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A fuzzy multi-objective linear programming with interval-typed triangular fuzzy numbers

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Abstract: A multi-objective linear programming problem (ITF-MOLP) is presented in this paper, in which coefficients of both the objective functions and constraints are interval-typed triangular fuzzy numbers. An algorithm of the ITF-MOLP is provided by introducing the cut set of interval-typed triangular fuzzy numbers and the dominance possibility criterion. In particular, for a given level, the ITF-MOLP is converted to the maximization of the sum of membership degrees of each objective in ITF-MOLP, whose membership degrees are established based on the deviation from optimal solutions of individual objectives, and the constraints are transformed to normal inequalities by utilizing the dominance possibility criterion when compared with two interval-typed triangular fuzzy numbers. Then the equivalent linear programming model is obtained which could be solved by Matlab toolbox. Finally several examples are provided to illuminate the proposed method by comparing with the existing methods and sensitive analysis demonstrates the stability of the optimal solution.

Keywords: Multi-objective linear programming, triangular fuzzy numbers, fuzzy optimization, dominance possibility criterion

MSC: 90C29,90C70

1 Introduction

Optimization problems include objectives and constraints. Deterministic optimization problems have been well studied, but they are limited and inadequate to exactly express the real problem [1]. In our daily life, many complicated problems involve uncertain data in economics, social sciences, medical diagnosis, natural sciences and many other fields. Accordingly, fuzzy optimal control and multi-objective linear programming with fuzzy parameters have been playing an increasing role in uncertain systems [2-6].

Recently, several researchers have considered the issues of expressing the coefficients of objectives and constraints in multi-objective linear programming. There are several well-known theories to describe uncertainty such as fuzzy set theory, possibility theory, probability theory and other mathematical tools. For instance, fuzzy linear programming with fuzzy coefficients are considered by many authors [7-9]. Wang and Wang [10] have proposed a fuzzy multi-objective linear programming with fuzzy-numbered cost coefficients and transformed the problem into a multi-objective problem with parametrically interval-valued cost coefficients by utilizing membership functions. Different algorithms [11, 12] are developed to solve multi-objective linear programming problems based on interval-valued cost coefficients. Combining fuzziness and randomness in an optimization problem, many models are considered such as fuzzy random chance-

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constrained programming model [13, 14] and multi-objective linear programming model with fuzzy random coefficients [15]. Liu and Liu [14] have established an expected value model and developed a hybrid intelligent algorithm of the fuzzy random multi-objective programming problem. Li et al. [15] have presented a genetic algorithm using the compromise approach for solving a fuzzy random multi-objective programming problem. However, the probability distributions of parameters may be unknown [16]. Considering uncertainties as interval analysis, multi-objective robust optimization approaches are developed which wouldn't involve any probability distribution [17]. In recent years, regarding coefficients as interval numbers [18, 19], many multi-objective programming with interval parameters [20-23] have been discussed in detail. The approach proposed by Hajiagha et al. [21] proves to be flexible especially in ill-defined information circumstance. Moreover, the approach has less computational complexity than the literature [23] when the number of objective functions increases.

As pointed out by Chiang [24], Transportation problems prove to be better to express the parameters as interval-valued fuzzy numbers instead of normal fuzzy numbers. To find solution of a linear multi-objective transportation problem with parameters represented as interval-valued fuzzy numbers, Gupta and Kumar [25] propose a linear ranking function method via signed distance among interval-valued fuzzy numbers and obtain the non-dominated solution of the transformed crisp linear programming model. As a general form of fuzzy number linear programming, Farhadinia [26] introduces a formulation of interval-valued trapezoidal fuzzy number linear programming problems and presents its primal simplex algorithm in fuzzy sense via signed distance ranking function.

The description of incomplete and vague information has received more and more attentions recently. Interval-valued trapezoidal fuzzy numbers [27-29] are introduced to express the attributes and applied to solve the actual multi-attribute group decision making problems. As the complexity of information in the real world is increasing, interval-valued fuzzy numbers, which have many advantages in decision making and multi-objective programming fields, still have its limits to process the vague information. In this paper, we offer the notion of interval-typed triangular fuzzy numbers, which could be regarded as an extension of interval-valued triangular fuzzy numbers and interval numbers respectively. Moreover, interval-typed triangular fuzzy numbers may be not interval-valued triangular fuzzy numbers. Like traditional fuzzy sets, Interval-typed triangular fuzzy numbers could also be exploited to extend many practical applications in reality [30]. Considering the fact that both interval-valued fuzzy numbers and interval numbers are more general and better to express incomplete and vague information [27-29],we try to propose an effective algorithm of multi-objective linear programming in which the coefficients of objective functions and constraints are stated as interval-typed triangular fuzzy numbers.

The rest of this paper is organized as follows. In section 2, we review some basic definitions related to interval-typed triangular fuzzy numbers and several useful operators. In section 3, we give multi-objective linear programming (ITF-MOLP) on the basis of interval-typed triangular fuzzy numbers and discuss some interesting properties. Particularly, the decomposition theorem about an interval-typed triangular fuzzy number is presented. In section 4, we establish an algorithm of the ITF-MOLP. By introducing the cut set of interval-typed triangular fuzzy numbers, the objective function is transformed into the maximization of the sum of membership degrees of each objective, where the membership degree of each objective is given based on the deviation from optimal solutions of individual objective. Utilizing the dominance possibility criterion, the comparison between two interval-typed triangular fuzzy numbers in constraints is transformed to normal inequalities. Accordingly, the ITF-MOLP is finally converted to a linear programming that could be solved by existing methods. Section 5 gives some illustrated examples. Moreover, the comparisons with existing approaches are made and the sensitive analysis concerning the optimal solution of linear programming is also investigated. Section 6 concludes the results and points out further research.

2 Preliminaries

Suppose that $\mathcal{R}$ is the set of all real numbers and $\mathcal{R}^+$ is the set of all positive real numbers. Some basic concepts used in this paper are given in this section.

Definition 2.1. *[31] A triangular fuzzy number A is defined as a triplet (a_1, a_2, a_3). The membership function $\mu_A(x)$ is defined as*

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & if\ a_1 \le x < a_2, \\ \frac{x-a_3}{a_2-a_3}, & if\ a_2 \le x < a_3, \\ 0, & otherwise \end{cases} \tag{1}$$

where $a_1, a_2, a_3 \in \mathcal{R}$ and $a_1 \le a_2 \le a_3$; its membership function $\mu_A(x)$ is fuzzy convex, showing that the membership degree of element x belonging to A; a_2 represents the value for which $\mu_A(a_2) = 1$, and a_1 and a_3 are the most extreme values on the left and on the right of the fuzzy number A respectively with membership $\mu_A(a_1) = \mu_A(a_3) = 0$. If $a_1 = a_2 = a_3$, then A is reduced to a real number.

Definition 2.2. *[30] Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. Then the operations with these fuzzy numbers are defined as follows:*

(i) $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$;
(ii) $A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$;
(iii) $A \times B = (a_1 \cdot b_1, a_2 \cdot b_2, a_3 \cdot b_3)$ for $a_i > 0, b_i > 0$ with $i = 1, 2, 3$;
(iv) $A/B = (a_1/b_3, a_2/b_2, a_3/b_1)$ for $a_i > 0, b_i > 0$ with $i = 1, 2, 3$;
(v) $\lambda A = (\lambda a_1, \lambda a_2, \lambda a_3)$ for any positive scalar $\lambda \in \mathcal{R}$.

Definition 2.3. *[30] Let I denote a finite index set, $\{A_i = (a_{i1}, a_{i2}, a_{i3}) | i \in I\}$ be a family of triangular fuzzy numbers, and $\bigvee$ and $\bigwedge$ represent the supremum and infimum operator on the real set $\mathcal{R}$ respectively. Then*

$$\bigvee_{i \in I} A_i = (\bigvee_{i \in I} a_{i1}, \bigvee_{i \in I} a_{i2}, \bigvee_{i \in I} a_{i3}), \tag{2}$$

$$\bigwedge_{i \in I} A_i = (\bigwedge_{i \in I} a_{i1}, \bigwedge_{i \in I} a_{i2}, \bigwedge_{i \in I} a_{i3}). \tag{3}$$

The level (h^L, h^U)-interval-valued trapezoidal fuzzy number is given as follows.

Definition 2.4. *[32] A level (h^L, h^U)–interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$, denoted by*

$$\tilde{\tilde{A}} = [\tilde{A}^L, \tilde{A}^U] = \langle (a_1^L, a_2^L, a_3^L, a_4^L; h^L), (a_1^U, a_2^U, a_3^U, a_4^U; h^U) \rangle \tag{4}$$

is an interval-valued fuzzy set on $\mathcal{R}$ with the lower trapezoidal fuzzy number $\tilde{A}^L$ expressing by

$$\tilde{A}^L(x) = \begin{cases} h^L \cdot \frac{x-a_1^L}{a_2^L-a_1^L}, & a_1^L \le x \le a_2^L, \\ h^L, & a_2^L \le x \le a_3^L, \\ h^L \cdot \frac{a_4^L-x}{a_4^L-a_3^L}, & a_3^L \le x \le a_4^L, \\ 0, & otherwise \end{cases} \tag{5}$$

and the upper trapezoidal fuzzy number $\tilde{A}^U$ expressing by

$$\tilde{A}^U(x) = \begin{cases} h^U \cdot \frac{x-a_1^U}{a_2^U-a_1^U}, & a_1^U \le x \le a_2^U, \\ h^U, & a_2^U \le x \le a_3^U, \\ h^U \cdot \frac{a_4^U-x}{a_4^U-a_3^U}, & a_3^U \le x \le a_4^U, \\ 0, & otherwise \end{cases} \tag{6}$$

where $a_1^L \le a_2^L \le a_3^L \le a_4^L$, $a_1^U \le a_2^U \le a_3^U \le a_4^U$, $0 < h^L \le h^U \le 1$, $a_1^U \le a_1^L$ *and* $a_4^L \le a_4^U$. *Moreover,* $\tilde{A}^L \subseteq \tilde{A}^U$.

