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Achim Kempf, Aidan Chatwin-Davies, Robert T. W. Martin

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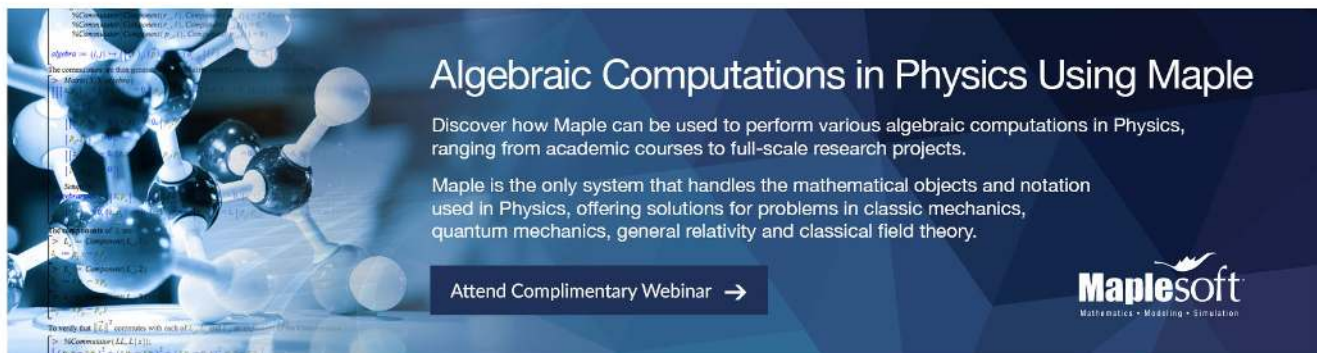
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## A fully covariant information-theoretic ultraviolet cutoff for scalar fields in expanding Friedmann Robertson Walker spacetimes

A. Kempf,<sup>1,2</sup> A. Chatwin-Davies,<sup>1</sup> and R. T. W. Martin<sup>3</sup>

<sup>1</sup>*Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*

<sup>2</sup>*Centre for Quantum Computing Technology, University of Queensland, Brisbane QLD 4072, Australia*

<sup>3</sup>*Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X1 - Rondebosch Cape Town, Western Cape 7701, South Africa*

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While a natural ultraviolet cutoff, presumably at the Planck length, is widely assumed to exist in nature, it is nontrivial to implement a minimum length scale covariantly. This is because the presence of a fixed minimum length needs to be reconciled with the ability of Lorentz transformations to contract lengths. In this paper, we implement a fully covariant Planck scale cutoff by cutting off the spectrum of the  $d'$  Alembertian. In this scenario, consistent with Lorentz contractions, wavelengths that are arbitrarily smaller than the Planck length continue to exist. However, the dynamics of modes of wavelengths that are significantly smaller than the Planck length possess a very small bandwidth. This has the effect of freezing the dynamics of such modes. While both wavelengths and bandwidths are frame dependent, Lorentz contraction and time dilation conspire to make the freezing of modes of trans-Planckian wavelengths covariant. In particular, we show that this ultraviolet cutoff can be implemented covariantly also in curved spacetimes. We focus on Friedmann Robertson Walker spacetimes and their much-discussed trans-Planckian question: The physical wavelength of each comoving mode was smaller than the Planck scale at sufficiently early times. What was the mode's dynamics then? Here, we show that in the presence of the covariant UV cutoff, the dynamical bandwidth of a comoving mode is essentially zero up until its physical wavelength starts exceeding the Planck length. In particular, we show that under general assumptions, the number of dynamical degrees of freedom of each comoving mode all the way up to some arbitrary finite time is actually finite. Our results also open the way to calculating the impact of this natural UV cutoff on inflationary predictions for the cosmic microwave background. © 2013 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4790482>]

### I. INTRODUCTION

Heuristic quantum gravity arguments suggest the existence of some form of a smallest length in nature. Assume, for example, one tried to resolve a distance with an uncertainty of less than a Planck length. Due to the uncertainty principle, this should imply a momentum uncertainty above the Planck momentum. This in turn, due to Einstein's equation, should lead to an uncertainty in curvature of the order of a Planckian size curvature radius. This curvature uncertainty foils the attempt to measure a distance with a precision below the Planck length. The difficulty with this intuition lies in expressing the existence of a minimum length covariantly, given that Lorentz contraction can contract any given length. That a minimum length is in fact not in contradiction with Lorentz contraction was shown in Ref. 1, see also Refs. 2–6. For a recent review on the introduction of a covariant minimum length in noncommutative geometry, see Ref. 7.

Here, we will consider the proposal in Refs. 8 and 9 that the notion of a smallest length in nature could manifest itself in that all physical fields possess no wavelength smaller than a fixed smallest wavelength. While the notion of a finite minimum wavelength is not covariant, it can be turned into a covariant notion, as we will discuss. It will, however, be instructive to first collect properties of fields, which possess no wavelength smaller than some fixed finite smallest wavelength, say at the Planck length.

To this end, we draw results from information theory. In information theory, the crucial equivalence between continuous and discrete information is established by Shannon sampling theory. It holds that bandlimited functions can be perfectly reconstructed anywhere from the values they take on any chosen discrete sets of points whose average spacing is given by the inverse of the bandwidth. For example, the amplitudes of 20 KHz bandlimited continuous music signals are usually sampled and recorded at the rate corresponding to this bandwidth. Then, using the Shannon sampling theory, the continuous music signal can be reconstructed perfectly for all times from only these discrete samples. Inaccuracies can only arise from errors made in taking the samples. While the samples need not be taken equidistantly, in engineering practice they usually are. The reason is that the more unevenly spaced the samples are taken, the more the reconstruction is sensitive to measurement inaccuracies.

If physical fields are bandlimited they too can be reconstructed everywhere from their amplitudes on any discrete lattice of sufficiently dense average spacing. It is worth pointing out a key difference to physical theories that are defined on a single lattice. In contrast to those theories, here the fields can be represented on continuous spacetime or, equivalently, on *any* discrete lattice whose average spacing is dense enough. The fact that no lattice is singled out implies, for example, that external symmetries such as translation and rotation invariance are not broken. With a minimum wavelength cutoff, spacetime could be in effect simultaneously continuous and discrete in mathematically the same way that information can be simultaneously continuous and discrete, see Ref. 10.

Concretely, functions with a minimum wavelength are called  $\Omega$ -bandlimited, where  $\Omega > 0$  is the magnitude of the highest frequency component they contain. On the real line,  $\mathbb{R}$ , the value that an  $\Omega$ -bandlimited function  $f$  takes at a point  $x \in \mathbb{R}$  is completely reconstructible, for example, from the values it takes on the discrete set of points  $\{x_n := \frac{n\pi}{\Omega}\}_{n \in \mathbb{Z}}$ , namely through the Shannon sampling formula:<sup>11</sup>

$$f(x) = \sum_{n \in \mathbb{Z}} f(x_n) \frac{\sin(\Omega(x - x_n))}{\Omega(x - x_n)}. \quad (1)$$

In  $\mathbb{R}^n$ , a bandlimit, or ultraviolet cutoff of this kind, can be established as a cutoff on the spectrum of the Laplacian,  $\Delta := -\sum_{j=1}^n \frac{d^2}{dx_j^2}$ . This is because the subspace of fields whose Fourier transforms have support only on the disc of radius  $\Omega$  in  $\mathbb{R}^n$  is that subspace of  $L^2(\mathbb{R}^n)$  spanned by the eigenfunctions of  $\Delta$  whose eigenvalues are less than or equal to  $\Omega^2$  in magnitude. More precisely, this subspace is the range of  $\chi_{[0, \Omega^2]}(\Delta)$  where  $\chi_{[0, \Omega^2]}$  is the characteristic function of the interval  $[0, \Omega^2]$  and  $\chi_{[0, \Omega^2]}(\Delta)$  is a projection defined by the functional calculus.

This ultraviolet cutoff on fields in flat space is naturally generalizable to scalar fields on curved space.<sup>9</sup> Given an arbitrary curved space, or Riemannian manifold, we can assume its Laplacian to be self-adjoint. If it possesses boundaries, we assume suitable boundary conditions have been chosen. We now restrict the space of physical scalar fields to be that subspace of square integrable functions on the manifold spanned by the eigenfunctions of the Laplacian whose eigenvalues are less than or equal to  $\Omega^2$ , where  $\Omega$  is the bandlimit or ultraviolet cutoff parameter. That is, define the space of physical fields on a manifold  $M$  to be  $B(M, \Omega) := \chi_{[-\Omega^2, \Omega^2]}(\Delta)L^2(M)$ , where  $\Delta$  is the Laplacian of the manifold  $M$ . This cutoff is covariant since the spectrum of the Laplacian is covariant. For a proof of the sampling property for certain classes of Riemannian manifolds, see, for example, Ref. 12.

In Ref. 9, to restore full covariance, a cutoff on the spectrum of the d'Alembertian on Minkowski space has been investigated. This cutoff is covariant because the spectrum of the d'Alembertian is scalar. Furthermore, it indeed provides a way to overcome the paradox that a fundamental minimum

length should not be able to coexist with Lorentz transformations since the latter can contract any length further. Namely, it was found that, in this scenario, wavelengths that are arbitrarily smaller than the Planck length continue to exist. However, the dynamics of modes of wavelengths that are significantly smaller than the Planck length then automatically possess a very small bandwidth. Their dynamics in effect freezes: It suffices to take samples at a very low rate in time to reconstruct the dynamics at all points in time. While both wavelengths and bandwidths are frame dependent, Lorentz contraction and time dilation conspire to make this behaviour covariant, as they have to because cutting the d'Alembertian's spectrum is covariant.

Here, we extend this preliminary analysis by showing that this ultraviolet cutoff can be implemented covariantly also in curved spacetimes. We study Friedmann Robertson Walker (FRW) spacetimes in particular, where we focus on the much-discussed trans-Planckian question: The physical wavelength of each comoving mode was smaller than the Planck scale at sufficiently early times. What was a mode's dynamics then? (See Ref. 13 for a review of the trans-Planckian question, Sec. V of Ref. 14, Refs. 15 and 16 for a cosmological introduction, and Ref. 17 for a more recent review. Various approaches to the trans-Planckian question are studied in Refs. 18–26 to name a few.) Here, we will show that in the presence of the covariant UV cutoff on the spectrum of the d'Alembertian, the dynamical bandwidth of a comoving mode is essentially zero up until its physical wavelength starts exceeding the Planck length. In particular, we show that under general assumptions, the number of dynamical degrees of freedom of each comoving mode all the way up to some arbitrary finite time is actually finite. Concretely, the number of samples in time that need to be taken from the beginning up to some finite time is finite. Our results should open the way to calculating also the impact of this natural UV cutoff on inflationary predictions for the cosmic microwave background (CMB).

In preparation, let us now recall certain results from the sampling theory of bandlimited functions on  $\mathbb{R}^n$  that we will later need.

