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A FUNCTIONAL CENTRAL LIMIT THEOREM FOR RANDOM MAPPINGS

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We consider the set of mappings of the integers $\{1, 2, \dots, n\}$ into $\{1, 2, \dots, n\}$ and put a uniform probability measure on this set. Any such mapping can be represented as a directed graph on n labelled vertices. We study the component structure of the associated graphs as $n \to \infty$. To each mapping we associate a step function on $[0, 1]$. Each jump in the function equals the number of connected components of a certain size in the graph which represents the map. We normalize these functions and show that the induced measures on $D[0, 1]$ converge to Wiener measure. This result complements another result by Aldous on random mappings.

1. Random mappings have been studied in some detail in recent years. Typically, for each $n > 0$, a uniform probability measure P_n is defined on T_n, the set of all maps from $\{1, 2, \dots, n\}$ into $\{1, 2, \dots, n\}$, by $P_n(\phi) = 1/n^n$ for each $\phi \in T_n$. In this context it is natural to investigate the limiting distributions, as $n \to \infty$, of various characteristics of random mappings. Most properties of a map $\phi \in T_n$ can also be described in terms of an associated directed graph G_ϕ on n vertices, labelled $1, 2, \dots, n$, which represents ϕ as follows. An edge from i to j exists in the associated graph G_ϕ if and only if $\phi(i) = j$. It is obvious that any limit law for some property of random mappings can be interpreted as a limit law for some characteristic of random directed graphs and vice versa. Limit distributions for many properties of the random graphs associated with random mappings have been computed (see [9], [10], [12]). In this paper we study only the component structure of the associated random graphs.

It is known (see [4]) that as $n \to \infty$, the expected number of connected components in the random graph G_ϕ is asymptotic to $(1/2)\ln n$. Nevertheless, the components of a typical graph G_ϕ, for $\phi \in T_n$, do not evenly partition the set $\{1, 2, \dots, n\}$. Kolchin [9] determined, for fixed m, the limiting distribution of the size of the mth largest connected component in a random mapping. It follows from this result that for any $0 < c < 1$, $\lim_{n \to \infty} P_n(\phi \in T_n$: the size of the largest component of G_ϕ is greater than $cn) > 0$, i.e., for large n, a typical element of T_n has a component whose size is comparable to n. On the other hand, Kolchin also showed that the limiting distribution of the number of components of fixed size k is Poisson with parameter $1/2k$. So, for example, $\lim_{n \to \infty} P_n(\phi \in T_n$: G_ϕ has a component of size one$) = 1 - 1/\sqrt{e}$.

Aldous [2] improved Kolchin's results by proving a global limit theorem for the component structure of a random mapping. We describe this result. For all $n > 0$, $i > 0$ and $\phi \in T_n$, define $M_i^n(\phi)$ to be the size of the ith largest

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component in G_ϕ if G_ϕ has at least i components, otherwise define $M_i^n(\phi) = 0$. Define the map L_n: $T_n \to \nabla$ by $L_n(\phi) = 1/n(M_1^n(\phi), M_2^n(\phi), \dots)$, where $\nabla = \{(x_1, x_2, \dots)$: $x_1, x_2, \dots \geq 0$ and $\sum_{i=1}^{\infty} x_i = 1\}$. Aldous showed that the induced measures $P_n \circ L_n^{-1}$ on ∇ converge weakly, as $n \to \infty$, to the Poisson–Dirichlet distribution on ∇ with parameter 1/2. This result establishes the limiting joint distribution of the sizes of the m largest components.

Nevertheless, some information is not contained in Aldous' theorem. The result does not yield the limiting distributions for the sizes of components of size $o(n)$. To recover this information we define, for each $n > 0$, a function X_n: $[0,1] \times T_n \to R$ by letting $X_n(t, \phi)$ equal the number of connected components in G_ϕ of size less than or equal to n^t, where $0 \leq t \leq 1$ and $\phi \in T_n$. The graph of $X_n(\cdot, \phi)$ is an increasing step function with jumps occurring at $\ln k/\ln n$ for $k = 1, 2, \dots, n$. The size of a jump at $\ln k/\ln n$ is equal to the number of connected components of size k in G_ϕ. Thus, for any $\phi \in T_n$, the component structure of G_ϕ is retrievable from the graph of $X_n(\cdot, \phi)$. We define the normalized functions Y_n: $[0,1] \times T_n \to R$ by

$$Y_n(t, \phi) = \frac{X_n(t, \phi) - (t/2)\ln n}{\sqrt{(1/2)\ln n}} \quad \text{for } 0 \leq t \leq 1 \text{ and } \phi \in T_n.$$

For fixed $\phi \in T_n$, the function $Y_n(\cdot, \phi)$ is an element of $D[0,1]$, the space of right-continuous functions with left limits on $[0,1]$. Thus Y_n induces a measure $P_n \circ Y_n^{-1}$ on $(D[0,1], \mathscr{D})$, where $\mathscr{D}$ denotes the σ-algebra generated by the Borel sets of $D[0,1]$ with respect to the Skorohod topology on $D[0,1]$ (see Billingsley [3]). We now state our result.

THEOREM 1. *The sequence of induced measures $P_n \circ Y_n^{-1}$ converges weakly to Wiener measure W on $(D[0,1], \mathscr{D})$ as $n \to \infty$.*

REMARKS. This is a global result which complements Aldous' theorem. One consequence of this result is a central limit theorem, when t is fixed, for the sequence of random variables $X_n(t, \cdot)$, since $Y_n(t, \cdot)$ converges weakly to the normal distribution $N(0, t)$ with mean 0 and variance t. When $t = 1$, we obtain the central limit theorem for the number of components in G_ϕ; this was first proved by Stepanov [12]. We also mention here that a similar Brownian motion result has been proved by DeLaurentis and Pittel [5] for random permutations. We obtain Theorem 1 by a different approach which can be generalized. In particular, by an argument similar to the proof given below, we are able to establish (see [7]) a functional central limit theorem for the Ewens sampling formula (see Kingman [8]) which arises in population genetics. The result for random permutations is then a special case of the result for the Ewens sampling formula.

To prove Theorem 1 we first define another sequence of functions $\overline{Y}_n$: $[0,1] \times T_n \to R$ such that

$$\lim_{n \to \infty} \sup_{\phi \in T_n} \rho\big(Y_n(\cdot, \phi), \overline{Y}_n(\cdot, \phi)\big) = 0, \tag{1}$$

where ρ is the Skorohod metric on $D[0,1]$. We establish that the *new* sequence of measures $P_n \circ \bar{Y}_n^{-1}$ converges weakly to Wiener measure. It then follows from (1) that the original sequence of measures, $P_n \circ Y_n^{-1}$, also converge to Wiener measure (see [3]).

