A FUNDAMENTAL MATRIX EQUATION FOR FINITE SETS[1]

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ABSTRACT. Let $S=\{x_1, x_2, \cdots, x_n\}$ be an n-set and let S_1, $S_2, \cdots, S_m$ be subsets of S. Let A of size m by n be the incidence matrix for these subsets of S. We now regard $x_1, x_2, \cdots, x_n$ as independent indeterminates and define $X=\text{diag}[x_1, x_2, \cdots, x_n]$. We then form the matrix product $AXA^T=Y$, where A^T denotes the transpose of the matrix A. The symmetric matrix Y has in its (i, j) position the sum of the indeterminates in $S_i \cap S_j$ and consequently Y gives us a complete description of the intersection patterns $S_i \cap S_j$. The specialization $x_1=x_2=\cdots=x_n=1$ of this basic matrix equation has been used extensively in the study of block designs. We give some other interesting applications of the matrix equation that involve subsets with various restricted intersection patterns.

1. **The matrix equation.** Let $S=\{x_1, x_2, \cdots, x_n\}$ be an n-set (a set of n elements) and let $S_1, S_2, \cdots, S_m$ be subsets of S. We set $a_{ij}=1$ if x_j is a member of S_i and we set $a_{ij}=0$ if x_j is not a member of S_i. The resulting (0, 1)-matrix

$$(1.1) \qquad A = [a_{ij}]$$

of size m by n is the familiar *incidence matrix* for the subsets $S_1, S_2, \cdots, S_m$ of S. It is clear that A characterizes the configuration of subsets.

Now let us regard $x_1, x_2, \cdots, x_n$ as independent indeterminates over the field of rational numbers and define

$$(1.2) \qquad X = \text{diag}[x_1, x_2, \cdots, x_n].$$

We then form the matrix product

$$(1.3) \qquad AXA^T = Y.$$

The matrix A^T denotes the transpose of the matrix A. The matrix Y is a symmetric matrix of order m. We know the structure of this matrix

explicitly. Thus the matrix Y has in its (i, i) position the sum of the indeterminates in S_i. More generally, the matrix Y has in its (i, j) position the sum of the indeterminates in $S_i \cap S_j$. It follows that the matrix Y gives us a complete description of the intersection patterns $S_i \cap S_j$ for the subsets $S_1, S_2, \cdots, S_m$ of S. For related material concerning intersection patterns we mention here the earlier investigations of Goodman [1], Hall [2], Kelly [3], and Ryser [6].

The matrix Y involves the indeterminates $x_1, x_2, \cdots, x_n$ and we write

$$(1.4) \qquad Y = Y(x_1, x_2, \cdots, x_n).$$

We may set $x_1 = x_2 = \cdots = x_n = 1$ and then (1.3) reduces to the classical equation

$$(1.5) \qquad AA^T = Y(1, 1, \cdots, 1).$$

But notice that $Y(1, 1, \cdots, 1)$ reveals only the cardinalities of the sets $S_i \cap S_j$.

The basic matrix equation (1.3) allows us to apply the powerful methods of matrix theory to the study of intersection patterns. It is of course difficult to make general statements concerning (1.3) without further constraints on the subsets. However, it is elementary to verify that

$$(1.6) \qquad \operatorname{rank}(Y) = \operatorname{rank}(A).$$

Indeed, an application of the standard theorems on rank to the matrix equation (1.3) implies

$$(1.7) \qquad \begin{aligned} \operatorname{rank}(AA^T) &= \operatorname{rank}(Y(1, 1, \cdots, 1)) \\ &\leqq \operatorname{rank}(Y(x_1, x_2, \cdots, x_n)) \leqq \operatorname{rank}(A). \end{aligned}$$

But an arbitrary real matrix Z has $\operatorname{rank}(ZZ^T) = \operatorname{rank}(Z)$, whence the result follows. We note further that in the special case $m = n$ we may take determinants in (1.3) and this gives

$$(1.8) \qquad \det(Y) = (\det(A))^2 \prod_{i=1}^{n} x_i.$$

2. **Applications.** We now proceed with applications of the matrix equation (1.3) for subsets with various restricted intersection patterns.

THEOREM 2.1. *Let $S_1, S_2, \cdots, S_m$ be subsets of the n-set*

$$S = \{x_1, x_2, \cdots, x_n\}$$

and let A of size m by n be the incidence matrix for these subsets of S. Suppose that the number of distinct nonempty set intersections of the form $S_i \cap S_j$ $(i \neq j)$ is less than n. Then there exists an integral nonzero diagonal

matrix E and an integral diagonal matrix D such that

$$(2.1) \qquad\qquad\qquad AEA^T = D.$$

Proof. We consider the matrix equation (1.3) for the subsets of Theorem 2.1. Let t denote the number of distinct nonzero elements in Y but not on the main diagonal of Y. These nonzero elements equated to zero give us a system of t linear homogeneous equations in the n unknowns $x_1, x_2, \cdots, x_n$. The matrix of this system of equations is a $(0, 1)$-matrix in its own right. The assumption $t < n$ implies that the linear homogeneous system has an integral solution $e_1, e_2, \cdots, e_n$ with at least one $e_i \neq 0$. This integral solution yields the desired diagonal matrix E.