## II. REVIEW OF BASIC SAMPLING THEORY OF BANDLIMITED FUNCTIONS

Much is known about the reconstruction and interpolation properties of bandlimited functions in one dimension. In essence, any bandlimited function can be reconstructed from any discrete set of points provided that set of points is sufficiently dense on the real line. Density is defined in the following way. Let  $\Lambda := \{x_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$  be a strictly increasing sequence of real numbers with a finite minimum spacing between its members. Such a set  $\Lambda$  is called a set of sampling if any bandlimited function can be reconstructed in a stable fashion from the values it takes on the points of  $\Lambda$ . More precisely,  $\Lambda$  is called a set of sampling if there are constants  $c, C > 0$  such that for every  $f \in B(\mathbb{R}, \Omega)$ ,

$$c \|f\|^2 \leq \sum_k |f(x_k)|^2 \leq C \|f\|^2. \quad (2)$$

The upper inequality always holds provided the sampling sequence  $\Lambda$  has a finite minimum spacing.<sup>27</sup> The double inequality shows that the linear map from  $B(\mathbb{R}, \Omega)$  to  $l^2(\mathbb{Z})$ , which maps  $f \in B(\mathbb{R}, \Omega)$  onto its sequence of sample values  $\{f(x_n)\}$ , is invertible. In particular, this implies that  $f$  can be perfectly reconstructed from its sample values and that a bounded error in the sample values leads to a bounded error in the reconstructed function, i.e.,  $f$  can be reconstructed from its sample values in a stable fashion. Now let  $n(r)$  be the minimum number of points of  $\Lambda$  in any subinterval of length  $r$ . The lower Beurling density of a sequence  $\Lambda$  is then defined as  $D(\Lambda) := \lim_{r \rightarrow \infty} \frac{n(r)}{r}$ . With this definition, the following theorem holds.<sup>28</sup>

### **Theorem 1:** (Beurling)

*The set of points  $\Lambda$  is a set of sampling for  $B(\mathbb{R}, \Omega)$  if  $D(\Lambda) > \frac{\Omega}{\pi}$ . Conversely, if  $\Lambda$  is a set of sampling then  $D(\Lambda) \geq \frac{\Omega}{\pi}$ .*

More generally, one can consider square integrable functions in  $\mathbb{R}^n$  whose Fourier transforms have support only in some compact  $n -$  dimensional set  $S$ . Call the space of such functions  $B(\mathbb{R}^n, S)$ . The necessity part of the previous theorem generalizes to these frequency limited functions:<sup>28</sup>

**Theorem 2:** (Landau) *If the set of points  $\Lambda$  is a set of sampling for  $B(\mathbb{R}^n, S)$  then  $D(\Lambda) \geq \frac{\mu(S)}{(2\pi)^n}$ .*

Here,  $\mu(S)$  denotes the Lebesgue measure, or volume of  $S$ .  $\Lambda$  is a set of points in  $\mathbb{R}^n$  with a finite minimum spacing between any two of its members, and  $n(r)$  is the smallest number of points of  $\Lambda$  in any  $n -$  dimensional ball of radius  $r$ . This theorem demonstrates that in order for a set of points to be a set of sampling for frequency limited functions, it must be sufficiently dense, where the minimum possible density is proportional to the volume of the compact set  $S$  on which the Fourier transforms of the functions in  $B(\mathbb{R}^n, S)$  have support.

Observe that if  $f \in B(\mathbb{R}^n, S)$ , then there is an  $M > 0$  such that  $S \subset M^n + \mathbf{x}$  where  $M^n$  is an  $n -$  dimensional cube of side length  $M$  and  $\mathbf{x} \in \mathbb{R}^n$ . It follows, by one-dimensional sampling in each coordinate, that  $\Lambda := \{\mathbb{Z}^n \frac{2\pi}{M}\}$  is a set of sampling for  $B(S)$ . This shows that sets of sampling always exist for any space of frequency limited functions. Of course, this example of a set of sampling will have density greater than that required by Landau's theorem, unless  $S$  is itself a cube.

### III. FLAT SPACETIME

Let  $M$  denote flat  $1 + d$  dimensional spacetime. Here, the d'Alembertian is simply  $-\frac{\partial^2}{\partial t^2} + \Delta$ , where  $\Delta$  is the spatial Laplacian. The eigenfunctions of the d'Alembertian are the plane waves  $e^{i(p_0 t - \mathbf{p} \cdot \mathbf{x})}$ , where  $p_0$  is a temporal frequency, and  $\mathbf{p}$  is the spatial frequency vector. In order to impose a covariant ultra-violet cutoff  $\Omega$ , we restrict our attention to the subspace  $B(M, \Omega)$  of square integrable functions on  $M$  spanned by the plane waves whose frequencies, or eigenvalues, obey the inequality

$$|p_0^2 - |\mathbf{p}|^2| \leq \Omega^2. \quad (3)$$

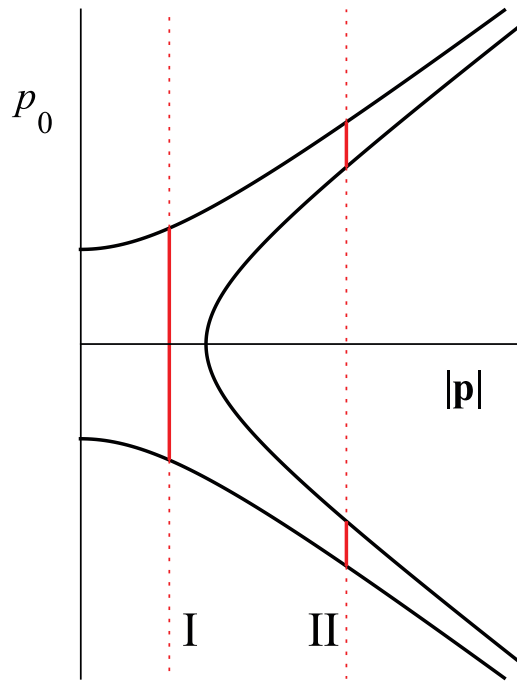
The set  $S$  of all  $p_0$  and  $\mathbf{p}$  satisfying this inequality is the closed interior of a region bounded by hyperboloids. This is an unbounded region, so there exist values of  $p_0$  and  $\mathbf{p}$  inside the region, which are individually arbitrarily large in magnitude. Unlike in the case of bandlimited functions in flat Euclidean space, there exists no discrete set of points  $\Lambda := \{x_n\}_{n \in \mathbb{Z}} \subset M$  of finite density, which is a set of sampling for  $B(M, \Omega)$ . This follows from the fact that  $B(M, \Omega)$  is the image of  $L^2(S)$  under the Fourier transform, where  $S$  is the region in  $\mathbb{R}^4$ , which obeys the inequality (3). It is straightforward to check that for any choice of  $1 \leq d \leq 3$  the volume of  $S$  is infinite, so that Landau's theorem implies that the minimum density a set of points needs to have in order to be a set of sampling for  $B(M, \Omega)$  is infinite.

Nevertheless, elements of  $B(M, \Omega)$  still have special reconstruction properties. Consider an arbitrary spatial mode, i.e., consider a field  $\phi(t, \mathbf{p})$  of fixed spatial momentum  $\mathbf{p}$ . Then the temporal frequencies  $p_0$  of that spatial mode  $\phi(t, \mathbf{p})$  are confined to the compact set  $S_{\mathbf{p}}$  described by the inequality

$$r_1 := \Re \left\{ \sqrt{|\mathbf{p}|^2 - \Omega^2} \right\} \leq |p_0| \leq \sqrt{|\mathbf{p}|^2 + \Omega^2} =: r_2. \quad (4)$$

$S_{\mathbf{p}}$  is a single interval  $[-r_2, r_2]$  for  $|\mathbf{p}| \leq \Omega$  and the union of two intervals  $[-r_2, -r_1] \cup [r_1, r_2]$  for  $|\mathbf{p}| > \Omega$  (see Figure 1). For the set  $\Lambda = \{t_n\}$  to be a set of sampling for a fixed spatial mode  $\mathbf{p}$ , by Landau's theorem,  $\Lambda$  must have a density at least as great as a minimum density, which is proportional to the length of  $S_{\mathbf{p}}$ . It is not hard to check that the length of  $S_{\mathbf{p}}$  vanishes in the limit as  $|\mathbf{p}| \rightarrow \infty$ . Therefore, each fixed spatial mode is temporally bandlimited and as a function of time belongs to  $B(\mathbb{R}, S_{\mathbf{p}})$ . In this simple case where  $S_{\mathbf{p}}$  is composed of at most two intervals, it is known how to explicitly construct sets of sampling, which achieve the minimum density required by Landau's theorem.<sup>29</sup> The fact that the volume of  $S_{\mathbf{p}}$  vanishes in the limit as  $|\mathbf{p}| \rightarrow \infty$  then means that the density of temporal degrees of freedom of a fixed spatial mode  $\mathbf{p}$  vanishes in this limit.



FIG. 1. Fixed spatial  $\mathbf{p}$  modes have finite temporal bandwidth.

This is the manifestation of the covariant ultra-violet cutoff in flat spacetime. Increasingly large spatial modes still exist, but their density of temporal degrees of freedom decreases to zero; they are “frozen out.” Further notice that this cutoff respects Lorentz symmetry, as it must since it is fully covariant. Suppose that one performs a Lorentz transformation to new time and space coordinates  $(t', \mathbf{x}')$ . If one had a temporal lattice of sample points  $\Lambda := \{t_n\}$  in the old coordinates, which was dense enough to reconstruct a fixed spatial mode  $\mathbf{p}$  in the old coordinates, then in the new coordinates the time coordinate is dilated, so that the image of the set of points  $\Lambda' = \{t'_n\}$  is less dense. However, under this coordinate change, the fixed spatial mode  $\mathbf{p}$  becomes a larger fixed spatial mode  $\mathbf{p}'$  due to length contraction. Consequently, the density a set of points in the time coordinate  $t'$  needs to have in order to be a set of sampling for the fixed spatial mode  $\mathbf{p}'$  is lower. In other words, the rate at which one needs to measure the values of a fixed spatial mode in the new coordinates in order to reconstruct it for all time is slower than the rate required in the old coordinates. This allows the image set  $\Lambda'$  of  $\Lambda$  under a Lorentz transformation to be both less dense and a set of sampling for the higher spatial frequency mode  $\mathbf{p}'$ . A further interesting observation is that since there is an upper bound for the temporal bandwidth volume of any fixed spatial mode, it is possible that there could be a set of points in time that is dense enough to be a set of sampling for *every* fixed spatial mode. Such sets of points  $\Lambda$  do indeed exist and are not difficult to construct. This means that if one knows the values that a covariantly bandlimited field takes on each spatial hypersurface corresponding to the fixed values of time belonging to  $\Lambda$ , then the field can be reconstructed everywhere for all space and time. Such a reconstruction formula can be written down explicitly. Namely, let  $\Lambda := \{x_n\} \cup \{y_n\}$ , where  $x_n := \frac{n\pi}{\sqrt{2}\Omega}$  and  $y_n := \frac{(n+\alpha)\pi}{\sqrt{2}\Omega}$ . Here,  $\alpha \in (0, 1)$  is arbitrary. The volume, or length of the set  $S_{\mathbf{p}}$  is always less than or equal to  $2\sqrt{2}\Omega$  for any fixed  $|\mathbf{p}| > 0$  so that by Landau’s theorem (Theorem 2), this set is dense enough to be a set of sampling for every fixed spatial mode  $\mathbf{p}$ . In fact,  $\Lambda$  is a set of sampling for each fixed spatial mode, and if  $\phi_{\mathbf{p}}$  is a fixed spatial mode, the reconstruction formula is given by<sup>29</sup>

$$\phi_{\mathbf{p}}(t) = \sum_{n \in \mathbb{Z}} \phi_{\mathbf{p}}(x_n)K(t - x_n) + \phi_{\mathbf{p}}(y_n)K(y_n - t), \quad (5)$$

where for  $|\mathbf{p}| < \Omega$ ,

$$K(t) = \frac{\cos(\sqrt{2}\Omega t - \alpha\sqrt{2}\Omega/2) - \cos(\alpha\sqrt{2}\Omega/2)}{\Omega t \sin(\alpha\sqrt{2}\Omega/2)}, \quad (6)$$

while for  $|\mathbf{p}| > \Omega$ ,

$$\begin{aligned} K(t) = & \frac{\cos\left(m_{\mathbf{p}}\alpha\sqrt{2}\Omega - (\sqrt{2}\Omega + \sqrt{\mathbf{p}^2 - \Omega^2})t\right)}{\sqrt{2}\Omega t \sin\left(m_{\mathbf{p}}\alpha\sqrt{2}\Omega\right)} \\ & - \frac{\cos\left(m_{\mathbf{p}}\alpha\sqrt{2}\Omega - ((2m_{\mathbf{p}} - 1)\sqrt{2}\Omega - \sqrt{\mathbf{p}^2 - \Omega^2})t\right)}{\sqrt{2}\Omega t \sin\left(m_{\mathbf{p}}\alpha\sqrt{2}\Omega\right)} \\ & + \frac{\cos\left((2m_{\mathbf{p}} - 1)\alpha\sqrt{2}\Omega/2 - ((2m_{\mathbf{p}} - 1)\sqrt{2}\Omega - \sqrt{\mathbf{p}^2 - \Omega^2})t\right)}{\sqrt{2}\Omega t \sin\left((2m_{\mathbf{p}} - 1)\alpha\sqrt{2}\Omega/2\right)} \\ & - \frac{\cos\left((2m_{\mathbf{p}} - 1)\alpha\sqrt{2}\Omega/2 - \sqrt{\mathbf{p}^2 - \Omega^2}t\right)}{\sqrt{2}\Omega t \sin\left((2m_{\mathbf{p}} - 1)\alpha\sqrt{2}\Omega/2\right)}. \end{aligned} \quad (7)$$