To show that the measures $P_n \circ \bar{Y}_n^{-1}$ converge to W we must check that the finite-dimensional distributions of the measures $P_n \circ \bar{Y}_n^{-1}$ converge weakly to those of Wiener measure, i.e., for any $0 = t_0 < t_1 < t_2 < \cdots < t_k \le 1$ and $a_1, a_2, \dots, a_k \in R$,

$$\lim_{n \to \infty} P_n\big(\bar{Y}_n(t_1) \le a_1, \bar{Y}_n(t_2) - \bar{Y}_n(t_1) \le a_2, \dots, \bar{Y}_n(t_k) - \bar{Y}_n(t_{k-1}) \le a_k\big)$$

$$(2) \qquad = \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \int_{-\infty}^{a_i} e^{-u^2/2(t_i - t_{i-1})}\, du,$$

where $\bar{Y}_n(t)$ denotes the random variable $\bar{Y}_n(t, \cdot)$ on T_n, and we must show that the sequence of measures $P_n \circ \bar{Y}_n^{-1}$ is tight on $(D[0,1], \mathscr{D})$. Given that the finite-dimensional distributions converge, it suffices to prove (see [3])

$$(3) \qquad E_n\big(\bar{Y}_n(t) - \bar{Y}_n(t_1)\big)^2\big(\bar{Y}_n(t_2) - \bar{Y}_n(t)\big)^2 \le \big(F(t_2) - F(t_1)\big)^{\alpha}$$

for any $n \in Z^+$ and $0 \le t_1 < t < t_2 \le 1$, where F is a strictly increasing continuous function on $[0,1]$ with $F(0) = 0$ and $\alpha > 1$.

To establish (2) and (3) above, we must compute expected values for certain random variables on T_n. In Section 2 we develop a tool for this purpose by modifying a technique first used in Shepp and Lloyd [11]. We then prove Theorem 1. We will often make use of Stirling's formula, so we remind the reader of the inequality

$$\sqrt{2\pi}\, n^{n+1/2} e^{-n} < n! < \sqrt{2\pi}\, n^{n+1/2} e^{-n}\left(1 + \frac{1}{6n}\right)$$

for all $n \ge 1$. We adopt some notational conventions. For $\alpha, \beta \in R^+$, the symbol $\Sigma_{k>\alpha}^{\beta}$ means the sum over all $k \in Z^+$ such that $\alpha < k \le \beta$, and we interpret this sum to be 0 whenever there is no $k \in Z^+$ such that $\alpha < k \le \beta$. Also, if $f(z) = \Sigma_{k=0}^{\infty} f_k z^k$ is a power series, then $[z^n] f(z) = f_n$, the coefficient of the z^n in $f(z)$.

2. In this section we construct a tool for the evaluation of the expectations which must be computed in order to prove Theorem 1. For $k \ge 1$, define $\alpha_k(\phi)$ to be the number of connected components of size k in G_ϕ, where $\phi \in \bigcup_{n=1}^{\infty} T_n$. Let A_n be the number connected mappings in T_n. A straightforward counting argument yields an expression for the joint distribution of $\alpha_1, \alpha_2, \dots, \alpha_n$ restricted to T_n in terms of $A_1, A_2, \dots, A_n$. For $m_1, m_2, \dots, m_n$, nonnegative integers such that $\Sigma_{k=1}^{n} k m_k = n$,

$$(4) \qquad P_n(\alpha_1 = m_1, \dots, \alpha_n = m_n) = \frac{n!}{n^n} \prod_{k=1}^{n} \left(\frac{A_k}{k!}\right)^{m_k} \frac{1}{m_k!}.$$

Note that $P_n(\alpha_1 = m_1, \dots, \alpha_n = m_n) = 0$ if $\sum_{k=1}^{n} k m_k \neq n$. The value of A_n for each $n > 0$ is given in Lemma 2.1, which we state without proof (see [4]).

LEMMA 2.1. *For each* $n \in Z^+$, $A_n = (n-1)! \sum_{k=0}^{n-1} n^k / k!$.

We now define an auxiliary system of sequence spaces. For $0 < z < 1$, let $\Omega_z = \{(m_1, m_2, \dots): m_i \text{ is a nonnegative integer for each } i \geq 1\}$ and let P_z be the product measure on Ω_z such that the distribution of the value of the kth coordinate in the product space Ω_z is Poisson with mean $(A_k/k!)(z/e)^k$. Define the random variable ν on Ω_z by $\nu(m_1, m_2, \dots) = \sum_{k=1}^{\infty} k m_k$. It follows from the next lemma that ν is finite P_z-almost surely for $0 < z < 1$.

LEMMA 2.2. *For* $0 < z < 1$ *and any nonnegative integer* n,

$$P_z(\nu = n) = \left(\frac{z}{e}\right)^n \frac{n^n}{n!} \frac{1}{S(z/e)},$$

where $S(z/e) = \sum_{m=0}^{\infty} (m^m/m!)(z/e)^m$.

PROOF. We begin by computing the probability generating function of ν with respect to the measure P_z. Since $m_1, m_2, \dots$ are independent Poisson variables with respect to P_z, for $|u| < 1$,

$$\begin{aligned} E_z(u^\nu) &= \prod_{k=1}^{\infty} E_z(u^{k m_k}) \\ &= \prod_{k=1}^{\infty} \exp\left[(u^k - 1)\frac{A_k}{k!}\left(\frac{z}{e}\right)^k\right] \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{A_k}{k!}\left(\left(\frac{uz}{e}\right)^k - \left(\frac{z}{e}\right)^k\right)\right) \\ &= S\left(\frac{uz}{e}\right) \Big/ S\left(\frac{z}{e}\right). \end{aligned}$$

