A GENERAL ALGEBRAIC APPROACH TO STEENROD OPERATIONS

by

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1. Introduction. Since the introduction of the Steenrod operations in the cohomology of topological spaces, it has become clear that similar operations exist in a variety of other situations. For example, there are Steenrod operations in the cohomology of simplicial restricted Lie algebras, in the cohomology of cocommutative Hopf algebras, and in the homology of infinite loop spaces (where they were introduced mod 2 by Araki and Kudo [3] and mod p, $p > 2$, by Dyer and Lashof [6]).

The purpose of this expository paper is to develop a general algebraic setting in which all such operations can be studied simultaneously. This approach allows a single proof, applicable to all of the above examples, of the basic properties of the operations, including the Adem relations. In contrast to categorical treatments of Steenrod operations, the elegant proofs developed by Steenrod [25-30] actually simplify somewhat in our algebraic setting. Further, even the most general existing categorical study of Steenrod operations, that of Epstein [7], cannot be applied to iterated loop spaces.

We emphasize that this is an expository paper. Although a number of new results and new proofs of old results are scattered throughout, the only real claim to originality lies in the basic context. We have chosen to give complete proofs of all results since a large number of minor simplifications in the arguments allows a substantial simplification of the theory as a whole. We have also included a number of topological results which should be well-known but appear not to be in the literature. In particular, in section 10, we give a quick complete calculation of the mod p cohomology Bockstein spectral sequence of $K(\pi, n)$'s and show that Serre's simple proof [23] of the axiomatization of the mod 2 Steenrod operations applies with only slight modifications to the case $p > 2$.

The general theory is presented in the first five sections. Most of the proofs in sections 1, 2, and 4 are based on those of Steenrod [25-30], and those of section 3 are simplifications of arguments of Dyer and Lashof [6]. Via acyclic models and a lemma due to Dold [5], the theory is applied to several simplicial categories and to topological spaces in sections 7 and 8. The standard properties of the Steenrod operations in spaces, except $P^o = 1$, drop out of the algebraic theory, and $P^o = 1$ is shown to follow from these properties. In contrast, we prove that $P^o = 0$ on the cohomology of simplicial restricted Lie algebras. The theory is applied to the cohomology of cocommutative Hopf algebras in section 11; the operations here are important in the study of the cohomology of the Steenrod algebra [13, 18]. The present analysis arose out of work on iterated loop spaces, but this application will appear elsewhere. The material of sections 6 and 9, which is peripheral to the study of Steenrod operations, is presented here with a view towards this application.

1. Algebraic preliminaries; equivariant homology

Let Λ be a commutative (ungraded) ring. By a Λ-complex, we understand a Z-graded differential Λ-module, graded by subscripts, with differential of degree minus one. We say that a Λ-complex K is positive if $K_q = 0$ for $q < 0$ and negative if $K_q = 0$ for $q > 0$. We use Z-graded complexes in order that our theory can be applied equally well to homology and to cohomology. The exposition will be geared to homology, where the notation is slightly simpler, and the notations appropriate for cohomology will be given in section 5. We give some elementary homological lemmas in this section; these extract the slight amount of information about the homology of groups that is needed for the development of the Steenrod operations.

If π is a group, we let $\Lambda\pi$ denote its group ring over Λ. We shall generally speak of $\Lambda\pi$-morphisms rather than π-equivariant Λ-morphisms, and we shall speak of π-morphisms when Λ is understood. Let Σ_r denote the symmetric group on r letters, and let $\pi \subset \Sigma_r$. For $q \in Z$, let $\Lambda(q)$ denote the $\Lambda\pi$-module which is Λ as a Λ-module and has the $\Lambda\pi$-action determined by $\sigma\lambda = (-1)^{qs(\sigma)}\lambda$, where $(-1)^{s(\sigma)}$ is the sign of $\sigma \in \pi$. If M is a $\Lambda\pi$-module, let M(q) denote the $\Lambda\pi$-module $M \otimes \Lambda(q)$ with the diagonal action $\sigma(m \otimes \lambda) = \sigma m \otimes \sigma\lambda$ (where $\otimes = \otimes_\Lambda$). If K is a Λ-complex, let $K^r = K \otimes \ldots \otimes K$, r factors K. Via permutation of factors, with the standard sign convention, K^r becomes a $\Lambda\pi$-complex for $\pi \subset \Sigma_r$, and $K^r(q)$ is defined.