In the above,  $m_{\mathbf{p}}$  is the largest integer for which  $(m_{\mathbf{p}} - 1)\sqrt{2}\Omega < \sqrt{\mathbf{p}^2 - \Omega^2}$ . It follows that if one is given a covariantly bandlimited field  $\varphi(t, \mathbf{x})$  and if one knows the values  $\{\varphi(t_n, \mathbf{x})\}_{t_n \in \Lambda; \mathbf{x} \in \mathbb{R}^3}$ , then this field can be reconstructed perfectly everywhere, for all space and time. This is because the knowledge of the values  $\{\varphi(t_n, \mathbf{x})\}_{t_n \in \Lambda; \mathbf{x} \in \mathbb{R}^3}$  determines the values  $\{\phi_{\mathbf{p}}(t_n)\}_{t_n \in \Lambda; \mathbf{p} \in \mathbb{R}^3}$  through the spatial Fourier transform. Here,  $\phi_{\mathbf{p}}(t) := \phi(t, \mathbf{p})$  is the spatial Fourier transform of  $\varphi(t, \mathbf{x})$ . These values together with the reconstruction formula (5) can be used to calculate  $\phi_{\mathbf{p}}(t)$  for all time  $t$  and for all fixed spatial modes  $\mathbf{p}$ . Taking the inverse Fourier transform then yields the original bandlimited field  $\varphi(t, \mathbf{x})$  for all values of space and time.

### A. Aside: Fixed temporal modes of bandlimited fields on flat spacetime

The conclusions in this section followed from the assumption that a spatial mode  $\mathbf{p}$  is fixed. One could instead choose to fix a temporal mode, or any combination of the temporal mode and spatial coordinate modes, and perform a similar analysis.

Note that while the situation is symmetric in 1 + 1 dimensional spacetime, i.e., whether one fixes a temporal or a spatial mode one obtains identical results, the situation is slightly different in 1 +  $k$  dimensional spacetime if  $k \geq 2$ .

For example, in 1 + 2 dimensional spacetime, the spatial bandwidth volume of a fixed temporal mode  $p_0$  is the area between two disks of radii  $r_1$  and  $r_2$  (see Eq. (4)). Moreover, this area is constant, independent of  $p_0$ . If  $k = 3$ , the spatial bandwidth volume is the volume between two spheres of radii  $r_i$ , and this volume diverges as  $p_0 \rightarrow \infty$ . While each fixed temporal mode has a finite spatial bandwidth volume, there is no uniform upper bound for this volume in 1 + 3 dimensions. Landau's theorem then implies that there is no set of spatial points  $\Lambda$ , which is a set of sampling for every fixed temporal mode  $p_0$  in 1 + 3 dimensional flat spacetime, and hence there is no reconstruction formula of the type (5) for a fixed temporal mode  $\phi_{p_0}(\mathbf{x})$  of a bandlimited function on this spacetime. (See Chap. 6 of Ref. 30 for details.)

## IV. EXPANDING FRW SPACETIMES

We are now prepared to begin our investigation of the covariant UV cutoff in expanding spacetimes. The line element for FRW spacetime can be expressed as  $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$  where  $t$  is proper time. The d'Alembertian in 1 + 3 dimensions is then a second order differential



linear operator given by

$$\tilde{\square} := -a^{-3}(t) \left( \frac{d}{dt} a^3(t) \frac{d}{dt} - a(t) \Delta \right), \quad (8)$$

where  $\Delta$  is the spatial Laplacian. This expression is defined to act on a suitable dense domain in  $L^2(\mathbb{R}^4, a^3(t) dt d\mathbf{x})$ .

Here, it is assumed that  $a(t) > 0$  is a positive function. Under Fourier transform in the spatial variables, we obtain the differential operator

$$\tilde{\square}_k := -a^{-3}(t) \left( \frac{d}{dt} a^3(t) \frac{d}{dt} + a(t) k^2 \right). \quad (9)$$

Here,  $k^2 := |\mathbf{k}|^2 \geq 0$  is the magnitude of the comoving spatial frequency vector. In order to construct a well-defined linear operator using this expression, we need to specify its domain.

For example, consider de Sitter spacetime, for which  $a(t) = e^{Ht}$ , where  $H$  is the Hubble constant. In this case, one can show that there is no unique self-adjoint operator associated with the expression (9). Instead one can use this expression to define a symmetric operator with equal and infinite deficiency indices on a dense domain in the Hilbert space. Once one has this symmetric operator, one can construct a family of self-adjoint extensions (self-adjoint operators, which extend the original symmetric operator). Each self-adjoint extension is a possible choice of self-adjoint d'Alembertian for this spacetime. Physical input is necessary to determine which choice is correct. Indeed, the covariant ultra-violet cutoff  $\Omega$  is imposed by projecting onto the subspace spanned by eigenfunctions to the d'Alembertian whose eigenvalues are less than or equal to  $\Omega^2$  in magnitude. This means that, in principle, one must implement the correct self-adjoint extension of the symmetric d'Alembertian operator before cutting off its spectrum. Fortunately, in some cases it will be sufficient to show that the choice of self-adjoint extension does not matter to answer the question at hand.

For the remainder of this paper, the time coordinate will be restricted to an interval  $[t_i, t_f]$ , where  $-\infty \leq t_i < t_f < \infty$ . For power law spacetime,  $a(t) = (Ht)^k$ , it is sensible to choose  $t_i = 0$  (indeed if  $k$  is not an integer, this scale factor is nonsensical for negative  $t$ ), but for spacetimes such as de Sitter, where  $a(t) = e^{Ht}$ , or something more general such as  $a(t) = \sin^2(t)e^t$ , we are free to choose  $t_i = -\infty$ . The restriction to times  $t_f$  that are finite is not necessary but it will be useful to determine a mode's total number of degrees up to some finite time.

Let us now consider a comoving spatial mode  $\phi(t, \mathbf{k})$  and investigate, as in the case of flat spacetime, the implications of the cutoff of the spectrum of the d'Alembertian for that mode. In other words, we consider operators defined using the expressions given by Eq. (9) acting on the Hilbert space  $\mathcal{H} := L^2([t_i, t_f]; a^3(t) dt)$  for fixed  $k := |\mathbf{k}|$ . Cutting off the spectrum of the d'Alembertian by  $\Omega^2$  on the full spacetime then amounts to cutting off the spectrum of each of the fixed- $k$  d'Alembertians  $\square_k$  by  $\Omega^2$ . As discussed above, in order to make this operation precise, we must use the expression (9) to define a particular self-adjoint operator for each comoving spatial momentum magnitude  $k$ . Since (9) is a second order Sturm-Liouville differential expression, we can do this by using the theory of ordinary differential operators.<sup>31</sup>

Define the linear manifold

$$\text{Dom}(\square_k^*) := \{\phi \in \mathcal{H} \mid \phi, a^3(t)\phi' \in AC[t_i, t_f]; \square_k \phi \in \mathcal{H}\}, \quad (10)$$

and define  $\text{Dom}(\hat{\square}_k)$  as the set of all  $\phi \in \text{Dom}(\square_k^*)$ , which have support contained in a compact subinterval of  $(t_i, t_f)$ . Here,  $AC[t_i, t_f]$  denotes the set of all absolutely continuous functions on  $[t_i, t_f]$ . The linear manifold  $\text{Dom}(\square_k^*)$  is the largest linear manifold in  $\mathcal{H}$  on which the formal expression (9) can be defined for fixed  $k$ . It is not difficult to verify that the operator  $\hat{\square}_k$  defined by  $\hat{\square}_k \phi = \tilde{\square}_k \phi$  for all  $\phi \in \text{Dom}(\hat{\square}_k)$  is a symmetric operator. Now let  $\square_k$  again denote the closure of  $\hat{\square}_k$ . It follows from Sec. 17 of Ref. 31 that the domain of  $\square_k$  is given by

$$\text{Dom}(\square_k) = \{\phi \in \text{Dom}(\square_k^*) \mid a^3(t_i)\phi'(t_i) = 0 = a^3(t_f)\phi'(t_f); \phi(t_i) = 0 = \phi(t_f)\}. \quad (11)$$

It can be shown that the operator  $\square_k^*$ , defined by  $\square_k^* \phi = \tilde{\square}_k \phi$  for all  $\phi \in \text{Dom}(\square_k^*)$  is the adjoint to  $\square_k$  (Sec. 17 of Ref. 31). It also follows from Sec. 17 of Ref. 31 that  $\square_k$  has deficiency indices  $(n, n)$  where  $0 \leq n \leq 2$ . Recall that the deficiency indices  $(n_+, n_-)$  of a symmetric operator  $A$  are defined to be  $n_{\pm} := \dim(\text{Ker}(A^* \mp i))$ , that  $A$  has a  $\mathcal{U}(n)$ -parameter family of self-adjoint extensions if and only if  $n_+ = n = n_-$ , and that if  $n_+ = 0 = n_-$ , then  $A$  is essentially self-adjoint. Here,  $\mathcal{U}(n)$  denotes the unitary group of  $n \times n$  unitary matrices. For more details of the theory of symmetric operators and their self-adjoint extensions, we refer the reader to Refs. 32 and 33, and to Ref. 31, for an introduction to symmetric differential operators.

### A. Deficiency indices and self-adjoint extensions

To investigate the effects of the covariant ultraviolet cutoff on the spacetimes discussed above, one must cut off the spectrum of self-adjoint extensions of the symmetric d'Alembertians  $\square_k$ . The deficiency indices and therefore the set of possible self-adjoint extensions of the d'Alembertian depend on the choice of scale factor function  $a(t)$  (although a few conclusions can even be drawn before making a specific choice of  $a(t)$ ).

As we will see Sec. IV A 1, the case where  $\square_k$  has deficiency indices  $(2, 2)$  will be generic and therefore of particular interest. If  $\square_k$  has deficiency indices  $(2, 2)$ , then by a theorem of Krein, the spectrum of any self-adjoint extension  $\square_k'$  is bounded below (p. 93 of Ref. 31). Furthermore, the resolvent  $(\square_k' - \lambda)^{-1}$ , with  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , will be a compact Hilbert-Schmidt operator (Sec. 19 of Ref. 31). Hence, the spectrum of such a resolvent consists of the closure of the set of eigenvalues whose only possible accumulation point is 0. It follows that the spectrum of any self-adjoint extension  $\square_k'$  of  $\square_k$  is purely discrete and has no finite accumulation point. Furthermore, one can show that each of the symmetric operators  $\square_k$  is simple, i.e., each has no eigenvalues. To see this, first recall the formula (11) for the domain of  $\square_k$ . Then by the existence-uniqueness theorem for ordinary differential equations (Sec. 16 of Ref. 31), it follows that if  $\phi \in \text{Dom}(\square_k)$  is a solution to  $\square_k \phi = \lambda \phi$ , then  $\phi \equiv 0$  since  $a^3(t_i)\phi'(t_i) = 0 = \phi(t_i)$ .

If  $\square_k$  is a simple symmetric operator with deficiency indices  $(2, 2)$ , it follows that the spectrum of any self-adjoint extension  $\square_k'$  of  $\square_k$  consists of eigenvalues of multiplicity at most 2, and that given any  $\lambda \in \mathbb{R}$  there is a self-adjoint extension  $\square_k'$  of  $\square_k$  for which  $\lambda$  is an eigenvalue of multiplicity 2 (Sec. 83 of Ref. 32).

