NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A FURTHER REMARK CONCERNING THE DISTRIBUTION OF THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE[1]

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1. **Introduction.** In our previous paper[1] it was found convenient to assume that the number m (of the variables of the quadratic form under consideration) is even. (Cf. p. 383, loc. cit.) This means that in the application to the mean square successive difference $n = m + 1$ must be odd. (Cf. p. 389, id.)

In this note we shall show that the distribution for an odd m (i.e. an even n) can be expressed by means of the distribution for an even m—the latter being already known, loc. cit.

Specifically, consider the distribution of $\gamma = \sum_{\mu=1}^{m} a_\mu x_\mu^2$, if the $x_1, \cdots, x_m$ are equidistributed over the surface $\sum_{\mu=1}^{m} x_\mu^2 = 1$. Denote the m-uplet $(a_1, \cdots, a_m)$ by A, then the distribution function of γ depends on A; denote that distribution by $\omega_A(\gamma)$. (Cf. p. 372 id., we write a_μ for the B_μ there.)

Now consider an m-uplet $A = (a_1, \cdots, a_m)$ and a p-uplet $B = (b_1, \cdots, b_p)$ and form the $m + p$-uplet $C = (a_1, \cdots, a_m, b_1, \cdots, b_p)$. Write $C = A + B$. Then we shall show that there exists a simple expression for $\omega_C(\gamma)$ in terms of $\omega_A(\gamma)$ and $\omega_B(\gamma)$.

For the specific application to the mean square successive difference, we can put $n = m + 1$, $A = (\cos(\pi\mu/n)$ for $\mu = 1, \cdots, \frac{1}{2}n - 1, \frac{1}{2}n + 1, \cdots, n - 1)$, $B = (0)$, $C = A + B = (\cos \pi\mu/n$ for $\mu = 1, \cdots, n - 1)$.

2. **The recursion formula.** We proceed as follows. $\omega_A(\gamma)$ can also be used to express the joint statistics of

$$\gamma = \sum_{\mu=1}^{m} a_\mu x_\mu^2 \quad \text{and} \quad \rho = \sum_{\mu=1}^{m} x_\mu^2,$$

or better, the volume of that part of the $x_1, \cdots, x_m$-space which corresponds to any given domain in the γ, ρ-plane. Thus the volume corresponding to a

[1] Cf. the paper by the same author, *Annals of Math. Stat.*, vol. 12(1941), pp. 367–395.

[2] Also Scientific Advisory Committee of the Ballistic Research Laboratory, Aberdeen Proving Ground.

given infinitesimal γ, ρ domain $d\gamma\, d\rho$ will clearly be

$$C_m \sqrt{\rho}^{\,m-1}\, d\sqrt{\rho}\cdot \omega_A\left(\frac{\gamma}{\rho}\right)\frac{d\gamma}{\rho},$$

where C_m is the (m − 1-dimensional) area of the $x_1, \cdots, x_m$-surface $\sum_{\mu=1}^{m} x_\mu^2 = 1$, (the unit sphere). I.e., this volume is

$$\tfrac{1}{2}C_m\,\omega_A\left(\frac{\gamma}{\rho}\right)\cdot \rho^{\frac{1}{2}m-2}\, d\gamma\, d\rho. \tag{1}$$

Similarly for

$$\zeta = \sum_{\nu=1}^{p} b_\nu u_\nu^2 \quad\text{and}\quad \sigma = \sum_{\nu=1}^{p} u_\nu^2$$

the volume corresponding to the infinitesimal ζ, σ domain $d\zeta\, d\sigma$ is

$$\tfrac{1}{2}C_p\,\omega_B\left(\frac{\zeta}{\sigma}\right)\cdot \sigma^{\frac{1}{2}p-2}\, d\zeta\, d\sigma. \tag{2}$$

Finally for $\theta = \gamma + \zeta = \sum_{\mu=1}^{m} a_\mu x_\mu^2 + \sum_{\nu=1}^{p} b_\nu u_\nu^2$ and $\tau = \rho + \sigma = \sum_{\mu=1}^{m} x_\mu^2 + \sum_{\nu=1}^{p} u^2$ the volume cosresponding to the infinitesimal θ, τ domain $d\theta\, d\tau$ is

$$\tfrac{1}{2}C_{m+p}\,\omega_{A+B}\left(\frac{\theta}{\tau}\right)\cdot \tau^{\frac{1}{2}(m+p)-2}\, d\theta\, d\tau. \tag{3}$$