Let I denote the Λ-free Λ-complex which has two basis elements e_o and e_1 of degree zero, one basis element e of degree one, and differential $d(e) = e_1 - e_o$. If Γ is a Hopf algebra over Λ and I is given the trivial Γ-module structure, $\gamma a = \varepsilon(\gamma)a$ for $\gamma \in \Gamma$ and $a \in I$, then the notion of a Γ-homotopy $h: f \simeq g$, where $f, g: K \rightarrow L$ are morphisms of Γ-complexes, is equivalent to the notion of a Γ-morphism $H: I \otimes K \rightarrow L$ such that $H(e_1 \otimes k) = f(k)$ and $H(e_o \otimes k) = g(k)$, where

$I \otimes K$ is given the diagonal Γ-action. In fact, H determines h by $h(k) = H(e \otimes k)$ and conversely.

With these notations, we have the following lemma. In all parts, $\Lambda\pi$ acts diagonally on tensor products.

<u>Lemma 1.1</u>. Let $\pi \subset \Sigma_r$ and let V be a positive $\Lambda\pi$-free complex.

(i) There exists a $\Lambda\pi$-morphism $h: I \otimes V \longrightarrow V \otimes I^r$ such that $h(e_o \otimes v) = v \otimes e_o^r$ and $h(e_1 \otimes v) = v \otimes e_1^r$ for all $v \in V$.

(ii) If $f, g: K \longrightarrow L$ are Λ-homotopic morphisms of Λ-complexes, then $1 \otimes f^r$, $1 \otimes g^r: V \otimes K^r \longrightarrow V \otimes L^r$ are $\Lambda\pi$-homotopic morphisms of $\Lambda\pi$-complexes.

(iii) If Λ is a field and K is a Λ-complex, then K is Λ-homotopy equivalent to $H(K)$ and $V \otimes K^r$ is $\Lambda\pi$-homotopy equivalent to $V \otimes H(K)^r$.

(iv) Let $v \in V$ satisfy $d(v \otimes 1) = 0$ in $V \otimes_\pi \Lambda$; let K be a Λ-complex and let $a, b \in K_q$ be homologous cycles. Then $v \otimes a^r$ and $v \otimes b^r$ are homologous cycles of $V \otimes_\pi K^r(q)$.

<u>Proof</u>. (i) Let $\varepsilon: I \longrightarrow \Lambda$ be the augmentation $\varepsilon(e_o) = 1 = \varepsilon(e_1)$, and let $J = \mathrm{Ker}(\varepsilon^r)$, $\varepsilon^r: I^r \longrightarrow \Lambda^r = \Lambda$. Define $k: V \longrightarrow V \otimes J$ by $k(v) = v \otimes (e_1^r - e_o^r)$. Since $H(J) = 0$, $H(V \otimes J) = 0$. Define a $\Lambda\pi$-homotopy $s: V \longrightarrow V \otimes J$ from k to the zero map by induction on degree as follows. Let $s_{-1} = 0$; given $s_{i-1}: V_{i-1} \longrightarrow (V \otimes J)_i$, we find easily that $d_i(k_i - s_{i-1}d_i) = 0$. Let $\{x_j\}$ be a $\Lambda\pi$-basis for V_i; for $x \in \{x_j\}$, choose $s_i(x)$ such that $d_{i+1}s_i(x) = k_i(x) - s_{i-1}d_i(x)$, and extend s_i to all of V_i by π-equivariance. The desired $\Lambda\pi$-morphism h is obtained by letting $h(e \otimes v) = s(v)$ for $v \in V$.

