REVUE FRANÇAISE D'AUTOMATIQUE, D'INFORMATIQUE ET DE RECHERCHE OPÉRATIONNELLE. RECHERCHE OPÉRATIONNELLE

MAHMUT PARLAR

WANG QINAN

A game theoretical analysis of the quantity discount problem with perfect and incomplete information about the buyer's cost structure

Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle, tome 29, nº 4 (1995), p. 415-439.

<http://www.numdam.org/item?id=RO_1995__29_4_415_0>

Recherche opérationnelle/Operations Research
(vol. 29, n° 4, 1995, p. 415 à 439)

A GAME THEORETICAL ANALYSIS OF THE QUANTITY DISCOUNT PROBLEM WITH PERFECT AND INCOMPLETE INFORMATION ABOUT THE BUYER'S COST STRUCTURE (*)

by Mahmut PARLAR (1) and QINAN WANG (2)

Communicated by Andrew K. S. JARDINE

Abstract. – *In this paper, we analyze the quantity discount problem by considering the competitive nature of the problem and the informational structure regarding the buyer's cost structure. We formulate the problem as a two-person nonzero-sum game and analyze the seller's optimal quantity discount schedule and the buyer's optimal order quantity by using Stackelberg equilibrium. We show that it is always possible for the seller and the buyer to gain from quantity discount. However, a quantity discount schedule under which the buyer orders more than his EOQ at the discounted price is necessary for the seller and the buyer to gain. The optimal quantity discount schedule when the seller knows the buyer's cost parameters is given by a single break point. When the seller does not know the buyer's cost parameters, an optimal quantity discount schedule may not exist. Two approaches have been developed for the seller to offer quantity discount in this case. The application of our analysis is discussed. Our results can be especially useful when the seller has many buyers.*

Keywords: Game theory, quantity discount.

Résumé. – *Nous analysons dans cet article le problème du rabais pour quantité en considérant la nature compétitive du problème et la structure informationnelle au regard de la structure des coûts de l'acheteur. Nous formulons le problème comme un jeu à deux personnes à somme nulle. Nous analysons le plan de rabais optimal du vendeur et le réapprovisionnement optimal de l'acheteur en utilisant l'équilibre de Stackelberg. Nous montrons qu'il est toujours possible pour le vendeur et l'acheteur de tirer un profit du rabais pour quantité. Cependant, un plan de rabais selon lequel l'acheteur commande davantage que son « EOQ » au prix de rabais est nécessaire pour que le vendeur et l'acheteur puisse faire un gain. Le plan de rabais optimal lorsque le vendeur connaît les paramètres de coût de l'acheteur est donné par un simple point de coupure. Dans le cas contraire, un plan de rabais optimal peut ne pas exister. Deux méthodes d'attaque sont développées alors pour le plan de rabais par le vendeur. Nous appliquons notre analyse, et nos résultats sont particulièrement utiles lorsque le vendeur a plusieurs acheteurs.*

Mots clés : Théorie des jeux, rabais pour quantité.

(*) This research was supported by the Natural Sciences and Engineering Research Council of Canada under Grant No. A5872. The authors wish to thank a referee for his insightful comments on the earlier versions of the paper.
Received July 1992.
(1) School of Business McMaster University Hamilton, Ontario, Canada, L8S 4M4.
E-mail: PARLAR@MCMASTER.CA
(2) Nanyang Business School, Nanyang Technological University, Singapore.

Recherche opérationnelle/Operations Research, 0399-0559/95/04/$ 4.00

1. INTRODUCTION

Traditional quantity discount models analyze primarily buyers' best reactions to quantity discount schedules provided by sellers. They minimize buyers' total buying cost and inventory related cost, assuming that sellers offer a quantity discount schedule and then accept orders, usually of larger sizes, that buyers place (Sethi, 1984; Hadley and Whitin, 1963; Peterson and Silver, 1979).

These models are useful for buyers to order in optimal quantities when replenishing their inventory. However, the models do not consider sellers' decisions of offering optimal quantity discount schedules. Recently, decision models have been developed solely from the sellers' perspective (Monahan, 1984; Lee and Rosenblatt, 1986; Dada and Srikanth, 1987). The models focus on sellers' best quantity discount schedules, assuming that buyers will cooperate as long as they will not be worse-off.

There are still two important issues of the quantity discount problem that should be considered. First, independent suppliers and buyers pursue their own interests in quantity discount. These interests are normally in conflict. Consider a situation where a buyer orders periodically from a seller. There are two decisions to make: order quantity and price. Order quantity is controlled by the buyer and price is determined by the seller. If the seller offers a quantity discount schedule to the buyer, the buyer orders the quantity that minimizes his total relevant costs. As such, the seller might want to take the buyer's reaction into consideration when offering his quantity discount schedule. Models developed exclusively from one party's point of view suppress the conflict of the seller and the buyer and fail to incorporate the competitive nature of the problem.

Second, sellers and buyers are invariably independent and opportunistic agents who are likely to have private information about their cost structures. Models on the quantity discount problem in the literature have always assumed that sellers and buyers have perfect information about each other's cost parameters. This appears to be unrealistic and is a major weakness for the models to be applied in practice. What one can do to get information about others' cost structures is usually to make some estimates of their cost parameters.

In this paper, we present an economic analysis of the quantity discount problem using a game-theoretic approach in which the seller and the buyer are treated as players in a two-person game. We analyze the seller's optimal quantity discount schedule and the buyer's optimal order quantity when they do not know each other's cost structure. The analysis is focused on sellers' quantity discount schedules because sellers usually lead in quantity discount by offering buyers quantity discount schedules.

The paper is organized as follows. In Section 2, we formulate the discount problem as a two-person nonzero-sum game in the normal form. In Section 3, we solve the problem with the model built in Section 2 by using Stackelberg equilibrium. Subsequently, we discuss the applications of our results in Section 4. Finally, main findings and possible extensions to this research are summarized in Section 5.

2. THE MODEL

We first consider the situation where a seller sells a single product to a single buyer who faces a deterministic and constant demand. This has been the setting of many discussions of the quantity discount problem (Monahan, 1988; Lee and Rosenblatt, 1986), probably because it is mathematically simple and can still provide useful insights into the problem.

We assume that the seller buys the product from another supplier or produces it himself at a large production rate that can be considered to be infinite for the purpose of this model. Currently, the seller offers the buyer the market price and the buyer orders his economic order quantity (EOQ) each time from the seller. In addition, the following conditions are assumed in our analysis: (1) both the seller's and the buyer's lead times are known with certainty; (2) no backlogging and lost-sales are allowed; (3) the seller and the buyer are motivated by profit maximization and cost savings, respectively; and (4) the players are rational and use only pure strategies.

We use the following notation.

A_b := the buyer's fixed ordering cost per order;

A_s := the seller's fixed cost of processing one order placed by the buyer;

A_e := the seller's fixed set-up cost or ordering cost per order when producing or ordering from an external supplier;

H_b := the buyer's inventory carrying cost per dollar per year;

H_s := the seller's inventory carrying cost per dollar per year;

D := the buyer's deterministic annual demand rate for the product;

C := the seller's unit acquisition cost (unit production cost or unit buying cost from the external supplier);

P := the unit maket price (the seller's initial unit selling price or the buyer's unit buying cost in the absence of discount);

Q := the buyer's EOQ under the unit market price P, *i.e.*, $Q = \sqrt{(2\, D\, A_b / P\, H_b)}$;

x := the factor by which the buyer will increase his ordering size, *i.e.*, he will order $(1 + x)\, Q$ units each time;

y := the factor by which the seller will decrease his unit selling price, *i.e.*, he will offer a unit selling price of $(1 - y)\, P$.

