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A FULL CLASSIFICATION OF CONTACT METRIC (k, μ)-SPACES

E. BOECKX

ABSTRACT. We show that a non-Sasakian contact metric manifold whose characteristic vector field belongs to the (k, μ)-nullity distribution is completely determined locally by its dimension and the values for k and μ. We present explicit examples for all possible dimensions and all possible (k, μ).

1. Introduction

In [2], the authors introduced the class of contact metric spaces $(M, \xi, \eta, \varphi, g)$ for which the characteristic vector field ξ belongs to the (k, μ)-nullity distribution for some real numbers k and μ. This means that the Riemann curvature tensor R satisfies

$$R(X, Y)\xi = k\ (\eta(Y)X - \eta(X)Y) + \mu\ (\eta(Y)hX - \eta(X)hY) \tag{1}$$

for all vector fields X and Y on M, where h denotes, up to a scaling factor, the Lie derivative of the structure tensor φ in the direction of ξ. For convenience, we will call such contact metric spaces *(k, μ)-spaces*. Clearly, Sasakian spaces belong to this class ($k = 1$ and $h = 0$), but the main interest is in the non-Sasakian contact metric manifolds with this curvature property.

The motivation for the study of (k, μ)-spaces was two-fold: on the one hand, non-Sasakian spaces of this type exist (the unit tangent sphere bundle of a flat Riemannian manifold with the usual contact metric structure being an example); on the other hand, this class of spaces is invariant under D-homothetic transformations (see Section 2).

The basic result in [2] is that the Riemann curvature tensor of a non-Sasakian (k, μ)-space is completely determined by the defining condition (1). The present author [3] used this fact to prove, first, that *every non-Sasakian (k, μ)-space is a locally homogeneous (hence analytic) contact metric space*, and, second, that *every non-Sasakian (k, μ)-space is locally φ-symmetric in the strong sense*. This means that the local reflections with respect to the integral curves of the characteristic vector field ξ are local isometries [5] (see also [4] and [6]).

By now, two classes of non-Sasakian (k, μ)-spaces are known. The first consists of the unit tangent sphere bundles of spaces of constant curvature, equipped with their natural contact metric structure. They satisfy the curvature condition (1) for

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$k = c(2-c)$ and $\mu = -2c$, where c is the constant sectional curvature. (If $c \neq 1$, the corresponding sphere bundle is not Sasakian.) Moreover, within the class of unit tangent sphere bundles, they are the only ones with this property ([2], see also [5]). D-homothetic transformations of these bundles provide further examples. Second, all three-dimensional unimodular Lie groups, except the commutative one, admit the structure of a left-invariant (k, μ)-space. (See [2], [4] and [9].)

To our knowledge, these examples are the only ones that have appeared in the literature, but they do not cover all possible combinations for k and μ for all dimensions. Hence the question, raised explicitly in [2]: Can one classify all (non-Sasakian) (k, μ)-spaces ? This is what we set out to do in this article.

After some preliminary notions and results, collected in Section 2, we prove that, for a non-Sasakian (k, μ)-space, not only the Riemann curvature but the complete local geometry is determined by its dimension and the numbers k and μ (Theorem 3). Next, we introduce an invariant for a (k, μ)-space which remains unchanged under a D-homothetic transformation. This invariant, together with the dimension of the manifold, then determines the local structure up to some D-homothetic transformation. In the last section, we give explicit examples of (k, μ)-spaces for any dimension and for all possible values for k and μ, thereby giving an affirmative answer (at least locally) to the above question.

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2. Preliminaries

In this section we collect the formulas and results we need. We begin with some basic facts on contact metric manifolds. We refer to [1] for a more detailed treatment. All manifolds in this note are assumed to be connected and smooth.

An odd-dimensional differentiable manifold M^{2n+1} has an *almost contact structure* if it admits a vector field ξ, a one-form η and a $(1, 1)$-tensor field φ satisfying

$$\eta(\xi) = 1 \quad \text{and} \quad \varphi^2 = -\mathrm{id} + \eta \otimes \xi.$$

In that case, one can always find a compatible Riemannian metric g, i.e., such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y on M. $(M, \xi, \eta, \varphi, g)$ is an *almost contact metric manifold.* If the additional property $d\eta(X, Y) = g(X, \varphi Y)$ holds, then $(M, \xi, \eta, \varphi, g)$ is called a *contact metric manifold.* As a consequence, the characteristic curves (i.e., the integral curves of the characteristic vector field ξ) are geodesics.

