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A functional relation for the Tornheim double zeta function

by

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1. Introduction

Definition 1.1. The *Tornheim double zeta function* $T(s,t,u)$, for $s,t,u \in \mathbb{C}$, $\Re(s+u)>1$, $\Re(t+u)>1$ and $\Re(s+t+u)>2$, is defined by

$$T(s,t,u) := \sum_{m,n=1}^{\infty} \frac{1}{m^s n^t (m+n)^u}. \tag{1.1}$$

This function $T(s,t,u)$ is a generalization of the Riemann zeta function $\zeta(s)$, $s \in \mathbb{C}$. Furthermore, $T(s,t,u)$ is continued meromorphically to $\mathbb{C}^3$ in [4]. By the definition, we have

$$T(s,t,u) = T(t,s,u), \qquad T(s,t,0) = \zeta(s)\zeta(t).$$

The case of $t=0$, that is $T(s,0,u)$, is called the *Euler–Zagier double zeta function* [10].

The values $T(a,b,c)$ for $a,b,c \in \mathbb{N}$ were first investigated by Tornheim [7] in 1950 and later Mordell [5] in 1958. Tornheim [7, Theorem 7] showed that $T(a,b,c)$ can be expressed as a polynomial in $\{\zeta(j) \mid 2 \le j \le a+b+c\}$ with rational coefficients when $a+b+c$ is odd, and that the same is true for $T(2r,2r,2r)$ and $T(2r-1,2r,2r+1)$ [7, Theorem 8], but he did not give the coefficients. Mordell [5, Theorem III] proved that $T(2r,2r,2r) = k_r \pi^{6r}$ for some rational number k_r. In 1985 Subbarao and Sitaramachandrarao [6, Theorem 4.1] explicitly determined $T(2p,2q,2r)+T(2q,2r,2p)+T(2r,2p,2q)$ $(p,q,r \in \mathbb{N})$. Then, by taking $p=q=r$, they gave an explicit formula for $T(2r,2r,2r)$ $(r \in \mathbb{N})$ [6, Remark 3.1]. In 1996 Huard, Williams and Zhang [3, Theorems 1–3] determined $T(r,0,N-r)$ $(r \in \mathbb{N}$, $N \in 2\mathbb{N}+1$, $1 \le r \le N-2)$, $T(p,q,N-p-q)$ $(p,q \in \mathbb{N} \cup \{0\}$, $N \in 2\mathbb{N}+1$, $1 \le p+q \le N-1$, $0 \le p,q \le N-2)$ and $T(r,r,r)$ $(r \in \mathbb{N})$. In 2002 Tsumura [8, Theorem 1]

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proved that $T(p,q,r)+(-1)^pT(p,r,q)+(-1)^{p+r}T(r,q,p)$ is a polynomial in $\{\zeta(k)\mid 2\le k\le p+q+r\}$ with rational coefficients for $p,q,r\in\mathbb{N}\cup\{0\}$ with $p+q\ge 2$ and $r\ge 2$. Recently, Espinosa and Moll provided an explicit formula for $T(x,y,z)$, $x,y,z\in\mathbb{R}$, in terms of integrals involving Hurwitz zeta functions (see [2, Proposition 2.1 and Theorem 2.4]). Also in 2006 Tsumura [9, Theorem 4.5] proved the following functional relation:

$$(1.2)\quad T(a,b,s)+(-1)^bT(b,s,a)+(-1)^aT(s,a,b)$$
$$= 2\sum_{\substack{j=0\\ j\equiv a\,(2)}}^{a}(2^{1-a+j}-1)\zeta(a-j)\sum_{l=0}^{j/2}\frac{(i\pi)^{2l}}{(2l)!}\binom{b-1+j-2l}{j-2l}\zeta(b+j+s-2l)$$
$$-4\sum_{\substack{j=0\\ j\equiv a\,(2)}}^{a}(2^{1-a+j}-1)\zeta(a-j)\sum_{l=0}^{(j-1)/2}\frac{(i\pi)^{2l}}{(2l+1)!}\sum_{\substack{k=0\\ k\equiv b\,(2)}}^{b}\zeta(b-k)$$
$$\times\binom{k-1+j-2l}{j-2l-1}\zeta(k+j+s-2l)$$

(where (2) means mod 2), for $a,b\in\mathbb{N}\cup\{0\}$, $b\ge 2$, $s\in\mathbb{C}$, except for the singular points of each side of this formula.

