A FURTHER GENERALIZATION OF THE KAKUTANI FIXED POINT THEOREM, WITH APPLICATION TO NASH EQUILIBRIUM POINTS

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Introduction

Kakutani's Fixed Point Theorem [3] states that in Euclidean n-space a closed point to (non-void) convex set map of a convex compact set into itself has a fixed point. Kakutani showed that this implied the minimax theorem for finite games. The object of this note is to point out that Kakutani's theorem may be extended to convex linear topological spaces, and implies the minimax theorem for continuous games with continuous payoff as well as the existence of Nash equilibrium points.

§1. The fixed point

Definition: Let X be a Hausdorff linear topological space, and let $S \subset X$. A point to ([redacted]-void) convex set mapping

$$\Phi: S \to X$$

is said to be closed if the graph, $\bigcup_{x \in S} (x, \Phi(x))$, is closed in $X \times X$. In terms of directed systems (or nets) [1] this may be stated as either of the implications[1]:

(a) $x_\delta \to x$, $y_\delta \in \Phi(x_\delta)$, $y_\delta \to y \Rightarrow y \in \Phi(x)$

(b) $x_\delta \to x$, $y_\delta \in \Phi(x_\delta)$, y a cluster point of $\{y_\delta\} \Rightarrow y \in \Phi(x)$

With this definition of a closed mapping we are able to extend a result of Bohnenblust and Karlin [2] to convex Hausdorff linear topological spaces; indeed we use the exact analogue of their proof. (A linear topological space X is said to be convex if there exists a base of convex

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[1] It is obvious that (b) $\Rightarrow$ (a). To see (a) $\Rightarrow$ (b) we must resort to redirected or "sub-directed" systems. Let y be a cluster point of $\{y_\delta\}$, $x_\delta \to x$, $y_\delta \in \Phi(x_\delta)$. Then let $\{U\}$ be the directed set of neighborhoods of y, directed by $U_1 \leq U_2 \equiv U_2 \subset U_1$. Set

$$\Delta' = \{(\delta, U) \mid y_\delta \in U\}, \quad (\delta_1, U_1) \leq (\delta_2, U_2) \equiv \delta_1 \leq \delta_2. \ U_1 \leq U_2.$$

Δ' is a directed set, since if $y_{\delta_1} \in U_1$, $y_{\delta_2} \in U_2 \ \exists \ \delta_0 \geq \delta_1, \delta_2$ and further a $\delta \geq \delta_0 \ni y_\delta \in U_1 \cap U_2$ since y is a cluster point. Now if we set $y_{(\delta, U)} = y_\delta$ for $(\delta, U) \in \Delta'$, clearly $y_{(\delta, U)} \underset{\Delta'}{\to} y$. Setting $x_{(\delta, U)} = x_\delta$ if $(\delta, U) \in \Delta'$ yields $x_{(\delta, U)} \underset{\Delta'}{\to} x$ clearly, and by (a) $y \in \Phi(x)$.

neighborhoods of the zero element (0) which define the topology of X; we shall always take a neighborhood in such a space to be one of the convex neighborhoods, and to be symmetric: $V = -V$ [6]).

<u>Theorem</u>: Given a closed point to convex set mapping Φ: $S \to S$ of a convex compact subset S of a convex Hausdorff linear topological space into itself there exists a fixed point $x \in \Phi(x)$.

(It is seen that this theorem duplicates the Tychonoff extension of Brouwer's theorem for Kakutani's theorem, and includes this in the same fashion that Kakutani's includes Brouwer's.)

Proof: Let V be a given <u>closed</u> neighborhood of 0. Since S is compact there exists a V-dense set $\{x_1, \ldots, x_n\} \subset S$, i.e., $\{x_1, \ldots, x_n\} \ni$

$$S \subset \bigcup_{i=1}^{n} (x_i + V).$$

Let S_V be the convex hull of $\{x_1, \ldots, x_n\}$ and let

$$\Phi_V(x) = (\Phi(x) + V) \cap S_V .$$

Now Φ_V is of course a point to convex set mapping[1] which in particular takes $S_V \to S_V$. Moreover Φ_V is a closed mapping. For, if

[1] $\Phi_V(x)$ is never void since there exists an $x_1 \in \Phi(x) + V$ whence $x_1 \in (\Phi(x) + V) \cap S_V = \Phi_V(x)$.

$x_\delta \to x$, $y_\delta \in \Phi_V(x_\delta)$, and $y_\delta \to y$ then

$$y_\delta \in (\Phi(x_\delta) + V) \cap S_V.$$

Hence there exist $z_\delta \in \Phi(x_\delta)$ and $v_\delta \in V$ such that $y_\delta = z_\delta + v_\delta \in S_V$. Since $z_\delta \in \Phi(x_\delta) \subset S$ and since S is compact, $\{z_\delta\}$ has a cluster point z in S. z belongs to $\Phi(x)$ since $x_\delta \to x$ and Φ is a closed mapping. Since $v_\delta = y_\delta - z_\delta$ and $y_\delta \to y$, $\{v_\delta\}$ must have $v = y - z$ as a cluster point. v belongs to V since $v_\delta \in V$ and V is closed. Thus $y = z + v \in \Phi(x) + V$. On the other hand $y \in S_V$ since $y_\delta \in \Phi_V(x_\delta) \subset S_V$, $y_\delta \to y$ and S_V is closed. Thus we have

$$y \in (\Phi(x) + V) \cap S_V = \Phi_V(x).$$

Hence by the Kakutani fixed point theorem for Euclidean spaces (since the relative topology on S_V is Euclidean), we have $x_V \in S_V$, $x_V \in \Phi_V(x_V)$.