On a contact metric manifold M, we define the $(1, 1)$-tensor h by

$$hX = \frac{1}{2}(\mathcal{L}_\xi \varphi)(X)$$

where $\mathcal{L}_\xi$ denotes Lie differentiation in the direction of ξ. The tensor h is self-adjoint, $h\xi = 0$, $\operatorname{tr} h = 0$ and $h\varphi = -\varphi h$. It holds $\nabla_X \xi = -\varphi X - \varphi h X$, or equivalently,

$$(\nabla_X \eta)(Y) = g(X, \varphi Y) - g(X, \varphi h Y). \tag{2}$$

If the vector field ξ on a contact metric manifold $(M, \xi, \eta, \varphi, g)$ is a Killing vector field, then the manifold is called a *K-contact manifold.* This is the case if and only if $h = 0$. Finally, if the Riemann curvature tensor satisfies

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi = \eta(Y)X - \eta(X)Y \tag{3}$$

for all vector fields X and Y on M, then the contact metric manifold is *Sasakian.* In that case, ξ is a Killing vector field, hence every Sasakian manifold is K-contact.

Next, we focus on the *(k, μ)-spaces*, the contact metric manifolds for which the curvature condition (1) holds. This class of manifolds was introduced in [2]. The authors prove:

THEOREM 1. *If $(M^{2n+1}, \xi, \eta, \varphi, g)$ is a (k, μ)-space, then $k \leq 1$. If $k = 1$, then $h = 0$ and $(M, \xi, \eta, \varphi, g)$ is a Sasakian manifold. If $k < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions, the eigendistributions of the tensor field h: $D(0) = \mathbb{R}\xi$, $D(\lambda)$ and $D(-\lambda)$, where 0, $\lambda = \sqrt{1-k}$ and $-\lambda$ are the (constant) eigenvalues of h.*

Moreover, if $k < 1$, the curvature tensor is completely determined by the defining condition (1).

THEOREM 2. *Let $(M^{2n+1}, \xi, \eta, \varphi, g)$ be a (k, μ)-space which is not Sasakian, i.e., $k < 1$. Then its Riemann curvature tensor R is given explicitly in its (0, 4)-form by*

$$\begin{aligned}
(4)\quad & g(R(X, Y)Z, W) \\
&= \left(1 - \frac{\mu}{2}\right)(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) \\
&\quad + g(Y, Z)g(hX, W) - g(X, Z)g(hY, W) \\
&\quad - g(Y, W)g(hX, Z) + g(X, W)g(hY, Z) \\
&\quad + \frac{1 - (\mu/2)}{1-k}(g(hY, Z)g(hX, W) - g(hX, Z)g(hY, W)) \\
&\quad - \frac{\mu}{2}(g(\varphi Y, Z)g(\varphi X, W) - g(\varphi X, Z)g(\varphi Y, W)) \\
&\quad + \frac{k - (\mu/2)}{1-k}(g(\varphi h Y, Z)g(\varphi h X, W) - g(\varphi h Y, W)g(\varphi h X, Z)) \\
&\quad + \mu g(\varphi X, Y)g(\varphi Z, W) \\
&\quad + \eta(X)\eta(W)((k - 1 + (\mu/2))g(Y, Z) + (\mu - 1)g(hY, Z))
\end{aligned}$$

$$- \eta(X)\eta(Z)\,((k-1+(\mu/2))g(Y,W) + (\mu-1)g(hY,W))$$
$$+ \eta(Y)\eta(Z)\,((k-1+(\mu/2))g(X,W) + (\mu-1)g(hX,W))$$
$$- \eta(Y)\eta(W)\,((k-1+(\mu/2))g(X,Z) + (\mu-1)g(hX,Z))$$

for all vector fields X, Y, Z and W on M.