In this paper, we prove the following result.

Theorem 1.2. *For all $a,b\in\mathbb{N}$ and $s\in\mathbb{C}$ except for the singular points, we have*

$$(1.3)\quad T(a,b,s)+(-1)^bT(b,s,a)+(-1)^aT(s,a,b)$$
$$=\frac{2}{a!\,b!}\sum_{k=0}^{\max(a,b)/2}\left\{a\binom{b}{2k}+b\binom{a}{2k}\right\}(a+b-2k-1)!(2k)!$$
$$\times\zeta(2k)\zeta(a+b+s-2k).$$

This functional relation is considerably simpler than that of Tsumura. We are not aware of a direct proof which shows that the right-hand sides of (1.2) and (1.3) are the same. "Mathematica 5.0" shows that they are equal for all $1\le a\le b\le 100$, $a,b\in\mathbb{N}$. It therefore seems unlikely that a non-trivial functional relation can be deduced by equating (1.2) and (1.3).

In Section 3, we obtain new proofs of formulas for the special values of $T(a,b,c)$, $a,b,c\in\mathbb{N}$ mentioned in the introduction by using the functional relation (1.3).

2. Proof of Theorem 1.2. Firstly, we define $\log t$, $t\in\mathbb{C}$, and t^s, $s,t\in\mathbb{C}$, by

$$\log t:=\log|t|+i\arg t,\qquad t^s:=e^{s\log t},\qquad 0\le\arg t<2\pi.$$

And for $s,t,u \in \mathbb{C}$, $\Re(s+u) > 1$, $\Re(t+u) > 1$ and $\Re(s+t+u) > 2$, we put

$$S(s,t,u) := \sum_{\substack{m\neq 0,\, n\neq 0\\ m+n\neq 0}} \frac{1}{m^s n^t (m+n)^u}. \tag{2.1}$$

LEMMA 2.1. *For all $a,b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$\begin{aligned} S(a,b,s) &= (1+e^{-\pi i(a+b+s)}) \\ &\quad \times (T(a,b,s) + (-1)^b T(b,s,a) + (-1)^a T(s,a,b)). \end{aligned} \tag{2.2}$$

Proof. Let

$$\begin{aligned}
T_1(a,b,s) &:= \sum_{m,n>0} \frac{1}{m^a n^b (m+n)^s} = T(a,b,s),\\
T_2(a,b,s) &:= \sum_{\substack{m<0,\, n>0\\ n>-m}} \frac{1}{m^a n^b (m+n)^s} = \sum_{\substack{m,n>0\\ n>m}} \frac{1}{(-m)^a n^b (n-m)^s}\\
&= (-1)^{-a} \sum_{m,k>0} \frac{1}{m^a (m+k)^b k^s} = (-1)^{-a} T(s,a,b),\\
T_3(a,b,s) &:= \sum_{\substack{m<0,\, n>0\\ -m>n}} \frac{1}{m^a n^b (m+n)^s} = \sum_{\substack{m,n>0\\ m>n}} \frac{1}{(-m)^a n^b (n-m)^s}\\
&= e^{-\pi i(a+s)} \sum_{n,k>0} \frac{1}{(n+k)^a n^b k^s} = e^{-\pi i(a+s)} T(b,s,a),\\
T_4(a,b,s) &:= \sum_{m,n<0} \frac{1}{m^a n^b (m+n)^s} = e^{-\pi i(a+b+s)} T(a,b,s),\\
T_5(a,b,s) &:= \sum_{\substack{m>0,\, n<0\\ -n>m}} \frac{1}{m^a n^b (m+n)^s} = \sum_{\substack{m,n>0\\ n>m}} \frac{1}{m^a (-n)^b (m-n)^s}\\
&= e^{-\pi i(b+s)} T(s,a,b),\\
T_6(a,b,s) &:= \sum_{\substack{m>0,\, n<0\\ m>-n}} \frac{1}{m^a n^b (m+n)^s} = \sum_{\substack{m,n>0\\ m>n}} \frac{1}{m^a (-n)^b (m-n)^s}\\
&= (-1)^{-b} T(b,s,a).
\end{aligned}$$