Now the closed neighborhoods of 0 form a natural directed set under

$$V_1 \leq V_2 \equiv V_2 \subset V_1$$

so that $\{x_V\}$ is a directed system. Hence we have a cluster point x of $\{x_V\}$. Using the process indicated in the footnote of page 2, we form

$$\Delta' = \left\{(V, U) \,\middle|\, x_V \in U, \quad U \text{ a neighborhood of } x\right\}$$

which is a directed set, and set

$$x_{(V,U)} = x_V \text{ for } (V, U) \in \Delta',$$

so that

(1) $$x_{(V,U)} \xrightarrow[\Delta']{} x$$

and

(2) for any V_0 there exist U and $V \geq V_0 \ni (V, U) \in \Delta'$.

Then we have $z_{(V,U)}$ such that

$$x_{(V,U)} - z_{(V,U)} \in V \text{ and } z_{(V,U)} \in \Phi(x_{(V,U)})$$

from $x_V \in \Phi(x_V) + V$, so that $z_{(V,U)} \xrightarrow[\Delta']{} x$ by (2). But then $x \in \Phi(x)$ since Φ is closed, and the theorem is proved.[1]

§ 2. Nash Equilibrium Points

Consider an n-person game played over a compact Hausdorff pure strategy spaces $A_1, \ldots, A_n$, in which the payoff to player i is

$$M_i(x_1, \ldots, x_n),$$

a real valued, continuous function over $A_1 \times A_2 \times \ldots \times A_n$. According to Nash [4], a set of mixed strategies $(f_1^o, \ldots, f_n^o)$ (which we interpret here as regular measures such that $f_i(A_i) = 1$) is an equilibrium point of the game if, for each i,

$$\int df_1^o \int df_2^o \ldots \int M_i df_n^o$$

is the maximum of all similar expressions in which f_i^o is replaced by an

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1 When Φ is a point-to-point mapping (Tychonoff's case) the approximation of Φ by a finite-dimensional point-to-convex set mapping Φ_V seems simpler and more direct than the usual approximation by a finite-dimensional point-to-point mapping which involves linear interpolation. The technique is due to Bohnenblust and Karlin.

arbitrary strategy f_i. By means of the preceding theorem we shall show that equilibrium points exist; we treat for convenience the case $n = 2$, the extension to the general case being obvious.

Let the players be I and II, the payoff to I be $M(x,y)$, to II be $N(x,y)$, both continuous on the compact Hausdorff space $A_I \times A_{II}$. Let S_I and S_{II} be the sets of mixed strategies for I and II. These are subsets of the spaces of functionals on the Banach spaces of continuous functions $C(A_I)$, $C(A_{II})$, on A_I and A_{II}, and are ω^* compact. Since the ω^* topology is convex, that is, the space $C(A_I)^*$ (under the ω^* topology) is a convex Hausdorff linear topological space, we may apply Theorem 1 to mappings on S_I or S_{II} or indeed to $S_I \times S_{II} \subset C(A_I)^* \times C(A_{II})^*$.

Set

$$E(f,g) = \int \int M df(x) dg(y)$$

$$F(f,g) = \int \int N df(x) dg(y)$$

which are, respectively, the expectations of I and II if I uses f, II uses g. We seek a pair (f^o, g^o) such that

$$E(f^o, g^o) = \sup_f E(f, g^o)$$

$$F(f^o, g^o) = \sup_g F(f^o, g).$$

For every f, set[1]

$$\mathcal{G}(f) = \hat{g}(F(f,g) = \sup_{g'} F(f,g'))$$

and for every g set

$$\mathcal{F}(g) = \hat{f}(E(f,g) = \sup_{f'} E(f',g)).$$

1 " $\hat{g}$ (...)" means "the set of g such that ..."

Both $\mathcal{F}(g)$ and $\mathcal{G}(f)$ are evidently convex, and are easily seen to be non-void and ω^* closed, hence the mapping

$$(f,g) \to (\mathcal{F}(g), \mathcal{G}(f))$$

is a point to convex set mapping of $S_I \times S_{II} \to S_I \times S_{II}$. Moreover it is easily seen that the mapping is closed since E and F are continuous functions on $S_I \times S_{II}$. Hence by the fixed point theorem there exists a pair (f^o, g^o) such that

$$(f^o, g^o) \in (\mathcal{F}(g^o), \mathcal{G}(f^o))$$

or

$$E(f^o,g^o) = \sup_f E(f,g^o)$$
$$F(f^o,g^o) = \sup_g F(f^o,g).$$

Thus equilibrium points exist. An obvious consequence is the existence of such points in the game in which only a convex ω^* closed subset of S_1 may be played. Another obvious consequence is the minimax theorem for the two person zero sum continuous game, wherein N = - M, for then

$$F(f,g) = -\int\int M df(x) dg(y),$$
$$F(f^o,g^o) = -\inf_g \int\int M df^o(x) dg(y),$$
$$E(f^o,g^o) = \sup_f \int\int M df(x) dg^o(y),$$

so that

$$\int\int M df^o(x) dg(y) \geq \int\int M df^o(x) dg^o(y) \geq \int\int M df(x) dg^o(y)$$

whence the minimax relation follows, and f^o and g^o appear as optimal strategies [5] .

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