Moreover, if $a_2^L = a_3^L$, $a_2^U = a_3^U$ *and* $h^L = h^U = 1$, *then* $\tilde{A}$ *is a normal interval-valued triangular fuzzy number.*

As for the definitions of interval-valued fuzzy numbers, one could also consult the references [27-29].In this paper, we introduce the definition of interval-typed triangular fuzzy numbers as follows.

Definition 2.5. *[30] An interval-typed triangular fuzzy number is a fuzzy interval* $[A^L, A^U]$, *where both the lower bound* $A^L = (a_1^L, a_2^L, a_3^L)$ *and the upper bound* $A^U = (a_1^U, a_2^U, a_3^U)$ *are triangular fuzzy numbers and* $a_2^L \le a_2^U$.

Moreover, if $a_1^U \le a_1^L$, $a_2^L = a_2^U$ *and* $a_3^L \le a_3^U$, *then* $[A^L, A^U]$ *is called an interval-valued triangular fuzzy number. If* $A^L = A^U$, *then* $[A^L, A^U]$ *is reduced to a triangular fuzzy number. If* $a_1^L = a_2^L = a_3^L$ *and* $a_1^U = a_2^U = a_3^U$, *then* $[A^L, A^U]$ *is reduced to an interval number. Moreover, if* $a_1^L = a_2^L = a_3^L = a_1^U = a_2^U = a_3^U$, *then* $[A^L, A^U]$ *is reduced to a numerical value.*

Remark. An interval-valued triangular fuzzy number is a special interval-typed triangular fuzzy number, however, an interval-typed triangular fuzzy number may be not an interval-valued triangular fuzzy number.

Definition 2.6. *[30] Let* $[A^L, A^U]$ *and* $[B^L, B^U]$ *be two interval-typed triangular fuzzy numbers. Then the operations with them are defined as follows:*

(i) $[A^L, A^U] + [B^L, B^U] = [A^L + B^L, A^U + B^U]$;

(ii) $[A^L, A^U] - [B^L, B^U] = [A^L - B^U, A^U - B^L]$;

(iii) $[A^L, A^U] \times [B^L, B^U] = [\bigwedge T_1, \bigvee T_1]$ *for* $T_1 = \{A^L \times B^L, A^L \times B^U, A^U \times B^L, A^U \times B^U\}$ *with* $a_i^L > 0$, $a_i^U > 0$, $b_i^L > 0$, $b_i^U > 0$, *for* $i = 1, 2, 3$, *where* $A^L = (a_1^L, a_2^L, a_3^L)$, $A^U = (a_1^U, a_2^U, a_3^U)$, $B^L = (b_1^L, b_2^L, b_3^L)$, $B^U = (b_1^U, b_2^U, b_3^U)$;

(iv) $[A^L, A^U]/[B^L, B^U] = [\bigwedge T_2, \bigvee T_2]$, *for* $T_2 = \{A^L/B^L, A^L/B^U, A^U/B^L, A^U/B^U\}$ *with* $a_i^L > 0$, $a_i^U > 0$, $b_i^L > 0$, $b_i^U > 0$, *for* $i = 1, 2, 3$, *where* $A^L = (a_1^L, a_2^L, a_3^L)$, $A^U = (a_1^U, a_2^U, a_3^U)$, $B^L = (b_1^L, b_2^L, b_3^L)$, $B^U = (b_1^U, b_2^U, b_3^U)$;

(v) $k[A^L, A^U] = [kA^L, kA^U]$ *for* $k \in \mathcal{R}^+$.

Definition 2.7. *[30] Let* $\{[A_i^L, A_i^U]\}_{i \in I}$ *be a collection of interval-typed triangular fuzzy numbers, where* I *denotes a finite index set. Then*

$$\bigvee_{i \in I} [A_i^L, A_i^U] = [\bigvee_{i \in I} A_i^L, \bigvee_{i \in I} A_i^U], \tag{7}$$

$$\bigwedge_{i \in I} [A_i^L, A_i^U] = [\bigwedge_{i \in I} A_i^L, \bigwedge_{i \in I} A_i^U]. \tag{8}$$

3 ITF-MOLP problem

As is well known, in classical transportation problem we always consider minimizing the costs of transporting several products. However, the complexity of the social environment in most real world problems requires the explicit consideration of objective functions other than cost [25]. Moreover, these objectives are frequently in conflict, measured in different scales and difficult to combine in one overall utility function. For instance, in real transportation problem the total transportation and implementation cost, the environmental impact and the distribution time need to be minimized respectively while the average delivery rate requires to be maximized.

Solving fuzzy linear programming problems, whose parameters in objects and constraints are considered as interval-valued fuzzy numbers meanwhile the decision variables are assumed to nonnegative crisp values, has received increasingly attention in recent years [25, 26, 33]. In this section, we consider a multi-objective linear programming problem (ITF-MOLP) whose all parameters except crisp decision variables are taken as interval-typed triangular fuzzy numbers. The ITF-MOLP problem could be considered as an extension of the multi-objective linear programming [25] whose parameters are assumed to be a special normal interval-valued triangular fuzzy numbers.

Next, we investigate an ITF-MOLP problem. An ITF-MOLP problem could be stated as follows:

$$max(min)\tilde{f}_1 = \sum_{j=1}^{n} \tilde{r}_{1j}x_j \tag{9}$$

$$max(min)\tilde{f}_2 = \sum_{j=1}^{n} \tilde{r}_{2j}x_j \tag{10}$$

$$\vdots$$

$$max(min)\tilde{f}_k = \sum_{j=1}^{n} \tilde{r}_{kj}x_j \tag{11}$$

s.t.

$$\begin{cases} \sum\limits_{j=1}^{n} \tilde{a}_{ij}x_j \preceq \tilde{b}_i, i = 1, 2, \cdots, m \\ \sum\limits_{j=1}^{n} \tilde{\varphi}_{tj}x_j \succeq \tilde{d}_t, t = 1, 2, \cdots, q \\ x_j \geq 0, j = 1, 2, \cdots, n. \end{cases} \tag{12}$$

Where $x_j \in \mathcal{R}$ denotes decision variable $(j = 1, 2, \cdots, n)$ and parameters $\tilde{r}_{lj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are all interval-typed triangular fuzzy numbers. Denote $\tilde{r}_{lj} = [\tilde{r}_{lj1}, \tilde{r}_{lj2}]$, $\tilde{a}_{ij} = [\tilde{a}_{ij1}, \tilde{a}_{ij2}]$, $\tilde{b}_i = [\tilde{b}_{i1}, \tilde{b}_{i2}]$, $\tilde{\varphi}_{tj} = [\tilde{\varphi}_{tj1}, \tilde{\varphi}_{tj2}]$, $\tilde{d}_t = [\tilde{d}_{t1}, \tilde{d}_{t2}]$, in which $\tilde{r}_{lj1}, \tilde{r}_{lj2}, \tilde{a}_{ij1}, \tilde{a}_{ij2}, \tilde{b}_{i1}, \tilde{b}_{i2}, \tilde{\varphi}_{tj1}, \tilde{\varphi}_{tj2}, \tilde{d}_{t1}$ and $\tilde{d}_{t2}$ are all triangular fuzzy numbers, $l = 1, \cdots, k; i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n;\ t = 1, 2, \cdots, q$.

From the viewpoint of linguistic model, maximizing or minimizing a certain objective $\tilde{f}_l$ means the maximization or minimization of the objective in fuzzy environment. Because of the existence of interval-typed triangular fuzzy numbers, the ITF-MOLP problem is not well-defined. That is, the meaning of maximizing or minimizing $\tilde{f}_l$, $(l = 1, 2, \cdots, k)$ is not clear and the constraints do not define a deterministic feasible set. Particularly, if the coefficients of multi-objective functions and constraints are interval numbers, then the ITF-MOLP degenerates to a multi-objective linear programming with interval coefficients [21].

To deal with the maximization or minimization of the multi-objectives and compare with interval-valued fuzzy numbers, many authors [25, 26] introduce a ranking method to compute the signed distance from interval-valued fuzzy number to y-axis as follows. Thus the multi-objective problem would be convenient for computation.

Lemma 3.1. [26] Let $\tilde{\tilde{A}}$ be a normal interval-valued fuzzy number. The signed distance of

$$\tilde{\tilde{A}} = [\tilde{A}^L, \tilde{A}^U] = \langle (a_1^L, a_2^L, a_3^L, a_4^L; h^L), (a_1^U, a_2^U, a_3^U, a_4^U; h^U) \rangle, \tag{13}$$

from $\mathcal{O}$(y–axis) is given as:

$$d(\tilde{\tilde{A}}, \mathcal{O}) = \frac{1}{8}[a_1^L + a_2^L + a_3^L + a_4^L + a_1^U + a_2^U + a_3^U + a_4^U]. \tag{14}$$

Definition 3.1. *[26] Let $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ be two normal interval-valued fuzzy numbers. Then the ranking of normal interval-valued trapezoidal fuzzy numbers is defined on the basis of the signed distance d, described in Lemma 3.1, as follows*

$$\tilde{\tilde{A}} \preceq \tilde{\tilde{B}} \quad \textit{if and only if} \quad d(\tilde{\tilde{A}}, \mathcal{O}) \leq d(\tilde{\tilde{B}}, \mathcal{O}); \tag{15}$$

$$\tilde{\tilde{A}} \succeq \tilde{\tilde{B}} \quad \textit{if and only if} \quad d(\tilde{\tilde{A}}, \mathcal{O}) \geq d(\tilde{\tilde{B}}, \mathcal{O}); \tag{16}$$

$$\tilde{\tilde{A}} = \tilde{\tilde{B}} \quad \textit{if and only if} \quad d(\tilde{\tilde{A}}, \mathcal{O}) = d(\tilde{\tilde{B}}, \mathcal{O}); \tag{17}$$

Noting that $I(\mathcal{R}) = \{[a^L, a^R] | a^L, a^R \in \mathcal{R}, a^L \leq a^R\}$ denote the set of all closed interval numbers on $\mathcal{R}$.