Imposing the covariant ultra-violet cutoff on the kind of spacetime discussed above means that one chooses a particular self-adjoint extension  $\square_k'$  and then considers the subspace,  $B_k(\Omega)$ , spanned by eigenfunctions of this d'Alembertian whose eigenvalues are less than or equal to  $\Omega^2$  in magnitude. The nature of the spectra of the  $\square_k'$  in this kind of spacetime implies that each of the subspaces  $B_k(\Omega)$  will be of finite dimension  $N_k < \infty$ . Let  $\{f_n\}_{n=1}^N$  be a basis for such a subspace. Since the  $f_n$  are linearly independent functions of time, almost all sets of  $N$  time points  $\{t_m\}_{m=1}^N$  in  $[t_i, t_f]$  are such that the matrix  $A$  with entries  $A_{nm} := f_n(t_m)$  is invertible. Letting  $\phi_{\mathbf{k}} = \sum_{n=1}^N c_n f_n$  denote the comoving spatial mode  $\mathbf{k}$  of a covariantly bandlimited field in this spacetime, it follows that  $\phi_{\mathbf{k}}(t_m) = \sum_{n=1}^N c_n f_n(t_m)$  so that

$$\phi_{\mathbf{k}}(t) = \sum_{n=1}^N \sum_{j=1}^N A_{nm}^{-1} \phi_{\mathbf{k}}(t_m) f_n(t). \quad (12)$$

Hence, each spatial mode of a covariantly bandlimited field in any FRW spacetime for which  $\square_k$  has deficiency indices  $(2, 2)$  is completely determined by the values it takes on a finite number of points in time, and has only a finite number of temporal degrees of freedom. Of course, this simple linear algebra argument applies to any finite dimensional subspace of functions.

#### 1. The deficiency indices are generically $(2, 2)$

As we will now show, certain generically satisfied conditions are sufficient to guarantee that the deficiency indices of  $\square_k$  are  $(2, 2)$ .

If  $t_i$  is finite, then  $\square_k$  will have deficiency indices (2, 2) provided the function  $a(t)$  is sufficiently well-behaved on the compact interval  $[t_i, t_f]$  (Sec. 17 of Ref. 31). In particular, this is the case if  $a(t)$  and  $1/a(t)$  are finite, positive, and differentiable on  $[t_i, t_f]$ . If  $t_i = -\infty$ , then applying Corollary 8 of Ref. 34 shows that a sufficient condition for  $\square_k$  to have deficiency indices (2, 2) is that

$$\frac{1}{k} \int_{-\infty}^{t_f} a(t) dt < \infty. \quad (13)$$

Note that we need to assume that  $k \neq 0$ , although this is not a significant restriction. Indeed, for fields with a finite positive mass,  $m > 0$ , the term  $k^2$  becomes  $k^2 + m^2$ , eliminating the zero mode as a special case. If each  $\square_k$  has equal deficiency indices (2, 2), then each  $\square_k$  has a  $\mathcal{U}(2)$ -parameter family of self-adjoint extensions.<sup>32</sup> The extensions can be constructed by extending the domain of  $\square_k$  by choosing appropriate boundary conditions at the end points of the interval  $[t_i, t_f]$  (Sec. 18 of Ref 31).

*Fixed assumptions about the cosmic expansion.* From now on, we will always assume that  $a(t)$  is positive, differentiable, and finite. Also, unless stated otherwise, we will assume for the remainder of the paper that either:

1.  $t_i > -\infty$  so that  $[t_i, t_f]$  is a compact interval and  $\square_k$  has deficiency indices (2, 2) for any  $k \geq 0$ , or
2.  $\int_{-\infty}^{t_f} a(t) dt < \infty$  so that  $\square_k$  has deficiency indices (2, 2) for any  $k > 0$ .

## B. Example: de Sitter spacetime

In de Sitter spacetime,  $a(t) = e^{Ht}$ , one can verify that in the case where  $t_i = -\infty$ , the operator  $\square_k$ , for  $k > 0$ , has deficiency indices (2, 2):

Switching to conformal time coordinates, let  $\eta(t) = \frac{1}{H}e^{-Ht}$ , i.e.,  $\eta'(t) = -1/a(t)$  and  $\eta \in [\eta_f, \infty)$ . It follows that  $a(\eta) = \frac{1}{H\eta}$  and that the operator  $\square_k$  takes the form

$$\square_k = -a^{-4}(\eta) (\partial_\eta a^2(\eta) \partial_\eta + a^2(\eta) k^2), \quad (14)$$

acting on a suitable dense domain in  $L^2([\eta_f, \infty), a^4(\eta) d\eta)$ . If they exist, any eigenfunctions of the adjoint operator  $\square_k^*$  must satisfy the following differential equation:

$$\phi''(\eta) - \frac{2}{\eta} \phi'(\eta) + \left( k^2 + \frac{\lambda}{H^2 \eta^2} \right) \phi(\eta) = 0. \quad (15)$$

Two linearly independent solutions are given by the Bessel functions

$$f_\lambda(\eta) = \eta^{3/2} J_{\beta(\lambda)}(k\eta), \quad (16)$$

and

$$g_\lambda(\eta) = \eta^{3/2} Y_{\beta(\lambda)}(k\eta), \quad (17)$$

where  $\beta(\lambda) := \sqrt{\frac{9}{4} - \frac{\lambda}{H^2}}$ . If  $\lambda$  is such that  $\beta(\lambda) \neq \mathbb{Z}$ , then the second linearly independent solution can be chosen to be  $h_\lambda(\eta) = \eta^{3/2} J_{-\beta(\lambda)}(k\eta)$  instead.

Since  $\square_k$  commutes with complex conjugation, it must have equal deficiency indices<sup>33</sup>, Theorem X.3], so we just need to check that if  $\lambda := (\frac{9}{4} - i) H^2$ ,  $\beta(\lambda) = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$ , then both solutions  $f_\lambda$  and  $h_\lambda$  are normalizable. This will show that  $2 = n_+ = n_-$ . Now for large  $\eta$ ,

$$J_\beta(k\eta) \sim \sqrt{\frac{2}{\pi k\eta}} \cos\left(k\eta - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{k\eta}\right), \quad (18)$$

so that  $f_\lambda(\eta)$  is asymptotically proportional to  $\eta \cos(k\eta - c)$  for large  $\eta$  (Chap. 11.6 of Ref. 35). It follows that  $f_\lambda$  and similarly  $g_\lambda$  are square integrable with respect to the measure  $a^4(\eta) d\eta = \frac{1}{H^4 \eta^4} d\eta$  so that  $\square_k$  has deficiency indices (2, 2) for  $k > 0$ , as expected.

As discussed above, it follows that the spaces  $B_k(\Omega)$  of fixed comoving spatial momentum modes of bandlimited fields have finite dimension  $N_k$  for any  $k > 0$ . It is remarkable that, therefore, any non-zero fixed comoving spatial mode in de Sitter spacetime with a finite end-time has only a finite number of degrees of freedom in time  $t \in (-\infty, 0]$ . Intuitively, this is plausible because, since the spacetime is expanding at an exponential rate, any fixed comoving spatial mode with magnitude  $k$  corresponds to extremely small proper wavelengths or high proper spatial frequencies for most of the time in  $t \in (-\infty, 0]$ . As was shown for the case of flat spacetime, the larger the proper spatial frequency mode, the smaller is its density of temporal degrees of freedom. Hence, since any fixed comoving spatial mode in de Sitter spacetime with a finite end-time corresponds to exponentially large proper spatial frequencies for most of  $t \in (-\infty, 0]$ , it is to be expected that such a comoving mode could have merely a finite number of temporal degrees of freedom.

### 1. The zero mode

As remarked previously, the zero mode is unphysical and can be safely ignored for any massive scalar field. Nevertheless, we include some facts here about the zero mode for the sake of completeness. In the case where  $k = 0$ , the ordinary differential equation (ODE) becomes

$$\phi''(\eta) - \frac{2}{\eta}\phi'(\eta) + \frac{\lambda}{H^2\eta^2}\phi(\eta) = 0, \quad (19)$$

which has two linearly independent solutions  $f_\lambda(\eta) = \eta^{q_1(\lambda)}$  and  $g_\lambda(\eta) = \eta^{q_2(\lambda)}$  where  $q_1(\lambda) = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{\lambda}{H^2}}$  and  $q_2(\lambda) = \frac{3}{2} + \sqrt{\frac{9}{4} - \frac{\lambda}{H^2}}$ . Again, choosing  $\lambda = (\frac{9}{4} - i)H^2$  we obtain that

$$f_\lambda(\eta) = \eta^{\frac{3+\sqrt{2}+i}{2} + \frac{i}{\sqrt{2}}}, \quad (20)$$

and

$$g_\lambda(\eta) = \eta^{\frac{3-\sqrt{2}+i}{2} + \frac{i}{\sqrt{2}}}. \quad (21)$$

It is now clear that  $f_\lambda$  is square integrable while  $g_\lambda$  is not (with respect to the measure). This shows that for de Sitter spacetime with  $t_i = -\infty$ ,  $\square_0$  has deficiency indices  $(1, 1)$ . Also note that if  $\lambda \geq \frac{9H^2}{4}$ , then the real part of both  $q_i(\lambda)$  is equal to 3, which shows that both solutions are non-normalizable. It then follows from Theorem 3, p. 92, of Ref. 31, that  $[\frac{9H^2}{4}, \infty)$  belongs to the continuous spectrum of  $\square_0$ . Moreover, the fact that  $[\frac{9}{4}, \infty)$  is in the continuous spectrum of the zero mode can be used to show that if  $\Omega^2 > \frac{9}{4}$  that  $N_k \rightarrow \infty$  as  $k \rightarrow 0$ . See, for example, Sec. 8.4.1 of Ref. 30.

Even though  $B_0(\Omega)$  has an infinite number of temporal degrees of freedom, the zero mode still has a finite temporal density of degrees of freedom. For simplicity, consider 1 + 1 dimensional de Sitter spacetime. Then, for an appropriate choice of self-adjoint extension of  $\square_0$ , it can be shown that if  $\phi \in B_0(\Omega)$  where  $\Omega^2 = B^2 + \frac{1}{4}$  then

$$\phi(\eta) = \sum_{n=0}^{\infty} \phi(\eta_n) K(\eta_n, \eta), \quad (22)$$

where  $\eta_n = e^{\frac{n}{B}}$  and

$$K(\eta_n, \eta) = \sqrt{\frac{\eta}{\eta_n}} \left( \frac{\sin(B\pi(\ln(\eta_n\eta)))}{B\pi \ln(\eta_n\eta)} + \frac{\sin(B\pi \ln(\eta/\eta_n))}{B\pi \ln(\eta/\eta_n)} \right), \quad (23)$$

(Sec. 8.4.4, p. 99, Ref. 30). This shows that the zero spatial mode of a covariantly bandlimited field is stably reconstructible from the values it takes on the set  $\Lambda := \{\eta_n\}_{n=0}^{\infty}$ .

## V. THE NUMBER OF TEMPORAL DEGREES OF FREEDOM OF A FIXED COMOVING SPATIAL MODE

We saw that under the assumptions above,  $\square_k$  has deficiency indices  $(2, 2)$  for any  $k > 0$ . Recall that as a consequence, no matter which self-adjoint extension  $\square'_k$  of  $\square_k$  is used to define  $B_k(\Omega)$ ,  $N_k := \dim(B_k(\Omega))$  will be finite.

In flat spacetime, we observed that large proper momentum spatial modes have less temporal bandwidth than smaller proper momentum modes. This suggests that also in expanding FRW spacetimes, large proper momentum spatial modes will have fewer temporal degrees of freedom than smaller proper momentum spatial modes. Studying this problem amounts to studying the spectrum of the second order differential operators  $\square_k$ .

### A. The freezing of comoving modes at early times

At early enough times, the proper wavelength of any comoving mode is arbitrarily small. From the study of the UV cutoff on Minkowski space, we therefore intuitively expect that the density of temporal degrees of freedom of a comoving mode drops for earlier and earlier times, until it reaches zero or a very small number and the mode therefore freezes.