A quantity discount scheme is a scenario in which the seller offers a lower unit selling price than the market price and the buyer orders each time a larger quantity than this EOQ under the market price. Using the notations defined above, it consists of a pair of ordering size $(1 + x)\, Q$ and unit selling price $(1 - y)\, P$, where x and y are such that

$$0 \le x, \tag{1}$$

$$0 \le y < 1. \tag{2}$$

The buyer's concern in this problem is his total inventory related and purchase cost of the product. The buyer's total annual relevant cost, denoted by TC, can be expressed as

$$\begin{aligned} TC &= (\text{Purchase cost}) + (\text{Ordering cost}) + (\text{Carrying cost}) \\ &= DP + A_b\, D/Q + QPH_b/2. \end{aligned} \tag{3}$$

If a discount scheme is formulated, the new total cost, TC_b, is

$$TC_b = DP\,(1 - y) + DA_b/[(1 + x)\, Q] + (1 + x)\,(1 - y)\, QPH_b/2.$$

What the buyer can gain under $(x,\, y)$ is $\pi_b\,(x,\, y) = TC - T\, C_b$, or

$$\pi_b\,(x,\, y) = y\, DP + [2 - 1/(1 + x) - (1 + x)\,(1 - y)]\, C_b/2.$$

where, $h_b = PH_b$ is the buyer's inventory holding cost per unit per year in the absence of discount and $C_b = \sqrt{(2\, D\, A_b\, h_b)}$ is the buyer's annual average inventory related costs when the seller offers the market unit price.

The seller's concern is also what he can gain from quantity discount. The relevant factors include his sales revenue, acquisition cost, order processing cost and inventory related cost which includes his inventory carrying cost and ordering cost from the external supplier. His annual profit, denoted by TP, can be expressed as

$$\begin{aligned} TP = {} & (\text{Sales revenue}) - (\text{Acquisition cost}) - (\text{Order processing cost}) \\ & - (\text{Inventory related cost}). \end{aligned}$$

The first three terms are, clearly, DP, DC, and $A_s D/Q$. To obtain the inventory related cost, the seller's replenishment policy or his optimal production batch size or ordering size has to be considered.

Suppose the seller orders a quantity at a time enough to meet N orders of size Q, where N is an integer. It has been shown that (Lee and Rosenblatt, 1986) the seller's total annual inventory related cost, denoted by TI, is

$$TI = [(N-1)/2]\, Q\, h_s + DA_e/(NQ), \tag{6}$$

where $h_s = CH_s$ is the seller's inventory holding cost per unit per year.

It is easy to verify that TI is a convex function of N. Thus, the optimal N is determined by $TI\,(N) \leq TI\,(N+1)$ and $TI\,(N) < TI\,(N-1)$. After some modification, these conditions turn out to be

$$\sqrt{(r+1/4)} - 1/2 \leq N < \sqrt{(r+1/4)} + 1/2, \tag{7}$$

where $r = Q_0^2/Q^2 = (A_e/h_s)/(A_b/h_b)$ and $Q_0 = \sqrt{(2\,A_e\,D/h_s)}$ is the seller's EOQ when facing a uniform demand of D units per year. Rewriting (7), we obtain

$$\begin{aligned} N^* &= 1 \quad \text{if} \quad r \leq 2, \\ &= 2 \quad \text{if} \quad 2 < r \leq 6, \\ &= 3 \quad \text{if} \quad 6 < r \leq 12, \ldots . \end{aligned} \tag{8}$$

The seller's optimal replenishing policy depends on r, a ratio (squared) of the seller's EOQ to the buyer's EOQ or a ratio of the seller's inventory related costs to the buyer's inventory related costs. If $r \leq 2$, the optimal N is 1. The seller orders what the buyer orders each time and carries no inventory (lot-for-lot policy). If $r > 2$, the lot-for-lot policy is not optimal and (8) should be used to determine the best replenishing policy. In both cases, his total inventory related cost is determined by (6).

Let N_0 denote the seller's initial optimal replenishing policy, *i.e.*, the seller orders $N_0\,Q$ units each time when the buyer orders Q units each time. The seller's annual profit, denoted by TP, when offering the market price to the buyer (no discount) is

$$TP = DP - DC - A_s\,D/Q - [(N_0 - 1)/2]\,Q\,h_s - (D\,A_e)/(N_0\,Q). \tag{9}$$

When a discount scheme is formulated, the buyer's order quantity is $(1+x)\,Q$ and the above analysis still holds. The seller's optimal ordering policy, denoted by N_x, is determined by

$$\sqrt{[r/(1+x)^2 + 1/4]} - 1/2 \leq N_x \leq \sqrt{[r/(1+x)^2 + 1/4]} + 1/2. \tag{10}$$

This can be rewritten in terms of N_0 and x as

$$N_x = N_0 - i \quad \text{when } x_i \leq x < x_{i+1}, \tag{11}$$

where $i = 0, 1, \ldots, N_0 - 1$, $x_0 = 0$, $x_i = \sqrt{\{r/[(N_0 - i)(N_0 - i + 1)]\}} - 1$ for $0 < i < N_0$, and $x_{N_0} = \infty$.

The seller's total inventory related cost is

$$TI_x = [(N_x - 1)/2](1 + x) Q h_s + D A_e/[N_x (1 + x) Q]. \tag{12}$$

At each x_i, $TI_x (N_0 - i + 1) = TI_x (N_0 - i)$, $0 < i < N_0$, and TI_x is continuous. However, TI_x is not differentiable at x_i. By differentiating TI_x in (x_i, x_{i+1}), we obtain

$$d TI_x/dx = (N_0 - i - 1)(Q/Q_0)(C_s/2)[1 - (1 + x_{i+1})^2/(1 + x)^2] < 0. \tag{13}$$

Therefore, TI_x decreases as x increases and the seller reduces his inventory related cost when the buyer increases his ordering size.

The seller's annual profit, denoted by TP_s, is

$$\begin{aligned} T P_s = {} & (1 - y) DP - DC - A_s D/[Q (1 + x)] \\ & - (N_x - 1)(1 + x) Q h_s/2 - DA_e/[N_x (1 + x) Q]. \end{aligned} \tag{14}$$

What the seller can gain under (x, y) is $\pi_s (x, y) = TP_s - TP$, or,

$$\begin{aligned} \pi_s (s, y) = {} & - y DP + [1 - 1/(1 + x)] A_s D/Q \\ & + (Q h_s/2)[(N_0 - 1) - (N_x - 1)(1 + x)] \\ & - (DA_e/Q) \{ 1/[(1 + x) N_x] - 1/N_0). \end{aligned} \tag{15}$$

Note that $\pi_s (x, y)$ is continuous in both x and y.

We have thus formulated the discount problem as a two-person game in the normal form. The buyer and the seller formulate a discount scheme (x, y), over which the buyer exerts his control through his ordering size (x) and the seller exerts his control through his unit selling price (y). The payoff functions for the buyer and the seller are given by $\pi_b (x, y)$ and $\pi_s (x, y)$ and their strategy spaces are given by (1) and (2), respectively. This is a nonzero-sum $(\pi_s + \pi_b \neq 0)$ game.