Indeed, an interval number $[a^L, a^R] \in I(\mathcal{R})$ is a special fuzzy number, whose membership degree could be stated as follows:

$$\mu(x) = \begin{cases} 1, \; if \; a^L \leq x \leq a^R \\ 0, \quad otherwise \end{cases} \tag{18}$$

The center, a^C of interval number $[a^L, a^R]$ is defined as follows:

$$a^C = \frac{a^L + a^R}{2} \tag{19}$$

Obviously, each one of a^C, a^L and a^R can be determined by two other scalars in Equation (19).

Definition 3.2. *Let $[a^L, a^R], [b^L, b^R] \in I(\mathcal{R})$. Then the operations between them are defined as follows:*
(i) $[a^L, a^R] + [b^L, b^R] = [a^L + b^L, a^R + b^R]$;
(ii)$[a^L, a^R] - [b^L, b^R] = [a^L - b^R, a^R - b^L]$;
(iii) $[a^L, a^R] \times [b^L, b^R] = \Big[min\{a^Lb^L, a^Lb^R, a^Rb^L, a^Rb^R\}, max\{a^Lb^L, a^Lb^R, a^Rb^L, a^Rb^R\}\Big]$

By Definition 3.2, the following properties are easily obtained.

Proposition 3.1. *Let $[b^L, b^R] \in I(\mathcal{R})$ and p be a scalar. Then the following statements hold.*
(i) $p - [b^L, b^R] = [p - b^R, p - b^L]$;
(ii)$p \cdot [b^L, b^R] = \begin{cases} [pb^L, pb^R], & if \; p \geq 0 \\ [pb^R, pb^L], & if \; p < 0 \end{cases}$

Proof. Let $p = [p, p]$. Then the above equations hold by Definition 3.2.

The λ-level (λ-cut cet) of a triangular fuzzy number $A = (a_1, a_2, a_3)$ is the interval number defined by

$$A_\lambda = \{x \in R | \mu_A(x) \geq \lambda\} = [A^L_\lambda, A^R_\lambda] \tag{20}$$

where $\lambda \in (0, 1]$. □

Proposition 3.2. *[34] Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. Then the following statements are satisfied.*
(i) $(A + B)_\lambda = A_\lambda + B_\lambda, \lambda \in (0, 1]$;
(ii) $(\mu A)_\lambda = \mu A_\lambda, \mu \in \mathcal{R}^+, \lambda \in (0, 1]$.

Proposition 3.3. *Let $A_i = (a^i_1, a^i_2, a^i_3)$ be a triangular fuzzy number, $u_i \in \mathcal{R}^+, i = 1, 2, \cdots, n$. Then the following statement holds:*

$$\left(\sum_{i=1}^n u_i A_i\right)_\lambda = \sum_{i=1}^n u_i (A_i)_\lambda, \; \lambda \in (0, 1]. \tag{21}$$

Proof. It is easily concluded by Proposition 3.2. □

The λ-cut cet of an interval-typed triangular fuzzy number $[A^L, A^U]$ is defined as $[(A^L)_\lambda, (A^U)_\lambda]$.

Definition 3.3. *Let T denote an index set, $[a_i, b_i], [c_i, d_i] \in I(\mathcal{R}), a_i \leq c_i, i \in T$. Then*

$$\bigvee_{i \in T} \Big[[a_i, b_i], [c_i, d_i]\Big] = \Big[\bigvee_{i \in T}[a_i, b_i], \bigvee_{i \in T}[c_i, d_i]\Big], \tag{22}$$

$$\bigwedge_{i\in T}\Big[[a_i, b_i], [c_i, d_i]\Big] = \Big[\bigwedge_{i\in T}[a_i, b_i], \bigwedge_{i\in T}[c_i, d_i]\Big], \tag{23}$$

where $\bigvee$ *and* $\bigwedge$ *represent the supremum and infimum operator on the real set* $\mathcal{R}$ *respectively.*

The decomposition theorem [34] of a fuzzy set, which characterizes the relationship among all the cut sets of a fuzzy set and the given fuzzy set, has been developed as follows:

Lemma 3.2. [34] Let a map $G : \mathcal{R} \to [0, 1]$ be a fuzzy set. Then

$$G = \bigvee_{\lambda\in[0,1]} (\lambda \wedge G_\lambda), \tag{24}$$

where G_λ is the λ-cut set of a fuzzy set G.

Similarly, the decomposition theorem about an interval-typed triangular fuzzy number could be obtained as follows.

Proposition 3.4. *Let* $[A^L, A^U]$ *be an interval-typed triangular fuzzy number. Then*

$$[A^L, A^U] = \bigvee_{\lambda\in[0,1]} (\lambda \wedge Z_\lambda), \tag{25}$$

where $Z_\lambda = [(A^L)_\lambda, (A^U)_\lambda]$.

Proof. Let $A^L = (a_1^L, a_2^L, a_3^L)$ and $A^U = (a_1^U, a_2^U, a_3^U)$ be two triangular fuzzy numbers , $a_2^L \le a_2^U$. By Definition 3.3 and Lemma 3.2, we have

$$A^L = \bigvee_{\lambda\in[0,1]} \Big(\lambda \wedge (A^L)_\lambda\Big)$$

and

$$A^U = \bigvee_{\lambda\in[0,1]} \Big(\lambda \wedge (A^U)_\lambda\Big),$$

where $(A^L)_\lambda$ and $(A^U)_\lambda$ are interval numbers.

Therefore

$$\begin{aligned}\bigvee_{\lambda\in[0,1]} (\lambda \wedge Z_\lambda) &= \bigvee_{\lambda\in[0,1]} \big[\lambda \wedge (A^L)_\lambda, \lambda \wedge (A^U)_\lambda\big] \\ &= \big[\bigvee_{\lambda\in[0,1]} (\lambda \wedge (A^L)_\lambda), \bigvee_{\lambda\in[0,1]} (\lambda \wedge (A^U)_\lambda)\big] \\ &= [A^L, A^U].\end{aligned}$$

□

4 An algorithm of ITF-MOLP

The algorithm of ITF-MOLP given in formulas (9 – 12) is a multi-stage procedure.

STEP1. The ITF-MOLP problem is decomposed to a set of k linear programming problems based on interval-typed triangular fuzzy numbers. Each problem optimizes one of the k objective functions associated with constraints set. For a given objective $l(l = 1, \cdots, k)$, this problem is given as follows:

$$max(min)\tilde{f}_l = \sum_{j=1}^{n} \tilde{r}_{lj} x_j \tag{26}$$

s.t.

$$\begin{cases} \sum_{j=1}^{n} \tilde{a}_{ij} x_j \preceq \tilde{b}_i, i = 1, 2, \cdots, m \\ \sum_{j=1}^{n} \tilde{\varphi}_{tj} x_j \succeq \tilde{d}_t, t = 1, 2, \cdots, q \\ x_j \geq 0, j = 1, 2, \cdots, n. \end{cases} \tag{27}$$

Where $\tilde{r}_{lj}, \tilde{a}_{ij}, \tilde{b}_i, \tilde{\varphi}_{tj}$ and $\tilde{d}_t$ are all interval-typed triangular fuzzy numbers. Denote $\tilde{r}_{lj} = [\tilde{r}_{lj1}, \tilde{r}_{lj2}]$, $\tilde{a}_{ij} = [\tilde{a}_{ij1}, \tilde{a}_{ij2}]$, $\tilde{b}_i = [\tilde{b}_{i1}, \tilde{b}_{i2}]$, $\tilde{\varphi}_{tj} = [\tilde{\varphi}_{tj1}, \tilde{\varphi}_{tj2}]$, $\tilde{d}_t = [\tilde{d}_{t1}, \tilde{d}_{t2}]$, in which $\tilde{r}_{lj1}, \tilde{r}_{lj2}, \tilde{a}_{ij1}, \tilde{a}_{ij2}, \tilde{b}_{i1}, \tilde{b}_{i2}, \tilde{\varphi}_{tj1}, \tilde{\varphi}_{tj2}, \tilde{d}_{t1}$ and $\tilde{d}_{t2}$ are all triangular fuzzy numbers, $i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n;\ t = 1, 2, \cdots, q$.

