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A Further Result Related to a Conjecture of R. Brück

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Abstract. In this paper, we investigate the uniqueness problem of a meromorphic function sharing one small function with its differential polynomial, and give a result which is related to a conjecture of R. Brück.

1. Introduction

In this paper, meromorphic function means meromorphic in the complex plane. We use the standard notations of Nevanlinna theory, which can be found in [12]. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r, a) = S(r, f)$. We say that two meromorphic functions f and g share a small function a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

Let $f(z)$ be a meromorphic function. It is known that the hyper order of $f(z)$,

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denoted by $\sigma_2(f)$, is defined by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In 1996, R. Brück [1] posed the following conjecture.

Brück Conjecture. Let f be a non-constant entire function such that the hyper order $\sigma_2(f)$ of f is not a positive integer and $\sigma_2(f) < +\infty$. If f and f' share a finite value a CM, then

$$\frac{f'-a}{f-a} = c,$$

where c is a nonzero constant.

In 1998, Gundersen and Yang [3] verified that the Conjecture is true when f is of finite order. In 1999, Yang [10] confirmed that the Conjecture is also true when f' is replaced by $f^{(k)}(k \geq 2)$ and f is of finite order. In recent years, many results have been published concerning the above conjecture, see [2, 5, 7, 8, 14, 15, 16, 17, 18] etc., and Zhang [17] was the first author who considers the case when f is a meromorphic function. We need the following definition.

Definition 1.1. Let l be a non-negative integer or infinite. Denote by $E_l(a,f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a,f) = E_l(a,g)$, we say that f and g share (a,l).

Remark. It is easy to see that f and g share (a,l) implies that f and g share (a,p) for $0 \leq p \leq l$. Also we note that f and g share the value a IM or CM if and only if f and g share $(a,0)$ or (a,∞), respectively. We also use $N_p(r,\frac{1}{f-a})$ to denote the counting function of the zeros of $f-a$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$.

Lahiri [5] improved the results of Zhang [17] by using the above definition and obtained the following Theorem:

Theorem A. *Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (1,2) and*

$$2\overline{N}(r,f) + N_2\left(r,\frac{1}{f^{(k)}}\right) + N_2\left(r,\frac{1}{f}\right) < (\lambda + o(1))T(r,f^{(k)}),$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-a}{f-a} = c$ for $c \in \mathbf{C}\setminus\{0\}$.

Let p be a positive integer and $a \in \mathbf{C}\bigcup\{\infty\}$. We use $N_{p)}(r,\frac{1}{f-a})$ to denote the counting function of the zeros of $f-a$, whose multiplicities are not greater than p, $N_{(p+1}(r, \frac{1}{f-a})$ to denote the counting function of the zeros of $f-a$ whose multiplicities are not less than $p+1$, and we use $\overline{N}_{p)}(r, \frac{1}{f-a})$ and $\overline{N}_{(p+1}(r, \frac{1}{f-a})$ to denote their corresponding reduced counting functions (ignoring multiplicities) respectively. Define

$$\delta_p(a, f) = 1 - \limsup_{r\to+\infty} \frac{N_p(r, \frac{1}{f-a})}{T(r, f)}.$$

It follows that $\delta_p(a, f) \geq \delta(a, f)$.

Let $L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f$. Zhang and Yang [16] obtained the following result which improves the results of [5, 8, 14, 18].