Equivalently to considering earlier and earlier times, we may of course also consider larger and larger  $k$ . In this subsection, we will therefore study the behaviour of  $N_k$  for large  $k$ . We will show that (under our fixed assumptions about the cosmic expansion  $a(t)$ ) there indeed exists a  $K > 0$  such that  $N_k = c \leq 2$  for all  $k \geq K$ . Here,  $K$  is independent of the choice  $\square'_k$  of self-adjoint extension used to define  $B_k(\Omega)$ , while the fixed constant  $c$  depends on the choice of self-adjoint extension. Again, the choice of self-adjoint extension is to be determined by physics.

Choose  $\lambda' \in \mathbb{R}$  such that  $|\lambda'| \leq \Omega^2$ . By Sec. 83 Ref. 32, there is a unique self-adjoint extension  $\square'_k$  of  $\square_k$ , which has  $\lambda'$  as an eigenvalue of multiplicity 2. Let  $f_{\lambda'}$  and  $g_{\lambda'}$  denote two linearly independent solutions to the ordinary differential equation  $\tilde{\square}_k \phi = \lambda' \phi$ . Here, the tilde over the  $\square_k$  is used to denote that we are not considering  $\tilde{\square}_k$  as a differential operator acting on any fixed domain, but as a differential expression acting on any functions for which this expression is defined.

*Lemma 1:* If a real number  $\lambda \neq \lambda'$  is an eigenvalue of  $\square'_k$  then  $0 = \Delta(\lambda; \lambda', k) := \langle f_\lambda, f_{\lambda'} \rangle \langle g_\lambda, g_{\lambda'} \rangle - \langle f_\lambda, g_{\lambda'} \rangle \langle g_\lambda, f_{\lambda'} \rangle$ .

*Proof:* If  $\lambda$  is another eigenvalue of  $\square'_k$  then there are  $c_1, c_2 \in \mathbb{C}$  such that  $c_1 f_\lambda + c_2 g_\lambda$  is an eigenvector of  $\square'_k$ , and hence it must be orthogonal to both  $f_{\lambda'}$  and  $g_{\lambda'}$ . Hence,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \langle c_1 f_\lambda + c_2 g_\lambda, f_{\lambda'} \rangle \\ \langle c_1 f_\lambda + c_2 g_\lambda, g_{\lambda'} \rangle \end{pmatrix} = (c_1, c_2) \begin{pmatrix} \langle f_\lambda, f_{\lambda'} \rangle & \langle f_\lambda, g_{\lambda'} \rangle \\ \langle g_\lambda, f_{\lambda'} \rangle & \langle g_\lambda, g_{\lambda'} \rangle \end{pmatrix}. \quad (24)$$

It follows that the determinant of the above matrix, which is  $\Delta(\lambda; \lambda', k)$ , vanishes.  $\square$

Using the method of Picard iterates that is often employed to prove the existence-uniqueness theorem for ordinary differential equations, it is not difficult to show that if the solutions  $f_\lambda$  and  $g_\lambda$  are chosen by imposing fixed initial conditions at some regular point in  $[t_i, t_f]$ , then  $f_\lambda(\eta)$  and  $g_\lambda(\eta)$  are entire as functions of  $\lambda$  for fixed  $k$  and  $\eta$  (pp. 51 – 56 of Ref. 31) and Sec. 2.3 of Ref. 36. Using this fact and Morera's theorem, it is straightforward to check that if  $f_\lambda$  and  $g_\lambda$  are chosen in this way, that  $\Delta(\lambda) := \Delta(\lambda; \lambda', k)$  for fixed  $\lambda'$  and  $k$  is an entire function of  $\lambda$ .

Since we assume  $t_f$  is finite, we can define a conformal time variable

$$\eta := \int_t^{t_f} a^{-1}(t') dt'. \quad (25)$$

It follows that  $\eta \in [0, \eta_i]$  where  $\eta_i := \eta(t_i)$ . For example, in de Sitter space with  $t_i = -\infty$ , we would get that  $\eta_i = +\infty$ . The line element in these coordinates is  $ds^2 = a^2(\eta) (-d\eta^2 + d\mathbf{x}^2)$ . Re-calculating the symmetric d'Alembertian for a fixed comoving spatial mode of magnitude  $k$  in

the  $\eta$  coordinates yields

$$\square_k = -a^{-4}(\eta) (\partial_\eta a^2 \partial_\eta + a^2 k^2), \quad (26)$$

acting on a suitable dense domain in the Hilbert space  $L^2([0, \eta_i]; a^4(\eta) d\eta)$ . Then,  $\tilde{\square}_k \phi(\eta) = \lambda \phi(\eta)$  takes the form

$$(p\phi')' + (rk^2 - q)\phi = 0 \quad (27)$$

with  $p(\eta) := a^2(\eta)$ ,  $r(\eta) = a^2(\eta)$ , and  $q(\eta) = -a^4(\eta)\lambda$ . Here, we have chosen  $q(\eta)$  so as to treat  $k^2$  as an ‘‘eigenvalue.’’ This ODE is already almost in Liouville normal form Sec. 2 of Ref. 37. To convert it to Liouville normal form, we let  $U(\eta) := \phi(\eta)(r(\eta)p(\eta))^{1/4} = a(\eta)\phi(\eta)$ . This yields the new ODE

$$-U'' + QU = k^2U \quad (28)$$

with the new potential

$$Q(\eta) = \frac{a''(\eta)}{a(\eta)} - \lambda a^2(\eta). \quad (29)$$

For large  $k$ , the potential  $Q$  becomes negligible, and it is intuitively clear that solutions  $U$  will behave asymptotically like sine or cosine. Therefore, the solutions  $\phi$  to the original ODE behave asymptotically like  $a^{-1}(\eta)$  multiplied with some linear combination of sine and cosine functions.

More precisely, by Sec. 3 of Ref. 37, let  $U_\lambda$  and  $V_\lambda$  be the solutions to the ODE (28), which obey the boundary conditions

$$\begin{pmatrix} U_\lambda(0) \\ U'_\lambda(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (30)$$

and

$$\begin{pmatrix} V_\lambda(0) \\ V'_\lambda(0) \end{pmatrix} = \begin{pmatrix} 0 \\ k \end{pmatrix}. \quad (31)$$

The following asymptotic formulae are valid for large  $k$ :

$$U_\lambda(\eta) \sim \cos(k\eta) + \mathcal{O}\left(\frac{1}{k}\right) \quad \text{and} \quad V_\lambda(\eta) \sim \sin(k\eta) + \mathcal{O}\left(\frac{1}{k}\right). \quad (32)$$

The exact form of the asymptotic series for  $U_\lambda$  and  $V_\lambda$  depends on the potential  $Q$  (Eq. (29)), which is determined by the scale factor  $a(\eta)$  and the eigenvalue  $\lambda$  (Sec. 3 of Ref. 37).

Now let  $f_\lambda = a^{-1}U_\lambda$  and  $g_\lambda = a^{-1}V_\lambda$ . Then  $f_\lambda$  and  $g_\lambda$  are two linearly independent solutions to the ODE (27), which obey the boundary conditions:

$$\begin{pmatrix} f_\lambda(0) \\ f'_\lambda(0) \end{pmatrix} = \begin{pmatrix} 1/a(0) \\ 0 \end{pmatrix} \quad (33)$$

and

$$\begin{pmatrix} g_\lambda(0) \\ g'_\lambda(0) \end{pmatrix} = \begin{pmatrix} 0 \\ k/a(0) \end{pmatrix}, \quad (34)$$

and which have the asymptotic behaviour

$$f_\lambda(\eta) \sim a^{-1}(\eta) \cos(k\eta) + \mathcal{O}\left(\frac{1}{k}\right) \quad \text{and} \quad g_\lambda(\eta) \sim a^{-1}(\eta) \sin(k\eta) + \mathcal{O}\left(\frac{1}{k}\right). \quad (35)$$

Now we define the function  $\Delta(\lambda) := \Delta(\lambda; \lambda', k)$  as before using this choice of  $f_\lambda$  and  $g_\lambda$  for each  $\lambda \in \mathbb{R}$ . As discussed previously,  $\Delta(\lambda)$  will be an entire function of  $\lambda$  for fixed  $\lambda'$  and  $k$ . By the asymptotic formulae for  $f_\lambda$ ,  $g_\lambda$ , it follows that as  $k \rightarrow \infty$ ,  $\langle f_\lambda, f_{\lambda'} \rangle$  approaches

$$\int_0^{\eta_i} \cos^2(k\eta) a^2(\eta) d\eta. \quad (36)$$



Note that the fixed assumption that  $\int_{t_i}^{t_f} a(t) dt < \infty$  becomes

$$\int_0^{\eta_i} a^2(\eta) d\eta < \infty \quad (37)$$

upon transforming to conformal time. As  $k \rightarrow \infty$ , the frequency of oscillation of  $\cos(k\eta)$  diverges, and intuitively this integral will converge to the integral of  $a^2(\eta)$  multiplied by  $\frac{1}{2}$ , the average value of  $\cos^2(\pi x)$  over one half period  $0 \leq x \leq 1$ :

*Lemma 2:* Suppose that  $a^2(\eta)$  is integrable on the interval  $[0, \eta_i]$ . Then

$$\lim_{k \rightarrow \infty} \int_0^{\eta_i} \cos^2(k\eta) a^2(\eta) d\eta = \frac{1}{2} \int_0^{\eta_i} a^2(\eta) d\eta = 1/2 \|1/a\|_{\mathcal{H}}^2. \quad (38)$$

In the statement of the above lemma,  $\mathcal{H} := L^2([0, \eta_i]; a^4(\eta) d\eta)$ , and  $\|\cdot\|_{\mathcal{H}}$  denotes the norm in this Hilbert space.

*Proof:* This is a straightforward consequence of the Riemann-Lebesgue lemma. Indeed, as remarked before, our fixed assumptions about the cosmic expansion  $a(t)$  imply that  $1/a \in \mathcal{H}$ , so that  $a^2(\eta) \in L^1[0, \eta_i]$ , and in particular  $a^2(\eta)\chi_{[0, \eta_i]}(\eta) \in L^1(\mathbb{R})$ . Using the trigonometric identity  $2\cos^2(x) = 1 + \cos(2x)$  yields

$$\lim_{k \rightarrow \infty} \int_0^{\eta_i} \cos^2(k\eta) a^2(\eta) d\eta = \frac{1}{2} \int_0^{\eta_i} a^2(\eta) d\eta + \frac{1}{4} \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} (e^{i2k\eta} + e^{-i2k\eta}) a^2(\eta)\chi_{[0, \eta_i]}(\eta) d\eta. \quad (39)$$

By the Riemann-Lebesgue lemma, the second limit vanishes.  $\square$

Similar arguments show that  $\langle g_\lambda, g'_\lambda \rangle$  converges to  $\frac{1}{2} \|1/a\|^2$  and that  $\langle f_\lambda, g_{\lambda'} \rangle$  converges to zero. It follows that for fixed  $\lambda$  and  $\lambda'$ ,  $\Delta(\lambda; \lambda', k)$  converges to  $\frac{1}{4} \|1/a\|^4$  as  $k \rightarrow \infty$ . This fact can be used to prove the main result of this section:

**Theorem 3:** Given any choice of self-adjoint extension  $\square'_k$  of  $\square_k$ , let  $B_k(\Omega) := \text{Ran}(\chi_{[-\Omega^2, \Omega^2]}(\square'_k))$ , which has finite dimension  $N_k$ . There is a  $K > 0$  such that  $N_k = c \leq 2$  for all  $k \geq K$  and for any choice of self-adjoint extension  $\square'_k$  used to define  $B_k(\Omega)$ .

*Proof:* Let  $\Omega$  be the bandlimit, or ultra-violet cutoff. Then for  $\lambda \in [-\Omega^2, \Omega^2]$  and fixed  $\lambda' \in [-\Omega^2, \Omega^2]$ , it follows that  $\langle f_\lambda, f_{\lambda'} \rangle$  converges uniformly to  $1/2 \|1/a\|^2$  for all  $\lambda$  in this interval as  $k \rightarrow \infty$ .

