therefore arbitrarily small as δ becomes small, or are shock lines that leave E_δ through the left or right side, and the strength of these must also be arbitrarily small as δ tends to zero. Similarly the state to the right of these shock lines must tend to u_r as δ tends to zero.

Since $\varepsilon(\delta)$ vanishes as $\delta \rightarrow 0$, the family of all shock lines not intersecting the left or right sides of E_δ must be the same, say k , since our system is strictly hyperbolic. Furthermore the speed of the approximated shock waves which constitute these shock lines must tend to s as $\delta \rightarrow 0$. Any approximate shock wave of family k and left and right states v_l and v_r respectively, will satisfy the approximate Lax inequalities

$$(3.14) \quad \lambda_{k-1}(v_l) < \tilde{s} + O(\delta) < \lambda_k(v_l),$$

$$(3.15) \quad \lambda_k(v_r) < \tilde{s} + O(\delta) < \lambda_{k+1}(v_r),$$

where \tilde{s} is the speed of the approximate shock wave. Now the result follows by applying (3.14) to the leftmost of the shock lines contained in E_δ and (3.15) to the rightmost. \square

After a certain time, we have that our approximation u_δ will consist of constant states with discontinuities that are moving apart. This has some similarity to the solution of a Riemann problem, and we will show that the limit function

also has such a property. We now let T_δ be the last time discontinuities in u_δ collide. After T_δ , u_δ will consist of a number of states; $\{u_i\}_1^M$ separated by discontinuities moving apart. We define the *real states* of u_δ to be those u_i such that u_{i-1} is connected to u_i by a different wave than the one connecting u_i to u_{i+1} . We label the real states of u_δ : $\{\bar{u}_i\}_1^N \subset \{u_i\}_1^M$. Concerning the real states of u_δ we have the following result.

Theorem 6. *For sufficiently small δ , let $\{\bar{u}_i\}_1^N$ be the real states of u_δ after T_δ . Then $N \leq n + 1$ (where n is the dimension of the system) and there exists some u_L, u_R such that*

$$(u_L, u_R) = [(u_0, \dots, u_{N'}) / (\varepsilon_1, \dots, \varepsilon_{N'})], \quad N \leq N' \leq n + 1,$$

where $|\bar{u}_i - u_i| \leq O(\delta)$ for $i \leq N$.

If (u_i, u_{i+1}) is a shock with speed s_i , then $(\bar{u}_i, \bar{u}_{i+1})$ is a single discontinuity moving with speed \bar{s}_i and $|\bar{s}_i - s_i| \leq O(\delta)$.

If (u_i, u_{i+1}) is a rarefaction wave, which we call $u(s)$ for $s = x/t$, then $(\bar{u}_i, \bar{u}_{i+1})$ is a series of approximate rarefaction waves $\{(\tilde{u}_j, \tilde{u}_{j+1})\}_{j=1}^k$. If $(\tilde{u}_j, \tilde{u}_{j+1})$ is separated by a discontinuity moving with speed s_j , then

$$|u(s_{ij}) - \tilde{u}_{ij}| \leq O(\delta) \quad \text{and} \quad |u(s_{ij}) - \tilde{u}_{i+1j}| \leq O(\delta).$$

Proof. We first show that $N \leq n + 1$. By construction of u_δ we have that the solution to the Riemann problem $(\bar{u}_i, \bar{u}_{i+1})$ consists of at most $n + 1$ states $\{v_i\}$ and that there exists a k such that for $j \leq k$, $|\bar{u}_i - v_j| \leq O(\delta)$, and for $j > k$, $|\bar{u}_{i+1} - v_j| \leq O(\delta)$. That is, all waves in the Riemann solution are small except the k -wave. Since we have no collisions in u_δ after T_δ , this wave will be either a single discontinuity or a single approximate rarefaction wave by genuine nonlinearity, and because (3.14) and (3.15) will hold for an approximate shock wave. The approximation u_δ is constructed so that the speed of this k -wave will be close to the speed of the approximate wave; for a shock the speeds will be the same, for a rarefaction the speed of the head or tail of the wave may differ from the correct speed by at most $O(\delta)$. For sufficiently small δ we have that $N \leq n + 1$ since $\bar{s}_i < \bar{s}_{i+1}$.