Theorem B. *Let f be a non-constant meromorphic function, $k \geq 1$ and $l \geq 0$ be integers. Let $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $L(f) - a$ share $(0, l)$. Then $f \equiv L(f)$ if one of the following assumptions holds,*

(1) $l \geq 2$ *and*

$$3\theta(\infty, f) + \delta_{2+k}(0, f) + \delta_2(0, f) + \delta(a, f) > 4,$$

(2) $l = 1$ *and*

$$\frac{7+k}{2}\theta(\infty, f) + \frac{1}{2}\delta_{1+k}(0, f) + \delta_2(0, f) + \delta_{2+k}(0, f) + \delta(a, f) > \frac{k}{2} + 5,$$

(3) $l = 0$ *and*

$$(2k+6)\theta(\infty, f) + \delta_2(0, f) + 2\delta_{1+k}(0, f) + \delta_{2+k}(0, f) + \theta(0, f) + \delta(a, f) > 2k + 10.$$

Definition 1.2. Let $p_0, p_1, \ldots p_k$ be non-negative integers. We call

$$M[f] = f^{p_0}(f')^{p_1} \cdots (f^{(k)})^{p_k}$$

a differential monomial in f with degree $d_M = p_0 + p_1 + \cdots + p_k$ and weight $\Gamma_M = p_0 + 2p_1 + \cdots + (k+1)p_k$, and

$$H[f] = \sum_{j=1}^{n} a_j M_j[f],$$

where a_j are small functions of f, is called a differential polynomial in f of degree $d = \max\{d_{M_j}, 1 \leq j \leq n\}$ and weight $\Gamma = \max\{\Gamma_{M_j}, 1 \leq j \leq n\}$, furthermore if $\deg(M_j) = d (j = 1, 2, \cdots n)$, then $H[f]$ is a homogeneous differential polynomial in f of degree d.

In this paper, we improve the above Theorems and obtain the following result.

Theorem 1.3. *Let f be a non-constant meromorphic function and $H[f]$ be a non-constant homogeneous differential polynomial of degree d and weight Γ satisfying*

$\Gamma \geq (k+2)d-2$. *Let* $a(z)$ *be a small meromorphic function of* f *such that* $a(z) \not\equiv 0, \infty$. *Suppose that* $f-a$ *and* $H[f]-a$ *share* $(0,l)$. *Then* $\frac{H[f]-a}{f-a} = C$ *for some non-zero constant* C *if one of the following assumptions holds,*

(i) $l \geq 2$ *and*

$$3\theta(\infty,f) + d\delta_{2+\Gamma-d}(0,f^d) + \delta_2(0,f) + \delta(a,f) > 4, \tag{1.1}$$

(ii) $l = 1$ *and*

$$\frac{7+\Gamma-d}{2}\theta(\infty,f) + \frac{d}{2}\delta_{1+\Gamma-d}(0,f^d) + d\delta_{2+\Gamma-d}(0,f^d) + \delta_2(0,f) + \delta(a,f) > \frac{\Gamma+9}{2}, \tag{1.2}$$

(iii) $l = 0$ *and*

$$\begin{aligned}[2(\Gamma-d)+6]\theta(\infty,f) + \delta_2(0,f) + d\delta_{2+\Gamma-d}(0,f^d) + 2d\delta_{1+\Gamma-d}(0,f^d) \\ + \theta(0,f) + \delta(a,f) > 2\Gamma+8.\end{aligned} \tag{1.3}$$

Especially, when $l = 0$, *i.e.* f *and* H *share a IM, if (1.3) holds, then* $f \equiv H[f]$.

2. Some Lemmas

Lemma 2.1.([11]) *Let* f *be a nonconstant meromorphic function,* k *be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f), \tag{2.1}$$

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f). \tag{2.2}$$

Suppose that F and G are two non-constant meromorphic functions such that F and G share the value 1 IM. Let z_0 be a 1-point of F of order p, a 1-point of G of order q. We denote by $N_L(r, \frac{1}{F-1})$ the counting function of those 1-points of F where $p > q$, by $N_E^{1)}(r, \frac{1}{F-1})$ the counting function of those 1-points of F where $p = q = 1$, by $N_E^{(2}$ the counting function of those 1 -points of F where $p = q \geq 2$; each point in these counting functions is counted only one time. Similarly, we can define $N_L(r, \frac{1}{G-1})$, $N_E^{1)}(r, \frac{1}{G-1})$ and $N_E^{(2}(r, \frac{1}{G-1})$.