By Lemma 2,  $\Delta(\lambda; \lambda', k)$  converges to the positive constant  $\frac{1}{4} \|1/a\|^4$  as  $k \rightarrow \infty$  for any fixed  $\lambda, \lambda'$ . Now keeping  $\lambda'$  fixed, consider  $\Delta_k(\lambda) := \Delta(\lambda; \lambda', k)$  as a net of continuous (in fact entire) functions of the variable  $\lambda$ . Given any compact interval  $I$ , since  $\Delta_k$  converges pointwise to a continuous (constant) function as  $k \rightarrow \infty$ , this convergence must be uniform on  $I$ .

Hence, there is a  $K > 0$  such that for all  $k \geq K$  we have that  $\Delta(\lambda; \lambda', k) > 0$  for all  $|\lambda| \leq \Omega^2$ . This shows that  $N_k = 2$  for all  $k \geq K$  (provided we define  $B_k(\Omega)$  using the self-adjoint extension  $\square'_k$ , which has  $\lambda'$  as an eigenvalue of multiplicity 2).

Now repeat the same argument again with  $\lambda' = 3\Omega^2 + \epsilon$  for some  $\epsilon > 0$ . Choose  $K > 0$  so large that for  $k \geq K$  we have that  $\square'_k$  has no eigenvalues in  $[-3\Omega^2, 3\Omega^2]$ . Then by Theorem 3 of Ref. 38,  $\Delta(\square_k)_t > 2\Omega^2$  for all  $t \in [-\Omega^2, \Omega^2]$ . Here,  $\Delta(\square_k)_t$  denotes the minimum uncertainty or standard deviation of the symmetric operator  $\square_k$ , taken over all states with expectation value  $t$ . It then follows from Theorem 2 of Ref. 38, that any self-adjoint extension of  $\square_k$  has at most 2 eigenvalues in the interval  $[-\Omega^2, \Omega^2]$ . This demonstrates that no matter which self-adjoint extension is used to define  $B_k(\Omega)$ , there is a  $K > 0$  independent of the choice of self-adjoint extension such that  $k > K$  implies that  $N_k = c \in \{0, 1, 2\}$ .  $\square$

Finally, we note that there always exists a self-adjoint extension of  $\square_k$  so that  $N_k$  becomes 0 for sufficiently large  $k$ .

### B. Example: de Sitter spacetime

In the case of de Sitter spacetime, we can obtain more information about the value  $K$  for which any comoving spatial mode with magnitude  $k \geq K$  has at most 2 degrees of freedom in time. Consider a de Sitter spacetime with scale factor  $a(t) = e^{Ht}$  that began expanding infinitely long ago in the past. We will consider the evolution up to a finite proper time  $t_f$ . In this case, it is convenient to modify the calculations in Sec. V A by defining conformal time in a different way. Here, define conformal time by  $\eta(t) = \frac{1}{H} e^{-Ht}$  so that  $\eta'(t) = -1/a(t)$  as before, but now  $\eta \in [\eta_f, \infty)$ , where  $\eta_f > 0$  is finite.

With these definitions, we can choose as our two linearly independent solutions to  $\square_k^* \phi = \lambda \phi$

$$f_\lambda(\eta) = \sqrt{\frac{k\pi}{2}} \eta^{\frac{3}{2}} J_{\beta(\lambda)}(k\eta) \sim \eta \cos\left(k\eta - \frac{\pi}{2}\beta(\lambda) - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{k\eta}\right), \quad (40)$$

and

$$g_\lambda(\eta) = \sqrt{\frac{k\pi}{2}} \eta^{\frac{3}{2}} Y_{\beta(\lambda)}(k\eta) \sim \eta \sin\left(k\eta - \frac{\pi}{2}\beta(\lambda) - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{k\eta}\right), \quad (41)$$

where  $\beta(\lambda) := \sqrt{\frac{9}{4} - \frac{\lambda}{H^2}}$ . Using properties of Bessel functions, one can show that  $\Delta(\lambda) = \Delta(\lambda; \lambda', k)$  is still entire as a function of  $\lambda$  for fixed  $\lambda'$  and  $k$ , and that it converges uniformly to  $\frac{1}{4} \|1/a\|^4$  as  $k \rightarrow \infty$  for  $\lambda$  in any fixed compact interval  $I$ , where

$$\|1/a\|^2 = \frac{1}{H^2} \int_{\eta_f}^{\infty} \frac{1}{\eta^2} d\eta = \frac{1}{H^2 \eta_f} = a(\eta_f)/H. \quad (42)$$

One can estimate the threshold  $K$  beyond which modes are dynamically frozen from the following considerations. For simplicity, fix  $\lambda' = 0$ , and consider the problem of finding the threshold for the self-adjoint extension  $\square_k'$  for which  $\lambda' = 0$  is an eigenvalue of multiplicity 2. Recall from Lemma 1 that  $\square_k'$  has no other eigenvalues in  $[-\Omega^2, \Omega^2]$  provided that  $\Delta(\lambda; \lambda' = 0, k)$  has no zeros in  $[-\Omega^2, \Omega^2]$ . Thus, we wish to determine the value  $K$  such that whenever  $k \geq K$ ,  $\Delta(\lambda; 0, k) \neq 0$  for all  $\lambda \in [-\Omega^2, \Omega^2]$ .

The inner products that define  $\Delta(\lambda; 0, k)$  are given by

$$\begin{aligned} \langle f_\lambda, f_0 \rangle &= \frac{k}{H^4} \left( \frac{\sin \delta}{\lambda/H^2} + \frac{\pi}{2} F(J, J) \right) & \langle f_\lambda, g_0 \rangle &= \frac{k}{H^4} \left( -\frac{\cos \delta}{\lambda/H^2} + \frac{\pi}{2} F(J, Y) \right), \\ \langle g_\lambda, f_0 \rangle &= \frac{k}{H^4} \left( \frac{\cos \delta}{\lambda/H^2} + \frac{\pi}{2} F(Y, J) \right) & \langle g_\lambda, g_0 \rangle &= \frac{k}{H^4} \left( \frac{\sin \delta}{\lambda/H^2} + \frac{\pi}{2} F(Y, Y) \right), \end{aligned} \quad (43)$$

where  $J$  and  $Y$  are placeholders for the Bessel  $J$  and Bessel  $Y$  functions, respectively, and where

$$F(A, B) := k\eta_f \frac{A_{\beta(\lambda)-1}(k\eta_f) B_{3/2}(k\eta_f) - A_{\beta(\lambda)}(k\eta_f) B_{1/2}(k\eta_f)}{\lambda/H^2} + \frac{A_{\beta(\lambda)}(k\eta_f) B_{\beta(\lambda)}(k\eta_f)}{\beta(\lambda) + 3/2} \quad (44)$$

for  $A, B \in \{J, Y\}$ . Notice that if we divide  $\Delta(\lambda; 0, k)$  by  $k^2/H^8$  and define new variables  $y := k\eta_f$  and  $\ell := \lambda/H^2$ , we have that

$$\frac{H^8}{k^2} \Delta(\lambda; 0, k) = G(y, \ell), \quad (45)$$

a function of only two variables. Since  $H \neq 0$ ,  $\Delta(\lambda)$  and  $\frac{H^8}{k^2} \Delta(\lambda)$  have the same zeros. So, equivalently to our initial problem, we may search for the threshold  $Z$  such that  $\frac{H^8}{k^2} \Delta(\lambda) = G(y, \ell)$  has no zeros on  $\ell \in [-\frac{\Omega^2}{H^2}, \frac{\Omega^2}{H^2}]$  for all  $y \geq Z$ . Hence, let

$$Z\left(\frac{\Omega}{H}\right) := \min_{y>0} \left\{ y \mid G(y, \ell) \neq 0 \quad \forall \ell \in \left[-\frac{\Omega^2}{H^2}, \frac{\Omega^2}{H^2}\right] \right\}. \quad (46)$$

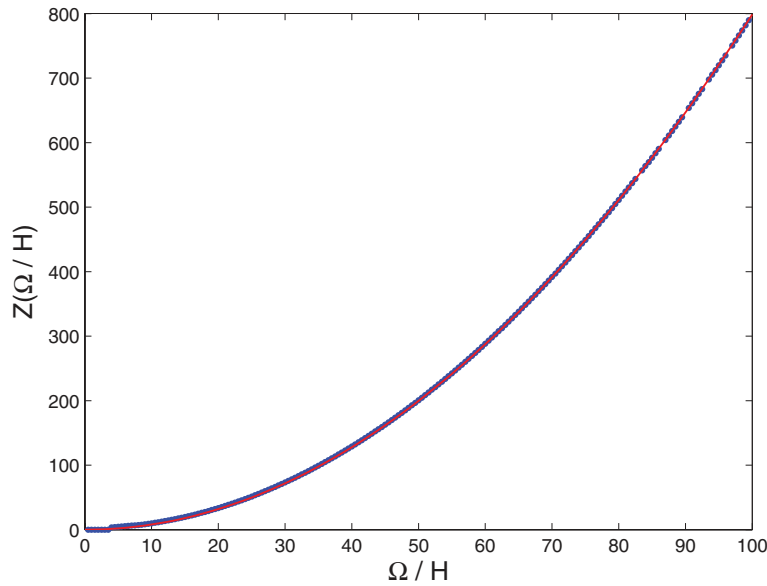


FIG. 2. Numerical simulation of  $Z(\Omega/H)$ . The red curve is a fit to the data by the curve  $Z = 0.07988(\Omega/H)^2$ .

Since  $y = k\eta_f$ , we therefore have that the threshold  $K$  is given by

$$K = \frac{1}{\eta_f} Z\left(\frac{\Omega}{H}\right). \quad (47)$$

Without knowing the functional form of  $Z$ , Eq. (46) indicates that  $Z$  is a monotonically increasing function of  $\Omega/H$ . Since  $\Delta$  approaches a constant function as  $k \rightarrow \infty$  (and hence as  $y \rightarrow \infty$ ), the  $\ell$ -zeros of  $G(y, \ell)$  are pushed farther and farther away from the origin as  $y$  increases. Increasing  $\Omega/H$  widens the interval over which  $\Delta$  must be nonzero, so larger values of  $\Omega/H$  produce larger thresholds  $Z(\Omega/H)$ .

Numerical analysis corroborates the preceding conclusion; a plot of  $Z(\Omega/H)$  with  $\ell$  restricted to lie in  $\ell \in [-\frac{\Omega^2}{H^2}, \min\{\frac{9}{4}, \frac{\Omega^2}{H^2}\}]$  is shown in Figure 2. When  $\ell \geq \frac{9}{4}$ , the orders of the Bessel functions in the definition of  $Z$  become imaginary, which considerably complicates their numerical analysis. We have circumstantially observed, however, that any  $\ell$ -zeros of  $G(y, \ell)$  for which  $\ell > \frac{9}{4}$  are greater in magnitude than the first zero for which  $\ell \leq \frac{9}{4}$ . Although it would be desirable to verify the behaviour of  $Z$  for larger values of  $\Omega/H$ , the computational task becomes highly nontrivial as  $\Omega/H$  increases. The current numerical analysis suggests that  $Z$  is proportional to  $(\Omega/H)^2$ .

One would expect that the modes in de Sitter space that are dynamical are those modes that cross the Planck length before the end of inflation. Conversely, the dynamically frozen modes should be those whose comoving wavelengths are so small that they never cross the Planck length during the period of inflation. The dependence of the threshold for mode freezing  $K$  on the parameters  $\eta_f$ ,  $\Omega$ , and  $H$  in Eq. (47) is indeed consistent with this expectation.

As  $\eta_f$  decreases toward 0,  $K$  increases; this is natural, as a small conformal end time  $\eta_f$  corresponds to a late proper end time  $t_f$ . The later the proper end time, the longer the period of inflation lasts, so the smaller a mode's comoving wavelength (and thus the larger  $k$ ) must be if it is to never cross the Planck length and become dynamical.

The dependence of  $K$  on the dimensionless ratio of the two length scales,  $\Omega/H$ , has an intuitive interpretation. Holding  $H$  fixed, increasing  $\Omega$  essentially corresponds to decreasing the value of the Planck length. This permits modes that were previously too small to cross the Planck length during inflation to make that crossing. Hence, the comoving wavelengths of modes that do not cross the Planck length decreases, or in other words, the threshold increases.