Then the objective function $\tilde{f}_l$ could be written as follows:

$$\tilde{f}_l = \Big[\sum_{j=1}^{n} \tilde{r}_{lj1} x_j, \sum_{j=1}^{n} \tilde{r}_{lj2} x_j\Big] \tag{28}$$

For a given λ-level, $\lambda \in (0, 1]$, the λ-cut set of the objective function $\tilde{f}_l$ is obtained by Propositions 3.3 and 3.4 as follows:

$$(\tilde{f}_l)_\lambda = \left[\sum_{j=1}^{n} (\tilde{r}_{lj1})_\lambda x_j, \sum_{j=1}^{n} (\tilde{r}_{lj2})_\lambda x_j\right] \tag{29}$$

Note that $(\tilde{r}_{lj1})_\lambda = [(\tilde{r}_{lj1})_\lambda^L, (\tilde{r}_{lj1})_\lambda^R]$ and $(\tilde{r}_{lj2})_\lambda = [(\tilde{r}_{lj2})_\lambda^L, (\tilde{r}_{lj2})_\lambda^R]$ in Equation (29). It follows from Definition 3.1 and Proposition 3.1 that

$$(\tilde{f}_l)_\lambda = \left[\left[\sum_{j=1}^{n} (\tilde{r}_{lj1})_\lambda^L x_j, \sum_{j=1}^{n} (\tilde{r}_{lj1})_\lambda^R x_j\right], \left[\sum_{j=1}^{n} (\tilde{r}_{lj2})_\lambda^L x_j, \sum_{j=1}^{n} (\tilde{r}_{lj2})_\lambda^R x_j\right]\right] \tag{30}$$

If the original objective is to maximize $\tilde{f}_l$, $l \in \{1, 2, \cdots, k\}$, then a decision maker with pessimistic and conservative attitude would require the maximization of the lower and center of “interval set" $(\tilde{f}_l)_\lambda$. From Definition 3.2 and Proposition 3.4, the solution of model (26) could be presented as the optimal solution of the following bi-objective problem based on a given λ–level:

$$max\left((\tilde{f}_l)_\lambda^L, (\tilde{f}_l)_\lambda^C\right) \tag{31}$$

where the lower bound of $(\tilde{f}_l)_\lambda$, denoted by $(\tilde{f}_l)_\lambda^L$, is given as,

$$(\tilde{f}_l)_\lambda^L = \omega_1 \sum_{j=1}^{n} (\tilde{r}_{lj1})_\lambda^L x_j + \omega_2 \sum_{j=1}^{n} (\tilde{r}_{lj2})_\lambda^L x_j, \tag{32}$$

the center of $(\tilde{f}_l)_\lambda$, denoted by $(\tilde{f}_l)_\lambda^C$, is given as,

$$(\tilde{f}_l)_\lambda^C = \omega_1 \sum_{j=1}^{n} (\tilde{r}_{lj1})_\lambda^C x_j + \omega_2 \sum_{j=1}^{n} (\tilde{r}_{lj2})_\lambda^C x_j. \tag{33}$$

And the weight ω_i satisfies the conditions:

$$\omega_1 + \omega_2 = 1, \omega_i \geq 0, i = 1, 2. \tag{34}$$

Remark. Note that $(\tilde{f}_l)_\lambda$ is not a classical interval set. Thus the lower bound of $(\tilde{f}_l)_\lambda$ considers the lower bounds of interval sets both $\left[\sum_{j=1}^{n} (\tilde{r}_{lj1})_\lambda^L x_j, \sum_{j=1}^{n} (\tilde{r}_{lj1})_\lambda^R x_j\right]$ and $\left[\sum_{j=1}^{n} (\tilde{r}_{lj2})_\lambda^L x_j, \sum_{j=1}^{n} (\tilde{r}_{lj2})_\lambda^R x_j\right]$, and the center of $(\tilde{f}_l)_\lambda$

considers the center points of interval sets $\left[\sum_{j=1}^{n}(\tilde{r}_{lj1})^L_\lambda x_j, \sum_{j=1}^{n}(\tilde{r}_{lj1})^R_\lambda x_j\right]$ and $\left[\sum_{j=1}^{n}(\tilde{r}_{lj2})^L_\lambda x_j, \sum_{j=1}^{n}(\tilde{r}_{lj2})^R_\lambda x_j\right]$. Moreover, parameters ω_1 and ω_2 represent the relative importance weight of $\left[\sum_{j=1}^{n}(\tilde{r}_{lj1})^L_\lambda x_j, \sum_{j=1}^{n}(\tilde{r}_{lj1})^R_\lambda x_j\right]$ and $\left[\sum_{j=1}^{n}(\tilde{r}_{lj2})^L_\lambda x_j, \sum_{j=1}^{n}(\tilde{r}_{lj2})^R_\lambda x_j\right]$, respectively. Particularly, an extremely conservative decision maker would select parameters $\omega_1 = 1$ and $\omega_2 = 0$.

If the original objective is to minimize $\tilde{f}_l$, $l \in \{1, 2, \cdots, k\}$, then the solution of model (26) could be given as the optimal solution of the following bi-objective problem:

$$min\left((\tilde{f}_l)^R_\lambda, (\tilde{f}_l)^C_\lambda\right) \tag{35}$$

Where

$$(\tilde{f}_l)^C_\lambda = \omega_1 \sum_{j=1}^{n}(\tilde{r}_{lj1})^C_\lambda x_j + \omega_2 \sum_{j=1}^{n}(\tilde{r}_{lj2})^C_\lambda x_j, \tag{36}$$

$$(\tilde{f}_l)^R_\lambda = \omega_1 \sum_{j=1}^{n}(\tilde{r}_{lj1})^R_\lambda x_j + \omega_2 \sum_{j=1}^{n}(\tilde{r}_{lj2})^R_\lambda x_j, \tag{37}$$

and both ω_1 and ω_2 are satisfied to Equation (34).

The comparison between fuzzy numbers can be carried out using dominance possibility criterion (DPC) or strong dominance possibility criterion ($SDPC$) [35, 36]. In this paper, we adopt dominance possibility criterion (DPC) to compare the relationship between two triangular fuzzy numbers. If both fuzzy numbers $\tilde{a} = (\underline{a}, a_0, \overline{a})$ and $\tilde{b} = (\underline{b}, b_0, \overline{b})$ are triangular fuzzy numbers, then

$$Poss(\tilde{a} \succeq_D \tilde{b}) = Poss(\tilde{b} \preceq_D \tilde{a}) = \begin{cases} 1, & if\ a_0 \geq b_0 \\ \frac{\bar{a}-\underline{b}}{(\bar{a}-a_0)+(b_0-\underline{b})}, & if\ b_0 \geq a_0, \bar{a} \geq \underline{b} \\ 0, & if\ \underline{b} \geq \bar{a} \end{cases} \tag{38}$$

$Poss(\tilde{a} \succeq \tilde{b})$ means the possibility that the maximum value of $\tilde{a}$ is greater than or equal to the minimum value of $\tilde{b}$ [36].

For $\mu \in (0, 1]$,

$$Poss(\tilde{a} \succeq_D \tilde{b}) \geq \mu$$

is equivalent to the inequalities below:

$$\begin{cases} \overline{a} \geq \underline{b} \\ (1-\mu)\bar{a} + \mu a_0 \geq (1-\mu)\underline{b} + \mu b_0. \end{cases} \tag{39}$$

Note that $\sum_{j=1}^{n} \tilde{a}_{ij}x_j \preceq_D \tilde{b}_i$ in (27) is well-defined. It follows that

$$\sum_{j=1}^{n} \tilde{a}_{ij}x_j = \left[\sum_{j=1}^{n} \tilde{a}_{ij1}x_j, \sum_{j=1}^{n} \tilde{a}_{ij2}x_j\right].$$

Let

$$\tilde{b}_i = [\tilde{b}_{i1}, \tilde{b}_{i2}].$$

Since $\sum_{j=1}^{n} \tilde{a}_{ij1}x_j, \sum_{j=1}^{n} \tilde{a}_{ij2}x_j, \tilde{b}_{i1}$, and $\tilde{b}_{i2}$ are all triangular fuzzy numbers, for a given tolerance measure μ, $\mu \in (0, 1]$, the constraint

$$\sum_{j=1}^{n} \tilde{a}_{ij}x_j \preceq_D \tilde{b}_i \tag{40}$$

could be transformed into the following conditions:

$$\begin{cases} Poss\left(\sum_{j=1}^{n} \tilde{a}_{ij2}x_j \preceq_D \tilde{b}_{i2}\right) \geq \mu \\ Poss\left(\sum_{j=1}^{n} \frac{\tilde{a}_{ij1}+\tilde{a}_{ij2}}{2} x_j \preceq_D \frac{\tilde{b}_{i1}+\tilde{b}_{i2}}{2}\right) \geq \mu \end{cases} \tag{41}$$

Here, $Poss\left(\sum_{j=1}^{n} \tilde{a}_{ij2}x_j \preceq_D \tilde{b}_{i2}\right)$ represents the possibility of the proposition that the upper bound $\sum_{j=1}^{n} \tilde{a}_{ij2}x_j$ of $\sum_{j=1}^{n} \tilde{a}_{ij}x_j$ is no larger than the upper bound $\tilde{b}_{i2}$ of $\tilde{b}_i$ under the dominance possibility criterion. Moreover, $Poss\left(\sum_{j=1}^{n} \frac{\tilde{a}_{ij1}+\tilde{a}_{ij2}}{2} x_j \preceq_D \frac{\tilde{b}_{i1}+\tilde{b}_{i2}}{2}\right)$ means the possibility of the proposition that the center $\sum_{j=1}^{n} \frac{\tilde{a}_{ij1}+\tilde{a}_{ij2}}{2} x_j$ of $\sum_{j=1}^{n} \tilde{a}_{ij}x_j$ is less than or equal to the center $\frac{\tilde{b}_{i1}+\tilde{b}_{i2}}{2}$ of $\tilde{b}_i$ under the DPC. In this way, we can appropriately characterize the linguistic term of the inequality (27).

And in the same way the constraint

$$\sum_{j=1}^{n} \tilde{\varphi}_{tj}x_j \succeq \tilde{d}_t \tag{42}$$

could be transformed into the following conditions:

$$\begin{cases} Poss(\sum_{j=1}^{n} \tilde{\varphi}_{tj1}x_j \succeq_D \tilde{d}_{t1}) \geq \mu \\ Poss(\sum_{j=1}^{n} \frac{\tilde{\varphi}_{tj1}+\tilde{\varphi}_{tj2}}{2} x_j \succeq_D \frac{\tilde{d}_{t1}+\tilde{d}_{t2}}{2}) \geq \mu \end{cases} \tag{43}$$

Finally, by solving the problem (26) based on Equations (26 – 43) for each objective function, the range of optimal objective functions are determined as

$$(\tilde{f}_l)^{\star}_{\lambda} \in \left[(\tilde{f}_l)^{\star L}_{\lambda}, (\tilde{f}_l)^{\star R}_{\lambda}\right],$$

where $(\tilde{f}_l)^{\star R}_{\lambda} = (\tilde{f}_l)^{\star L}_{\lambda} + 2((\tilde{f}_l)^{\star C}_{\lambda} - (\tilde{f}_l)^{\star L}_{\lambda})$.