Lemma 2.2.([13]) *Let* F *and* G *are two nonconstant meromorphic functions,*

$$\Delta = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.3}$$

If F and G share 1 IM and $\Delta \not\equiv 0$, then

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) \le N(r,\Delta) + S(r,F) + S(r,G). \tag{2.4}$$

Lemma 2.3. *Let $H[f]$ be a non-constant differential polynomial. Let z_0 be a pole of f of order p and neither a zero nor a pole of coefficients of $H[f]$. Then z_0 is a pole of $H[f]$ with order at most $pd + (\Gamma - d)$.*

Proof. Let

$$H[f] = \sum_{j=1}^{n} a_j H_j[f], \qquad H_j[f] = f^{p_0}(f')^{p_1}\cdots(f^{(k)})^{p_k},$$

$$d_{M_j} = p_0 + p_1 + \cdots + p_k, \qquad \Gamma_{M_j} = p_0 + 2p_1 + \cdots + (k+1)p_k.$$

Let z_0 be a pole of f of order p, then z_0 be a pole of $H_j[f]$ of order $pd_{M_j} + (\Gamma_{M_j} - d_{M_j})$. Because $d = \max\{d_{M_j}, 1 \le j \le n\}$, $\Gamma = \max\{\Gamma_{M_j}, 1 \le j \le n\}$ and z_0 neither be a zero nor be a pole of a_j, then z_0 is a pole of H[f] with order at most $pd + (\Gamma - d)$. □

Lemma 2.4. *Let f be a transcendental meromorphic function, $H[f]$ is a homogeneous differential polynomial in f of degree d and weight Γ. If $H[f] \not\equiv 0$, we have*

$$N\left(r, \frac{1}{H}\right) \le T(r,H) - dT(r,f) + dN\left(r, \frac{1}{f}\right) + S(r,f). \tag{2.5}$$

$$N\left(r, \frac{1}{H}\right) \le (\Gamma - d)\overline{N}(r,f) + dN\left(r, \frac{1}{f}\right) + S(r,f). \tag{2.6}$$

Proof. By the first fundamental theorem and the lemma of logarithmic derivatives, we have

$$\begin{aligned} N\left(r, \frac{1}{H}\right) &= T(r,H) - m\left(r, \frac{1}{H}\right) + O(1) \\ &\le T(r,H) - \left(m\left(r, \frac{1}{f^d}\right) - m\left(r, \frac{H}{f^d}\right)\right) + O(1) \\ &= T(r,H) - \left(T\left(r, \frac{1}{f^d}\right) - N\left(r, \frac{1}{f^d}\right)\right) + S(r,f) \\ &= T(r,H) - dT(r,f) + dN\left(r, \frac{1}{f}\right) + S(r,f). \qquad (2.7) \end{aligned}$$

This proves (2.5). From Lemma 3, we have

$$\begin{aligned} T(r,H) &= m(r,H)+N(r,H) \\ &\leq m\left(r,\frac{H}{f^d}\right)+m(r,f^d)+N(r,H) \\ &\leq dm(r,f)+dN(r,f)+(\Gamma-d)\overline{N}(r,f)+S(r,f) \\ &= dT(r,f)+(\Gamma-d)\overline{N}(r,f)+S(r,f). \end{aligned}$$

Combining with (2.7), we obtain (2.6). □

Lemma 2.5. *Let f be a non-constant meromorphic function, $H[f]$ is a homogeneous differential polynomial in f of degree d and weight Γ, and let p be a positive integer. If $H[f]\not\equiv 0$ and $\Gamma \geq (k+2)d-(p+1)$, we have*

$$N_p\left(r,\frac{1}{H}\right) \leq T(r,H)-dT(r,f)+N_{p+\Gamma-d}\left(r,\frac{1}{f^d}\right)+S(r,f), \tag{2.8}$$

$$N_p\left(r,\frac{1}{H}\right) \leq (\Gamma-d)\overline{N}(r,f)+N_{p+\Gamma-d}\left(r,\frac{1}{f^d}\right)+S(r,f). \tag{2.9}$$