(ii) Let $t: I \otimes K \longrightarrow L$ determine a Λ-homotopy from f to g. Then the following composite is a $\Lambda\pi$-morphism which determines a $\Lambda\pi$-homotopy from $1 \otimes f^r$ to $1 \otimes g^r$:

$$I \otimes V \otimes K^r \xrightarrow{h \otimes 1} V \otimes I^r \otimes K^r \xrightarrow{1 \otimes u} V \otimes (I \otimes K)^r \xrightarrow{1 \otimes t^r} V \otimes L^r,$$

where $u: I^r \otimes K^r \longrightarrow (I \otimes K)^r$ is the evident shuffling isomorphism.

(iii) Define $f: H(K) \longrightarrow K$ by sending each element of a basis for $H(K)$ to a chosen representative cycle. $K \cong \operatorname{Im} f \oplus \operatorname{Coker} f$ as a Λ-complex and $\operatorname{Coker} f$ is acyclic and therefore contractible since Λ is a field. The first half follows and implies the second half by (ii).

(iv) Define a morphism of Λ-complexes $f: I \longrightarrow K$, of degree q, by $f(e_1) = a$, $f(e_o) = b$, and $f(e) = (-1)^q c$, where $d(c) = a-b$ in K (the sign ensures that $df(e) = (-1)^q fd(e)$). Let $F: I \otimes V \longrightarrow V \otimes K^r(q)$ be the composite $I \otimes V \xrightarrow{h} V \otimes I^r \xrightarrow{1 \otimes f^r} V \otimes K^r(q)$. A check of signs shows that f^r is a $\Lambda\pi$-morphism, hence that F is a morphism of $\Lambda\pi$-complexes of degree qr. By (i), we find that

$$F(e_i \otimes v) = (1 \otimes f^r)(v \otimes e_i^r) = (-1)^{qr \deg v} v \otimes f(e_i)^r , \quad i = 0 \text{ or } 1.$$

Since π operates trivially on I and $d(v \otimes 1) = 0$ in $V \otimes_\pi \Lambda$, we have that $d(e \otimes v) = (e_1 - e_o) \otimes v$ in $I \otimes_\pi V$. Thus, in $V \otimes_\pi K^r(q)$,

$$dF(e \otimes v) = (-1)^{qr} F(e_1 \otimes v - e_o \otimes v) = (-1)^{qr(\deg v + 1)} (v \otimes a^r - v \otimes b^r),$$

and this proves the result.

We now consider the cyclic group π of prime order p. We recall the definition of the standard $\Lambda\pi$-free resolution $W = W(p, \Lambda)$ of Λ.

<u>Definition 1.2.</u> Let π be the cyclic group of prime order p with generator α. Let W_i be $\Lambda\pi$-free on one generator e_i, $i \geq 0$. Let $N = 1 + \alpha + \ldots + \alpha^{p-1}$ and $T = \alpha - 1$ in $\Lambda\pi$. Define a differential d, augmentation ε, and coproduct ψ on W by the formulas

(1) $$d(e_{2i+1}) = Te_{2i} \quad \text{and} \quad d(e_{2i}) = Ne_{2i-1}\,; \quad \varepsilon(\alpha^j e_o) = 1\,;$$

$$\psi(e_{2i+1}) = \sum_{j+k=i} e_{2j} \otimes e_{2k+1} + \sum_{j+k=i} e_{2j+1} \otimes \alpha e_{2k} \quad \text{and}$$

$$\psi(e_{2i}) = \sum_{j+k=i} e_{2j} \otimes e_{2k} + \sum_{j+k=i-1} \; \sum_{0 \leq r < s < p} \alpha^r e_{2j+1} \otimes \alpha^s e_{2k+1}\,.$$

Then W is a differential $\Lambda\pi$-coalgebra and a $\Lambda\pi$-free resolution of Λ. When necessary for clarity, we shall write $W(p, \Lambda)$ for W. Of course,