STEP2. Consider the lth objective again. If the objective is a maximization type, its membership function could be defined as follows:

$$\tilde{\mu}_l(x) = \begin{cases} 1, & if\ (\tilde{f}_l(x))^L_{\lambda} \vee (\tilde{f}_l(x))^C_{\lambda} \geq (\tilde{f}_l)^{\star R}_{\lambda} \\ \frac{(\tilde{f}_l(x))^L_{\lambda} \vee (\tilde{f}_l(x))^C_{\lambda} - (\tilde{f}_l)^{\star L}_{\lambda}}{(\tilde{f}_l)^{\star R}_{\lambda} - (\tilde{f}_l)^{\star L}_{\lambda}}, & if\ (\tilde{f}_l)^L_{\lambda} \vee (\tilde{f}_l(x))^C_{\lambda} \leq (\tilde{f}_l)^{\star R}_{\lambda} \end{cases} \tag{44}$$

In the same way, for a minimization type objective, the membership function can be given as follows:

$$\tilde{\mu}_l(x) = \begin{cases} 1, & if\ (\tilde{f}_l(x))^R_{\lambda} \wedge (\tilde{f}_l(x))^C_{\lambda} \leq (\tilde{f}_l)^{\star L}_{\lambda} \\ \frac{(\tilde{f}_l)^{\star R}_{\lambda} - (\tilde{f}_l(x))^R_{\lambda} \wedge (\tilde{f}_l(x))^C_{\lambda}}{(\tilde{f}_l)^{\star R}_{\lambda} - (\tilde{f}_l)^{\star L}_{\lambda}}, & if\ (\tilde{f}_l)^{\star L}_{\lambda} \leq (\tilde{f}_l(x))^R_{\lambda} \wedge (\tilde{f}_l(x))^C_{\lambda} \end{cases} \tag{45}$$

Lemma 4.1. *From Equations* (44) *and* (45) *it always holds that* $\tilde{\mu}_l(x) \leq 1$.

Proof. Suppose that $\tilde{\mu}_l(x) > 1$. Then based on Equation (44), it follows that

$$\tilde{\mu}_l(x) > 1 \Rightarrow \frac{(\tilde{f}_l(x))^L_{\lambda} \vee (\tilde{f}_l(x))^C_{\lambda} - (\tilde{f}_l)^{\star L}_{\lambda}}{(\tilde{f}_l)^{\star R}_{\lambda} - (\tilde{f}_l)^{\star L}_{\lambda}} > 1$$

$\Rightarrow (\tilde{f}_l(x))^L_{\lambda} \vee (\tilde{f}_l(x))^C_{\lambda} - (\tilde{f}_l)^{\star L}_{\lambda} > (\tilde{f}_l)^{\star R}_{\lambda} - (\tilde{f}_l)^{\star L}_{\lambda}$
$\Rightarrow (\tilde{f}_l(x))^L_{\lambda} \vee (\tilde{f}_l(x))^C_{\lambda} > (\tilde{f}_l)^{\star R}_{\lambda}$
$\Rightarrow (\tilde{f}_l(x))^L_{\lambda} > (\tilde{f}_l)^{\star R}_{\lambda}\ \ or\ \ (\tilde{f}_l(x))^C_{\lambda} > (\tilde{f}_l)^{\star R}_{\lambda}$
$\Rightarrow (\tilde{f}_l(x))^R_{\lambda} > (\tilde{f}_l)^{\star R}_{\lambda}$.

This contradicts with the optimality of $(\tilde{f}_l)^{\star R}_\lambda$.

Based on Equation (45), it follows that

$$\tilde{\mu}_l(x) > 1 \Rightarrow \frac{(\tilde{f}_l)^{\star R}_\lambda - (\tilde{f}_l(x))^R_\lambda \wedge (\tilde{f}_l(x))^C_\lambda}{(\tilde{f}_l)^{\star R}_\lambda - (\tilde{f}_l)^{\star L}_\lambda} > 1$$

$\Rightarrow (\tilde{f}_l)^{\star R}_\lambda - (\tilde{f}_l(x))^R_\lambda \wedge (\tilde{f}_l(x))^C_\lambda > (\tilde{f}_l)^{\star R}_\lambda - (\tilde{f}_l)^{\star L}_\lambda$
$\Rightarrow (\tilde{f}_l(x))^R_\lambda \wedge (\tilde{f}_l(x))^C_\lambda < (\tilde{f}_l)^{\star L}_\lambda$
$\Rightarrow (\tilde{f}_l(x))^R_\lambda < (\tilde{f}_l)^{\star L}_\lambda \;\; or \;\; (\tilde{f}_l(x))^C_\lambda < (\tilde{f}_l)^{\star L}_\lambda$
$\Rightarrow (\tilde{f}_l(x))^L_\lambda < (\tilde{f}_l)^{\star L}_\lambda$.

This contradicts with the optimality of $(\tilde{f}_l)^{\star L}_\lambda$. It completes the proof. □

STEP3. For a given λ-level, $\lambda \in (0, 1]$, the problem (9 – 12) is transformed into the linear programming based on interval-typed fuzzy numbers as follows:

$$max\{\tilde{\mu}_1(x), \tilde{\mu}_2(x), \cdots, \tilde{\mu}_k(x)\} \tag{46}$$

s.t.

$$\begin{cases} \tilde{\mu}_l(x) \le 1, l = 1, 2, \cdots, k \\ \sum_{j=1}^{n} \tilde{a}_{ij} x_j \preceq \tilde{b}_i, i = 1, 2, \cdots, m \\ \sum_{j=1}^{n} \tilde{\varphi}_{tj} x_j \succeq \tilde{d}_t, t = 1, 2, \cdots, q \\ x_j \ge 0, j = 1, 2, \cdots, n. \end{cases} \tag{47}$$

STEP4. For a given λ-level, $\lambda \in [0, 1]$, the problem (46) is transformed into the form below:

$$max \sum_{l=1}^{k} \tilde{\mu}_l(x) \tag{48}$$

s.t.

$$\begin{cases} \tilde{\mu}_l(x) \le 1, l = 1, 2, \cdots, k \\ \sum_{j=1}^{n} \tilde{a}_{ij} x_j \preceq \tilde{b}_i, i = 1, 2, \cdots, m \\ \sum_{j=1}^{n} \tilde{\varphi}_{tj} x_j \succeq \tilde{d}_t, t = 1, 2, \cdots, q \\ x_j \ge 0, j = 1, 2, \cdots, n. \end{cases} \tag{49}$$

STEP5. For a given λ-level and μ-tolerance measure, $\lambda, \mu \in (0, 1]$, the problem (48) is transformed into the single objective programming as follows:

$$max \sum_{l=1}^{k} \tilde{\mu}_l(x) \tag{50}$$

s.t.

$$\begin{cases} \tilde{\mu}_l(x) \le 1, l = 1, 2, \cdots, k \\ Poss(\sum_{j=1}^{n} \tilde{a}_{ij2} x_j \preceq_D \tilde{b}_{i2}) \ge \mu \\ Poss(\sum_{j=1}^{n} \frac{\tilde{a}_{ij1}+\tilde{a}_{ij2}}{2} x_j \preceq_D \frac{\tilde{b}_{i1}+\tilde{b}_{i2}}{2}) \ge \mu \\ Poss(\sum_{j=1}^{n} \tilde{\varphi}_{tj1} x_j \succeq_D \tilde{d}_{t1}) \ge \mu \\ Poss(\sum_{j=1}^{n} \frac{\tilde{\varphi}_{tj1}+\tilde{\varphi}_{tj2}}{2} x_j \succeq_D \frac{\tilde{d}_{t1}+\tilde{d}_{t2}}{2}) \ge \mu \\ x_j \ge 0, j = 1, 2, \cdots, n. \end{cases} \tag{51}$$

Therefore, the optimal solution of the linear programming (50 – 51) could be obtained by Matlab toolbox through Matlab software on the computer platform.

5 Numerical examples

In this section, some numerical examples of ITF-MOLP problem are investigated.

Example1: Consider the following ITF-MOLP:

$Max\tilde{f}_1 = [(0.8, 1, 1.2), (2.5, 3, 3.5)]x_1 + [(-1.3, -1, -0.8), (1.2, 1.5, 1.7)]x_2$

$Max\tilde{f}_2 = [(0.3, 0.5, 0.7), (1.8, 2, 2.2)]x_1 + [(-1.7, -1.5, -1.3), (-1.2, -1, -0.8)]x_2$

s.t.

$$\begin{cases} [(0.8, 1, 1.2), (1.8, 2, 2.2)]x_1 + [(1.3, 1.5, 1.6), (2.8, 3, 3.3)]x_2 \preceq [(3.8, 4, 4.5), (5, 6, 7)] \\ [(0.8, 1, 1.5), (2.5, 3, 3.2)]x_1 + [(2.2, 2.5, 3), (3, 3.5, 3.8)]x_2 \preceq [(10, 12, 13), (11, 13, 13.5)] \\ x_1 \geq 0, x_2 \geq 0 \end{cases}.$$

The problem is decomposed to two linear programmings based on interval-typed triangular fuzzy numbers:

Problem 1.

$Max\tilde{f}_1 = [(0.8, 1, 1.2), (2.5, 3, 3.5)]x_1 + [(-1.3, -1, -0.8), (1.2, 1.5, 1.7)]x_2$

s.t.

$$\begin{cases} [(0.8, 1, 1.2), (1.8, 2, 2.2)]x_1 + [(1.3, 1.5, 1.6), (2.8, 3, 3.3)]x_2 \preceq [(3.8, 4, 4.5), (5, 6, 7)] \\ [(0.8, 1, 1.5), (2.5, 3, 3.2)]x_1 + [(2.2, 2.5, 3), (3, 3.5, 3.8)]x_2 \preceq [(10, 12, 13), (11, 13, 13.5)] \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Problem 2.