Obviously we have

$$\sum_{j=1}^{6} T_j(a,b,s) = S(a,b,s).$$

This implies (2.2). We can also see that the convergence of $S(a,b,s)$ is equivalent to the convergence of $T(a,b,s)$. ∎

LEMMA 2.2 ([11]). *For $\Re(s) > 1$, $\Re(t) > 1$ and $\Re(u) > 1$, we have*

$$S(s,t,u) = \int_0^1 \sum_{m\neq 0} \frac{e^{2\pi imx}}{m^s} \sum_{n\neq 0} \frac{e^{2\pi inx}}{n^t} \sum_{l\neq 0} \frac{e^{-2\pi ilx}}{l^u}\,dx. \tag{2.3}$$

Proof. By putting $l = m+n$, we have

$$\begin{aligned} S(s,t,u) &= \sum_{\substack{m,n,l\neq 0\\ m+n=l}} \frac{1}{m^s n^t l^u} = \sum_{m,n,l\neq 0} \int_0^1 \frac{e^{2\pi i(m+n-l)x}}{m^s n^t l^u}\,dx \\ &= \int_0^1 \sum_{m,n,l\neq 0} \frac{e^{2\pi i(m+n-l)x}}{m^s n^t l^u}\,dx = \int_0^1 \sum_{m\neq 0} \frac{e^{2\pi imx}}{m^s} \sum_{n\neq 0} \frac{e^{2\pi inx}}{n^t} \sum_{l\neq 0} \frac{e^{-2\pi ilx}}{l^u}\,dx. \end{aligned}$$

Changing the order of summation and integration is justified by absolute convergence. ∎

We denote by $B_j(x)$ the Bernoulli polynomial of order j defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{j=0}^{\infty} B_j(x)\,\frac{t^j}{j!}, \qquad |t| < 2\pi.$$

It is known (see [1, p. 266, (22), and p. 267, (24)]) that

$$B_{2j} := B_{2j}(0) = (-1)^{j+1} 2(2j)!(2\pi)^{-2j}\zeta(2j), \qquad j \in \mathbb{N}, \tag{2.4}$$

$$B_j(x) = -\frac{j!}{(2\pi i)^j} \lim_{K\to\infty} \sum_{\substack{k=-K\\ k\neq 0}}^{K} \frac{e^{2\pi ikx}}{k^j}, \qquad j \in \mathbb{N}. \tag{2.5}$$

For $k \in \mathbb{Z}$, $j \in \mathbb{N}$ we have

$$\int_0^1 e^{-2\pi ikx} B_j(x)\,dx = \begin{cases} 0, & k = 0, \\ -(2\pi ik)^{-j} j!, & k \neq 0. \end{cases} \tag{2.6}$$

In fact, the case of $k = 0$ is obvious, and in the case of $k \neq 0$, we get (2.6) by using (2.5). Next we quote [1, p. 276, 19(b)], for $p + q \geq 2$, which is

$$\begin{aligned} &B_p(x)B_q(x) \\ &= \sum_{k=0}^{\max(p,q)/2} \left\{ p\binom{q}{2k} + q\binom{p}{2k} \right\} \frac{B_{2k}B_{p+q-2k}(x)}{p+q-2k} - (-1)^p\,\frac{p!q!}{(p+q)!}\,B_{p+q}. \end{aligned} \tag{2.7}$$