Consider the impact of an increase in the buyer's order quantity on the seller. If $r \leq 2$, the seller uses the lot-for-lot policy. Any increase in the buyer's order quantity does not alter this policy. His gain by inducing the buyer to order large quantities comes from only the decrease in his ordering and order processing costs associated with the buyer's reduced frequency of ordering. If $r > 2$, the seller should not use the lot-for-lot policy initially. When the buyer increases his ordering size, he orders or produces in batches including fewer and fewer number of orders of the buyer. His gain by inducing the buyer to order larger quantities comes from not only the decrease in his ordering and order processing cost but also the reduction in this inventory carrying cost.

If the seller uses the lot-for-lot policy initially, $N_0 = 1$ and $N_x = 1$. For any $x > 0$, (15) becomes

$$\pi_s(x, y) = -y\,DP + [1 - 1/(1 + x)]\,(A_e + A_s)\,D/Q. \tag{16}$$

This is the case discussed by Monahan (1984). He assumed implicitly that the seller is always using the lot-for-lot policy and carries no inventory. As pointed out by Lee and Rosenblatt (1986) and Joglekar (1988), the lot-for-lot policy is normally inappropriate. As shown above, the seller's initial optimal replenishing policy depends on his own EOQ and the buyer's EOQ. The seller, as a major intermediary, usually has better inventory facilities than the buyer. His EOQ is often much larger than the buyer's EOQ.

In the modelling of the quantity discount problem, the buyer's inventory carrying cost formula has been used to obtain the seller's saving in inventory carrying cost (Lal and Staelin, 1984; Dada and Srikanth, 1987). It can be seen from our analysis that this extension may be used only if the seller's order quantity is determined independently of the buyer's ordering size. In the discussion of a single buyer, this is normally not the case. This translation requires an additional condition $H_b > H_s$ for the existence of an optimal solution (Dada and Srikanth, 1987).

By the nature of the problem, no one will play the game if quantity discount makes him worse-off, hence $\pi_b(x, y) \geq 0$ and $\pi_s(x, y) \geq 0$. After some algebra, these conditions become

$$y \geq y_b = [x^2\,(C_b/2)/(1 + x)]/[DP + (1 + x)\,C_b/2], \tag{17}$$

$$\begin{aligned} y \leq y_s = {} & [1 - 1/(1 + x)]\,A_s/(PQ) + [Q\,h_s/(2\,DP)] \\ & \times [(N_0 - 1) - (N_x - 1)\,(1 + x)] \\ & - [A_e/(PQ)]\,\{1/[(1 + x)\,N_x] - 1/N_0\}. \end{aligned} \tag{18}$$

We note that both (17) and (18) are continuous and increasing in x. The set of such points is shown by the area bounded by solid curves in figure.

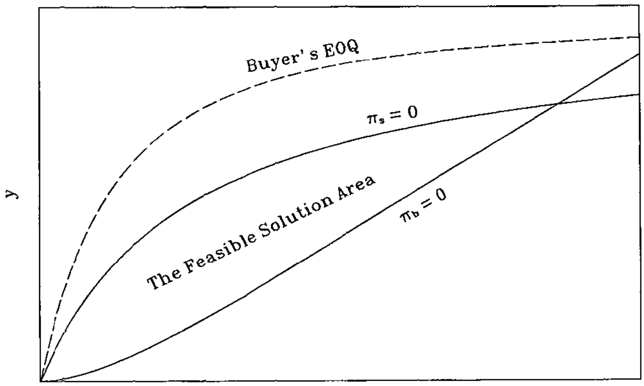

Figure. – The feasible solution area.

The area bounded by solid curves in figure characterizes the feasible solution area for the problem. Any point on the boundary represents a discount scheme which makes one gain and the other neither lose nor gain, except the two end points at which no one gains or loses; any point inside this area represents a discount scheme which makes both players gain; and any point outside the area makes at least one lose.

LEMMA 1: *The feasible solution area is always non-empty.*

Proof: See Appendix I.

Q.E.D.

For any product that the seller and the buyer are trading, it is always possible and worthwhile for both the seller and the buyer to exploit the benefit of quantity discount. We discuss in the following the seller's optimal quantity discount schedule and the buyer's optimal order quantity.

3. SOLUTION TO THE PROBLEM

We consider independent sellers and buyers who pursue their own interests. Because their interests are usually in conflict, they rarely work in full cooperation as assumed by some authors (Goyal, 1976; Banerjee, 1986a; Kohli and Park, 1989). We discuss non-cooperative solutions to the quantity discount problem.

Quantity discount schedules are usually given by suppliers. Buyers always order each time the quantity that minimizes their total relevant costs. Therefore, sellers act as leaders and buyers act as followers in quantity discount. The determination of seller's quantity discount schedules should take buyers' reaction into consideration. This type of decision problem is characterized by Stackelberg equilibrium, which gives the leader's optimal decision by maximizing his payoff after taking account of the follower's response to his decision (Basar and Olsder, 1982, pp. 176-183). In what follows, we focus on the seller's quantity discount schedule by using Stackelberg equilibrium.

3.1. Informational structure

It has been generally assumed that sellers and buyers know each other's cost parameters (Goyal, 1976; Dada and Srikanth, 1987; Jucker and Rosenblatt, 1985). This is normally unrealistic for independent suppliers and buyers because revealing one's cost parameters usually puts one into a position of disadvantage in bargaining and competition. They are likely to have private information about their cost parameters.

In this paper, we discuss the quantity discount problem when the seller and the buyer do not know each other's cost structure. Because the buyer determines his order quantity according to the quantity discount schedule provided by the seller, he does not need information about the seller's cost parameters to make his ordering decision. However, to determine his optimal quantity discount schedule, the seller has to take the buyer's reaction into consideration and needs information about the buyer's cost parameters.

The seller's decision depends on his own payoff function was well as the buyer's payoff function. The payoff functions are determined by the market demand, the seller's and the buyer's cost structures, the seller's selling price and the buyer's order quantity. We assume that each of the seller and the buyer knows his own cost parameters and the market demand with certainty. We also assume that the seller knows the buyer's EOQ under the market price from the buyer's previous orders. Conversely, the only unknown factor for the seller's decision making is the buyer's cost parameters. From the buyer's payoff function, it can be seen that information is summarized in a single parameter C_b, or the buyer's annual inventory related cost under the market price. The value of C_b determines the position of the lower boundary of the feasible solution area (*Fig.*).

We analyze the seller's quantity discount schedule in two cases. First, C_b is known to the seller with certainty. In this case the seller has perfect information about the buyer's cost structure. Second, the seller does not know C_b with certainty. In this case the seller has incomplete information about the buyer's cost structure.

3.2. Reaction curve

If the seller's decision y is given, the buyer will choose the order quantity x that maximizes his payoff $\pi_b(x, y)$. When y goes through its domain $[0, 1)$, the buyer's decision will form a curve over $0 \leq y < 1$. This curve is called the buyer's reaction curve, which gives the buyer's best responses to the seller's all possible decisions (Basar and Olsder, 1982). Similarly, the seller's reaction curve is his best responses to the buyer's possible decisions.