$Max\tilde{f}_2 = [(0.3, 0.5, 0.7), (1.8, 2, 2.2)]x_1 + [(-1.7, -1.5, -1.3), (-1.2, -1, -0.8)]x_2$

s.t.

$$\begin{cases} [(0.8, 1, 1.2), (1.8, 2, 2.2)]x_1 + [(1.3, 1.5, 1.6), (2.8, 3, 3.3)]x_2 \preceq [(3.8, 4, 4.5), (5, 6, 7)] \\ [(0.8, 1, 1.5), (2.5, 3, 3.2)]x_1 + [(2.2, 2.5, 3), (3, 3.5, 3.8)]x_2 \preceq [(10, 12, 13), (11, 13, 13.5)] \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Both problems 1 and 2 are then solved based on the above method.

Consider the problem 1. This problem is further transformed into the following model:

$$max((\tilde{f}_1)^L_\lambda, (\tilde{f}_1)^C_\lambda)$$

s.t.

$$\begin{cases} (\tilde{f}_1)^L_\lambda = \omega_1[(0.8 + \lambda(1 - 0.8))x_1 + (-1.3 + \lambda(-1 - (-1.3)))x_2]+ \\ \omega_2[(2.5 + \lambda(3 - 2.5))x_1 + (1.2 + \lambda(1.5 - 1.2))x_2] \\ (\tilde{f}_1)^C_\lambda = \omega_1[\frac{1}{2}(0.8 + \lambda(1 - 0.8) + 1.2 - \lambda(1.2 - 1))x_1 + \frac{1}{2}(-1.3 + \lambda(-1 - (-1.3))- \\ 0.8 - \lambda(-0.8 - (-1)))x_2] + \omega_2[\frac{1}{2}(2.5 + \lambda(3 - 2.5) + 3.5 - \lambda(3.5 - 3))x_1+ \\ \frac{1}{2}(1.2 + \lambda(1.5 - 1.2) + 1.7 - \lambda(1.7 - 1.5))x_2] \\ Poss((1.8, 2, 2.2)x_1 + (2.8, 3, 3.3)x_2 \preceq_D (5, 6, 7)) \geq \mu \\ Poss(\frac{1}{2}[(0.8, 1, 1.2) + (1.8, 2, 2.2)]x_1 + \frac{1}{2}[(1.3, 1.5, 1.6) + (2.8, 3, 3.3)]x_2 \\ \preceq_D \frac{1}{2}[(3.8, 4, 4.5) + (5, 6, 7)]) \geq \mu \\ Poss((2.5, 3, 3.2)x_1 + (3, 3.5, 3.8)x_2 \preceq_D (11, 13, 13.5)) \geq \mu \\ Poss(\frac{1}{2}[(0.8, 1, 1.5) + (2.5, 3, 3.2)]x_1 + \frac{1}{2}[(2.2, 2.5, 3) + (3, 3.5, 3.8)]x_2 \\ \preceq_D \frac{1}{2}[(10, 12, 13) + (11, 13, 13.5)]) \geq \mu \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Where λ, μ, ω_1 and ω_2 are all given, $\lambda, \mu \in (0, 1]$, $\omega_1 + \omega_2 = 1$, $\omega_1 \geq 0$, $\omega_2 \geq 0$.

By (39), the model above can be equivalently transformed into the following form:

$$max((\tilde{f}_1)^L_\lambda, (\tilde{f}_1)^C_\lambda)$$

s.t.

$$\begin{cases} (\tilde{f}_1)^L_\lambda = [\omega_1(0.8 + 0.2\lambda) + \omega_2(2.5 + 0.5\lambda)]x_1 + [\omega_1(-1.3 + 0.3\lambda) + \omega_2(1.2 + 0.3\lambda)]x_2 \\ (\tilde{f}_1)^C_\lambda = (\omega_1 + 3\omega_2)x_1 + [\frac{1}{2}\omega_1(-2.1 + 0.1\lambda) + \frac{1}{2}\omega_2(2.9 + 0.1\lambda)]x_2 \\ 1.8x_1 + 2.8x_2 \le 5 \\ (1-\mu)(1.8x_1 + 2.8x_2) + \mu(2x_1 + 3x_2) \le (1-\mu)7 + \mu 6 \\ \frac{0.8+1.8}{2}x_1 + \frac{1.3+2.8}{2}x_2 \le \frac{4.5+7}{2} \\ (1-\mu)[\frac{0.8+1.8}{2}x_1 + \frac{1.3+2.8}{2}x_2] + \mu[\frac{1+2}{2}x_1 + \frac{1.5+3}{2}x_2] \le (1-\mu)\frac{4.5+7}{2} + \mu\frac{4+6}{2} \\ 2.5x_1 + 3x_2 \le 13.5 \\ (1-\mu)(2.5x_1 + 3x_2) + \mu(3x_1 + 3.5x_2) \le (1-\mu)13.5 + \mu 13 \\ \frac{0.8+2.5}{2}x_1 + \frac{2.2+3}{2}x_2 \le \frac{13+13.5}{2} \\ (1-\mu)[\frac{0.8+2.5}{2}x_1 + \frac{2.2+3}{2}x_2] + \mu[\frac{1+3}{2}x_1 + \frac{2.5+3.5}{2}x_2] \le (1-\mu)\frac{13+13.5}{2} + \mu\frac{12+13}{2} \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$

If we choose $\lambda = \mu = 0.7$, $\omega_1 = \omega_2 = 0.5$, then the above model is stated as follows:

$$max((\tilde{f}_1)^L_\lambda, (\tilde{f}_1)^C_\lambda)$$

s.t.

$$\begin{cases} (\tilde{f}_1)^L_\lambda = 1.895x_1 + 0.16x_2 \\ (\tilde{f}_1)^C_\lambda = 2x_1 + 0.235x_2 \\ 1.8x_1 + 2.8x_2 \le 5 \\ (0.2\mu + 1.8)x_1 + (0.2\mu + 2.8)x_2 \le 7 - \mu \\ 1.3x_1 + 2.05x_2 \le 5.75 \\ (0.2\mu + 1.3)x_1 + (0.2\mu + 2.05)x_2 \le 5.75 - 0.75\mu \\ 2.5x_1 + 3x_2 \le 13.5 \\ (0.5\mu + 2.5)x_1 + (0.5\mu + 3)x_2 \le 13.5 - 0.5\mu \\ 1.65x_1 + 2.6x_2 \le 13.25 \\ (0.35\mu + 1.65)x_1 + (0.4\mu + 2.6)x_2 \le 13.25 - 0.75\mu \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$

Problem 2 is further transformed into the following model:

$$max((\tilde{f}_2)^L_\lambda, (\tilde{f}_2)^C_\lambda)$$

s.t.

$$\begin{cases} (\tilde{f}_2)^L_\lambda = \omega_1[(0.3 + \lambda(0.5 - 0.3))x_1 + (-1.7 + \lambda(-1.5 - (-1.7)))x_2] + \\ \qquad \omega_2[(1.8 + \lambda(2 - 1.8))x_1 + (-1.2 + \lambda(-1 - (-1.2)))x_2] \\ (\tilde{f}_2)^C_\lambda = \omega_1[\frac{1}{2}(0.3 + \lambda(0.5 - 0.3) + 0.7 - \lambda(0.7 - 0.5))x_1 + \\ \qquad \frac{1}{2}(-1.7 + \lambda(-1.5 - (-1.7)) + (-1.3) - \lambda(-1.3 - (-1.5)))x_2] + \omega_2[\frac{1}{2}(1.8 + \lambda(2 - 1.8) \\ \qquad +2.2 - \lambda(2.2 - 2))x_1 + \frac{1}{2}(-1.2 + \lambda(-1 - (-1.2)) + (-0.8) - \lambda(-0.8 - (-1)))x_2] \\ Poss((1.8, 2, 2.2)x_1 + (2.8, 3, 3.3)x_2 \preceq_D (5, 6, 7)) \ge \mu \\ Poss(\frac{1}{2}[(0.8, 1, 1.2) + (1.8, 2, 2.2)]x_1 + \frac{1}{2}[(1.3, 1.5, 1.6) + (2.8, 3, 3.3)]x_2 \\ \qquad \preceq_D \frac{1}{2}[(3.8, 4, 4.5) + (5, 6, 7)]) \ge \mu \\ Poss((2.5, 3, 3.2)x_1 + (3, 3.5, 3.8)x_2 \preceq_D (11, 13, 13.5)) \ge \mu \\ Poss(\frac{1}{2}[(0.8, 1, 1.5) + (2.5, 3, 3.2)]x_1 + \frac{1}{2}[(2.2, 2.5, 3) + (3, 3.5, 3.8)]x_2 \\ \qquad \preceq_D \frac{1}{2}[(10, 12, 13) + (11, 13, 13.5)]) \ge \mu \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$

By utilizing Matlab toolbox on computer platform, the optimal solution of problem 1 is $x^* = (2.7778, 0)$. Therefore, $(\tilde{f}_1)^{*L}_\lambda = 5.2639$, $(\tilde{f}_1)^{*C}_\lambda = 5.5556$ and $(\tilde{f}_1)^{*R}_\lambda = 5.8473$. Similarly, Problem 2 can be solved and the optimal solution is $x^* = (2.7778, 0)$. Therefore, $(\tilde{f}_2)^{*L}_\lambda = 3.3056$, $(\tilde{f}_2)^{*C}_\lambda = 3.4722$ and $(\tilde{f}_2)^{*R}_\lambda = 3.8054$.