$W(p,\Lambda) = W(p, Z) \otimes \Lambda$. The structure of $W(p, Z_p)$ shows that $H_*(\pi; Z_p) = H(W(p, Z_p) \otimes_\pi Z_p)$ is given, with its Bockstein operation β and coproduct ψ, by the formula

(2) $H_*(\pi; Z_p)$ has Z_p-basis $\{e_i \mid i \geq 0\}$ such that $\beta(e_{2i}) = e_{2i-1}$ and

$$\psi(e_i) = \sum_{j+k=i} e_j \otimes e_k \text{ if } p = 2 \text{ or } i \text{ is odd}, \quad \psi(e_{2i}) = \sum_{j+k=i} e_{2j} \otimes e_{2k} \text{ if } p > 2.$$

We embed π in Σ_p by $\alpha(1, \ldots, p) = (p, 1, \ldots, p-1)$, where Σ_p acts on $\{1, \ldots, p\}$. We then have the following lemma.

<u>Lemma 1.3</u>. Let $W^{(n)} = \sum_{i \leq n} W_i$ be the n-skeleton of $W = W(p, Z_p)$. Let G be any set of left coset representatives for π in Σ_p. Let K be a Z_p-module with totally ordered basis $\{x_j \mid j \in J\}$. Let $A \subset K^p$ have basis $\{x_j^p \mid j \in J\}$ and let $B \subset K^p$ have basis $\{x_{\gamma(j_p)} \mid \gamma \in G,\ j_1 \leq \ldots \leq j_p,\ j_1 < j_p\}$. Then

$$H(W^{(n)} \otimes_\pi K^p) = (\bigoplus_{i=0}^{n} e_i \otimes A) \oplus (e_o \otimes B) \oplus (\operatorname{Ker} d_n \otimes B), \quad d_n: W_n \longrightarrow W_{n-1}.$$

<u>Proof.</u> It is easy to see that K^p is isomorphic as a $Z_p\pi$-module to $A \oplus (Z_p\pi \otimes B)$, where π acts trivially on A and acts on $Z_p\pi \otimes B$ by its left action on $Z_p\pi$. Since $H(W^{(n)} \otimes_\pi A) = H(W^{(n)} \otimes_\pi Z_p) \otimes A$ and $H(W^{(n)} \otimes_\pi Z_p\pi \otimes B) = H(W^{(n)}) \otimes B$, the result follows.

Recall that if π is any subgroup of Σ and if $\gamma \in N(\pi)$, the normalizer of π in Σ, then conjugation by γ defines a homomorphism $\gamma_*: H(\pi; M) \longrightarrow H_*(\pi; M)$ for any $\Lambda\Sigma$-module M. γ_* is the map induced on homology from $\gamma_\# \otimes \gamma : W \otimes_\pi M \longrightarrow W \otimes_\pi M$ where W is any $\Lambda\pi$-free resolution of Λ and $\gamma_\# : W \longrightarrow W$ is any morphism of Λ-complexes such that $\gamma_\#(\sigma w) = \gamma\sigma\gamma^{-1}\gamma_\#(w)$ for $\sigma \in \pi$ and $w \in W$. (It is easy to verify that $\gamma_\#$ exists and that γ_* is independent of the choice of W and of $\gamma_\#$.) Clearly $\gamma_* = 1$ if $\gamma \in \pi$ since we may then define $\gamma_\#(w) = \gamma w$ so that

$$(\gamma_\# \otimes_\pi \gamma)(w \otimes m) = \gamma w \otimes \gamma m = w \otimes m, \quad w \in W \text{ and } m \in M.$$

If $\pi \subset \rho \subset \Sigma$ and $\gamma \in N(\pi) \cap N(\rho)$, then the following diagram commutes:

$$\begin{array}{ccc} H_*(\pi; M) & \xrightarrow{j_*} & H_*(\rho\,; M) \\ \big\downarrow \gamma_* & & \big\downarrow \gamma_* \\ H_*(\pi; M) & \xrightarrow{j_*} & H_*(\rho\,; M) \end{array}$$

In fact, if W is any $\Lambda\rho$-free resolution of Λ, then W is also a $\Lambda\pi$-free resolution of Λ, and the above diagram results from the observation that j_* is induced from $W \otimes_\pi M \longrightarrow W \otimes_\rho M$. In particular, $j_* = j_*\gamma_*$ if $\gamma \in \rho$.