The last equality is obtained from the identity $\sum_{k=1}^{\infty} (A_k/k!)(z/e)^k = \ln S(z/e)$. This identity is established by a combinatorial generating function argument. The series $S(x) = \sum_{k=0}^{\infty} (k^k/k!) x^k$ is the exponential generating function for the number of mappings of $\{1, 2, \dots, k\}$ into $\{1, 2, \dots, k\}$. From this it is easy to verify (see [1]) that $\exp(\sum_{r=1}^{\infty} (A_r/r!) x^r) = S(x)$. Therefore,

$$P_z(\nu = n) = [u^n] E_z(u^\nu) = [u^n] S\left(\frac{uz}{e}\right) \Big/ S\left(\frac{z}{e}\right) = \left(\frac{z}{e}\right)^n \frac{n^n}{n!} \frac{1}{S(z/e)}$$

for $n \geq 0$. □

REMARK. Here is the idea behind the construction of this auxiliary system of sequence spaces. For n fixed, the random variables $\alpha_1, \dots, \alpha_n$ restricted to T_n must be dependent since $\sum_{k=1}^{n} k\alpha_k(\phi) = n$ for all $\phi \in T_n$. Using the construction given above we can avoid computational difficulties which arise from the dependence of $\alpha_1, \dots, \alpha_n$ restricted to T_n. Roughly speaking, in the space Ω_z with the product measure P_z, an infinite sequence $(m_1, m_2, \dots)$ corresponds to choosing m_1 components of size 1, m_2 components of size 2, etc., independently accordingly to Poisson distributions with parameters $(A_1/1!)(z/e), (A_2/2!)(z/e)^2, \dots,$ respectively. The random variable $\nu(m_1, m_2, \dots) = \sum_{k=1}^{\infty} km_k$ determines the random "size" of the "graphs" with component-type vector $(m_1, m_2, \dots)$. By letting the size of the graphs vary, we have gained independence of the variables that count the number of components of each size. The probability measure P_z on Ω_z is related to the measure P_n on T_n as follows. For $(m_1, m_2, \dots) \in \Omega_z$ such that $\sum_{k=1}^{\infty} km_k = n$,

$$
\begin{aligned}
&P_z((m_1, m_2, \dots) | \nu = n) \\
&= \frac{\prod_{k=1}^{\infty}(A_k/k!)^{m_k}(z/e)^{km_k}\exp((-A_k/k!)(z/e)^k)1/m_k!}{(z/e)^n(n^n/n!)\exp(-\ln S(z/e))} \\
&= \frac{n!}{n^n}\prod_{k=1}^{n}\left(\frac{A_k}{k!}\right)^{m_k}\frac{1}{m_k!} \\
&= P_n(\alpha_1 = m_1, \dots, \alpha_n = m_n).
\end{aligned} \tag{5}
$$

As we have noted, to prove Theorem 1 we must compute the expectations of various functions on T_n which are determined by the component structure of elements of T_n. To do this we use a transform which relates the expectations of functions defined on Ω_z to the expectations of functions defined on T_n. Let Ψ be any function defined on Ω_z, then Ψ determiones a function Ψ_n on T_n as follows. For $\phi \in T_n$, let $\Psi_n(\phi) = \Psi(\alpha_1(\phi), \alpha_2(\phi), \dots, \alpha_n(\phi), 0, 0, \dots)$. Let $E_z(\Psi)$ denote the expectation of Ψ with respect to P_z and let $E_n(\Psi_n)$ denote the expectation of Ψ_n with respect to P_n. Using (5), we compute

$$
\begin{aligned}
E_z(\Psi) &= \sum_{n=0}^{\infty} P_z(\nu = n)E_z(\Psi | \nu = n) \\
&= \sum_{n=1}^{\infty} P_z(\nu = n)E_n(\Psi_n) + P_z(\nu = 0)\Psi(\bar{0}) \\
&= \sum_{n=1}^{\infty} \frac{n^n}{n!}\left(\frac{z}{e}\right)^n \frac{E_n(\Psi_n)}{S(z/e)} + \frac{\Psi(\bar{0})}{S(z/e)},
\end{aligned}
$$

where $\bar{0} = (0, 0, \dots)$. Thus

$$S\left(\frac{z}{e}\right)E_z(\Psi) = \sum_{n=1}^{\infty} \frac{n^n}{n!}\left(\frac{z}{e}\right)^n E_n(\Psi_n) + \Psi(\bar{0}). \tag{6}$$

We note from (6), that $E_n(\Psi_n)$ is the coefficient of z^n in $(e^n n!/n^n)S(z/e)E_z(\Psi)$. We now turn to the proof of Theorem 1.

Proof of Theorem 1. The first step is to define functions $\bar{Y}_n$: $[0,1] \times T_n \to R$, each of which depends on a parameter z_n, by

$$\bar{Y}_n(t, \phi) = \frac{X_n(t, \phi) - \sum_{k=1}^{n^t} A_k/k!(z_n/e)^k}{\sqrt{(1/2)\ln n}}$$

for $0 \le t \le 1$ and $\phi \in T_n$. The parameter is chosen so that $z_n \in (0, 1)$ and so that

$$\sum_{k=1}^{n} \left| \frac{A_k e^{-k}}{k!}(z_n^k - 1) \right| < 1.$$

Thus, for all $0 \le t \le 1$,

$$\left| \sum_{k=1}^{n^t} \frac{A_k}{k!}\left(\frac{z_n}{e}\right)^k - \frac{t}{2}\ln n \right|$$

$$\le \sum_{k=1}^{n} \left| \frac{A_k e^{-k}}{k!}(z_n^k - 1) \right| + \sum_{k=1}^{n} \left| \frac{A_k e^{-k}}{k!} - \frac{1}{2k} \right| + \left| \sum_{k=1}^{n^t} \frac{1}{2k} - \left(\frac{t}{2}\right)\ln n \right| \tag{7}$$

$$\le 2 + \sum_{k=1}^{n} \left| \frac{A_k e^{-k}}{k!} - \frac{1}{2k} \right|.$$

To bound the right side of (7) we first note that

$$\sum_{j=0}^{k-1} \frac{k^j e^{-k}}{j!} = \text{prob}(U_1 + \cdots + U_k \le k - 1),$$

where $U_1, \dots, U_k$ are i.i.d. Poisson random variables with parameter 1. It then follows from the Berry–Esseen theorem [6] and Lemma 2.1 that for $k \ge 1$,

$$\left| \frac{A_k e^{-k}}{k!} - \frac{1}{2k} \right| = \frac{1}{k} \left| \sum_{j=0}^{k-1} \frac{k^j e^{-k}}{j!} - \frac{1}{2} \right| \le \frac{8}{k^{3/2}}.$$