In the next section, we will need two more results from [2] concerning the covariant derivatives of the tensors h and φ:

$$(5) \qquad (\nabla_X h)Y = ((1-k)g(X,\varphi Y) - g(X,\varphi hY))\,\xi - \eta(Y)\,((1-k)\varphi X + \varphi hX) - \mu\eta(X)\varphi hY,$$

$$(6) \qquad (\nabla_X \varphi)Y = (g(X,Y) + g(X,hY))\,\xi - \eta(Y)(X + hX).$$

(Note that these formulas are valid for all possible values of k, including $k = 1$.)

Finally, we recall the notion of a *D-homothetic transformation* of a contact metric manifold (M,ξ,η,φ,g). For a (strictly) positive constant a, this transformation associates to (M,ξ,η,φ,g) another contact metric manifold $(M,\bar\xi,\bar\eta,\bar\varphi,\bar g)$ related to the first one by

$$\bar\eta = a\,\eta, \quad \bar\xi = \frac{1}{a}\,\xi, \quad \bar\varphi = \varphi, \quad \bar g = a\,g + a(a-1)\,\eta\otimes\eta.$$

A D-homothetic transformation with constant a transforms a (k,μ)-space into a $(\bar k,\bar\mu)$-space where

$$(7) \qquad \bar k = \frac{k + a^2 - 1}{a^2} \quad \text{and} \quad \bar\mu = \frac{\mu + 2a - 2}{a}$$

(see [2]). In particular, we see that a D-homothetic transformation of a Sasakian space $(k = 1)$ is again Sasakian $(\bar k = 1)$.

3. Equivalence problem

THEOREM 3. *Let $(M^{2n+1},\xi,\eta,\varphi,g)$ and $(M'^{2n+1},\xi',\eta',\varphi',g')$ be two non-Sasakian (k,μ)-spaces. Then they are locally isometric as contact metric spaces. In particular, if they are simply connected and complete, they are globally isometric.*

Proof. Let $p \in M$, $p' \in M'$ and put $\lambda = \sqrt{1-k} > 0$. Then we have the decomposition

$$T_pM = D_p(\lambda) \oplus D_p(-\lambda) \oplus \mathbb{R}\xi_p$$

and a similar one for $T_{p'}M'$. Now, let $(e_1,\ldots,e_n)$ be any orthonormal basis of $D_p(\lambda)$ and $(e'_1,\ldots,e'_n)$ any orthonormal basis of $D_{p'}(\lambda)$. Define the linear isometry F: $T_pM \to T_{p'}M'$ by

$$F\xi_p = \xi'_{p'}, \quad Fe_i = e'_i, \quad F\varphi e_i = \varphi' e'_i$$

for $i = 1, \dots, n$. It follows that

$$F^*\eta'_{p'} = \eta_p, \quad F^*h'_{p'} = h_p, \quad F^*\varphi'_{p'} = \varphi_p.$$

As a consequence, we see from the expression (4) for the Riemann curvature tensor R that $F^*R'_{p'} = R_p$. Further, as follows from the formulas (2), (5) and (6) and an easy induction, we also find, for all $\ell \in \mathbb{N}$,

$$F^*\nabla'^{\ell}\eta'_{p'} = \nabla^{\ell}\eta_p, \quad F^*\nabla'^{\ell}h'_{p'} = \nabla^{\ell}h_p, \quad F^*\nabla'^{\ell}\varphi'_{p'} = \nabla^{\ell}\varphi_p.$$

On the one hand, this implies that $F^*\nabla'^{\ell}R'_{p'} = \nabla^{\ell}R_p$ for all $\ell \in \mathbb{N}$, so (M, g) and (M', g') are locally isometric ([7]). On the other hand, the local isometry also maps the structure tensors (ξ, η, φ) to (ξ', η', φ'). Hence, both (k, μ)-spaces are locally isometric as contact metric spaces. □

Remark. Adapting the above proof, one can give an alternative demonstration for the local homogeneity of a non-Sasakian (k, μ)-space, at least when the structure is known to be analytic [3].