For a given y, we obtain

$$d\pi_b/dx = [1/(1+x)^2 - (1-y)]\, C_b/2, \tag{19}$$

$$d^2\, \pi_b/dx^2 = -C_b/(1+x)^3 < 0. \tag{20}$$

Thus, π_b is concave in x for $x \geq 0$. The optimal x, for a given y, is determined by setting (19) equal to zero, which gives

$$x = 1/\sqrt{(1-y)} - 1. \tag{21}$$

This is the buyer's reaction curve. It is easy to see that this is the buyer's EOQ formula for a given y.

On the other hand, $d\pi_s/dy = -DP < 0$ for any given x. Therefore, π_s is strictly decreasing in y and attains its maximum at $y = 0$ for any given x. The seller's reaction curve is $y = 0$. This is reasonable because the seller would never lower his selling price if the buyer's order quantity is given in advance. The buyer does not have the potential to act as the leader in the quantity discount problem.

3.3. Quantity discount

A quantity discount schedule is characterized by a direct association between discount and order quantity. Different restrictions on discount and order quantity can form different quantity discount schedules. Three popular (all-unit, incremental and carload lot) quantity discount schedules have been discussed by Jucker and Rosenblatt (1985). They showed that an all-unit quantity discount schedule is general enough to admit others as special cases. Thus, we use only all-unit quantity discount schedules in our analysis. In an all-unit quantity discount schedule, all units in an order are eligible for the appropriate discount.

An all-unit quantity discount schedule defines price as a non-increasing function of order quantity. Generally, it is defined by using a step function as: the seller offers a unit selling price P_i for any order between Q_i and Q_{i+1}, where $i = 1, 2, \ldots, n$, $P_1 > P_2 > \ldots > P_n$, $Q_1 < Q_2 < \ldots < Q_n$ and $Q_{n+1} = \infty$. In our notations, a quantity discount schedule is that the seller offers a unit selling price $(1 - y_i) P$ for any order between $Q(1 + x_i)$ and $Q(1 + x_{i+1})$, where $i = 0, 1, \ldots, n$, $0 = y_0 < y_1 < \ldots < y_n$, $0 = x_0 < x_1 < \ldots < x_n$ and $x_{n+1} = \infty$. Each (x_i, y_i) $(i > 0)$ is referred to as a break point.

LEMMA 2: *The seller should not offer any discount to the buyer without using a quantity discount schedule.*

Proof: See Appendix II.

Q.E.D.

If the seller offers a discount without imposing any restriction on the order quantity eligible for the discount, the buyer orders according to his EOQ and the seller loses. Thus, the seller should not offer any discount without using a quantity discount schedule. A quantity discount schedule is necessary for the seller to gain from quantity discount.

Lemma 2 also suggests that if the seller is to offer a quantity discount schedule, he has to make the quantity for each break point greater than the buyer's EOQ under the discounted price. In other words, each break point (x, y) has to satisfy the condition

$$(1 + x)\sqrt{(1 - y)} > 1. \tag{22}$$

Otherwise $(1 + x)\sqrt{(1 - y)} \leq 1$ or $(1 + x) Q \sqrt{(1 - y)}$, where $(1 + x) Q$ is the quantity at the break point and $Q/\sqrt{(1 - y)}$ is the buyer's EOQ under the unit selling price $(1 - y) P$. The buyer orders his EOQ under the discounted price and the seller loses. We note that every point in the feasible solution area satisfies condition (22).

PROPOSITION 1: *A quantity discount schedule that makes the buyer order more than his EOQ under the discounted price is necessary and sufficient for the seller and the buyer to exploit the benefit of quantity discount.*

Proof: The necessity condition has been established by Lemma 2.

From Lemma 1, we have at least one point in the feasible solution area. For any point in the feasible solution area, assign it to be the single break point of a quantity discount schedule. The buyer orders the quantity at the break point, and both the seller and the buyer gain. Therefore, a quantity discount schedule can be used to achieve any possible quantity discount scheme for independent suppliers and buyers.

Q.E.D.

A quantity discount schedule provides a necessary and good mechanism for the seller and the buyer to gain from quantity discount. This might explain why quantity discount schedules have been so widely used in practice. In the following, we discuss the determination of quantity discount schedules for the seller when he acts as the leader by using Stackelberg equilibrium. Because the buyer's reaction to a quantity discount schedule depends on the break points, the analysis will be different from that when no quantity discount schedule is used. We consider only points in the feasible solution area as possible break points of quantity discount schedules.

3.3.1. *Perfect Information*

We first consider the case where the buyer's cost structure or C_b is known to the buyer with certainty. When a single buyer is considered, only one order quantity will be selected. Because the seller knows the buyer's reaction to a break point, a quantity discount schedule with only one break point is adequate. Such a schedule in our notations is defined as: the seller offers no discount for any order $0 \leq x < x_1$ and a discount y_1 for any order $x_1 \leq x$, where $x_1 > 0$ and $y_1 > 0$.

The seller gains nothing if the buyer orders the initial order quantity Q or $x = 0$. If he wants to gain from quantity discount, the seller has to provide sufficient incentive for the buyer to order more than Q units each time. Let $w \geq 0$ the least amount of gain that the buyer is interested to change his order quantity. The buyer orders more than his EOQ under the market price or Q units each time if $\pi_b \geq w$.

The buyer's reaction (order quantity) to a quantity discount schedule with a single break point (x_1, y_1) is then $x = 0$ if $\pi_b(x_1, y_1) < w$ and $x = x_1$ if $\pi_b(x_1, y_1) \geq w$. On the other hand, the seller gains zero if the buyer orders Q units each time $(x = 0)$ and $\pi_s(x_1, y_1)$ if the buyer orders $(1 + x_1) Q$ units each time $(x = x_1)$. The buyer's payoff is zero when $\pi_b(x_1, y_1) < w$ and $\pi_s(x_1, y_1)$ if $\pi_b(x_1, y_1) \geq w$.

The seller's optimal quantity discount schedule by using Stackelberg equilibrium is given as follows.

PROPOSITION 2: *If the seller knows the buyer's cost parameters, his optimal quantity discount schedule has only one break point* (x_1, y_1) *given by*

$$y_1 = [2\,w + C_b\,x_1^2/(1+x_1)]/[2\,DP + C_b(1+x_1)], \tag{23}$$

where x_1 *is obtained by maximizing* π_s *or*

$$\begin{aligned} \operatorname*{Max}_{x \geq 0} \{ & -y_1\,(x)\,DP + [1 - 1/(1+x)]\,A_s\,D/Q \\ & + (Q\,h_s/2)\,[(N_0 - 1) - (N_x - 1)\,(1+x)] \\ & - (DA_e/Q)\,[1/(1+x)\,N_x - 1/N_0]\,\}. \end{aligned} \tag{24}$$

Proof: Because the seller knows the buyer's cost parameters and, thus, the buyer's reaction to any break point, a quantity discount schedule with only one break point is adequate for the purpose of maximizing his payoff.

The seller is to find the break point (x_1, y_1) that maximizes his payoff. According to Lemma 1 and 2, such a point exists for an appropriate w. Because the buyer's reaction is $x = 0$ if $\pi_b\,(x_1, y_1) < w$ and $x = x_1$ if $\pi_b\,(x_1, y_1) \geq w$, the seller's optimal break point (x_1, y_1) is determined by maximizing his payoff function with respect to both x and y, subject to $\pi_b \geq w$. The constraint has to be tight in maximizing π_s because, as can be seen from (15) and (5), the seller will set $x = \infty$ and make the buyer lose otherwise.