Proof. From (2.6), we have

$$\begin{aligned} N_p\left(r,\frac{1}{H}\right)+\sum_{j=p+1}^{\infty}\overline{N}_{(j}\left(r,\frac{1}{H}\right) &\leq (\Gamma-d)\overline{N}(r,f)+N_{p+\Gamma-d}\left(r,\frac{1}{f^d}\right) \\ &\quad +\sum_{j=p+\Gamma-d+1}^{\infty}\overline{N}_{(j}\left(r,\frac{1}{f^d}\right)+S(r,f). \end{aligned}$$

Since $\Gamma \geq (k+2)d-(p+1)$, then we have

$$\begin{aligned} N_p\left(r,\frac{1}{H}\right) &\leq (\Gamma-d)\overline{N}(r,f)+N_{p+\Gamma-d}\left(r,\frac{1}{f^d}\right)+\sum_{j=p+\Gamma-d+1}^{\infty}\overline{N}_{(j}\left(r,\frac{1}{f^d}\right) \\ &\quad -\sum_{j=p+1}^{\infty}\overline{N}_{(j}\left(r,\frac{1}{H}\right)+S(r,f) \\ &\leq (\Gamma-d)\overline{N}(r,f)+N_{p+\Gamma-d}\left(r,\frac{1}{f^d}\right)+S(r,f). \end{aligned}$$

Thus (2.9) holds. By the same arguments as above, we obtain (2.8) from (2.5).□

3. Proof of Theorem 1.3

Let $F = \frac{H[f]}{a}, G = \frac{f}{a}$. From the conditions of Theorem 1.3, we know that F and G share $(1, l)$ except the zeros and poles of a(z). From the proof of Lemma 2.4, we have

$$T(r, F) = O(T(r, f)) + S(r, f), \quad T(r, G) = T(r, f) + S(r, f). \tag{3.1}$$

It is obvious that f is a transcendental meromorphic function. Let Δ be defined by (2.3). We distinguish two cases.

Case 1. $\Delta \equiv 0$. Integrating (2.3), yields

$$\frac{1}{G-1} = \frac{C}{F-1} + D, \tag{3.2}$$

where C and D are constants and $C \neq 0$. If there exists a pole z_0 of f with multiplicity p which is not zero or pole of a, then z_0 is a pole of F with multiplicity $pd + (\Gamma - d)$, a pole of G with multiplicity p. This contradicts (3.2) as H contains at least one derivative. Therefore, we have

$$\overline{N}(r, F) = \overline{N}(r, G) = \overline{N}(r, f) = S(r, f). \tag{3.3}$$

(3.2) also shows that F and G share the value 1 CM.

Next, we will prove $D = 0$.

Suppose $D \neq 0$, then we have

$$\frac{1}{G-1} = \frac{D(F - 1 + \frac{C}{D})}{F-1}. \tag{3.4}$$

So, we have

$$\overline{N}\left(r, \frac{1}{D(F-1+\frac{C}{D})}\right) = \overline{N}\left(r, \frac{G-1}{F-1}\right) = S(r, f). \tag{3.5}$$

Subcase 1.1. If $\frac{C}{D} \neq 1$, then by using (3.3), (3.5) and the second fundamental theorem, we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1+\frac{C}{D}}\right) + S(r, F) \\ &\leq \overline{N}\left(r, \frac{1}{F}\right) + S(r, F) \leq (1 + o(1))T(r, F). \end{aligned}$$

This gives that

$$T(r,F)=\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)=N_1\left(r,\frac{1}{F}\right)+S(r,F).$$

So we have

$$T(r,H)=N\left(r,\frac{1}{H}\right)+S(r,f)=N_1\left(r,\frac{1}{H}\right)+S(r,f).$$