So the right side of (7) is less than 26 and

$$\begin{aligned}\lim_{n\to\infty} \sup_{\phi\in T_n} \rho\big(Y_n(\cdot,\phi), \overline{Y}_n(\cdot,\phi)\big) &\le \lim_{n\to\infty} \sup_{\phi\in T_n} \sup_{t\in[0,1]} |Y_n(t,\phi) - \overline{Y}_n(t,\phi)| \\ &= \lim_{n\to\infty} \sup_{t\in[0,1]} \frac{|\sum_{k=1}^{n^t}(A_k/k!)(z_n/e)^k - (t/2)\ln n|}{\sqrt{(1/2)\ln n}} \\ &= 0.\end{aligned}$$

Thus it suffices to show that the measures $P_n \circ \overline{Y}_n^{-1}$ converge to Wiener measure. We do this by showing the convergence of the finite-dimensional distributions of measures $P_n \circ \overline{Y}_n^{-1}$ to those of W and by establishing the bound

$$E_n\big(\overline{Y}_n(t) - \overline{Y}_n(t_1)\big)^2\big(\overline{Y}_n(t_2) - \overline{Y}_n(t)\big)^2 \le 640(t_2 - t_1)^{3/2} \le (75t_2 - 75t_1)^{3/2}$$

for any $n \in Z^+$ and $0 \le t_1 < t < t_2 \le 1$.

Convergence of the finite-dimensional distributions. To show that the finite-dimensional distributions of $P_n \circ \overline{Y}_n^{-1}$ converge to those of W, we show that for any $0 \le t < t' \le 1$ and any $a, b \in R$,

$$\begin{aligned}&\lim_{n\to\infty} P_n\big(\overline{Y}_n(t) \le a, \overline{Y}_n(t') - \overline{Y}_n(t) \le b\big) \\ &\quad = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{a} e^{-u^2/2t}\,du \,\frac{1}{\sqrt{2\pi(t'-t)}} \int_{-\infty}^{b} e^{-u^2/2(t'-t)}\,du.\end{aligned}$$

The argument extends in an obvious way to the general case.

Case 1: $0 \le t < t' < 1$. For each $n > 0$, let

$$a_n = a\sqrt{(1/2)\ln n} + \sum_{k=1}^{n^t} (A_k/k!)(z_n/e)^k$$

and let $b_n = b\sqrt{(1/2)\ln n} + \sum_{k>n^t}^{n^{t'}}(A_k/k!)(z_n/e)^k$, where z_n is the parameter which appears in the definition of $\overline{Y}_n$. Define the indicator function I_{a_n} on $\bigcup_{m=1}^{\infty} T_m$ by $I_{a_n}(\phi) = 1$ for all $\phi \in \bigcup_{m=1}^{\infty} T_m$ such that $\sum_{k=1}^{n^t}\alpha_k(\phi) \le a_n$ and $I_{a_n}(\phi) = 0$ otherwise. Likewise, define I_{b_n} on $\bigcup_{m=1}^{\infty} T_m$ by $I_{b_n}(\phi) = 1$ if $\sum_{k>n^t}^{n^{t'}}\alpha_k(\phi) \le b_n$ and $I_{b_n}(\phi) = 0$ otherwise. Extend I_{a_n} to a function on Ω_{z_n} by letting $I_{a_n}(m_1, m_2, \dots) = 1$ if $\sum_{k=1}^{n^t} m_k \le a_n$ and $I_{a_n}(m_1, m_2, \dots) = 0$ otherwise and similarly extend the definition of I_{b_n} on Ω_{z_n}.

Observe that $P_n(\overline{Y}_n(t) \le a, \overline{Y}_n(t') - \overline{Y}_n(t) \le b) = E_n(I_{a_n} I_{b_n})$. It follows from (6) that

$$E_n\big(I_{a_n} I_{b_n}\big) = \big[(z_n)^n\big] \frac{n!e^n}{n^n} E_{z_n}\big(I_{a_n} I_{b_n}\big) S\Big(\frac{z_n}{e}\Big).$$

Furthermore, the functions I_{a_n} and I_{b_n} restricted to Ω_{z_n} are independent random variables with respect to the product measure P_{z_n} since they depend on disjoint coordinates in the product space Ω_{z_n}. So

$$E_{z_n}(I_{a_n} I_{b_n}) = E_{z_n}(I_{a_n}) E_{z_n}(I_{b_n}) = P_{z_n}\left(\sum_{k=1}^{n^t} m_k \le a_n\right) P_{z_n}\left(\sum_{k>n^t}^{n^{t'}} m_k \le b_n\right).$$

Let $\mu(n, z) = \sum_{k=1}^{n^t}(A_k/k!)(z/e)^k$ and $\mu'(n, z) = \sum_{k>n^t}^{n^{t'}}(A_k/k!)(z/e)^k$, then it follows from the construction of (Ω_{z_n}, P_{z_n}) that the sums $\sum_{k=1}^{n^t} m_k$ and $\sum_{k>n^t}^{n^{t'}} m_k$ are Poisson random variables with parameters $\mu(n, z_n)$ and $\mu'(n, z_n)$, respectively. Thus

$$P_n(\bar{Y}_n(t) \le a, \bar{Y}_n(t') - \bar{Y}_n(t) \le b)$$

$$= [(z_n)^n] \frac{n! e^n}{n^n} E_{z_n}(I_{a_n}) E_{z_n}(I_{b_n}) S\left(\frac{z_n}{e}\right)$$

$$= [(z_n)^n] \frac{n! e^n}{n^n} \exp(-\mu(n, z_n) - \mu'(n, z_n))$$

$$\times \sum_{k=0}^{a_n} \frac{(\mu(n, z_n))^k}{k!} \sum_{j=0}^{b_n} \frac{(\mu'(n, z_n))^j}{j!} \sum_{m=0}^{\infty} \frac{m^m}{m!}\left(\frac{z_n}{e}\right)^m$$

$$= [(z_n)^n] \frac{n! e^n}{n^n} \sum_{l=0}^{n^T} \frac{(-\mu(n, z_n) - \mu'(n, z_n))^l}{l!}$$

(8)
$$\times \sum_{k=0}^{a_n} \frac{(\mu(n, z_n))^k}{k!} \sum_{j=0}^{b_n} \frac{(\mu'(n, z_n))^j}{j!} \sum_{m=0}^{n} \frac{m^m}{m!}\left(\frac{z_n}{e}\right)^m$$

$$+ [(z_n)^n] \frac{n! e^n}{n^n} \sum_{l>n^T}^{\infty} \frac{(-\mu(n, z_n) - \mu'(n, z_n))^l}{l!}$$

(9)
$$\times \sum_{k=0}^{a_n} \frac{(\mu(n, z_n))^k}{k!} \sum_{j=0}^{b_n} \frac{(\mu'(n, z_n))^j}{j!} \sum_{m=0}^{n} \frac{m^m}{m!}\left(\frac{z_n}{e}\right)^m,$$

where $0 < T < 1 - t'$. To compute $\lim_{n\to\infty} P_n(\bar{Y}_n(t) \le a, \bar{Y}_n(t') - \bar{Y}_n(t) \le b)$, we will show that the limit of expression (9) is 0 and that expression (8) converges to the correct value as $n \to \infty$.