## VI. THE IMPACT OF THE COVARIANT ULTRAVIOLET CUTOFF ON THE SPECTRUM OF QUANTUM FIELD FLUCTUATIONS

So far, we discussed how the presence of the covariant ultra-violet cutoff in expanding FRW spacetimes would affect the kinematics of individual modes of scalar fields on the spacetime. Let us now begin to investigate how the covariant cutoff would affect the full dynamics of scalar quantum fields. In particular, the presence of this natural cutoff in nature could manifest itself with a potentially measurable effect in the cosmic microwave background (CMB). It is, therefore, of great interest to implement the covariant ultraviolet cutoff in the standard model of cosmic inflation.

In order to determine how the covariant ultra-violet cutoff affects the dynamics of a scalar quantum field, it is convenient to use the path integral formalism since it is manifestly covariant. In the path integral picture, the assumption that there exists this covariant ultraviolet cutoff in nature means that the set of scalar fields that one integrates over in the quantum field theoretic path integral is restricted to the space  $B(M, \Omega)$  of covariantly bandlimited scalar fields on the spacetime  $M$ .

### A. The two-point function

The dynamics of a free scalar field  $\hat{\phi}$  are determined by the Feynman propagator, which is given in the path integral picture through

$$G_F(x, x') = \frac{\int \phi(x)\phi(x')e^{iS[\phi]} \mathcal{D}[\phi]}{\int e^{iS[\phi]} \mathcal{D}[\phi]}, \quad (48)$$

and in the interaction picture of the operator formalism through:

$$G_F(x, x') = \langle 0|T\hat{\phi}(x)\hat{\phi}(x')|0\rangle. \quad (49)$$

Here,  $G_F(x, x')$  is ambiguous up to a choice of vacuum, which must be made on the basis of physical input.

Denoting the coordinates  $x = (t, \mathbf{x})$ , the key object of interest for inflationary predictions for the CMB is  $G_F(t = t', \mathbf{p})$ , the equal-time spatial Fourier transform of the two-point function. This is because  $G_F(t = t', \mathbf{p})$  yields the fluctuation spectrum of the modes of a quantum field  $\hat{\phi}$ , and it is this type of fluctuation spectrum that determines the spectrum of the scalar and tensor fluctuations that are imprinted in the temperature and polarization spectra of the CMB.<sup>39</sup> In the equation of motion for  $G_F$ ,

$$(\square_x - m^2)G_F(x, x') = i\delta^4(x - x'), \quad (50)$$

$\square_x$  is the d'Alembertian with respect to unprimed coordinates. The presence of homogeneous solutions requires a choice of boundary condition on  $G_F(x, x')$ , which in turn corresponds to the choice of the vacuum. Technically, the choice of boundary condition fixes a self-adjoint extension  $\square'$  of  $\square$ .

### B. The two-point function in Minkowski space

We can here only begin our study of the impact of the covariant cutoff on the spectrum of quantum field fluctuations, namely by considering the simple case of 1 + 3 dimensional Minkowski space, where the d'Alembertian reads  $\square_x = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ . In this case, since  $\square_x = \square_{x-x'}$ , and with Eq. (50), we can treat  $G_F$  as a function exclusively of the separation  $x - x'$ . Fourier transforming Eq. (50) with respect to  $x - x'$ , yields

$$G_F(p) = \frac{i}{(2\pi)^2} \frac{1}{p_0^2 - |\mathbf{p}|^2 - m^2 + i\epsilon}. \quad (51)$$

Through the Fourier transform, the choice of boundary condition becomes a choice of pole prescription. The introduction of Feynman's  $i\epsilon$  which implies the limit  $\epsilon \rightarrow 0^+$  after integrations is the well-known choice of pole prescription that yields the Feynman propagator on Minkowski space.

One recovers the equal time and momentum dependent two-point function of interest by performing the inverse Fourier transform with respect to  $p_0$ :

$$G_F(t - t', \mathbf{p}) = \frac{i}{(2\pi)^{5/2}} \int_{-\infty}^{\infty} dp_0 \frac{e^{ip_0(t-t')}}{p_0^2 - |\mathbf{p}|^2 - m^2 + i\epsilon}. \quad (52)$$

Setting  $t = t'$ , using standard methods of contour integration and defining  $\omega := \sqrt{|\mathbf{p}|^2 + m^2}$ , one has

$$G_F(t = t', \mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{2\omega}. \quad (53)$$

How does the above calculation change if one assumes the covariant cutoff? Let  $G_F^c(x, x')$  be the composition of  $G_F(x, x')$  and the projector  $P_{B(M, \Omega)}$  that projects onto  $B(M, \Omega)$ , the space of covariantly bandlimited functions, so that

$$G_F^c(x, x') := P_{B(M, \Omega)} G_F(x, x') = \frac{\int_{B(M, \Omega)} \phi(x) \phi(x') e^{iS[\phi]} \mathcal{D}[\phi]}{\int_{B(M, \Omega)} e^{iS[\phi]} \mathcal{D}[\phi]}. \quad (54)$$

If we perform a full inverse Fourier transform on Eq. (51), we obtain the following integral representation of  $G_F(x - x')$ :

$$G_F(x - x') = \frac{i}{(2\pi)^4} \int dp_0 d^3\mathbf{p} \frac{e^{i(p_0(t-t') - \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}'))}}{p_0^2 - |\mathbf{p}|^2 - m^2 + i\epsilon}. \quad (55)$$

Recall that the plane waves  $e^{i(p_0(t-t') - \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}'))}$  are the eigenfunctions of  $\square_{x-x'}$  with corresponding eigenvalues  $p_0^2 - |\mathbf{p}|^2$ . Equation (55) is therefore manifestly a linear combination of eigenfunctions of the d'Alembertian, and so the action of the projector  $P_{B(M, \Omega)}$  gives

$$G_F^c(x - x') = \frac{i}{(2\pi)^4} \int_{|p_0^2 - |\mathbf{p}|^2| \leq \Omega^2} dp_0 d^3\mathbf{p} \frac{e^{i(p_0(t-t') - \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}'))}}{p_0^2 - |\mathbf{p}|^2 - m^2 + i\epsilon}. \quad (56)$$

Performing a spatial Fourier transform on the previous equation to obtain the two-point function of cosmological interest, we find that

$$G_F^c(t - t', \mathbf{p}') = \frac{i}{(2\pi)^{11/2}} \int_{\mathbb{R}^3} d^3(\mathbf{x} - \mathbf{x}') e^{i\mathbf{p}' \cdot (\mathbf{x} - \mathbf{x}')} \int_{|p_0^2 - |\mathbf{p}|^2| \leq \Omega^2} dp_0 d^3\mathbf{p} \frac{e^{i(p_0(t-t') - \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}'))}}{p_0^2 - |\mathbf{p}|^2 - m^2 + i\epsilon}, \quad (57)$$

whence

$$G_F^c(t = t', \mathbf{p}) = \frac{i}{(2\pi)^{5/2}} \int_{\mathcal{I}(\mathbf{p})} dp_0 \frac{1}{p_0^2 - |\mathbf{p}|^2 - m^2 + i\epsilon}. \quad (58)$$

The interval  $\mathcal{I}(\mathbf{p})$  has two qualitatively different forms depending on the value of  $|\mathbf{p}|$ . Referring to Eq. (3) and Figure 1, we have that

- I. if  $|\mathbf{p}| \leq \Omega$ , then  $\mathcal{I}(\mathbf{p}) = \left[ -\sqrt{|\mathbf{p}|^2 + \Omega^2}, \sqrt{|\mathbf{p}|^2 + \Omega^2} \right]$ , or
- II. if  $|\mathbf{p}| > \Omega$ , then  $\mathcal{I}(\mathbf{p}) = \left[ -\sqrt{|\mathbf{p}|^2 + \Omega^2}, -\sqrt{|\mathbf{p}|^2 - \Omega^2} \right] \cup \left[ \sqrt{|\mathbf{p}|^2 - \Omega^2}, \sqrt{|\mathbf{p}|^2 + \Omega^2} \right]$ .

Evaluating the integral in Eq. (58), we ultimately find that

$$G_F^c(t = t', \mathbf{p}) = \begin{cases} \frac{1}{(2\pi)^{3/2}} \frac{1}{2\omega} - \frac{i}{(2\pi)^{5/2}} \frac{1}{\omega} \ln \left| \frac{r_2 + \omega}{r_2 - \omega} \right| & |\mathbf{p}| \leq \Omega \\ \frac{1}{(2\pi)^{3/2}} \frac{1}{2\omega} - \frac{i}{(2\pi)^{5/2}} \frac{1}{\omega} \left( \ln \left| \frac{r_2 + \omega}{r_2 - \omega} \right| - \ln \left| \frac{\omega + r_1}{\omega - r_1} \right| \right) & |\mathbf{p}| > \Omega \end{cases}, \quad (59)$$

where  $r_2 := \sqrt{|\mathbf{p}|^2 + \Omega^2}$  and  $r_1 := \sqrt{|\mathbf{p}|^2 - \Omega^2}$ . Details of the aforementioned calculation may be found in the Appendix. A plot of the absolute values of both  $G_F(t = t', \mathbf{p})$  and  $G_F^c(t = t', \mathbf{p})$  are shown in Figure 3. Notice that the integral in Eq. (58) (which defines  $G_F^c(t = t', \mathbf{p})$ ) is the same

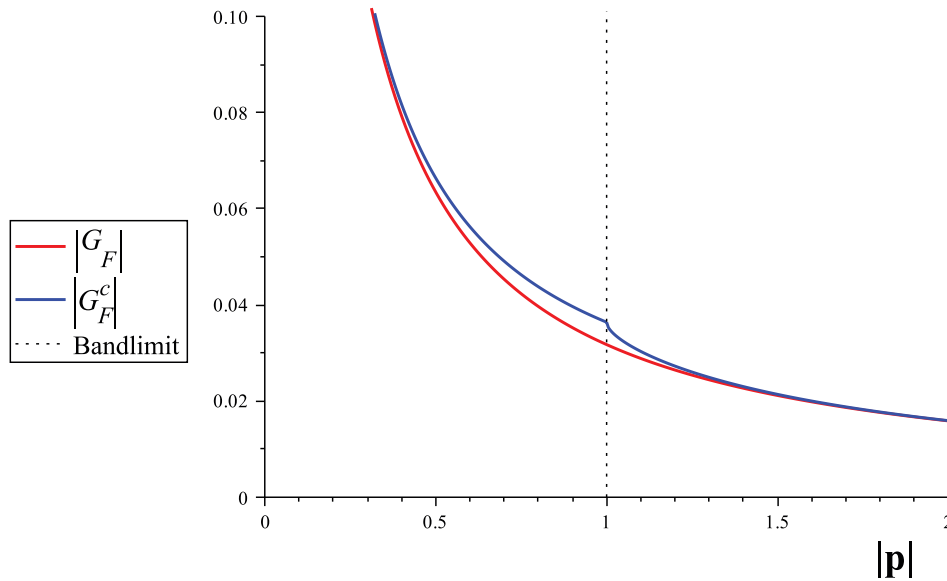


FIG. 3. Plots of the two-point function  $G_F$  (red) and the covariantly bandlimited two-point function  $G_F^c$  (blue). The plotting parameters are  $\Omega = 1$  and  $m = 0.01$ .

integral as in Eq. (52) (which defines  $G_F(t = t', \mathbf{p})$ ) less an integration over  $\mathbb{R} \setminus \mathcal{I}(\mathbf{p})$ , i.e.,

$$G_F^c(t = t', \mathbf{p}) = G_F(t = t', \mathbf{p}) - \frac{i}{(2\pi)^{5/2}} \int_{\mathbb{R} \setminus \mathcal{I}(\mathbf{p})} dp_0 \frac{1}{p_0^2 - \omega^2 + i\epsilon}. \quad (60)$$

The poles of these integrands are at the points  $p_0 = \pm \omega$ , which always lie in the interval  $\mathcal{I}(\mathbf{p})$ , hence the integral over  $\mathbb{R} \setminus \mathcal{I}(\mathbf{p})$  always yields a real number. Consequently, the difference between  $G_F$  and  $G_F^c$  is always a purely imaginary correction term, as is evident in Eqs. (59) and (60). Since  $G_F(t = t', \mathbf{p})$  is a real-valued function, the magnitude of  $G_F^c(t = t', \mathbf{p})$  will always be larger for all  $\mathbf{p}$ , which is indeed observed in Figure 3. The physical consequence of this effect is that the covariantly bandlimited theory predicts larger quantum fluctuations of the field  $\hat{\phi}$  than the standard theory. Intuitively, the discarding of high-frequency contributions to the two-point function eliminated destructive interference caused by these modes.