We digress briefly and describe some material from a forthcoming paper by Ryser [6]. Let W denote a nonzero symmetric matrix of order $m \geq 2$ with nonnegative integral elements and with 0's in all of the main diagonal positions. Let A be a $(0, 1)$-matrix of size m by n and let each of the column sums of A be greater than 1. Then we say that A *represents* W provided that there exists a diagonal matrix D of order m such that

$$(2.2) \qquad\qquad\qquad AA^T = D + W.$$

Thus A may be regarded as the incidence matrix for subsets $S_1, S_2, \cdots, S_m$ of an n-set S with the intersection cardinalities $|S_i \cap S_j|$ $(i \neq j)$ prescribed by W. Extraneous elements are not allowed in the subsets in the sense that each of the elements of S is required to occur in at least two of the subsets. It is entirely elementary to verify that there always exists a simple "canonical matrix" with column sums 2 that represents W. We designate by $C(W)$ the class of all $(0, 1)$-matrices A that represent W. Each of the matrices in $C(W)$ has exactly m rows. But the number of columns may undergo considerable variation from one matrix to another.

A *line* of a matrix designates either a row or a column of the matrix. A *triangle* of a $(0, 1)$-matrix A is a submatrix of A of order 3 with all line sums equal to 2. The following simple lemma on triangles is derived in [5]. Let A be a $(0, 1)$-matrix of size m by n. Suppose that every element of AA^T is positive and that A contains no triangles. Then A contains a column of m 1's. This lemma implies the following theorem concerning an arbitrary class $C(W)$ [6]. Every matrix in the class $C(W)$ contains a triangle or else the class contains exactly one matrix apart from column permutations without triangles.

We return to the incidence matrix A of Theorem 2.1. Suppose now that each of the column sums of A is greater than 1. Then the matrix of our system of linear homogeneous equations does not contain a column of 0's. Hence it follows that in this case the diagonal matrix E contains both

positive and negative e_i's. We associate the integer e_i with column i of A and call $|e_i|$ the *multiplicity* of column i of A. We now rewrite the matrix equation (2.1) in the following form

$$(2.3) \qquad\qquad A_1 A_1^T - A_2 A_2^T = D.$$

The matrix A_1 consists of the columns of A with associated e_i positive and each column is repeated with its appropriate multiplicity. Similarly, the matrix A_2 consists of the columns of A with associated e_j negative and each column is repeated with its appropriate multiplicity. The ordering of the columns within A_1 and A_2 is immaterial. Thus (2.3) tells us that A_1 and A_2 are both members of a certain class $C(W)$.

Now let us suppose further that the matrix A is without triangles. Then both of the matrices A_1 and A_2 are also without triangles. Hence since A_1 and A_2 belong to the same class $C(W)$ we may conclude that A_1 and A_2 are the same apart from column permutations. Thus the matrix A has two identical columns.

The following theorem is a direct consequence of the preceding discussion.

THEOREM 2.2. *Let A be a $(0, 1)$-matrix of size m by n. Let the columns of A be distinct and let each of the column sums of A be greater than* 1. *Suppose further that A contains no triangles. Then $n \leq m(m-1)/2$.*

PROOF. Let us assume that $n > m(m-1)/2$. Then the hypotheses of Theorem 2.1 are satisfied and equations (2.1) and (2.3) are valid. Furthermore, the assumption that A contains no triangles implies that A has two identical columns, contrary to hypothesis.

It is a remarkable fact that equality is attained in Theorem 2.2 for general m. We illustrate this for the case $m=5$. The following matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & * & * & * & * & * & * \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & * & * & * \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & * \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

is of size 5 by 10. Let the asterisks denote 0's. Then the columns of the matrix may be regarded as all of the 2-subsets of a 5-set arranged in a natural ordering. This matrix has many triangles. But it is easy to verify inductively that every triangle must contain an asterisk. We now replace all of the asterisks by 1's. The new matrix is then without triangles and fulfills all of our additional requirements as well.

The proof of the following theorem uses techniques similar to those employed by van Lint and Ryser [4] in their study of block designs with repeated blocks.

THEOREM 2.3. *Let A be a $(0, 1)$-matrix of order n and let A satisfy the matrix equation*

$$(2.4) \qquad\qquad AEA^T = D.$$

The matrices D and E are real (or complex) diagonal matrices of order n and D is nonsingular. Then A is a permutation matrix.

PROOF. Since the matrix D is nonsingular we may write

$$(2.5) \qquad\qquad AEA^TD^{-1} = EA^TD^{-1}A = I,$$

where I is the identity matrix of order n. Hence it follows that

$$(2.6) \qquad\qquad A^TD^{-1}A = E^{-1}.$$

An inspection of the main diagonal of (2.6) implies

$$(2.7) \qquad A^T(1/d_1, 1/d_2, \cdots, 1/d_n)^T = (1/e_1, 1/e_2, \cdots, 1/e_n)^T.$$

We remark that in (2.7) we have made strong use of the fact that A is a $(0, 1)$-matrix. We now multiply (2.7) by AE and this gives

$$(2.8) \qquad\qquad A(1, 1, \cdots, 1)^T = (1, 1, \cdots, 1)^T.$$

Thus each of the row sums of A equals 1. But A is a nonsingular $(0, 1)$-matrix and hence A is a permutation matrix.

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