The absolute value of expression (9) is less than or equal to

$$R_n = \frac{n! e^n}{n^n} \sum_{l>n^T} \frac{(\mu(n,1) + \mu'(n,1))^l}{l!} \sum_{k=0}^{a_n} \frac{(\mu(n,1))^k}{k!}$$

$$\times \sum_{j=0}^{b_n} \frac{(\mu'(n,1))^j}{j!} \sum_{m=0}^{n} \frac{m^m e^{-m}}{m!}.$$

To estimate R_n, we first note that Stirling's formula yields

$$\sum_{m=0}^{n} \frac{m^m e^{-m}}{m!} \le 1 + \sum_{m=1}^{n} \frac{1}{\sqrt{2\pi n}} \le \sqrt{n}.$$

We also recall that $A_k e^{-k}/k! = (1/k)\sum_{j=0}^{k-1} k^j e^{-k}/j! < 1/k$, so for all n sufficiently large,

$$\mu(n,1) + \mu'(n,1) = \sum_{k=1}^{n^{t'}} \frac{A_k e^{-k}}{k!} < \sum_{k=1}^{n^{t'}} \frac{1}{k} \le 2t' \ln n.$$

Using these two bounds, we have

$$\begin{aligned} R_n &\le \frac{n! e^n}{n^{n-1/2}} \sum_{l > n^T} \frac{(2t' \ln n)^l}{l!} \sum_{k=0}^{a_n} \frac{(2t' \ln n)^k}{k!} \sum_{j=0}^{b_n} \frac{(2t' \ln n)^j}{j!} \\ &< \frac{n! e^n}{n^{n-1/2}} \sum_{l > n^T} \frac{(2t' \ln n)^l}{l!} n^{4t'} \\ &< \frac{n! e^n}{n^{n-1/2}} \frac{(2t' \ln n)^{\lceil n^T \rceil}}{\lceil n^T \rceil !} \sum_{l=0}^{\infty} \left(\frac{2t' \ln n}{n^T} \right)^l n^{4t'} \\ &< 2n \left(\frac{2et' \ln n}{\lceil n^T \rceil} \right)^{\lceil n^T \rceil} \sum_{l=0}^{\infty} \left(\frac{2t' \ln n}{n^T} \right)^l n^{4t'}. \end{aligned}$$

The last inequality is obtained by using Stirling's formula. Now note that for sufficiently large n, $(2t' \ln n)/n^T \le (2et' \ln n)/\lceil n^T \rceil \le \frac{1}{2}$, so substituting this bound into the above inequality yields

$$R_n \le 2n^{4t'+1} \left(\tfrac{1}{2}\right)^{n^T} \sum_{l=0}^{\infty} \left(\tfrac{1}{2}\right)^l \le 4n^5 \left(\tfrac{1}{2}\right)^{n^T}.$$

Thus $R_n \to 0$ as $n \to \infty$, and hence the absolute value of the expression (9) goes to 0 as $n \to \infty$.

To compute the limit of expression (8) recall that (8) is equal to $[(z_n)^n](n! e^n/n^n) Q_n(z_n) \sum_{m=0}^{n} (m^m/m!)(z_n/e)^m$, where

$$Q_n(z) = \sum_{l=0}^{n^T} \frac{(-\mu(n,z) - \mu'(n,z))^l}{l!} \sum_{k=0}^{a_n} \frac{(\mu(n,z))^k}{k!} \sum_{j=0}^{b_n} \frac{(\mu'(n,z))^j}{j!}.$$

The degree of $Q_n(z)$ is less than $n^T n^{t'} + a_n n^t + b_n n^{t'}$, so for all large n, $\deg Q_n(z) \le n^{T'}$, where $T + t' < T' < 1$, since $a_n = O(\ln n)$ and $b_n = O(\ln n)$. If we write $Q_n(z) = \sum_{j=0}^{d_n} c_{j,n} z^j$, where d_n denotes the degree of $Q_n(z)$, then expression (8) is equal to $\sum_{j=0}^{d_n} c_{j,n}((n-j)^{n-j}e^j n!/(n-j)!n^n)$. It follows from Stirling's formula that for $0 \le j \le d_n$,

$$1 \le \left((n-j)^{n-j}e^j n!/(n-j)!n^n\right) \le \sqrt{n/(n-n^{T'})}\,(1+1/n).$$

Thus (8) is bounded between $Q_n(1) = \sum_{j=0}^{d_n} c_{j_n}$ and $Q_n(1)\sqrt{n/(n-n^{T'})}\,(1+1/n)$, and the limit of (8) as $n \to \infty$ equals $\lim_{n\to\infty} Q_n(1)$.

To compute $\lim_{n\to\infty} Q_n(1)$, we write

$$\begin{aligned} Q_n(1) &= \exp(-\mu(n,1)-\mu'(n,1)) \sum_{k=0}^{a_n} \frac{(\mu(n,1))^k}{k!} \sum_{j=0}^{b_n} \frac{(\mu'(n,1))^j}{j!} \\ &\quad - \sum_{l>n^T}^{\infty} \frac{(-\mu(n,1)-\mu'(n,1))^l}{l!} \sum_{k=0}^{a_n} \frac{(\mu(n,1))^k}{k!} \sum_{j=0}^{b_n} \frac{(\mu'(n,1))^j}{j!}. \end{aligned} \tag{10}$$