Lemma 1.4. Let π be cyclic of prime order $p > 2$ and let $q \in Z$. Consider $j_*: H_*(\pi; Z_p(q)) \longrightarrow H_*(\Sigma_p; Z_p(q))$. Then

(i) If q is even, $j_*(e_i) = 0$ unless $i = 2t(p-1)$ or $i = 2t(p-1) - 1$.

(ii) If q is odd, $j_*(e_i) = 0$ unless $i = (2t+1)(p-1)$ or $i = (2t+1)(p-1) - 1$.

Proof. Let k generate the multiplicative subgroup of Z_p, $k^{p-1} = 1$. Let Σ_p operate on Z_p and define $\gamma \in \Sigma_p$ by $\gamma(i) = ki$. Then $\gamma\alpha\gamma^{-1} = \alpha^k$ and γ is an odd permutation in $N(\pi)$. Define $\gamma_\# : W \longrightarrow W$ by

$$\gamma_\#(e_{2i}) = k^i e_{2i}; \quad \gamma_\#(e_{2i+1}) = k^i \sum_{j=0}^{k-1} \alpha^j e_{2i+1}; \quad \gamma_\#(\sigma e_i) = \gamma\sigma\gamma^{-1}\gamma_\#(e_i), \quad \sigma \in \pi$$

Then $\gamma_\# d = d\gamma_\#$ and $\gamma_\# \otimes \gamma$ induces the conjugation γ_* on $H_*(\pi; Z_p(q))$. Since $\gamma \in \Sigma_p$, $j_*\gamma_* = j_*$, hence $j_*(e_i - \gamma_* e_i) = 0$ for all i. γ operates by $(-1)^q$ on $Z_p(q)$ and therefore

$$\gamma_*(e_{2i}) = (-1)^q k^i e_{2i} \quad \text{and} \quad \gamma_*(e_{2i+1}) = (-1)^q k^{i+1} e_{2i+1} .$$

Thus $j_*(e_{2i}) = 0$ unless $1 - (-1)^q k^i \equiv 0 \mod p$ and $j_*(e_{2i+1}) = 0$ unless $1 - (-1)^q k^{i+1} \equiv 0 \mod p$. Clearly $k^i \equiv 1 \mod p$ if and only if $i = t(p-1)$ for some t and $k^i \equiv -1 \mod p$ if and only if $2i = (2t+1)(p-1)$ for some t. The result follows easily.

2. The definition and elementary properties of the operations

We now define a large algebraic category $\mathcal{C}(p, n)$ on which the Steenrod operations will be defined. Steenrod operations will be obtained for particular categories of interest by obtaining functors to $\mathcal{C}(p, n)$. The interest of the integer n in the following definition lies solely in the applications to iterated loop spaces. For all other known applications, only the case $n = \infty$ is relevant.

Definitions 2.1. Let Λ be a commutative ring, let r be an integer, and let π be a subgroup of Σ_r. Let W be a $\Lambda\pi$-free resolution of Λ, let V be a $\Lambda\Sigma_r$-free resolution of Λ, and let $j: W \longrightarrow V$ be a morphism of $\Lambda\pi$-complexes over Λ. Assume that $W_o = \Lambda\pi$ with generator e_o. Let $0 \leq n \leq \infty$ and let $W^{(n)}$ and $V^{(n)}$ denote the n-skeletons of W and V. Define a category $\mathcal{C}(\pi, n, \Lambda)$ as follows. The objects of $\mathcal{C}(\pi, n, \Lambda)$ are pairs (K, θ), where K is a homotopy associative differential Λ-algebra and $\theta: W^{(n)} \otimes K^r \longrightarrow K$ is a morphism of $\Lambda\pi$-complexes such that