Solving $\pi_b = w$ for y, we obtain (23). Substituting the obtained y into π_s and maximizing π_s in x, we obtain x_1 by (24).

Q.E.D.

COROLLARY 1: *If the seller knows the buyer's cost parameters, the buyer gets at most* w *from quantity discount.*

Proof: This is obvious from Proposition 2.

Q.E.D.

If the seller knows the buyer's cost parameters, the seller has more control of quantity discount than the buyer because he maximizes his own payoff by letting the buyer gain w. When $w = 0$, the buyer gains nothing and the seller obtains all the benefit from quantity discount. This is the case discussed by many writers (Monahan, 1984; Lee and Rosenblatt, 1986), although a quantity discount schedule with a positive w seems to be more appropriate because the buyer normally needs some incentive to respond positively to a quantity discount schedule.

Because $\pi_b \geq w$ is always tight when maximizing π_s, decreasing w usually increases the seller's gain. The seller gets his maximum gain when $w = 0$. It is then desirable for him to set w as small as possible. But the buyer may not cooperate by ordering his initial order quantity, in which case the seller gains nothing. The problem in this case may be considered as a bargaining problem over w for the seller and the buyer. However, the quantity discount problem is normally not resolved through bargaining in practice. It is usually the seller who takes the initiative by offering a quantity discount schedule. We treat w as an exogenous parameter in our model. Nevertheless, we will demonstrate that it is often worthwhile for the seller to give the buyer some payoff. When w is unknown in developing a quantity discount schedule, he should carefully select $w > 0$ to get the buyer to be engaged in a quantity discount.

Because N_x is a discrete function of x, finding x_1 by maximizing π_s given by (24) analytically is usually cumbersome. To avoid this difficulty, the following non-linear programming model can be used.

$$\left.\begin{array}{l} \underset{x,y,N}{\mathrm{MAX}} \ \{ -y\, DP + [1 - 1/(1+x)]\, A_s\, D/Q \\ \qquad +(Q\, h_s/2)\, [(N_0 - 1) - (N-1)\,(1+x)] \\ \qquad -(DA_e/Q)\, [1/(1+x)\, N - 1/N_0] \}, \\ \mathrm{S.T.}\ :\ y\, DP + [2 - 1/(1+x) - (1+x)\,(1-y)]\, C_b/2 - w \geq 0, \\ \qquad x < \surd\{ r/[N\,(N-1)] \}, \\ \qquad x \geq \surd\{ r/[N\,(N+1)] \}, \\ \qquad N \geq 1 \text{ and integer}, \\ \qquad y > 0 \text{ and } y < 1. \end{array}\right\} \tag{25}$$

This model can be easily solved by using a non-linear programming package such as GINO (Liebman *et al.*, 1984).

3.3.2. *Incomplete information*

The case where the seller has perfect information about the buyer's cost structure is a simplified version of reality. Practically, independent suppliers and buyers are likely to have private information about their cost parameters. In this section, we discuss the determination of the seller's quantity discount schedule when he does not know the buyer's cost structure or C_b with certainty.

In this case, the seller does not know the buyer's reaction to a break point with certainty. A quantity discount schedule with many break points may provide more possibility that the buyer will respond positively so the seller can gain. Therefore, a quantity discount schedule with a single break point may no longer be adequate for the maximization of his payoff function. Quantity discount schedules with many break points should be considered. In the latter case, the buyer also has an opportunity to select an order quantity at one of the break points that maximizes his own payoff. The seller does not have the same control of the problem when he has perfect information about the buyer's cost parameters.

Solving $\pi_b \geq w$, we obtain

$$C_b \geq C(x, y) = 2(w - y\,DP)/[2 - 1/(1+x) - (1+x)(1-y)]. \quad (26)$$

For a quantity discount schedule with a single break point (x_1, y_1), the buyer's reaction is $x = x_1$ if $C_b \geq C(x_1, y_1)$ and $x = 0$ otherwise. Thus, the seller's payoff is $\pi_s(x_1, y_1)$ if $C_b \geq C(x_1, y_1)$ and 0 otherwise. Assume that the seller knows the probability distribution of C_b and is to maximize his expected payoff. The optimal break point is determined by maximizing $\pi_s(x_1, y_1)\,P\{C_b \geq C(x_1, y_1)\}$ with respect to x_1 and y_1, $x_1 > 0$ and $y_1 > 0$.

For a quantity discount schedule with two break point (x_1, y_1) and (x_2, y_2) $x_1 < x_2$ and $y_1 < y_2$, let $C_{bk} = C(x_k, y_k)$, $\pi_b(x_i, y_i) \leq \pi_b(x_j, y_j)$, $i \neq j$, $k = 1, 2$. The buyer's reaction is $x = x_j$ if $C_b \geq C_{bj}$; $x = x_i$ if $C_{bi} \leq C_b < C_{bj}$; and $x = 0$ otherwise. The seller's expected payoff is $\pi_s(x_i, y_i)\,P\{C_{bi} \leq C_b < C_{bj}\} + \pi_s(x_j, y_j)\,P\{C_b \geq C_{bj}\}$. The optimal quantity discount schedule with two break points is determined by maximizing the expected payoff with respect to x_1, x_2, y_1 and y_2 subject to $x_1 < x_2$, $y_1 < y_2$, and $\pi_b(x_i, y_i) \leq \pi_b(x_j, y_j)$.

For quantity discount schedules with more than two break points, the seller's expected payoff can be found similarly to a quantity discount schedule with two break points. The optimal break points are determined by a maximization problem.

However, to find the optimal quantity discount schedule for the seller in this case is difficult. First, for quantity discount schedules with many beak points, the buyer's reaction depends on which break point gives him the most payoff. In the case of two break points above, we assumed $\pi_b(x_i, y_i) \leq \pi_b(x_j, y_j)$ in order to determine the buyer's reaction. This condition will become cumbersome for quantity discount schedules with more break points. Second, the optimal number of break points is unknown. An optimal quantity discount schedule may not exist at all.

Because of this difficulty, we develop two approaches (heuristic) for the seller to develop quantity discount schedules in the following.

(i) For each value of C_b, a single optimal quantity discount schedule has been obtained in our analysis. This optimal schedule is determined by a single break point (x, y) that maximizes the seller's payoff while letting the buyer gain w. For all values of C_b, we obtain y as a function of x. Because both player's payoff functions are continuous, this function is generally a continuous curve.

This function is also a quantity discount schedule in the general form. It represents all optimal solutions for C_b and is obtained without knowing C_b. However, if this is offered to the buyer, the buyer may not order the quantity determined by the optimal quantity discount schedule when the seller knows his cost parameters. The buyer will order the quantity on the schedule that maximizes his payoff. This schedule is usually not the optimal quantity discount schedule when the seller does not know the buyer's cost structure.

However, this schedule represents the seller's best decision for each C_b. When w is small, for example, $w = 0$, the seller obtains all the gain. Even if the buyer orders the quantity on the schedule that maximizes his own payoff, it is also likely to give the seller a significant payoff. Assume that the seller is able to estimate the range for the buyer's average annual inventory related cost C_b, denoted by $C_L \le C_b \le C_U$. A quantity discount schedule can be developed as follows.

(a) Select the number of break points, denoted by M, for the quantity discount schedule. The selection of M could be based on the seller's past experiences or industrial norms.