The absolute value of the second term on the right side of (10) is less than or equal to R_n and goes to 0 as $n \to \infty$. The first term on the right side of (10) equals $P(Z_n \le a_n)P(Z_n' \le b_n)$, where Z_n and Z_n' are Poisson random variables with parameters $\mu(n,1) = \sum_{k=1}^{n^t} A_k e^{-k}/k!$ and $\mu'(n,1) = \sum_{k>n^t}^{n^{t'}} A_k e^{-k}/k!$, respectively. So

$$\begin{aligned} \lim_{n\to\infty} Q_n(1) &= \lim_{n\to\infty} P(Z_n \le a_n)P(Z_n' \le b_n) \\ &= \lim_{n\to\infty} P\left(\frac{Z_n - \mu(n,z_n)}{\sqrt{(1/2)\ln n}} \le a\right) P\left(\frac{Z_n' - \mu'(n,z_n)}{\sqrt{(1/2)\ln n}} \le b\right). \end{aligned}$$

To compute one of the limits above, we write

$$\frac{Z_n - \mu(n,z_n)}{\sqrt{(1/2)\ln n}} = \frac{Z_n - \mu(n,1)}{\sqrt{\mu(n,1)}}\left(\frac{\sqrt{\mu(n,1)}}{\sqrt{(1/2)\ln n}}\right) + \frac{\mu(n,1)-\mu(n,z_n)}{\sqrt{(1/2)\ln n}}.$$

By the choice of the parameter z_n,

$$\lim_{n\to\infty}\left|\frac{\mu(n,1)-\mu(n,z_n)}{\sqrt{(1/2)\ln n}}\right| \le \lim_{n\to\infty}\frac{1}{\sqrt{(1/2)\ln n}} = 0.$$

Also, from inequality (7),

$$\left|\mu(n,1) - \left(\frac{t}{2}\right)\ln n\right| \le |\mu(n,1)-\mu(n,z_n)| + \left|\mu(n,z_n) - \left(\frac{t}{2}\right)\ln n\right| \le 27$$

and so $\sqrt{\mu(n,1)/((1/2)\ln n)} \to \sqrt{t}$ as $n \to \infty$. Thus $(Z_n - \mu(n,z_n))/\sqrt{(1/2)\ln n}$

and $(\sqrt{t}(Z_n - \mu(n,1)))/\sqrt{\mu(n,1)}$ converge in distribution to the same limit. Since $\mu(n,1) \to \infty$ as $n \to \infty$, the normalized Poisson variable

$$\left(\sqrt{t}\,(Z_n - \mu(n,1))\right)/\sqrt{\mu(n,1)}$$

converges in distribution to the normal distribution $N(0, t)$ and hence $(Z_n - \mu(n, z_n))/\sqrt{(1/2)\ln n}$ converges in distribution to $N(0, t)$. In particular

$$\lim_{n\to\infty} P\left(\frac{Z_n - \mu(n, z_n)}{\sqrt{(1/2)\ln n}} \le a\right) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{a} e^{-u^2/2t}\, du$$

and by the same argument,

$$\lim_{n\to\infty} P\left(\frac{Z_n' - \mu'(n, z_n)}{\sqrt{(1/2)\ln n}} \le b\right) = \frac{1}{\sqrt{2\pi(t'-t)}} \int_{-\infty}^{b} e^{-u^2/2(t'-t)}\, du.$$

This establishes the limit for expression (10) and completes the proof for this case.

CASE 2: $0 \le t < t' = 1$. For $0 < \varepsilon < 1/2 \wedge 1 - t$ and $n > 0$, we have

$$\begin{aligned} & P_n\left(\overline{Y}_n(t) \le a, \overline{Y}_n(1) - \overline{Y}_n(t) \le b\right) \\ & \quad \le P_n\left(\overline{Y}_n(t) \le a, \overline{Y}_n(1-\varepsilon) - \overline{Y}_n(t) \le b + \sqrt[4]{\varepsilon}\right) \\ & \quad\quad + P_n\left(\left|\overline{Y}_n(1) - \overline{Y}_n(1-\varepsilon)\right| \ge \sqrt[4]{\varepsilon}\right) \end{aligned} \tag{11}$$

and

$$\begin{aligned} & P_n\left(\overline{Y}_n(t) \le a, \overline{Y}_n(1) - \overline{Y}_n(t) \le b\right) \\ & \quad \ge P_n\left(\overline{Y}_n(t) \le a, \overline{Y}_n(1-\varepsilon) - \overline{Y}_n(t) \le b - \sqrt[4]{\varepsilon}\right) \\ & \quad\quad - P_n\left(\left|\overline{Y}_n(1) - \overline{Y}_n(1-\varepsilon)\right| \ge \sqrt[4]{\varepsilon}\right). \end{aligned} \tag{12}$$

Fixing ε, we can compute, using Case 1, the limit of the first term in each bound given above for $P_n(\overline{Y}_n(t) \le a, \overline{Y}_n(1) - \overline{Y}_n(t) \le b)$. To bound the second term in both cases we use Chebyshev's inequality,

$$P_n\left(|\overline{Y}_n(1) - \overline{Y}_n(1-\varepsilon)| \ge \sqrt[4]{\varepsilon}\right) \le \frac{E_n\left(\overline{Y}_n(1) - \overline{Y}_n(1-\varepsilon)\right)^2}{\sqrt{\varepsilon}}.$$

To bound $E_n(\overline{Y}_n(1) - \overline{Y}_n(1-\varepsilon))^2$ we begin by defining γ_n^ε on Ω_{z_n} by $\gamma_n^\varepsilon(m_1, m_2, \dots) = \sum_{k > n^{1-\varepsilon}}^{n} m_k$. We extend $\overline{Y}_n(1) - \overline{Y}_n(1-\varepsilon)$ in the usual way to a function on Ω_{z_n} and note that $\overline{Y}_n(1) - \overline{Y}_n(1-\varepsilon) = (\gamma_n^\varepsilon - \mu_n^\varepsilon(z_n))/\sqrt{(1/2)\ln n}$ on

Ω_{z_n} where $\mu_n^\varepsilon(z) = \sum_{k>n^{1-\varepsilon}}^n (A_k/k!)(z/e)^k$. Thus, by (6),

$$
\begin{aligned}
E_n\big(\bar{Y}_n(1) - \bar{Y}_n(1-\varepsilon)\big)^2 &= \big[(z_n)^n\big]\frac{n!e^n}{n^n}E_{z_n}\big(\bar{Y}_n(1) - \bar{Y}_n(1-\varepsilon)\big)^2 S\Big(\frac{z_n}{e}\Big) \\
&= \big[(z_n)^n\big]\frac{2n!e^n}{n^n \ln n}E_{z_n}\big(\gamma_n^\varepsilon - \mu_n^\varepsilon(z_n)\big)^2 S\Big(\frac{z_n}{e}\Big) \qquad (13)\\
&= \big[(z_n)^n\big]\frac{2n!e^n}{n^n \ln n}\mu_n^\varepsilon(z_n) S\Big(\frac{z_n}{e}\Big).
\end{aligned}
$$