Let $p=1$, then from assumption we have

$$\Gamma\geq(k+2)d-2=(k+2)d-(p+1).$$

Thus from (2.8) in Lemma 2.5, we get

$$T(r,H)=N_1\left(r,\frac{1}{H}\right)+S(r,f)\leq T(r,H)-dT(r,f)+N_{1+\Gamma-d}\left(r,\frac{1}{f^d}\right)+S(r,f).$$

So we have

$$dT(r,f)\leq N_{1+\Gamma-d}\left(r,\frac{1}{f^d}\right)+S(r,f).$$

This gives that

$$dT(r,f)=N_{1+\Gamma-d}\left(r,\frac{1}{f^d}\right)+S(r,f).$$

So we have

$$\delta_{2+\Gamma-d}(0,f^d)=\delta_{1+\Gamma-d}(0,f^d)=0.$$

Since (3.3), we get

$$\theta(\infty,f)=1. \tag{3.6}$$

Subcase 1.1.1. $l\geq 2$.

From $\delta_2(0,f)+\delta(a,f)>1$ and the definition of deficiency, we have

$$T(r,f)>N_2\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a}\right). \tag{3.7}$$

Using the second fundamental theorem of Nevanlinna and (3.3), we have

$$\begin{aligned}T(r,f) &\leq \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}(r,f)+S(r,f)\\ &= \overline{N}\left(r,1f\right)+\overline{N}\left(r,\frac{1}{f-a}\right)+S(r,f).\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8), we have

$$N_2\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a}\right)<T(r,f)\leq\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-a}\right)+S(r,f).$$

So we have

$$N_2\left(r,\frac{1}{f}\right)=S(r,f),N\left(r,\frac{1}{f-a}\right)=S(r,f).$$

This gives that

$$\overline{N}\left(r,\frac{1}{f}\right)=S(r,f),\overline{N}\left(r,\frac{1}{f-a}\right)=S(r,f).$$

From (3.8), we get a contradiction.

Subcase 1.1.2. $l=1$.

When $d\geq 2$, by using (1.2) and the definition of deficiency, we get a contradiction. When $d=1$, using the similar method in subcase 1.1.1, we get a contradiction.

Subcase 1.1.3. $l=0$.

By using (1.3) and the definition of deficiency , we get a contradiction.

Subcase 1.2. If $\frac{C}{D}=1$, then from (3.4), we have

$$\frac{1}{G-1}\equiv C\frac{F}{F-1}.$$

This gives us that

$$\left(G-1-\frac{1}{C}\right)F\equiv-\frac{1}{C}.$$

Using that $F=\frac{H}{a}$ and $G=\frac{f}{a}$, we get

$$f-a\left(1+\frac{1}{C}\right)\equiv-\frac{a^2}{C}\cdot\frac{1}{H}. \tag{3.9}$$

Using (3.3) (3.9), Lemma 2.3 and the first fundamental theorem, we get

$$\begin{aligned}(d+1)T(r,f) &= T\left(r,\frac{1}{f^d(f-(1+\frac{1}{C})a)}\right)+O(1)\\ &= T\left(r,-\frac{CH}{f^da^2}\right)+O(1)\\ &= N\left(r,\frac{H}{f^d}\right)+S(r,f)\\ &\leq dN\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq (d+o(1))T(r,f),\end{aligned}$$

which is a contradiction, hence D=0. This gives from (3.2) that

$$\frac{F-1}{G-1} \equiv C.$$

So we get $\frac{H[f]-a}{f-a} = C (C \neq 0)$.
Next, we will prove $C = 1$ when $l = 0$.
Suppose $C \neq 1$, then we have

$$G \equiv \frac{1}{C}(F - 1 + C)$$

and

$$N\left(r, \frac{1}{G}\right) = N\left(r, \frac{1}{F-1+C}\right). \tag{3.10}$$

By the second fundamental theorem and (3.3) (3.10), we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1+C}\right) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f) \\ &= N_1\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right). \end{aligned}$$