(i) The restriction of θ to $e_o \otimes K^r$ is Λ-homotopic to the iterated product $K^r \longrightarrow K$, associated in some fixed order, and

(ii) θ is $\Lambda\pi$-homotopic to a composite $W^{(n)} \otimes K^r \xrightarrow{j \otimes 1} V^{(n)} \otimes K^r \xrightarrow{\phi} K$, where ϕ is a morphism of $\Lambda\Sigma_r$-complexes.

A morphism $f: (K, \theta) \longrightarrow (K', \theta')$ in $\mathcal{C}(\pi, n, \Lambda)$ is a morphism of Λ-complexes $f: K \longrightarrow K'$ such that the diagram

$$\begin{array}{ccc} W^{(n)} \otimes K^r & \xrightarrow{\quad\theta\quad} & K \\ \Big\downarrow 1 \otimes f^r & & \Big\downarrow f \\ W^{(n)} \otimes (K')^r & \xrightarrow{\quad\theta'\quad} & K' \end{array}$$

is $\Lambda\pi$-homotopy commutative. A morphism f is said to be perfect if $\theta'(1 \otimes f^r) = f\theta$, with no homotopy required, and $\mathcal{P}(\pi, n, \Lambda)$ denotes the subcategory of $\mathcal{C}(\pi, n, \Lambda)$ having the same objects (K, θ) and all perfect morphisms between them. Λ is itself an object of $\mathcal{C}(\pi, n, \Lambda)$, with

$\theta = \varepsilon \otimes 1 : W^{(n)} \otimes \Lambda^r \longrightarrow \Lambda^r = \Lambda$, and an object $(K, \theta) \in \mathcal{C}(\pi, n, \Lambda)$ is said to be unital if K has a two-sided homotopy identity e such that $\eta : \Lambda \longrightarrow K$, $\eta(1) = e$, is a morphism in $\mathcal{C}(\pi, n, \Lambda)$. The tensor product of objects (K, θ) and (L, θ') in $\mathcal{C}(\pi, n, \Lambda)$ is the pair $(K \otimes L, \tilde{\theta})$, where $\tilde{\theta}$ is the composite

$$W^{(n)} \otimes (K \otimes L)^r \xrightarrow{\psi \otimes U} W^{(n)} \otimes W^{(n)} \otimes K^r \otimes L^r \xrightarrow{1 \otimes T \otimes 1} W^{(n)} \otimes K^r \otimes W^{(n)} \otimes L^r \xrightarrow{\theta \otimes \theta'} K \otimes L$$

Here U is the evident shuffling isomorphism, $T(x \otimes y) = (-1)^{\deg x \deg y} y \otimes x$, and $\psi : W \longrightarrow W \otimes W$ is any fixed $\Lambda\pi$-morphism over Λ; conditions (i) and (ii) are clearly satisfied by the pair $(K \otimes L, \tilde{\theta})$. An object $(K, \theta) \in \mathcal{C}(\pi, n, \Lambda)$ is said to be a Cartan object if the product $K \otimes K \longrightarrow K$ is a morphism in $\mathcal{C}(\pi, n, \Lambda)$. When π is cyclic of prime order p, we agree to choose W to be the explicit resolution $W(p, \Lambda)$ of Definition 1.2, and we abbreviate $\mathcal{C}(\pi, n, Z_p)$ to $\mathcal{C}(p, n)$ and $\mathcal{P}(\pi, n, Z_p)$ to $\mathcal{P}(p, n)$. An object $(K, \theta) \in \mathcal{C}(p, n)$ is said to be reduced mod p if (K, θ) is obtained by reduction mod p from an object $(\tilde{K}, \tilde{\theta}) \in \mathcal{C}(\pi, n, Z)$ such that $\tilde{K}$ is a flat Z-module.