Using model (50), the problem above is transformed into the following problem:

$$max \quad \mu_1(x) + \mu_2(x) = \frac{2x_1 + 0.235x_2 - 5.2639}{2(5.5556 - 5.2639)} + \frac{1.25x_1 - 1.25x_2 - 3.3056}{2(3.4722 - 3.3056)}$$

s.t.

$$\begin{cases} 5.2629 \le 2x_1 + 0.235x_2 \le 5.8473 \\ 3.3056 \le 1.25x_1 - 1.25x_2 \le 3.8054 \\ 1.8x_1 + 2.8x_2 \le 5 \\ 1.94x_1 + 2.94x_2 \le 6.3 \\ 1.3x_1 + 2.05x_2 \le 5.75 \\ 1.44x_1 + 2.19x_2 \le 5.2250 \\ 2.5x_1 + 3x_2 \le 13.5 \\ 2.85x_1 + 3.35x_2 \le 13.15 \\ 1.65x_1 + 2.6x_2 \le 13.25 \\ 1.895x_1 + 2.88x_2 \le 12.7250 \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$

The optimal solution of the above linear programming is obtained as follows: $x^* = (2.7778, 0)$. And $\mu_1(x^*) + \mu_2(x^*) = 1$.

If we adopt different λ-levels of ITF-MOLP and μ-tolerance measure, the optimal solutions of the given problem are easily obtained by the algorithm proposed in this paper.

Example 2.[37] In a competitive business environment, a company produces two products I and II. Product I is manufactured approximately 1 hour by both machines A and B. Product II is manufactured approximately 2 hours and 1 hours by machine A and B, respectively. Subject to many factors such as machine breakdown, waiting for material, bottleneck, the available time of machine A is approximately 4 hours and 2 hours for machine B. In addition, product I is needed to be mixed after processing on both machines A and B. The estimated mixing time for product I is 2 hours. The available time for mixing is approximately 3 hours. The prices for product I and II are 2 and 1 dollar(s) per kilogram, respectively. The management of the company wants to determine how much to produce for each product to maximize the total revenue.

Let x_1 be the amount of product I to be produced and x_2 the amount of product II to be produced.

Therefore, we have the following linear programming problem:

$$Max \quad \tilde{\tilde{Z}} = \tilde{\tilde{2}}x_1 + \tilde{\tilde{1}}x_2 \tag{52}$$

s.t.

$$\begin{cases} \tilde{\tilde{1}}x_1 + \tilde{\tilde{2}}x_2 \preceq \tilde{\tilde{4}} \\ \tilde{\tilde{1}}x_1 + \tilde{\tilde{1}}x_2 \preceq \tilde{\tilde{2}} \\ \tilde{\tilde{2}}x_1 \preceq \tilde{\tilde{3}} \\ x_1, x_2 \ge 0. \end{cases} \tag{53}$$

Van Hop [37] assumes that all parameters are in form of normal-symmetric-triangular fuzzy numbers with the left and right spreads equal to 0.5. In this regards, the above parameters can be represented as the following

(1,1)-interval-valued trapezoidal fuzzy numbers:

$$
\begin{aligned}
\tilde{\tilde{c}}_1 &= \tilde{\tilde{2}} = \langle(1.5, 2, 2, 2.5; 1), (1.5, 2, 2, 2.5; 1)\rangle,\\
\tilde{\tilde{c}}_2 &= \tilde{\tilde{1}} = \langle(0.5, 1, 1, 1.5; 1), (0.5, 1, 1, 1.5; 1)\rangle,\\
\tilde{\tilde{a}}_{11} &= \tilde{\tilde{1}} = \langle(0.5, 1, 1, 1.5; 1), (0.5, 1, 1, 1.5; 1)\rangle,\\
\tilde{\tilde{a}}_{12} &= \tilde{\tilde{2}} = \langle(1.5, 2, 2, 2.5; 1), (1.5, 2, 2, 2.5; 1)\rangle,\\
\tilde{\tilde{a}}_{21} &= \tilde{\tilde{1}} = \langle(0.5, 1, 1, 1.5; 1), (0.5, 1, 1, 1.5; 1)\rangle,\\
\tilde{\tilde{a}}_{22} &= \tilde{\tilde{1}} = \langle(0.5, 1, 1, 1.5; 1), (0.5, 1, 1, 1.5; 1)\rangle,\\
\tilde{\tilde{a}}_{31} &= \tilde{\tilde{2}} = \langle(1.5, 2, 2, 2.5; 1), (1.5, 2, 2, 2.5; 1)\rangle,\\
\tilde{\tilde{a}}_{32} &= \tilde{\tilde{0}} = \langle(0, 0, 0, 0; 1), (0, 0, 0, 0; 1)\rangle,\\
\tilde{\tilde{b}}_1 &= \tilde{\tilde{4}} = \langle(3.5, 4, 4, 4.5; 1), (3.5, 4, 4, 4.5; 1)\rangle,\\
\tilde{\tilde{b}}_2 &= \tilde{\tilde{2}} = \langle(1.5, 2, 2, 2.5; 1), (1.5, 2, 2, 2.5; 1)\rangle,\\
\tilde{\tilde{b}}_3 &= \tilde{\tilde{3}} = \langle(2.5, 3, 3, 3.5; 1), (2.5, 3, 3, 3.5; 1)\rangle.
\end{aligned}
$$

Van Hop [37] has solved the above problem and obtained the optimal solution $X^{H\star} = (x_1^{H\star}, x_2^{H\star}) = (1.57, 0.94)$. Applying the signed distance ranking function to the above problem together with introducing the slack variables, Farhadinia [26] has given the optimal solution $X^{F\star} = (x_1^\star, x_2^\star) = (1.5, 0.5)$.

Let parameter $\omega_1 = \omega_2 = 0.5$. The levels λ and μ are very important parameters in the proposed models. It is necessary to know variation in the range of the optimal solution with the change of the parameters. Utilizing our proposed method, we obtain different optimal solutions of Example 2 with different values of λ and μ, which is shown in Table 1. The candidate values of levels λ and μ are respectively selected in the ranges $[0.5, 0.9]$ and $[0.5, 1]$ based on the decision makers' opinion. Moreover, we investigate the sensitive analysis with different levels λ and μ as shown in Table 2.

Table 1: Results of the optimal solution under different levels

λ	μ	$x_1^\star$	$x_2^\star$
0.9	1	1.5	0.5
0.9	0.95	1.5316	0.5453
0.8	0.9	1.5641	0.5938
0.8	0.85	1.5974	0.6458
0.7	0.8	1.6316	0.7018
0.7	0.75	1.6667	0.7619
0.6	0.7	1.7027	0.8267
0.6	0.65	1.7397	0.8966
0.5	0.6	1.7778	0.9722
0.5	0.5	1.8571	1.0476

The results shown in Tables 1 and 2 could be explained as follows. Firstly, it can be seen from Table 1 that the optimal solution $X^\star = (x_1^\star, x_2^\star) = (1.5, 0.5)$ when the levels λ and μ are set as 0.9 and 1 respectively. The optimal solution accords with the optimal solution given by Farhadinia [26]. Secondly, the optimal solution values would vary under different levels of λ and μ, which allows decision makers' objective attitude in the process of solving the linear programming problem. These results could help decision makers identify desired schemes. Finally, Table 2 tells us that for a small perturbation of the levels λ and μ, the variation of the optimal solution is very small. For example, when $\lambda = 0.9$ and $\mu = 0.95$, the optimal solution in Example 2 is $X^\star = (1.5316, 0.5453)$. Meanwhile, the optimal solution is $X^\star = (1.5316, 0.5452)$ as $\lambda = 0.9 + 10^{-4}$ and $\mu =$

$0.95 + 10^{-4}$, and the corresponding optimal solution is $X^{*} = (1.5317, 0.5454)$ while $\lambda = 0.9 - 10^{-4}$ and $\mu = 0.95 - 10^{-4}$. Therefore, the perturbation of the levels of λ and μ would not lead to unstable tendency of the optimal solution, which shows the robustnes of the results.

Table 2: Sensitive analysis

λ	μ	x_1^*	x_2^*
0.9	0.95	1.5316	0.5453
$0.9 + 10^{-4}$	$0.95 + 10^{-4}$	1.5316	0.5452
$0.9 - 10^{-4}$	$0.95 - 10^{-4}$	1.5317	0.5454
0.7	0.8	1.6316	0.7018
$0.7 + 10^{-4}$	$0.8 + 10^{-4}$	1.6315	0.7016
$0.7 - 10^{-4}$	$0.8 - 10^{-4}$	1.6316	0.7019
0.6	0.7	1.7027	0.8267
$0.6 + 10^{-4}$	$0.7 + 10^{-4}$	1.7026	0.8266
$0.6 - 10^{-4}$	$0.7 - 10^{-4}$	1.7028	0.8268

Note that the level of constraint violation (or satisfaction) provides one tool for comparing optimal solutions. If we investigate the tightly of constraints of the above linear programming in optimal solutions. For example, we select $X^{*} = (1.5, 0.5)$, $X_1^{*} = (1.5316, 0.5453)$ and $X_2^{*} = (1.5641, 0.5938)$ with different levels in our proposed algorithm, take $X^{F*} = (1.5, 0.5)$ given by Farhadinia [26] and discuss $X^{H*} = (1.57, 0.94)$ obtained by Van Hop [37]. The results are summarized in Table 3. As pointed out by Farhadinia [26], it is shown in Table 3 that the feasibility level of constraint violation in X^{*} and X^{F*} are intuitionally more controllable than that in Van Hop's solution X^{H*}. In addition, the feasibility level of constraint violation in X_1^{*} and X_2^{*} are also better than Van Hop's optimal solution. From the obtained results, it is clear that our proposed algorithm could gives better solution than Farhadinia and Van Hop's methods.