The last equality holds since, with respect to P_{z_n}, the function γ_n^ε is a Poisson variable on Ω_{z_n} with parameter $\mu_n^\varepsilon(z_n)$. Using Stirling's formula and the bound $(A_k/k!)e^{-k} \le 1/k$, we have

$$
\begin{aligned}
&\frac{2\big[(z_n)^n\big]n!e^n}{n^n \ln n}\mu_n^\varepsilon(z_n)S(z_n/e) \\
&= \frac{2n!}{n^n \ln n}\sum_{k>n^{1-\varepsilon}}^{n}\frac{A_k}{k!}\frac{(n-k)^{n-k}}{(n-k)!} \\
&\le \frac{2}{\ln n}\sum_{k>n^{1-\varepsilon}}^{n-1}\frac{2}{k}\sqrt{\frac{n}{n-k}} + \frac{4\sqrt{2\pi}}{\sqrt{n}\ln n} \\
&\le \frac{4}{\ln n}\left[\sum_{k>n^{1-\varepsilon}}^{3n/4}\frac{1}{k} + \frac{4}{3}\sum_{k>3n/4}^{n-1}\frac{1}{n\sqrt{1-k/n}} + \sqrt{\frac{2\pi}{n}}\right] \\
&\le \frac{4}{\ln n}\left[\varepsilon \ln n + \frac{4}{3}\int_{3/4}^{1}\frac{dx}{\sqrt{1-x}} + \sqrt{\frac{2\pi}{n}}\right] \\
&\le 5\varepsilon
\end{aligned}
$$

for all sufficiently large n. Substituting this bound into (13) and using Chebyshev's inequality we have $\limsup_{n\to\infty} P_n(|\bar{Y}_n(1) - \bar{Y}_n(1-\varepsilon)| \ge \sqrt[4]{\varepsilon}) \le 5\sqrt{\varepsilon}$. Now take limits in (11) and (12) to get

$$
\begin{aligned}
&\limsup_{n\to\infty} P_n\big(\bar{Y}_n(t) \le a, \bar{Y}_n(1) - \bar{Y}_n(\varepsilon) \le b\big) \\
&\le \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{a} e^{-u^2/2t}\,du \frac{1}{\sqrt{2\pi(1-\varepsilon-t)}}\int_{-\infty}^{b} e^{-u^2/2(1-\varepsilon-t)}\,du + 5\sqrt{\varepsilon}
\end{aligned}
$$

and

$$
\begin{aligned}
&\liminf_{n\to\infty} P_n\big(\bar{Y}_n(t) \le a, \bar{Y}_n(1) - \bar{Y}_n(t) \le b\big) \\
&\ge \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{a} e^{-u^2/2t}\,du \frac{1}{\sqrt{2\pi(1-\varepsilon-t)}}\int_{\infty}^{b} e^{-u^2/2(1-\varepsilon-t)}\,du - 5\sqrt{\varepsilon}.
\end{aligned}
$$

Let $\varepsilon \to \infty$ to obtain the desired limit in this case.

The bound for $E_n(\overline{Y}_n(t) - \overline{Y}_n(t_1))^2(\overline{Y}_n(t_2) - \overline{Y}_n(t))^2$. We begin by noting that there are two cases where $E_n(\overline{Y}_n(t) - \overline{Y}_n(t_1))^2(\overline{Y}_n(t_2) - \overline{Y}_n(t))^2 = 0$. First, for n fixed, if $\ln(k-1)/\ln n \le t_1 \le t < \ln k/\ln n$ for some $2 \le k \le n$ then $\overline{Y}_n(t, \phi) = \overline{Y}_n(t_1, \phi)$ for all $\phi \in T_n$ and the expectation

$$E_n\big(\overline{Y}_n(t) - \overline{Y}_n(t_1)\big)^2\big(\overline{Y}_n(t_2) - \overline{Y}_n(t)\big)^2 = 0.$$

Similarly, if $\ln(k-1)/\ln n \le t \le t_2 < \ln k/\ln n$ for some $2 \le k \le n$ then $\overline{Y}_n(t, \phi) = \overline{Y}_n(t_2, \phi)$ for all $\phi \in T_n$ and the expectation is 0. Thus the expectation will be nonzero only if $\ln(k-1)/\ln n \le t_1 < \ln k/\ln n$ and $\ln(k+1)/\ln n \le t_2$ for some $2 \le k \le n-1$. In this case

$$t_2 - t_1 \ge \frac{\ln(k+1) - \ln k}{\ln n} \ge \frac{1}{k \ln n} \ge \frac{1}{2n^{t_1} \ln n}. \tag{14}$$

Thus, to avoid trivialities, we assume in the calculations below that inequality (14) holds.

To compute the bound recall that

$$E_n\big(\overline{Y}_n(t) - \overline{Y}_n(t_1)\big)^2\big(\overline{Y}_n(t_2) - \overline{Y}_n(t)\big)^2$$
$$= \frac{[(z_n)^n]n!e^n}{n^n} E_{z_n}\big(\overline{Y}_n(t) - \overline{Y}_n(t_1)\big)^2\big(\overline{Y}_n(t_2) - \overline{Y}_n(t)\big)^2 S\left(\frac{z_n}{e}\right).$$

The functions $\overline{Y}_n(t) - \overline{Y}_n(t_1)$ and $\overline{Y}_n(t_2) - \overline{Y}_n(t)$ are independent random variables on (Ω_{z_n}, P_{z_n}), so [cf. (13)]

$$\begin{aligned}
&\frac{[(z_n)^n]n!e^n}{n^n} E_{z_n}\big(\overline{Y}_n(t) - \overline{Y}_n(t_1)\big)^2\big(\overline{Y}_n(t_2) - \overline{Y}_n(t)\big)^2 S\left(\frac{z_n}{e}\right) \\
&= \frac{4[(z_n)^n]n!e^n}{n^n(\ln n)^2} E_{z_n}\left(\sum_{k>n^{t_1}}^{n^t} m_k - \mu_{t_1}^t(n, z_n)\right)^2 \\
&\quad \times E_{z_n}\left(\sum_{k>n^t}^{n^{t_2}} m_k - \mu_t^{t_2}(n, z_n)\right)^2 S\left(\frac{z_n}{e}\right) \\
&= \frac{4[(z_n)^n]n!e^n}{n^n(\ln n)^2} \mu_{t_1}^t(n, z_n)\mu_t^{t_2}(n, z_n) S\left(\frac{z_n}{e}\right),
\end{aligned} \tag{15}$$

where $\mu_{t_1}^t(n, z_n) = \sum_{k>n^{t_1}}^{n^t}(A_k/k!)(z_n/e)^k = E_{z_n}(\sum_{k>n^{t_1}}^{n^t} m_k)$ and $\mu_t^{t_2}(n, z_n) = \sum_{k>n^t}^{n^{t_2}}(A_k/k!)(z_n/e)^k = E_{z_n}(\sum_{k>n^t}^{n^{t_2}} m_k)$. We proceed to expand the right side of (15).