(b) Divide the estimated range for C_b into $M - 1$ sub-ranges such that $C_L = C_1 < C_2 < \ldots < C_M = C_U$.

(c) Solve (x, y) for each C_i, $i = 1, 2, \ldots, M$, by using (25).

(d) Let the solution form (c) for C_i be denoted by (x_i, y_i), $i = 1, 2, \ldots, M$. A quantity discount schedule with M break points is obtained such that a discount of y_i is offered for $x_i \le x < x_{i-1}$, $i = M + 1$, $M, \ldots, 1$, $x_{M+1} = 0$, $y_{M+1} = 0$, and $x_0 = \infty$.

We will demonstrate by a numerical example in the next section that a quantity discount schedule developped by the above heuristic can give the seller a significant payoff.

(ii) The quantity discount schedule developed in (i) does not guarantee the seller any amount of gain. The seller may want to ensure that it gets at least a certain amount of payoff by offering a quantity discount schedule. Let u denote the amount that the seller wants to gain. The seller has to set up the quantity discount schedule such that $\pi_s(x, y) \geq u$. Solving it for y, we obtain

$$y \leq y_u = y_s - u/(DP). \tag{27}$$

Because y_u is less than y_s and continuous and increasing in x, it is a quantity discount schedule in the general form. The buyer orders a quantity on this schedule if it makes him gain and Q units otherwise. Since the buyer always orders the quantity that maximizes his payoff, constraint (27) is tight if he orders more than Q units each time. The seller gains u in this case and zero otherwise. A quantity discount schedule in the step function form can be obtained by selecting break points directly from y_u in this case.

4. APPLICATIONS OF THE RESULTS

If the seller is able to make a single accurate estimation about the buyer's annual inventory related cost, the optimal quantity discount schedule with a single break point should be used. Otherwise the approaches developed in the incomplete information section should be used.

4.1. Comparison of our model with some models in the literature

Quantity discount models have been developed by assuming that the buyer will cooperate as long as he will not be worse-off. In our analysis, we assume that the buyer always orders the quantity that maximizes his own payoff. To illustrate the difference between these two approaches, we first compare our model with the models developed by Monahan (1986) and Lee and Rosenblatt (1986) by using a numerical example provided by Rosenblatt and Lee (1985). The comparison is limited to the perfect information case because their models have been developed under this assumption.

Example: $D = 100$, $P = \$10$, $A_s = \$0$, $A_b = \$1200$, $A_e = \$1200$, $H_b = 0.5$, $h_s = \$2.5$.

When no discount is offered, $Q = 219$, $Q_0 = 310$, $r = 2$ and $N_0 = 1$. The seller's optimal inventory resplenishing policy is the lot-for-lot policy and this policy does not change when the buyer increases his order quantity.

When a quantity discount is considered, by using Monahan's model or Lee and Rosenblatt's model, the improved ordering size is 310 units and the discount is \$0.6645. The seller gains \$93.97 and the buyer gains \$50.85.

If our model is used, with $w = \$50.85$, the buyer's improved order quantity is 410 units with a 13.45% discount from the seller. The buyer gains \$50.85 and the seller gains \$120.18. If the seller is satisfied with the gain by using Monahan's model or Lee and Rosenblatt's model, the seller should offer a 16.75% discount to induce the buyer to order 420 units each time. The seller gains \$93.97 and the buyer gains \$104.21. □

Our model generally provides a more efficient, more flexible and perhaps more equitable solution to the quantity discount problem. This is because our model considers any quantity discount scheme that makes both the seller and the buyer gain. Their models restrict the solution of the problem to a single curve, *i.e.*, $\pi_b = 0$.

It is also interesting to consider the model developed by Rosenblatt and Lee (1985) because they considered the buyer's reaction explicitly in the determination of the seller's quantity discount schedule. A linear quantity discount schedule was used in their analysis. For the above numerical example, the optimal quantity discount schedule given by them is $p = 10 - 0.0059\,x$, where p is the unit selling price and x is the ordering size.

It seems that the buyer was not considered as an independent agent who always maximizes his payoff in their analysis. Otherwise, it is easy to verify that the buyer will order an infinitely large number of units each time to maximize his payoff and the seller loses with the above quantity discount schedule. We have considered independent suppliers and buyers who always pursue their own interests. This appears to be more realistic. We also found that linear quantity discount schedules are usually not efficient for sellers to maximize their payoff from quantity discount.

4.2. A numerical example

We now discuss the applications of our results. We focus on the case where the seller has incomplete information about the buyer's cost structure and first demonstrate our results by using the above numerical example.

Example: We assume that the seller has only the following information in the above example: $D = 100$, $P = \$10$, $A_s = \$0$, $Q = 40\sqrt{30}$, $A_e = \$1200$, $h_s = \$2.5$. The seller has estimated that the buyer's inventory carrying cost per dollar per year is from 0.3 to 0.7. Hence, the estimated range for C_b is from $\$120\sqrt{30}$ to $\$280\sqrt{30}$. Note that $C_b = QPH_b$.

We consider a quantity discount schedule with five break points. By dividing the estimate range of C_b into four sub-ranges with equal length and solving the non-linear program (25) for each C_i, we obtain the break points for $w = 0$ as follows.

C_b	x	y	π_s	π_b
$120\sqrt{30}$	1.0619	0.1071	174.95	0
$160\sqrt{30}$	0.9208	0.1050	157.54	0
$200\sqrt{30}$	0.8292	0.1028	145.45	0
$240\sqrt{30}$	0.7646	0.1008	136.51	0
$280\sqrt{30}$	0.7164	0.0990	129.61	0

A quantity discount schedule with five break points is obtained as

Order quantity	Discount (%)
219-376	0.00
377-386	9.90
387-400	10.08
401-420	10.28
421-451	10.50
452-∞	10.71

If the buyer's actual annual inventory related cost is $\$200\sqrt{30}$, the buyer's optimal order quantity is 377 units. The buyer's payoff is \$28.29 and the seller's payoff is \$129.61.

If the seller is willing to let the buyer gain at least \$50.85, the amount given by Monahan's model, $w = 50.85$, and the seller's quantity discount schedule with five break points is as follows.

Order quantity	Discount (%)
219-383	0.00
384-394	12.69
395-409	13.05
410.430	13.45
431-463	13.90
464-∞	14.38

The buyer's optimal quantity under the quantity discount schedule is 384 units if his actual annual inventory related cost is $\$200\sqrt{30}$. The buyer's payoff is \$72.63 and the seller's payoff is \$107.78.

If the seller wants to ensure that he gets, for example, $u = \$129.61$, a quantity discount schedule is obtained by using (27) as $y = 0.1\sqrt{30}\,x/(1+x) - 0.1291$. The buyer will order 406 units each time, getting a 12.22% discount. The seller gains \$129.61 and the buyer gains \$31.81. □

The seller can get a significant payoff by using the heuristic developed in our analysis. The buyer can also improve his profitability significantly.

In the numerical example, the number of break points was arbitrarily selected. The seller should select the number of break points based on industrial norms or his experiences in practice. We note that the break points in the example are close to each other. The seller may provide a quantity discount schedule with fewer number of break points. This will also limit the buyer's choices to select an order quantity. If the first break point is eliminated in the above quantity discount schedules, the buyer will order each time the quantity at the second break point. The seller gains more and the buyer gains less. However, if $C_b >$ \$260 $\sqrt{30}$, the buyer may order 219 units each time and the seller gains nothing. The seller has to balance these two tradeoffs.