In fact, from Figure 1 it had to be expected that there will be a cusp in the graph of  $G_F^c$  in Figure 3. Referring to Figure 1, the tangent line to the horizontally opening hyperbola that bounds the region  $|p_0^2 - |\mathbf{p}|^2| \leq \Omega^2$  is vertical at the cutoff value  $|\mathbf{p}| = \Omega$ . This point is the boundary across which the form of  $\mathcal{I}(\mathbf{p})$  qualitatively changes and is the location of the cusp.

## VII. CONCLUSIONS AND OUTLOOK

We investigated the consequences of a possible covariant minimum wavelength in nature in the form of a cutoff on the spectrum of the d'Alembertian. In this scenario, wavelengths smaller than the Planck length do exist but the dynamics of such modes is in effect frozen due to their exceedingly small temporal bandwidth. The information density in such modes is low in a literally information-theoretic sense.

In particular, we showed that comoving modes in expanding spacetimes unfreeze, i.e., that they grow temporal degrees of freedom and develop nontrivial dynamics, only after their proper wavelength starts exceeding the Planck length. Later, as in standard inflationary cosmology, once a comoving mode has outgrown the Hubble horizon, it again loses its dynamics, namely because its two oscillatory solutions turn into a constant and a quickly decaying solution. (As in the usual inflationary scenario, the mode then re-acquires nontrivial dynamics after inflation ends and the mode re-enters the Hubble horizon.)



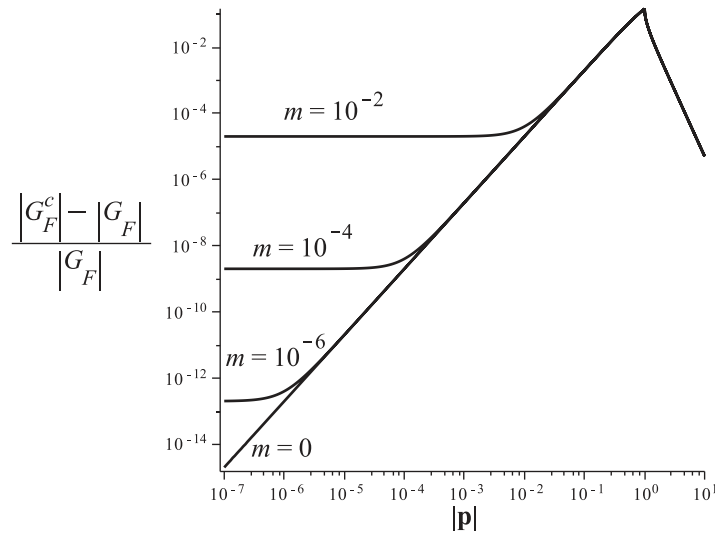


FIG. 4. Plot of the relative difference in magnitudes of  $G_F(t = t', \mathbf{p})$  and  $G_F^c(t = t', \mathbf{p})$  in Minkowski space for different field masses  $m$ . The cutoff is set to  $\Omega = 1$ .

The fact that soon after their unfreezing the modes' dynamics freezes up again, namely upon Hubble horizon crossing, could be crucial. It implies that, in principle, precision measurements of the CMB and its polarization may provide evidence for or against the existence of a covariant bandlimit in nature. This in turn would imply the enticing prospect of experimental access to Planck scale physics. Indeed, according to the standard model of inflationary cosmology, the quantum fluctuations that seeded fluctuations in the CMB's temperature and polarization spectrum (and therefore cosmic structure formation) were frozen upon their Hubble horizon crossing during inflation. The Hubble horizon during inflation, however, is generally thought to have been only five or six orders of magnitude larger than the Planck length. This means that the physics of the Planck scale, such as potentially a covariant UV cutoff, could conceivably have an imprint in the CMB that is not all too much suppressed. This is because the magnitude of the effect can be expected to be proportional to some power,  $\alpha$ , of the dimensionless ratio,  $\sigma$ , of the two basic length scales, the Planck length and the Hubble length during inflation. Here, as we mentioned,  $\sigma$  is known to be no larger than about  $10^{-5}$  in most realistic models of inflation. In order to determine the experimental prospects, it will be important to determine  $\alpha$ .

To obtain a first indication of the size of the effects, let us consider Minkowski space. Here, the effect is very small, indicating a value of  $\alpha$  close to 2. To see this, let us consider Figure 4 which shows the relative difference between  $G_F(t = t', \mathbf{p})$  and  $G_F^c(t = t', \mathbf{p})$ . At  $|\mathbf{p}| = 10^{-5}\Omega$ , experiments would need to be able to measure the two-point function to the tremendous precision of  $2 \times 10^{-9}\%$  or better to be sensitive to the covariant cutoff. For a typical inflationary spacetime such as de Sitter spacetime, or a more realistic cosmological model such as power law inflation, the effect of a covariant cutoff may well be much larger than in the case of Minkowski space, and  $\alpha$  may be closer to 1, given that inflation tends to amplify quantum fluctuations.

It is difficult, therefore, to predict the value of  $\alpha$  and therefore to determine how large an effect the covariant cutoff possesses in an expanding FRW spacetime at this stage. A detailed calculation of  $\alpha$  for the covariant UV cutoff in the inflationary scenario is in progress.

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## APPENDIX: TWO-POINT FUNCTION CALCULATION DETAILS

In this appendix, we discuss the details of how we calculate the covariantly bandlimited two-point function in Minkowski space given by Eq. (58). Recall that the region of integration  $\mathcal{I}(\mathbf{p})$  assumes one of two qualitatively different forms depending on whether  $|\mathbf{p}| \leq \Omega$  or  $|\mathbf{p}| > \Omega$ . We must consider each case separately.

Consider the first case. Here,  $\mathcal{I}(\mathbf{p}) = [-\sqrt{|\mathbf{p}|^2 + \Omega^2}, \sqrt{|\mathbf{p}|^2 + \Omega^2}]$ . Recall that the poles of the integrand in Eq. (58) occur at  $p_0 = \pm \omega = \pm \sqrt{|\mathbf{p}|^2 + m^2}$ , so if  $m < \Omega$ , the poles lie within the bounds of integration, and if  $m > \Omega$ , the poles lie outside the bounds. Only the first case is physical, as we cannot have masses whose magnitudes lie beyond the Planck scale, to which  $\Omega$  is set. Thus, further assume that  $m < \Omega$ . We can then evaluate the integral in Eq. (58) as follows:

$$G_F^c(t = t', \mathbf{p}) = \frac{i}{(2\pi)^{5/2}} \int_{-r_2}^{r_2} dp_0 \frac{1}{p_0^2 - \omega^2} \quad (\text{A1})$$

$$= \frac{i}{(2\pi)^{5/2}} \left[ \int_{-\infty}^{\infty} dp_0 \frac{1}{p_0^2 - \omega^2} - \int_{r_2}^{\infty} dp_0 \frac{1}{p_0^2 - \omega^2} - \int_{-\infty}^{-r_2} dp_0 \frac{1}{p_0^2 - \omega^2} \right] \quad (\text{A2})$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{2\omega} - \frac{2i}{(2\pi)^{5/2}} \lim_{R \rightarrow \infty} \int_{r_2}^R dp_0 \frac{1}{p_0^2 - \omega^2}, \quad (\text{A3})$$

where  $r_2 := \sqrt{|\mathbf{p}|^2 + \Omega^2}$ . We have that

$$\lim_{R \rightarrow \infty} \int_{r_2}^R dp_0 \frac{1}{p_0^2 - \omega^2} = \lim_{R \rightarrow \infty} \frac{1}{2\omega} (\ln |p_0 - \omega| - \ln |p_0 + \omega|) \Big|_{r_2}^R \quad (\text{A4})$$

$$= \frac{1}{2\omega} \ln \left( \frac{r_2 + \omega}{r_2 - \omega} \right), \quad (\text{A5})$$

so for  $|\mathbf{p}| \leq \Omega$  and  $m < \Omega$ ,

$$G_F^c(t = t', \mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{2\omega} - \frac{i}{(2\pi)^{5/2}} \frac{1}{\omega} \ln \left( \frac{r_2 + \omega}{r_2 - \omega} \right). \quad (\text{A6})$$

For completeness, let us also calculate  $G_F^c(t = t', \mathbf{p})$  for  $m > \Omega$ . In this case, the poles of the integrand lie outside the bounds of integration, so we can evaluate the integral using the fundamental theorem of calculus.

$$G_F^c(t = t', \mathbf{p}) = \frac{i}{(2\pi)^{5/2}} \int_{-r_2}^{r_2} dp_0 \frac{1}{p_0^2 - \omega^2} \quad (\text{A7})$$

$$= \frac{i}{(2\pi)^{5/2}} \frac{1}{2\omega} (\ln |p_0 - \omega| - \ln |p_0 + \omega|) \Big|_{-r_2}^{r_2} \quad (\text{A8})$$

$$= -\frac{i}{(2\pi)^{5/2}} \frac{1}{\omega} \ln \left( \frac{\omega + r_2}{\omega - r_2} \right). \quad (\text{A9})$$

Consider now the second case. Here,  $\mathcal{I}(\mathbf{p}) = [-\sqrt{|\mathbf{p}|^2 + \Omega^2}, -\sqrt{|\mathbf{p}|^2 - \Omega^2}] \cup [\sqrt{|\mathbf{p}|^2 - \Omega^2}, \sqrt{|\mathbf{p}|^2 + \Omega^2}]$ . Again, the poles lie within the bounds of integration if  $m < \Omega$

and outside the bounds if  $m > \Omega$ . Assuming  $m < \Omega$  and proceeding similarly to before, we have

$$G_F^c(t = t', \mathbf{p}) = \frac{i}{(2\pi)^{5/2}} \left[ \int_{-r_2}^{-r_1} dp_0 \frac{1}{p_0^2 - \omega^2} + \int_{r_1}^{r_2} dp_0 \frac{1}{p_0^2 - \omega^2} \right] \quad (\text{A10})$$

$$= \frac{i}{(2\pi)^{5/2}} \left[ \int_{-\infty}^{\infty} dp_0 \frac{1}{p_0^2 - \omega^2} - \int_{-\infty}^{-r_2} dp_0 \frac{1}{p_0^2 - \omega^2} - \int_{-r_1}^{r_1} dp_0 \frac{1}{p_0^2 - \omega^2} - \int_{r_2}^{\infty} dp_0 \frac{1}{p_0^2 - \omega^2} \right] \quad (\text{A11})$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{2\omega} - \frac{i}{(2\pi)^{5/2}} \frac{1}{\omega} \left[ \ln \left( \frac{r_2 + \omega}{r_2 - \omega} \right) - \ln \left( \frac{\omega + r_1}{\omega - r_1} \right) \right], \quad (\text{A12})$$

where  $r_1 := \sqrt{|\mathbf{p}|^2 - \Omega^2}$ . In the case where  $m > \Omega$ ,

$$G_F^c(t = t', \mathbf{p}) = \frac{i}{(2\pi)^{5/2}} \left[ \int_{-r_2}^{-r_1} dp_0 \frac{1}{p_0^2 - \omega^2} + \int_{r_1}^{r_2} dp_0 \frac{1}{p_0^2 - \omega^2} \right] \quad (\text{A13})$$

$$= \frac{2i}{(2\pi)^{5/2}} \int_{r_1}^{r_2} dp_0 \frac{1}{p_0^2 - \omega^2} \quad (\text{A14})$$

$$= -\frac{i}{(2\pi)^{5/2}} \frac{1}{\omega} \left[ \ln \left( \frac{\omega + r_2}{\omega - r_2} \right) - \ln \left( \frac{\omega + r_1}{\omega - r_1} \right) \right]. \quad (\text{A15})$$

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