Now $\theta = \gamma + \zeta$, $\tau = \rho + \sigma$ connect (1), (2), (3) as follows:

$$\tfrac{1}{2}C_{m+p}\,\omega_{A+B}\left(\frac{\theta}{\tau}\right)\tau^{\frac{1}{2}(m+p)-2} = \int_0^\tau d\rho\cdot\int d\gamma\cdot \tfrac{1}{2}C_m\,\omega_A\left(\frac{\gamma}{\rho}\right)\rho^{\frac{1}{2}m-2}\cdot \tfrac{1}{2}C_p\,\omega_B\left(\frac{\theta-\gamma}{\tau-\rho}\right)(\tau-\rho)^{\frac{1}{2}p-2}.$$

This gives (either by simply putting $\tau = 1$, or else by replacing θ, γ, ρ by $\tau\theta$, $\tau\gamma$, $\tau\rho$)

$$\omega_{A+B}(\theta) = \frac{C_m C_p}{2C_{m+p}}\int_0^1 d\rho\cdot\int d\gamma\cdot\omega_A\left(\frac{\gamma}{\rho}\right)\omega_B\left(\frac{\theta-\gamma}{1-\rho}\right)\rho^{\frac{1}{2}m-2}(1-\rho)^{\frac{1}{2}p-2}.$$

To determine $\frac{C_m C_p}{2C_{m+p}}$ apply to this $\int d\theta \cdots$. Then

$$1 = \frac{C_m C_p}{2C_{m+p}}\int_0^1 d\rho\cdot \rho^{\frac{1}{2}m-1}(1-\rho)^{\frac{1}{2}p-1}$$

$$= \frac{C_m C_p}{2C_{m+p}} B[\tfrac{1}{2}m, \tfrac{1}{2}p] = \frac{C_m C_p}{2C_{m+p}}\,\frac{\Gamma[\frac{1}{2}m]\Gamma[\frac{1}{2}p]}{\Gamma[\frac{1}{2}(m+p)]}.$$

Accordingly:

$$\omega_{A+B}(\theta) = \frac{\Gamma[\frac{1}{2}(m+p)]}{\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2}p)}\int_0^1 d\rho\cdot\int d\gamma\cdot\omega\left(\frac{\gamma}{\rho}\right)\omega\left(\frac{\theta-\gamma}{1-\rho}\right)\rho^{\frac{1}{2}m-2}(1-\rho)^{\frac{1}{2}p-2}. \tag{I}$$

3. **The special case.** Let us now return to the special case mentioned at the end of 1—the application to the mean square successive difference.

There $p = 1$ and $B = (0)$, so that the "distribution" of ζ is concentrated at the point 0. Hence $\omega_B(\zeta)$ is an "improper" distribution, concentrated in the same way.[3] Using C and A as described at the end of 1, the above formula becomes (now $m = n - 2$, $p = 1$)

$$\text{(II)} \qquad \omega_{A+(0)}(\theta) = \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-2)]\Gamma[\frac{1}{2}]} \int_0^1 d\rho \cdot \omega_A\left(\frac{\theta}{\rho}\right) \rho^{\frac{1}{2}n-3}(1-\rho)^{-\frac{1}{2}}.$$

It would have been equally easy, of course, to establish (II) directly.

Putting $\rho = 1/t$ gives

$$\text{(III)} \qquad \omega_{A+(0)}(\theta) = \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-2)]\Gamma[\frac{1}{2}]} \int_1^\infty dt \cdot \omega_A(\theta t) t^{-\frac{1}{2}(n-3)}(t-1)^{-\frac{1}{2}}.$$

Since $\omega_A(\gamma)$ vanishes for $|\gamma| > \cos(\pi/n)$, we may replace this integral $\int_1^\infty$ by $\int_1^{\cos(\pi/n)/|\theta|}$

Formula (III) can be used for numerical work, and also to extend the formula (3) on p. 391, loc. cit., to even values of n.

CONVEXITY PROPERTIES OF GENERALIZED MEAN VALUE FUNCTIONS

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In an article appearing in the *Annals of Mathematical Statistics*[1] it was pointed out that while the mean value functions appearing below have been studied and used since 1840, there appeared to have been no attempt made to investigate the behavior of their second derivatives.

Consider (1) the unit weight or simple sample form

$$\varphi(t) \equiv \left(\frac{x_1^t + x_2^t + \cdots + x_n^t}{n}\right)^{1/t},$$

in which the x_i are positive numbers and in which t may take any real value; (2) the weighted sample form

$$\omega(t) \equiv \left(\frac{c_1 x_1^t + c_2 x_2^t + \cdots + c_n x_n^t}{c_1 + c_2 + \cdots + c_n}\right)^{1/t},$$

[3] Dirac's famous "delta function." It could be described by a Stieltjes integral.

[1] Nilan Norris, "Convexity properties of generalized mean value functions," *Annals of Math. Stat.*, Vol. 8 (1937), pp. 118-120.