The first step is to bound the terms in the expansion by using Stirling's formula and the inequality $A_k e^{-k}/k! \le 1/k$ as follows:

$$
\begin{aligned}
&\frac{4[(z_n)^n]n!e^n}{n^n(\ln n)^2}\mu^t_{t_1}(n, z_n)\mu^{t_2}_t(n, z_n)S\left(\frac{z_n}{e}\right)\\
&= \frac{4}{(\ln n)^2}\left[\sum_{j>n^{t_1}}^{n^t}\sum_{k>n^t}^{n/4\wedge n^{t_2}}\frac{A_jA_kn!(n-j-k)^{n-j-k}}{j!k!n^n(n-j-k)!}\right.\\
&\qquad\left.+\sum_{j>n^{t_1}}^{n^t}\sum_{k>n/4\vee n^t}^{(n-j)\wedge n^{t_2}}\frac{A_jA_kn!(n-k-j)^{n-j-k}}{j!k!n^n(n-j-k)!}\right]\\
&\le \frac{8}{(\ln n)^2}\sum_{j>n^{t_1}}^{n^t}\sum_{k>n^t}^{n/4\wedge n^{t_2}}\frac{1}{jk}\sqrt{\frac{n}{n-j-k}}\\
&\quad+\frac{8}{(\ln n)^2}\sum_{j>n^{t_1}}^{n^t}\sum_{k>n/4\vee n^t}^{(n-j-1)\wedge n^{t_2}-1}\frac{1}{jk}\sqrt{\frac{n}{n-j-k}}\\
&\quad+\frac{8}{(\ln n)^2}\sum_{\substack{j>n^{t_1}\\ \text{s.t. } n-j>n/4}}^{n^t}\frac{\sqrt{2\pi}\sqrt{n}}{j(n-j)}.
\end{aligned}\tag{16}
$$

Next we bound each term on the right side of (16). First,

$$
\begin{aligned}
&\frac{8}{(\ln n)^2}\sum_{j>n^{t_1}}^{n^t}\sum_{k>n^t}^{n/4\wedge n^{t_2}}\frac{1}{jk}\sqrt{\frac{n}{n-j-k}}\\
&\le \frac{8\sqrt{2}}{(\ln n)^2}\sum_{j>n^{t_1}}^{n^t}\sum_{k>n^t}^{n^{t_2}}\frac{1}{jk}\\
&\le \frac{8\sqrt{2}}{(\ln n)^2}\left((t-t_1)\ln n+\frac{1}{n^{t_1}}\right)\left((t_2-t)\ln n+\frac{1}{n^t}\right)\\
&\le 72\sqrt{2}\,(t_2-t_1)^2.
\end{aligned}
$$

The last inequality follows from (14). Next,

$$
\begin{aligned}
&\frac{8}{(\ln n)^2}\sum_{j>n^{t_1}}^{n^t}\sum_{k>n/4\vee n^t}^{n-j-1\wedge n^{t_2}-1}\frac{1}{jk}\sqrt{\frac{n}{n-j-k}}\\
&\le \frac{8}{(\ln n)^2}\sum_{j>n^{t_1}}^{n^t}\sum_{k>n/4\vee n^t}^{(n-j-1)\wedge n^{t_2}-1}\frac{4}{jn}\frac{1}{\sqrt{1-j/n-k/n}}\\
&\le \frac{32}{(\ln n)^2}\sum_{j>n^{t_1}}^{n^t}\frac{1}{j}\int_{n^t/n}^{(1-j/n)\wedge(n^{t_2}/n)}\frac{dx}{\sqrt{1-j/n-x}}
\end{aligned}
$$

$$\leq \frac{32}{(\ln n)^2} \sum_{j>n^{t_1}}^{n^t} \frac{1}{j} \int_{n^t/n-j}^{1\wedge(n^{t_2}/n-j)} \frac{du}{\sqrt{1-u}}$$

$$\leq \frac{64}{(\ln n)^2} \sum_{j>n^{t_1}}^{n^t} \frac{1}{j} \left(\sqrt{1 - \frac{n^t}{n-j}} - \sqrt{1 - \left(1 \wedge \frac{n^{t_2}}{n-j}\right)} \right).$$

$$\leq \frac{64\sqrt{t_2 - t}}{(\ln n)^{3/2}} \sum_{j>n^{t_1}}^{n^t} \frac{1}{j}$$

$$\leq \frac{64\sqrt{t_2 - t}}{(\ln n)^{3/2}} \left((t_1 - t)\ln n + \frac{1}{n^{t_1}} \right)$$

$$\leq 192(t_2 - t_1)^{3/2}.$$

To obtain this bound we have used the inequality $\sqrt{1-x} - \sqrt{1-y} \leq \sqrt{\ln(y/x)}$ for all $0 < x < y \leq 1$. Finally

$$\frac{8}{(\ln n)^2} \sum_{\substack{j>n^{t_1} \\ \text{s.t. } n-j>n/4}}^{n^t} \frac{\sqrt{2\pi n}}{j(n-j)} \leq \frac{32\sqrt{2\pi}}{\sqrt{n}(\ln n)^2} \sum_{j>n_1^t}^{n^t} \frac{1}{j}$$

$$\leq \frac{96\sqrt{2\pi}(t - t_1)}{\sqrt{n^{t_1}} \ln n} \leq 192\sqrt{\pi}(t_2 - t_1)^{3/2}.$$

It follows that the right side of (16) is less than or equal to $640(t_2 - t_1)^{3/2}$. This establishes the bound for $E_n(\overline{Y}_n(t) - \overline{Y}_n(t_1))^2(\overline{Y}_n(t_2) - \overline{Y}_n(t))^2$ and completes the proof. □

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