Table 3: Level of linear programming constraint violation in optimal solutions

	The left-hand side of constraint	The right-hand side of constraint
First constraint in X^{*}	$\overline{\langle(2, 3, 3, 4; 1)\rangle}$	$\overline{\langle(3.5, 4, 4, 4.5; 1)\rangle}$
First constraint in X^{1*}	$\overline{\langle(1.5838, 2.6222, 2.6222, 3.6607; 1)\rangle}$	$\overline{\langle(3.5, 4, 4, 4.5; 1)\rangle}$
First constraint in X^{2*}	$\overline{\langle(1.6728, 2.7517, 2.7517, 3.8307; 1)\rangle}$	$\overline{\langle(3.5, 4, 4, 4.5; 1)\rangle}$
First constraint in X^{F*}	$\overline{\langle(2, 3, 3, 4; 1)\rangle}$	$\overline{\langle(3.5, 4, 4, 4.5; 1)\rangle}$
First constraint in X^{H*}	$\overline{\langle(3.135, 4.39, 4.39, 5.645; 1)\rangle}$	$\overline{\langle(3.5, 4, 4, 4.5; 1)\rangle}$
Second constraint in X^{*}	$\overline{\langle(1, 2, 2, 3; 1)\rangle}$	$\overline{\langle(1.5, 2, 2, 2.5; 1)\rangle}$
Second constraint in X^{1*}	$\overline{\langle(1.0385, 2.0769, 2.0769, 3.1154; 1)\rangle}$	$\overline{\langle(1.5, 2, 2, 2.5; 1)\rangle}$
Second constraint in X^{2*}	$\overline{\langle(1.0790, 2.1579, 2.1579, 3.2369; 1)\rangle}$	$\overline{\langle(1.5, 2, 2, 2.5; 1)\rangle}$
Second constraint in X^{F*}	$\overline{\langle(1, 2, 2, 3; 1)\rangle}$	$\overline{\langle(1.5, 2, 2, 2.5; 1)\rangle}$
Second constraint in X^{H*}	$\overline{\langle(1.255, 2.51, 2.51, 3.765; 1)\rangle}$	$\overline{\langle(1.5, 2, 2, 2.5; 1)\rangle}$
Third constraint in X^{*}	$\overline{\langle(2.25, 3, 3, 3.75; 1)\rangle}$	$\overline{\langle(2.5, 3, 3, 3.5; 1)\rangle}$
Third constraint in X^{1*}	$\overline{\langle(2.2974, 3.0632, 3.0632, 3.8290; 1)\rangle}$	$\overline{\langle(2.5, 3, 3, 3.5; 1)\rangle}$
Third constraint in X^{2*}	$\overline{\langle(2.3462, 3.1282, 3.1282, 3.9103; 1)\rangle}$	$\overline{\langle(2.5, 3, 3, 3.5; 1)\rangle}$
Third constraint in X^{F*}	$\overline{\langle(2.25, 3, 3, 3.75; 1)\rangle}$	$\overline{\langle(2.5, 3, 3, 3.5; 1)\rangle}$
Third constraint in X^{H*}	$\overline{\langle(2.355, 3.14, 3.14, 3.925; 1)\rangle}$	$\overline{\langle(2.5, 3, 3, 3.5; 1)\rangle}$

where, for notation convenience, we represent $\langle(a_1, a_2, a_3, a_4; h), (a_1, a_2, a_3, a_4; h)\rangle$ by $\overline{\langle(a_1, a_2, a_3, a_4; h)\rangle}$.

Example 3. Consider the following linear programming:

$$Min\ \tilde{\tilde{Z}} = \tilde{\tilde{c}}_1 x_1 + \tilde{\tilde{c}}_2 x_2 \tag{54}$$

s.t.

$$\begin{cases} \tilde{\tilde{a}}_{11} x_1 + \tilde{\tilde{a}}_{12} x_2 \preceq \tilde{\tilde{b}}_1 \\ \tilde{\tilde{a}}_{21} x_1 + \tilde{\tilde{a}}_{22} x_2 \preceq \tilde{\tilde{b}}_2 \\ x_1,\ x_2 \geq 0. \end{cases} \tag{55}$$

Where the values of parameters are given as follows,

$$\begin{aligned} \tilde{\tilde{c}}_1 &= \left[(-5,-3,-2),(-10,-3,-1)\right], \\ \tilde{\tilde{c}}_2 &= \left[(8.5,10,11),(8,10,13.5)\right], \\ \tilde{\tilde{a}}_{11} &= \left[(0.75,1,1.2),(0.5,1,1.4)\right], \\ \tilde{\tilde{a}}_{12} &= \left[(0.75,1,1.1),(0.5,1,1.3)\right], \\ \tilde{\tilde{a}}_{21} &= \left[(2.5,3,3.7),(2,3,4)\right], \\ \tilde{\tilde{a}}_{22} &= \left[(2.5,3,3.7),(2,3,4.5)\right], \\ \tilde{\tilde{b}}_1 &= \left[(6,7,7.5),(5.5,7,8)\right], \\ \tilde{\tilde{b}}_2 &= \left[(6,7,8),(5,7,8.5)\right]. \end{aligned}$$

Utilizing our proposed method, the linear programming problem could be converted into the following programming:

$$Min\ (\tilde{Z})^C_\lambda \wedge (\tilde{Z})^R_\lambda \tag{56}$$

s.t.

$$\begin{cases} (\tilde{Z})^C_\lambda = \left[0.5\omega_1(-7+\lambda) + 0.5\omega_2(-11+5\lambda)\right]x_1 + \left[0.5\omega_1(19.5+0.5\lambda) + 0.5\omega_2(21.5-1.5\lambda)\right]x_2 \\ (\tilde{Z})^R_\lambda = \left[\omega_1(-2-\lambda) + \omega_2(-1-2\lambda)\right]x_1 + \left[\omega_1(11-\lambda) + \omega_2(13.5-3.5\lambda)\right]x_2 \\ 0.5x_1 + 0.5x_2 \leq 8 \\ (0.5+0.5\mu)x_1 + (0.5+0.5\mu)x_2 \leq 8-\mu \\ 0.625x_1 + 0.625x_2 \leq 7.75 \\ (0.625+0.375\mu)x_1 + (0.625+0.375\mu)x_2 \leq 7.75-0.75\mu \\ 2x_1 + 2x_2 \leq 8.5 \\ (2+\mu)x_1 + (2+\mu)x_2 \leq 8.5-1.5\mu \\ 2.25x_1 + 2.25x_2 \leq 8.25 \\ (2.25+0.75\mu)x_1 + (2.25+0.75\mu)x_2 \leq 8.25-1.25\mu \\ x_1,\ x_2 \geq 0. \end{cases} \tag{57}$$

Applying the signed distance ranking function together with the slack variables x_3 and x_4, Farhadinia [26] transforms the above problem into the following problem:

$$Min\ \tilde{Z} = \tilde{c}_1 x_1 + \tilde{c}_2 x_2 + \tilde{0} x_3 + \tilde{0} x_4 \tag{58}$$

s.t.

$$\begin{cases} 7.85x_1 + 7.65x_2 + x_3 = 55 \\ 24.2x_1 + 24.7x_2 + x_4 = 55.5 \\ x_1,\ x_2,\ x_3,\ x_4 \geq 0. \end{cases} \tag{59}$$

Then by Matlab software, we could obtain the optimal solutions with different parameters ω_1, ω_2, λ and μ, which is shown in Table 4. With the increase of λ and μ, there is a decreasing trend for the optimal solution. Particularly, when levels $\lambda = 0.9$ and $\mu = 1$, the optimal solution is $X^{*} = (2.3333, 0.0000)$. This approximately agrees with the optimal solution $X^{F*} = (2.2934, 0.0000)$ obtained by Farhadinia's signed distance method [26]. Table 5 presents the sensitive analysis with different levels of the parameters. It is shown that the optimal solution is stable, whose variety may be ignored with the perturbation of the levels of λ and μ.

Table 4: The optimal solutions with different parameters

ω_1	ω_2	λ	μ	x_1^*	x_2^*
1	0	0.9	1	2.3333	0.0000
1	0	0.8	0.9	2.4359	0.0000
0.7	0.3	0.8	0.8	2.5439	0.0000
0.7	0.3	0.7	0.7	2.6577	0.0000
0.6	0.4	0.6	0.6	2.7778	0.0000
0.6	0.4	0.6	0.5	2.9048	0.0000

Table 5: Sensitive analysis

ω_1	ω_2	λ	μ	x_1^*	x_2^*
0.7	0.3	0.8	0.95	2.3840	0.0000
0.7	0.3	$0.8 + 10^{-4}$	$0.95 + 10^{-4}$	2.3839	0.0000
0.7	0.3	$0.8 - 10^{-4}$	$0.95 - 10^{-4}$	2.3841	0.0000
0.6	0.4	0.7	0.75	2.6000	0.0000
0.6	0.4	$0.7 + 10^{-4}$	$0.75 - 10^{-4}$	2.6001	0.0000
0.6	0.4	$0.7 - 10^{-4}$	$0.75 + 10^{-4}$	2.5999	0.0000

6 Conclusions

In this paper, we consider multi-objective linear programming problems on the basis of interval-typed triangular fuzzy numbers. An algorithm of the proposed problem is established. By introducing the λ–cut set of interval-typed triangular fuzzy numbers and the dominance possibility criterion to compare with two interval-typed fuzzy numbers, the problem is equivalently transformed into maximization of the sum of membership degrees of each original objective when λ level is given in advance. These membership degrees are obtained based on the deviation from optimal solutions of individual objectives, and the constraints are transformed to classical inequalities by utilizing the dominance possibility criterion. The linear programming model could be solved by Matlab toolbox. It could be noted that the proposed algorithm for a problem constituting k objectives turns out solving $(2k + 1)$ linear programming problems. Three illustrated examples are given to validate the feasibility of the proposed algorithm.

Further studies will explore generalized multi-objective linear programming based on interval-typed fuzzy numbers, in which the coefficients of multi-objective functions and constraints may be stated as different typed fuzzy numbers such as interval-typed trapezoidal fuzzy numbers and interval-valued fuzzy numbers. It would be interesting to consider different tolerance measures for each constraint. Also, in order to more efficiently cope with optimal decision making and optimal control in uncertain systems, we will develop the application of the proposed multi-objective linear programming models.

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