Next, we associate to a non-Sasakian (k, μ)-manifold $(M, \xi, \eta, \varphi, g)$ the number

$$I_M = \frac{1 - \mu/2}{\sqrt{1-k}} = \frac{1 - \mu/2}{\lambda}.$$

Then we have:

PROPOSITION 4. *Let $(M_i, \xi_i, \eta_i, \varphi_i, g_i)$, $i = 1, 2$, be two non-Sasakian (k_i, μ_i)-spaces of the same dimension. Then $I_{M_1} = I_{M_2}$ if and only if, up to a D-homothetic transformation, the two spaces are locally isometric as contact metric spaces. In particular, if both spaces are simply connected and complete, they are globally isometric up to a D-homothetic transformation.*

Proof. Recall that a (k, μ)-space is transformed under the D-homothetic transformation with constant $a > 0$ in a $(\bar{k}, \bar{\mu})$-space with

$$\bar{k} = \frac{k + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a},$$

or equivalently,

$$\bar{\lambda} = \sqrt{1 - \bar{k}} = \frac{\sqrt{1-k}}{a} = \frac{\lambda}{a}, \quad 1 - \bar{\mu}/2 = \frac{1 - \mu/2}{a}.$$

Hence, a D-homothetic transformation preserves the invariant I_M.

Conversely, if $(M, \xi, \eta, \varphi, g)$ is a (k, μ)-manifold and $I_M = \frac{1-\bar{\mu}/2}{\sqrt{1-\bar{k}}}$ for some numbers $\bar{k}, \bar{\mu}$ ($\bar{k} < 1$), then applying a D-homothetic transformation to $(M, \xi, \eta, \varphi, g)$ with constant $a = \sqrt{1-k}/\sqrt{1-\bar{k}}$ we get a $(\bar{k}, \bar{\mu})$-space.

The proposition now follows easily: if $I_{M_1} = I_{M_2}$, apply a D-homothetic transformation to $(M_1, \xi_1, \eta_1, \varphi_1, g_1)$ with constant $a = \sqrt{1-k_1}/\sqrt{1-k_2}$ to obtain a (k_2, μ_2)-space $(M_1, \xi_1/a, a\eta_1, \varphi_1, ag_1 + a(a-1)\,\eta_1 \otimes \eta_1)$. By the previous theorem, this is locally isometric as a contact metric space to $(M_2, \xi_2, \eta_2, \varphi_2, g_2)$. □

COROLLARY 5. *Let $(M, \xi, \eta, \varphi, g)$ be a non-Sasakian (k, μ)-space. Then it is locally isometric, up to a D-homothetic transformation, to the unit tangent sphere bundle of some space of constant curvature (different from 1) if and only if $I_M > -1$.*

Proof. As mentioned in the introduction, $T_1M(c)$ is a (k, μ)-space for $k = c(2-c)$ and $\mu = -2c$. The corresponding invariant is given by $I_{T_1M(c)} = (1+c)/|1-c|$. When c varies over the reals, the invariant takes every value strictly greater than -1. □

4. The local classification of non-Sasakian (k, μ)-spaces

From the results in the previous section, it follows that we know all non-Sasakian (k, μ)-spaces locally as soon as we have, for every odd dimension $2n+1$ and for every possible value for the invariant I, one (k, μ)-space $(M, \xi, \eta, \varphi, g)$ with $I_M = I$.

For $I > -1$ and for every odd dimension $2n+1$, we have such an example, namely the unit tangent sphere bundle $T_1M^{n+1}(c)$ of a space of constant curvature c, $c \neq 1$, for the appropriate c. See [2] or [5] for more details and explicit formulas for these contact metric spaces.

In this section, we present examples for any odd dimension $2n+1$ and for arbitrary $I \leq -1$. Our examples are left-invariant contact metric structures on Lie groups.