Proof of Theorem 1.2. Firstly, we assume $a, b \geq 2$, $1 + e^{-\pi i(a+b+s)} \neq 0$ and $\Re(s) > 1$. By using (2.6) and (2.7), we have

$$\begin{aligned}
&-\int_0^1 B_a(x)B_b(x)\sum_{l\neq 0}\frac{e^{-2\pi ilx}}{l^s}\,dx\\
&\quad= -\int_0^1 \sum_{k=0}^{\max(a,b)/2}\left\{a\binom{b}{2k}+b\binom{a}{2k}\right\}\frac{B_{2k}B_{a+b-2k}(x)}{a+b-2k}\sum_{l\neq 0}\frac{e^{-2\pi ilx}}{l^s}\,dx\\
&\quad= \sum_{k=0}^{\max(a,b)/2}\left\{a\binom{b}{2k}+b\binom{a}{2k}\right\}\frac{(a+b-2k-1)!B_{2k}}{(2\pi i)^{a+b-2k}}\\
&\qquad\times\zeta(a+b+s-2k)(1+e^{-\pi i(a+b+s)}).
\end{aligned}$$

Because of (2.2), (2.4) and (2.5), we obtain (1.3) in this region. By analytic continuation, we have (1.3) for all $a, b \in \mathbb{N}$, $a, b \geq 2$ and $s \in \mathbb{C}$ except for the singular points of each side of this formula.

Next we consider the case of $a = 1$, $b \geq 2$. For $a, b \in \mathbb{N}$, $a, b \geq 2$ and $s \in \mathbb{C}$ except for the singular points, we define $K(a, b, s)$ by the right-hand side of (1.3). We quote some basic properties [3, (1.5)] proved by easy computations, for $s, t, u \in \mathbb{C}$ except for the singular points:

$$\tag{2.8}\begin{cases}
T(s,t-1,u+1)+T(s-1,t,u+1)=T(s,t,u),\\
T(s,t+1,u-1)-T(s-1,t+1,u)=T(s,t,u),\\
T(s+1,t,u-1)-T(s+1,t-1,u)=T(s,t,u).
\end{cases}$$

For $b \geq 2$, we have

$$\begin{aligned}
K(2,b,s) =\ &T(2,b,s)+(-1)^bT(b,s,2)+(-1)^2T(s,2,b)\\
=\ &T(1,b,s+1)+(-1)^bT(b,s+1,1)+(-1)T(s+1,1,b)\\
&+T(2,b-1,s+1)+(-1)^{b-1}T(b-1,s+1,2)\\
&+(-1)^2T(s+1,2,b-1)
\end{aligned}$$

by (2.8) and the result in the case $a, b \geq 2$ which we have already shown. Hence we have to show

$$K(2,b,s) = K(1,b,s+1) + K(2,b-1,s+1), \qquad b \geq 2.$$

In fact we have

$$\begin{aligned}
&\frac{2}{b!}\left\{\binom{b}{2k}+b\binom{1}{2k}\right\}+\frac{2}{2!\,(b-1)!}\left\{2\binom{b-1}{2k}+(b-1)\binom{2}{2k}\right\}\\
&\qquad\qquad= \frac{2}{2!\,b!}\left\{2\binom{b}{2k}+b\binom{2}{2k}\right\}(b+1-2k), \qquad 0 \leq k \leq b/2.
\end{aligned}$$

In the cases of $k = 0, 1, b/2$, we have this equation immediately. For $2 \leq k \leq$

$(b-1)/2$, we obtain it by

$$b\binom{b-1}{l} = \frac{b(b-1)\cdots(b-l+1)(b-l)}{l!} = (b-l)\binom{b}{l}, \qquad 0 \le l \le b.$$