If the seller's estimation of the buyer's inventory related cost is reliable, the estimated range of C_b is small. A quantity discount schedule with fewer number of break points might give the seller more payoff. Otherwise, a quantity discount schedule with more break points might give the seller more insurance to get a positive payoff.

4.3. Many buyers and joint solution

Our results when the seller has incomplete information about the buyer's cost structure can also be applied when he has many buyers and does not know their cost structures.

When the seller has more than one buyer, his inventory replenishing policy depends on the size and timing of the buyers' orders. Let S denote the number of buyers and use the notations above with a subscript i for buyer i. If the seller orders a quantity each time to meet N_i orders from buyer i, his order quantity is $Q_S = \Sigma\, N_i\, Q_i$. The seller's average inventory carrying cost can be expressed as $\Sigma\,[(N_i - 1)\, Q_i\, h_s/2]$ and his ordering cost is $(\Sigma\, D_i)\, A_e/(\Sigma\, N_i\, Q_i)$. The seller's total inventory related cost is

$$
\begin{aligned}
TI &= \Sigma\,(N_i - 1)\, Q_i\, h_s/2 + \Sigma\, D_i\, A_e/(\Sigma\, N_i\, Q_i) \\
&= Q_S\, h_s/2 + (\Sigma\, D_i)\, A_e/Q_S - \Sigma\, Q_i\, h_s/2. \qquad (28)
\end{aligned}
$$

The seller's optimal order quantity should include an integer number of orders from each buyer. However, when the buyers are many and small, we can approximate it by the quantity that minimizes TI by (28). By minimizing

TI with respect to Q_S, we obtain that the optimal Q_S is the seller's EOQ or $Q_S^* = \sqrt{[2(\Sigma D_i) A_e/h_s]}$. Substituting Q_S^* into (28), we get

$$TI^{**} = \sqrt{[2(\Sigma D_i) A_e/h_s]} - \Sigma Q_i h_s/2. \tag{29}$$

The seller's payoff from quantity discount is then given by

$$\begin{aligned} \pi_s &= -\Sigma D_i P y_i + \Sigma (A_s D_i/Q_i)[1 - 1/(1 + x_i)] + \Sigma x_i Q_i h_s/2 \\ &= \Sigma \{-D_i P y_i) + (D_i/Q_i)[1 - 1/(1 + x_i)] + x_i Q_i h_s/2\} \\ &= \Sigma \pi_{s_i}, \end{aligned} \tag{30}$$

where $\pi_{s_i} = -(D_i P y_i) + (D_i/Q_i)[1 - 1/(1 + x_i)] + x_i Q_i h_s/2$ is the seller's gain from buyer i.

Because π_{s_i} is independent, a quantity discount schedule can be developed for each buyer by using the results above. However, a supplier normally offers a common quantity discount schedule to all buyers. When the seller offers a quantity discount schedule to each buyer, the difference in the seller's decision problem is that he has a different payoff function (or the upper boundary of the feasible solution area) for each buyer. Therefore, a common quantity discount schedule can be developed for all buyers by using the average of π_{s_i} or

$$\pi_s = -\bar{D} P y + (A_s \bar{D}/\bar{Q})[1 - 1/(1 + x)] + x \bar{Q} h_s/2, \tag{31}$$

where $\bar{D} = (\Sigma D_i)/S$ and $\bar{Q} = (\Sigma D_i)/S$. Using π_s given by (31) and π_b given by (5), a quantity discount schedule can be developed by using the procedures developed in our analysis.

In the literature, several models have been developed to address the joint solution of the problem (Goyal, 1976; Banerjee, 1986a). By letting $\pi = \pi_b + \pi_s$, we have

$$\begin{aligned} \pi = {} & [1 - 1/(1 + x)](A_b + A_s) D/Q + (D A_e/Q)[1/N_0 - 1/(1 + x) N_x] \\ & + (Q h_s/2)[(N_0 - 1) - (N_x - 1)(1 + x)] \\ & + C_b [1 - 1/(1 + x) - (1 + x)(1 - y)]/2. \end{aligned} \tag{32}$$

Differentiating π with respect to y, we get

$$\partial \pi/\partial y = (Q h_b/2)(1 + x) > 0. \tag{33}$$

Hence, π increases as y increases. To maximize the joint payoff, the seller should set y as high as possible or his unit selling price as low as possible.

At the extreme, the seller should sell the product to the buyer at a price of zero and his profit becomes negative. This is unrealistic. Even if there is some collaborative agreement for them to cooperate, this will put the seller into a position of great disadvantage in bargaining. Such a joint solution is highly artificial.

When quantity discount schedules are used, how much the seller and the buyer can gain jointly depends on the quantity discount schedule provided by the seller. In the above numerical example, the joint gain for the seller and the buyer when they work together but no one will be worse-off is $301.31. In this case, the buyer gets all the gain and the seller gains nothing. This solution is also difficult to implement. However, using the quantity discount schedules provided above, the seller and the buyer can get more than fifty percent of this gain.

5. CONCLUSIONS AND POSSIBLE EXTENSIONS

In the present paper, we have discussed the quantity discount problem using the game-theoretic approach. We considered two important aspects of the problem that have not been adequately addressed: the competitive nature of the problem and the seller's information about the buyer's cost structure. The main conclusions are briefly summarized as follows.

(1) It is always possible for the seller and the buyer to benefit from quantity discount.

(2) Quantity discount schedules are necessary and sufficient for independent suppliers and buyers to exploit the benefit of quantity discount.

(3) The optimal quantity discount schedule when the seller knows the buyer's cost structure is given by a quantity discount schedule with a single break point. Our model generally gives better results than quantity discount models that assume the buyer will cooperate as long as he will not be worse-off.

(4) When the buyer's cost structure is unknown to the seller, a quantity discount schedule with a single break point is generally inadequate. In this case, an otpimal quantity discount schedule is difficult to find. Two approaches have been developed in our analysis for the seller to develop quantity discount schedules in this case.

(5) Our results when the seller does not know the buyer's cost structure can be applied when the seller has many buyers.

(6) Quantity discount schedules can significantly improve the profitability of independent suppliers and buyers.

In this research, we provide some realistic solutions to the quantity discount problem. However, we have restricted our analysis to a single product and considered the seller's replenishing policy solely based on the buyer's order quantity. In reality, suppliers and buyers normally trade many products. Sellers, as buyers from other suppliers, many also determine their order quantity from other suppliers for the purpose of obtaining quantity discount. These considerations represent some interesting and challenging extensions to our research.

APPENDIX I; Proof of Lemma 1

LEMMA 1: *The feasible solution area is always non-empty.*

Proof: We consider $H_s > 0$ and $A_e > 0$. Let $Y(x) = y_s - y_b$. Note that $Y(x)$ is the difference between the upper bound and the lower bound of the feasible solution area, $Y(x)$ is continuous in x and $Y(0) = 0$.