By Lemma 2.5 for $p = 1$, we have

$$dT(r,f) \leq N_{1+\Gamma-d}\left(r, \frac{1}{f^d}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f).$$

From the above formula and the definition of deficiency, we have

$$d\delta_{1+\Gamma-d}(0, f^d) + \theta(0,f) \leq 1. \tag{3.11}$$

So we have

$$d\delta_{2+\Gamma-d}(0, f^d) + \delta_2(0,f) \leq 1, \ d\delta_{1+\Gamma-d}(0, f^d) \leq 1. \tag{3.12}$$

Combining (3.11) (3.12) (3.6) with the assumptions of Theorem 1.3, we get a contradiction.
So $C = 1$ and $G \equiv F$, i.e. $f \equiv H[f]$.
This is just the conclusion of this theorem.

Case 2. $\Delta \not\equiv 0$.
By a similar method that used in the proof of Theorem B[16], we get

$$\begin{aligned} T(r,F)+T(r,G) &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F-1}\right) \\ &+ \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G-1}\right) \\ &- N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right) &+ \overline{N}\left(r,\frac{1}{G-1}\right) \leq \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right) \\ &+ \overline{N}(r,G)+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ &+ 2N_E^{(2}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{3.14}$$

Subcase 2.1. $l \geq 2$. It is easy to see that

$$\begin{aligned} 3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right) &+ 2N_E^{(2}\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ &\leq N\left(r,\frac{1}{G-1}\right)+S(r,f). \end{aligned} \tag{3.15}$$

From (3.13) (3.14) and (3.15), we have

$$\begin{aligned} T(r,F)+T(r,G) &\leq 3\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right) \\ &+ N_2\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{G-1}\right)+S(r,f). \end{aligned}$$

Noting that

$$N_2\left(r,\frac{1}{F}\right)=N_2\left(r,\frac{a}{H}\right)\leq N_2\left(r,\frac{1}{H}\right)+S(r,f).$$

Let $p=2$, then from assumption we have

$$\Gamma \geq (k+2)d-2 > (k+2)d-(p+1).$$

Thus, from (2.8) in Lemma 2.5 we obtain that

$$\begin{aligned} T(r,H)+T(r,f) &\leq 3\overline{N}(r,f)+T(r,H)-dT(r,f)+N_{2+\Gamma-d}\left(r,\frac{1}{f^d}\right) \\ &+ N_2\left(r,\frac{1}{f}\right)+T(r,f)-m\left(r,\frac{1}{f-a}\right)+S(r,f). \end{aligned}$$

So we have

$$dT(r,f) \le 3\overline{N}(r,f) + N_{2+\Gamma-d}\left(r,\frac{1}{f^d}\right) + N_2\left(r,\frac{1}{f}\right) - m\left(r,\frac{1}{f-a}\right) + S(r,f).$$

This gives that

$$3\theta(\infty,f) + d\delta_{2+\Gamma-d}(0,f^d) + \delta_2(0,f) + \delta(a,f) \le 4.$$

Which contradicts the assumption (1.1) of Theorem 1.3.

Subcase 2.2. $l = 1$. Noting that

$$\begin{aligned} 2N_L\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{G-1}\right) &+ 2N_E^{(2}\left(r,\frac{1}{G-1}\right) + N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ &\le N\left(r,\frac{1}{G-1}\right) + S(r,f) \end{aligned}$$

and

$$\begin{aligned} N_L\left(r,\frac{1}{F-1}\right) &\le \frac{1}{2}N\left(r,\frac{F}{F'}\right) \le \frac{1}{2}T\left(r,\frac{F}{F'}\right) = \frac{1}{2}T\left(r,\frac{F'}{F}\right) + O(1) \\ &\le \frac{1}{2}N\left(r,\frac{F'}{F}\right) + S(r,f) \\ &\le \frac{1}{2}\left(\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F)\right) + S(r,f) \\ &\le \frac{1}{2}\left(\overline{N}\left(r,\frac{1}{H}\right) + \overline{N}(r,f)\right) + S(r,f) \\ &\le \frac{1}{2}\left[(\Gamma-d+1)\overline{N}(r,f) + N_{1+\Gamma-d}\left(r,\frac{1}{f^d}\right)\right] + S(r,f). \end{aligned}$$