We can prove (1.3) for the case of $a = b = 1$ similarly. ∎

3. New proofs of known formulas. In this section, from our theorem we deduce formulas for the special values of $T(a,b,c)$ $(a,b,c \in \mathbb{N})$ mentioned in the introduction. By taking $a = 2p$, $b = 2q$, $s = 2r$ in (1.3), we have

$$\begin{aligned} &T(2p,2q,2r) + T(2q,2r,2p) + T(2r,2p,2q) \\ &\quad = \frac{2}{(2p)!\,(2q)!} \sum_{k=0}^{\max(p,q)} \left\{ 2p\binom{2q}{2k} + 2q\binom{2p}{2k} \right\} (2p+2q-2k-1)!(2k)! \\ &\qquad \times \zeta(2k)\zeta(2p+2q+2r-2k). \end{aligned}$$

This formula coincides with [6, Theorem 4.1]. (There is a misprint in [6, Theorem 4.1], "min" is to be replaced by "max".) Putting $a = b = s = r$ in (1.3) we have, after easy computations of binomial coefficients,

$$T(r,r,r) = \frac{4}{1+2(-1)^r} \sum_{k=0}^{r/2} \binom{2r-2k-1}{2k-1} \zeta(2k)\zeta(3r-2k).$$

This formula is [3, Theorem 3].

For $a,b,c \in \mathbb{N}$, we define $N(a,b,c)$ as half of the right-hand side of (1.3). We recall the harmonic product formula

$$T(a,0,b) + T(b,0,a) = \zeta(a)\zeta(b) - \zeta(a+b).$$

Putting $s = 0$ in (1.3) and multiplying by $(-1)^a$, we obtain

$$(-1)^a\zeta(a)\zeta(b) + (-1)^{a+b}T(b,0,a) + T(a,0,b) = 2(-1)^a N(a,b,0).$$

When $a + b \in 2\mathbb{N}+1$, we can remove $T(b,0,a)$ by summing the above two formulas. Hence

$$\text{(3.1)} \qquad T(a,0,b) = -\frac{\zeta(a+b)}{2} + \frac{1+(-1)^b}{2}\,\zeta(a)\zeta(b) + (-1)^a N(a,b,0)$$

for all $a, b \ge 2$, $a + b \in 2\mathbb{N}+1$. Next by changing the variables in (1.3), we obtain

$$\begin{cases} (-1)^b T(a,b,c) + T(b,c,a) + (-1)^c T(c,a,b) = 2N(b,c,a), \\ (-1)^a T(a,b,c) + (-1)^c T(b,c,a) + T(c,a,b) = 2N(c,a,b). \end{cases}$$

In the case of $a + b + c \in 2\mathbb{N}+1$, we can remove $T(b,c,a)$ and $T(c,a,b)$ by multiplying the former equality by $(-1)^b$ and the latter by $(-1)^a$, and summing the resulting formulas. Hence we have

$$\text{(3.2)} \qquad T(a,b,c) = (-1)^b N(b,c,a) + (-1)^a N(c,a,b), \qquad a+b+c \in 2\mathbb{N}+1.$$

By putting $s = t = 1$ in the first equation of (2.8), we obtain

$$T(1, 1, u) = 2T(1, 0, u + 1).$$

Hence we can calculate $T(1, 0, c + 1)$ if $c + 1 \in 2\mathbb{N}$. Therefore we obtain another proof of [3, Theorems 1, 2]. Moreover we get

$$T(p, q, r) + (-1)^p T(p, r, q) + (-1)^{p+r} T(r, q, p) = 2(-1)^p N(p, r, q)$$

by taking $a = p$, $b = r$ and $s = q$ in (1.3), and multiplying by $(-1)^p$. Hence we obtain another proof of [8, Theorem 1], because $N(p, q, r)$ is a polynomial in $\{\zeta(k) \mid 2 \le k \le p + q + r\}$ with rational coefficients for $p, q, r \in \mathbb{N} \cup \{0\}$ with $p + q \ge 2$ and $r \ge 2$.

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