We can now define the Steenrod operations in the homology $H(K)$ of an object $(K, \theta) \in \mathcal{C}(p, n)$. Observe that if $x \in H(K)$ and $0 \leq i \leq n$, then $e_i \otimes x^p$ is a well-defined element of $H(W^{(n)} \otimes_\pi K^p) \cong H(W^{(n)} \otimes_\pi H(K)^p)$ by Lemmas 1.1 and 1.3; here (iv) of Lemma 1.1 applies, since $\pi \subset \Sigma_p$ contains only even permutations, and shows that $e_i \otimes x^p$ is represented by $e_i \otimes a^p \in W^{(n)} \otimes_\pi K^p$ for any representative cycle a of x.

<u>Definitions 2.2.</u> Let $(K, \theta) \in \mathcal{C}(p, n)$ and let $x \in H_q(K)$. For $0 \leq i \leq n$, define $D_i(x) \in H_{pq+i}(K)$ by $D_i(x) = \theta_*(e_i \otimes x^p)$, $\theta_* : H(W^{(n)} \otimes_\pi K^p) \longrightarrow H(K)$. If $p = 2$, define $P_s : H_q(K) \longrightarrow H_{q+s}(K)$ for $s \leq q+n$ by the formula

(i) $\qquad P_s(x) = 0$ if $s < q$; $P_s(x) = D_{s-q}(x)$ if $s \geq q$.

If $p > 2$, define $P_s : H_q(K) \longrightarrow H_{q+2s(p-1)}(K)$ for $2s(p-1) \leq q(p-1)+n$ and define

$\beta P_s : H_q(K) \longrightarrow H_{q+2s(p-1)-1}(K)$ for $2s(p-1) \leq q(p-1)+n+1$ by the formulas

(ii) $P_s(x) = 0$ if $2s < q$; $P_s(x) = (-1)^s \nu(q) D_{(2s-q)(p-1)}(x)$ if $2s \geq q$ and

$\beta P_s(s) = 0$ if $2s \leq q$; $\beta P_s(x) = (-1)^s \nu(q) D_{(2s-q)(p-1)-1}(x)$ if $2s > q$,

where $\nu(2j+\varepsilon) = (-1)^j (m!)^{\varepsilon}$, j any integer, $\varepsilon = 0$ or 1, $m = \frac{1}{2}(p-1)$,

or, equivalently since $(m!)^2 \equiv (-1)^{m+1}$ mod p , $\nu(q) = (-1)^{q(q-1)m/2} (m!)^q$

Observe that, if $n = \infty$, the P_s and, if $p > 2$, the βP_s are defined for all integers s and that βP_s is a single symbol which is not a priori related to any Bockstein operation. The P_s and βP_s are appropriately defined for applications to homology; as shown in section 5, the appropriate formulation for cohomology is obtained by a simple change of notation.

The following proposition contains most of the elementary properties of the D_i, P_s, and βP_s. In particular, if $p > 2$, it shows that the P_s and βP_s account for all non-trivial operations D_i and that βP_s is the composition of P_s and the Bockstein β provided that (K, θ) is reduced mod p.

Proposition 2.3. Let $(K, \theta) \in \mathcal{C}(p, n)$ and consider $D_i : H_q(K) \longrightarrow H_{pq+i}(K)$.

(i) If $f: K \longrightarrow K'$ is a morphism in $\mathcal{C}(p,n)$, then $D_i f_* = f_* D_i$.

(ii) If $i < n$, then D_i is a homomorphism.

(iii) D_o is the p-th power operation in the algebra $H(K)$ and if (K, θ) is unital, then $D_i(e) = 0$ for $i \neq 0$, where $e \in H_o(K)$ is the identity.

(iv) If $p > 2$ and $i < n$, then $D_i = 0$ unless either

(a) q is even and $i = 2t(p-1)$ or $i = 2t(p-1) - 1$ for some t, or

(b) q is odd and $i = (2t+1)(p-1)$ or $i = (2t+1)(p-1) - 1$ for some t.