To prove the second statement we use induction on N ; the number of real states in u_δ after T_δ . In case $N = 2$ we have just seen that the theorem holds. Assume it to be true for N . By “near” we will in the following mean $O(\delta)$. By construction of u_δ and what was said in the last paragraph we have states $\{v_l, v_r\}$ such that (v_l, v_r) is near $(\bar{u}_N, \bar{u}_{N+1})$, and that the Riemann problem (v_l, v_r) is solved by a j -wave, where $j > N - 1$. By the induction hypothesis, we also have states (u_1, \dots, u_N) near $(\bar{u}_1, \dots, \bar{u}_N)$ such that the Riemann problem (u_1, u_N) is solved by (u_1, \dots, u_N) . We now have states u_N, v_l , and v_r , such that u_N is near v_l , and v_l is connected to v_r by a j -wave. Assume that we can find a state u_{N+1} near v_r , and that u_N is connected to u_{N+1} by a j -wave. Then

$$|\bar{u}_{N+1} - u_{N+1}| \leq |\bar{u}_{N+1} - v_r| + |v_r - u_{N+1}| \leq 2O(\delta),$$

and the induction step will be completed.

To show the existence of such a state we consider two cases: First we assume that the j -wave is a rarefaction wave. In this case v_r lies on an orbit starting from v_l of the vector field of the right eigenvector r_j . Orbits starting from nearby points will stay close to the orbit starting from v_l for small rarefaction

strengths. Such a state can therefore be found if the wave is sufficiently weak. This will be the case if $T.V.(u_0)$ is sufficiently small.

If the j -wave is a shock, then v_l, v_r will satisfy the Hugoniot relation

$$(3.16) \quad s_j(v_r - v_l) = f(v_r) - f(v_l).$$

By continuity of f , we can find a state u_{N+1} near v_r , and a speed \hat{s}_N near s_j , such that (3.16) holds for the triplet $\{u_N, u_{N+1}, \hat{s}_N\}$. By the continuity of the eigenvalues of df this triplet will also satisfy (3.13). The proof of the statement for rarefaction waves is similar. \square

There is an immediate corollary of this:

Corollary 4. *Let $u = \lim_{\delta \rightarrow 0} u_\delta$, $\lim_{x \rightarrow -\infty} u_0(x) = u_L$, and $\lim_{x \rightarrow \infty} u_0(x) = u_R$. As $t \rightarrow \infty$, u will consist of a finite number of constant states $\{u_i\}_1^N$, separated by rarefaction waves or shocks, where $N \leq n + 1$. These states are the states in the solution to the Riemann problem (u_L, u_R) , and they will be separated by the same waves as the corresponding states in the solution to the Riemann problem.*

Proof. By the proof of the preceding theorem, we can define a function $\bar{u}_\delta(x, t)$ for $t > T_\delta$ such that \bar{u}_δ consists of N constant states separated by shocks or rarefaction waves. Let $\|\cdot\|_1$ denote the $L_1^{\text{loc}}(\mathbb{R})$ norm. As $\delta \rightarrow 0$ the difference $\|u_\delta - \bar{u}_\delta\|_1 \rightarrow 0$. Therefore

$$\|u - \bar{u}_\delta\|_1 \leq \|u - u_\delta\|_1 + \|u_\delta - \bar{u}_\delta\|_1.$$

Here both terms on the right tend to zero as $\delta \rightarrow 0$. Note that u does not necessarily become equal to some \bar{u}_δ in finite time. \square

We remark that the equivalent of Corollary 4 was shown to hold for the Glimm construction by Liu in [15].

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