Let $\mathfrak{g}$ be a $(2n+1)$-dimensional Lie algebra with basis $\{\xi, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ and the Lie bracket given by

$$
\begin{aligned}
[\xi, X_1] &= -\frac{\alpha\beta}{2} X_2 - \frac{\alpha^2}{2} Y_1, \\
[\xi, X_2] &= \frac{\alpha\beta}{2} X_1 - \frac{\alpha^2}{2} Y_2, \\
[\xi, X_i] &= -\frac{\alpha^2}{2} Y_i, \qquad i \geq 3, \\
[\xi, Y_1] &= \frac{\beta^2}{2} X_1 - \frac{\alpha\beta}{2} Y_2, \\
[\xi, Y_2] &= \frac{\beta^2}{2} X_2 + \frac{\alpha\beta}{2} Y_1,
\end{aligned}
$$

$$
\begin{aligned}
[\xi, Y_i] &= \frac{\beta^2}{2} X_i, \qquad i \geq 3, \\
[X_1, X_i] &= \alpha X_i, \qquad i \neq 1, \\
[X_i, X_j] &= 0, \qquad i, j \neq 1, \\
[Y_2, Y_i] &= \beta Y_i, \qquad i \neq 2, \\
[Y_i, Y_j] &= 0, \qquad i, j \neq 2, \\
[X_1, Y_1] &= -\beta X_2 + 2\xi, \\
[X_1, Y_i] &= 0, \qquad i \geq 2, \\
[X_2, Y_1] &= \beta X_1 - \alpha Y_2, \\
[X_2, Y_2] &= \alpha Y_1 + 2\xi, \\
[X_2, Y_i] &= \beta X_i, \qquad i \geq 3, \\
[X_i, Y_1] &= -\alpha Y_i, \qquad i \geq 3, \\
[X_i, Y_2] &= 0, \qquad i \geq 3, \\
[X_i, Y_j] &= \delta_{ij} (-\beta X_2 + \alpha Y_1 + 2\xi), \qquad i, j \geq 3
\end{aligned}
$$

for real numbers α and β. A straightforward computation shows that this defines indeed a Lie algebra. The associated Lie group G is not unimodular as soon as $\dim \mathfrak{g} \geq 5$ and not both α and β are zero, because $\operatorname{tr} \operatorname{ad}_{X_1} = (n-1)\alpha$ and $\operatorname{tr} \operatorname{ad}_{Y_2} = (n-1)\beta$.

Next we define a left-invariant contact metric structure on G as follows:

- the basis $\{\xi, X_1, \dots, X_n, Y_1, \dots, Y_n\}$ is orthogonal,
- the characteristic vector field is given by ξ,
- the one-form η is the metric dual of ξ,
- the $(1, 1)$-tensor field φ is determined by

$$\varphi(\xi) = 0, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i.$$

A lengthy but routine calculation shows that $(G, \xi, \eta, \varphi, g)$ is a (k, μ)-space. For our present purpose, we suppose that $\beta^2 > \alpha^2$. Then

$$k = 1 - \frac{(\beta^2 - \alpha^2)^2}{16}, \quad \mu = 2 + \frac{\alpha^2 + \beta^2}{2}.$$

Hence,

$$\lambda = \frac{\beta^2 - \alpha^2}{4} \neq 0,$$

so the (k, μ)-space is not Sasakian. Further,

$$I_G = -\frac{\beta^2 + \alpha^2}{\beta^2 - \alpha^2} \leq -1.$$

For the appropriate choice of $\beta > \alpha \geq 0$, I_G attains any real value smaller than or equal to -1.

Remark. For the three-dimensional case we obtain the Lie algebra

$$[\xi, X] = -\frac{\alpha^2}{2} Y,$$
$$[\xi, Y] = \frac{\beta^2}{2} X,$$
$$[X, Y] = 2\xi$$

which corresponds to a unimodular Lie group. For $\beta > \alpha > 0$, we obtain a left-invariant contact metric structure on $SL(2, \mathbb{R})$ with $I_{SL(2,\mathbb{R})} < -1$; if $\beta > \alpha = 0$, we have a left-invariant contact metric structure on $E(1, 1)$ with $I_{E(1,1)} = -1$. (See also [2] and [8].) To complete the picture in the three-dimensional case, we note that left-invariant contact metric structures which are (k, μ)-spaces are further possible on $SU(2)$ with $I_{SU(2)} > 1$, on $E(2)$ with $I_{E(2)} = 1$ and on $SL(2, \mathbb{R})$ with $-1 < I_{SL(2,\mathbb{R})} < 1$ ([2]).

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Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium eric.boeckx@wis.kuleuven.ac.be