Using the same method as Subcase 2.1, we get

$$\begin{aligned} dT(r,f) \le\ & \frac{\Gamma-d+7}{2}\overline{N}(r,f) + N_{2+\Gamma-d}\left(r,\frac{1}{f^d}\right) + \frac{1}{2}N_{1+\Gamma-d}\left(r,\frac{1}{f^d}\right) \\ & + N_2\left(r,\frac{1}{f}\right) - m\left(r,\frac{1}{f-a}\right) + S(r,f). \end{aligned}$$

Which contradicts with (1.2) of Theorem 1.3.

Subcase 2.3. $l = 0$. Noting that

$$\begin{aligned} N_L\left(r,\frac{1}{F-1}\right) + 2N_L\left(r,\frac{1}{G-1}\right) &+ 2N_E^{(2}\left(r,\frac{1}{G-1}\right) + N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ &\le N\left(r,\frac{1}{G-1}\right) + S(r,f). \end{aligned} \tag{3.16}$$

From Lemma 2.5, we have

$$\begin{aligned} N_L\left(r,\frac{1}{F-1}\right) &\leq N\left(r,\frac{F}{F'}\right)\\ &\leq N\left(r,\frac{F'}{F}\right)+S(r,f)\\ &\leq \left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)\right)+S(r,f)\\ &\leq \left(\overline{N}\left(r,\frac{1}{H}\right)+\overline{N}(r,f)\right)+S(r,f)\\ &\leq N_{1+\Gamma-d}\left(r,\frac{1}{f^d}\right)+(\Gamma-d+1)\overline{N}(r,f)+S(r,f). \end{aligned}$$

So we have

$$\begin{aligned} 2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right) &\leq 2N_{1+\Gamma-d}\left(r,\frac{1}{f^d}\right)+2(\Gamma-d+1)\overline{N}(r,f)\\ &+ \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f). \end{aligned} \tag{3.17}$$

Combining (3.13) (3.14) (3.16) with (3.17), we have

$$\begin{aligned} T(r,H)+T(r,f) &\leq N_2\left(r,\frac{1}{H}\right)+N_2\left(r,\frac{1}{f}\right)+3\overline{N}(r,f)+N\left(r,\frac{1}{f-a}\right)\\ &+2N_{1+\Gamma-d}\left(r,\frac{1}{f^d}\right)+2(\Gamma-d+1)\overline{N}(r,f)\\ &+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f). \end{aligned} \tag{3.18}$$

From (2.8), we have

$$N_2\left(r,\frac{1}{H}\right)\leq T(r,H)-dT(r,f)+N_{2+\Gamma-d}\left(r,\frac{1}{f^d}\right)+S(r,f).$$

Substituting this into (3.18), we have

$$\begin{aligned} dT(r,f) &\leq N_2\left(r,\frac{1}{f}\right)+2(\Gamma-d+3)\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\\ &+ 2N_{1+\Gamma-d}\left(r,\frac{1}{f^d}\right)+N_{2+\Gamma-d}\left(r,\frac{1}{f^d}\right)-m\left(r,\frac{1}{f-a}\right)+S(r,f). \end{aligned}$$

So we have

$$\begin{aligned} &\delta_2(0,f)+\theta(0,f)+2(\Gamma-d+3)\theta(\infty,f)+d\delta_{2+\Gamma-d}(0,f^d)\\ &+2d\delta_{1+\Gamma-d}(0,f^d)+\delta(a,f)\leq 2\Gamma+8. \end{aligned}$$

Which contradicts the assumption of Theorem 1.3.
Now the proof has been completed. □

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