By differentiating $Y(x)$ with respect to x at $x = 0$, we obtain

$$\begin{aligned} dY/dx|_{x=0} &= A_s/(PQ) + (1/DP)\,[DA_e/(N_0\,Q) - (N_0 - 1)\,Q\,h_s/2] \\ &\geq [(N_0 - 1)\,Q/DP]\,\{\,DA_e/[N_0\,(N_0 - 1)\,Q^2] - h_s/2)\,\} \\ &\quad \text{since}\; A_s/(PQ) \geq 0 \\ &> 0\;\text{since}\; Q < Q_0/\surd[N_0\,(N_0 - 1)]\;\text{and}\; A_e\,D/Q_0 = Q_0\,h_s/2. \end{aligned}$$

$Y(x)$ is increasing at $x = 0$. Because $Y(0) = 0$, there is an $x_0 > 0$ such that $Y(x) > 0$ in $(0,\ x_0)$.

Q.E.D.

APPENDIX II. Proof of Lemma 2

LEMMA 2: *The seller should not offer any discount to the buyer without using a quantity discount schedule.*

Proof: If the seller does not use a quantity discount schedule and acts as the leader by offering a discount y to the buyer, the buyer's uses his reaction function (21) or his EOQ to determine his order quantity.

By substituting (21) into the seller's payoff function, we obtain

$$\begin{aligned}\pi_s = & -y\,DP + [1 - \surd(1-y)]\,A_s\,D/Q \\ & + (Q\,h_s/2)[(N_0 - 1) - (N_y - 1)/\surd(1-y)] \\ & - (DA_e/Q)\,[\surd(1-y)/N_y - 1/N_0],\end{aligned} \tag{A2-1}$$

where N_Y is the seller's optimal replenishing policy when offering a discount y. By substituting (21) into (11), we obtain

$$N_y = N_0 - i \quad \text{when} \;\; y_i \le y < y_{i+1}, \tag{A2-2}$$

where $i = 0,\ 1,\ \ldots,\ N_0 - 1,\ y_0 = 0,\ y_{N_0} = y_0$ and

$$y_i = 1 - (N_0 - i + 1)\,(N_0 - i) \qquad \text{for} \quad /r0 < i < N_0.$$

Because π_s is continuous in y for $0 \le y < 1$ and differentiable in each interval $(y_i,\ y_{i+1})$, we obtain by differentiating π_s with respect to y in each interval that

$$\begin{aligned} d\pi_s/dy = & -DP + (A_s\,D/Q)\,(1-y)^{-1/2}/2 - (Q\,h_s/2)\,(N_y - 1)\,(1-y)^{-3/2}/2 \\ & + (DA_e/Q)\,(1-y)^{-1/2}/(2\,N_y) \\ \le & -DP + (A_s\,D/Q)\,(1-y)^{-1/2}/2 + (DA_e/Q)\,(1-y)^{-1/2}/(2\,N_y) \\ \le & -\{(1-y)\,DP - A_s\,D/[(1+x)\,Q]/2 \\ & \quad - DA_e/[2\,N_y\,(1+x)\,Q)]\}/(1-y) \\ \le & -\{(1-y)\,DP - A_s\,D/[(1+x)\,Q] \\ & \quad - DA_e/[N_y\,(1+x)\,Q)]\}/(1-y) \\ \le & -(TP_s + DC)/(1-y). \end{aligned} \tag{A2-3}$$

We use (21) or $1 + x = 1/\surd(1-y)$ to obtain (A2-3).

We assume $TP > 0$ or the seller gets a positive profit for trading the product initially. For any $y > 0$, if $TP_s \le 0$, $\pi_s = TP_s - TP < 0$; if $TP_s > 0$, $d\pi_s/dy < 0$ and π_s is strictly decreasing. Because π_s is continuous for $y \ge 0$ and $\pi_s\,(0) = 0$, $\pi_s < 0$. The seller loses for any $y > 0$ and he should not off any discount.

It is obvious that the Stackelberg equilibrium when the seller acts as the leader is (0, 0) in this case.

Q.E.D.

REFERENCES

A. BANERJEE, A Joint Lot-Size Model for Purchaser and Vendor, *Decision Sci.*, 1986a, *17*, pp. 292-311.

A. BANERJEE, Notes on "A Quantity Discount Pricing Model to Increase Vendor's Profit, *Management Sci.*, 1986b, *32 (11)*, pp. 1513-1517.

T. BASAR and G. J. OLSDER, *Dynamic Noncooperative Game Theory*, Academic Press, New York, 1982.

M. DADA and K. N. SRIKANTH, Princing Policies for Quantity Discounts, *Management Sci.*, 1987, *33 (10)*, pp. 1247-1252.

S. K. GOYAL, An Integrated Inventory Model for a Single Supplier-Single Customer Problem, *International Journal of Production Research*, 1976, *15 (1)*, pp. 107-111.

S. K. GOYAL, Comment on: A Generalized Quantity Discount Princing Model to Increase Supplier's Profits, *Management Sci.*, 1987 *33 (12)*, pp. 1635-1636.

S. K. GOYAL, A Joint Economic-Lot-Size Model for Purchaser and Vendor: A Comment, *Decision Sciences*, 1988, *19 (1)*, pp. 236-241.

G. HADLEY and T. M. WHITIN, *Analysis of Inventory Systems*, Prentice-Hall, Englewood Cliffs, New York, 1963.

JOGLEKAR P. N., Comments on A Quantity Discount Princing Model to Increase Vendor Profits, *Management Sci.*, 1988, *34 (11)*, pp. 1391-1398.

P. JOGLEKAR and S. THARTHARE, The Individually Responsible and Rational Decision Approach to Economic Lot Sizes for One Vendor and Many Purchasse, *Decision Sciences*, 1990, *21*, pp. 492-506.

J. V. JUCKER and M. J. ROSENBLATT, Single-Period Inventory Models: Behavioral Implications and A New Solution Procedure, *Naval Research Logistics Quarterly*, 1985, *32 (4)*, pp. 537-550.

R. KOHLI and H. PARK, A Cooperative Game Theory Model of Quantity Discounts, *Management Sci.*, 1989, *35 (6)*, pp. 693-707.

R. LAL and R. STAELIN, An Approach for Developing An Optimal Discount Pricing Policy, *Management Sci.*, 1984, *30 (12)*, pp. 1524-1539.

H. L. LEE and M. J. ROSENBLATT, A Generalized Quantity Discount Pricing Model to Increase Vendor's Profit, *Management Sci.*, 1986, *32 (9)*, pp. 1177-1185.

J. LIEBMAN, L. LASDON, L. SCHRAGE and A. WAREN, *Modeling and Optimization with GINO*, The Scientific Press, U.S.A., 1986.

J. P. MONAHAN, A Quantity Discount Pricing Model to Increase Vendor's Profit, *Management Sci.*, 1984, *30 (6)*, pp. 720-726.

J. P. MONAHAN, On "Comments on A Quantity Discount Pricing Model to Increase Vendor Profits, *Management Sci.*, 1988, *34 (11)*, pp. 1398-1340.

M. PARLAR and Q. WANG, Discounting Decisions in a Supplier-Buyer Relationship with a Linear Buyer's Demand, *IIE Transactions*, 1994, *26 (2)*, pp. 34-41.

R. PETERSON and E. A. SILVER, Decision Systems for Inventory Management and Production Planning, Wiley, New York, 1979.

M. J. ROSENBLATT and H. L. LEE, Improving Profitability with Quantity Discounts under Fixed Demand, *IIE Transactions*, 1985, *17 (4)*, pp. 388-395.

S. P. SETHI, A Quantity Discount Lot Size Model with Disposals, *International Journal of Production Research*, 1984, *22 (1)*, pp. 31-39.