(v) If (K, θ) is reduced mod p and β is the Bockstein, then

(a) $\beta D_{2i} = D_{2i-1}$ if either $p > 2$ or q is even, and $2i < n$

(b) $\beta D_{2i+1} = D_{2i}$ if $p = 2$ and q is odd, and $2i+1 < n$.

Proof. Part (i) is immediate from the definitions and from Lemma 1.1 (iv), and part (iii) is immediate from the definitions.

(ii) Let $a, b \in K_q$ be cycles and define $\Delta(a, b) = (a+b)^p - a^p - b^p \in K^p$. $\Delta(a, b)$ is a sum of monomials involving both a's and b's, and π permutes such monomials freely. Let $c \in K^p$ be a sum of monomials whose permutations under π give each monomial of $\Delta(a, b)$ exactly once. Then $\Delta(a, b) = Nc$. If i is odd, then $d(e_{i+1} \otimes c) = e_i \otimes Nc$ and if i is even, then $d(T^{p-2}e_{i+1} \otimes c) = e_i \otimes Nc$ in $W^{(n)} \otimes_\pi K^p$, $i < n$, since $T^{p-1} = N$ in $Z_p\pi$. Thus $e_i \otimes \Delta(a, b)$ is a boundary and therefore D_i is a homomorphism, $i < n$.

(iv) In the notations of Definition 2.1, we have that θ is homotopic to a composite $W^{(n)} \otimes_\pi K^p \xrightarrow{j \otimes 1} V^{(n)} \otimes_\pi K^p \longrightarrow V^{(n)} \otimes_{\Sigma_p} K^p \xrightarrow{\phi} K$. Since nothing is changed by tensoring with two copies of $Z_p(q)$, this composite can equally well be written as

$$W^{(n)}(q) \otimes_\pi K^p(q) \xrightarrow{j \otimes 1} V^{(n)}(q) \otimes_{\Sigma_p} K^p(q) \xrightarrow{\phi} K.$$

Let $a \in K_q$ be a cycle. Then, bythe definition of $K^p(q)$, a^p is a basis for a trivial Σ_p-subcomplex of $K^p(q)$. Therefore, if $j(e_i) = d(f)$ in $V^{(n)} \otimes_{\Sigma_p} Z_p(q) = V^{(n)}(q) \otimes_{\Sigma_p} Z_p$, then $d(f \otimes a^p) = j(e_i) \otimes a^p$ in $V^{(n)}(q) \otimes_{\Sigma_p} K^p(q)$. For $i < n$, j induces $j_*: H_i(\pi; Z_p(q)) \longrightarrow H_i(\Sigma_p; Z_p(q))$, and the desired conclusion now follows immediately from Lemma 1.4.

(v) Let (K, θ) be the mod p reduction of $(\tilde{K}, \tilde{\theta})$. Let $a \in \tilde{K}_q$ satisfy $d(a) = pb$. An easy calculation demonstrates that, in $\tilde{K}^p$,

$$d(a^p) = pNba^{p-1} \text{ if } p > 2 \text{ or } q \text{ is even;}$$

$$d(a^2) = 2Tab \text{ if } p = 2 \text{ and } q \text{ is odd.}$$

In the former case, if $2i < n$, then, in $W(p, Z)^{(n)} \otimes_\pi \tilde{K}^p$,

$$d(e_{2i} \otimes a^p) = e_{2i-1} \otimes Na^p + pe_{2i} \otimes Nba^{p-1}$$
$$\equiv p[e_{2i-1} \otimes a^p + d(T^{p-2}e_{2i+1} \otimes ba^{p-1})] \bmod p^2,$$

since $T^{p-1} \equiv N \bmod p$. In the latter case, if $2i+1 < n$, then

$$d(e_{2i+1} \otimes a^2) = e_{2i} \otimes Ta^2 - 2e_{2i+1} \otimes Tab \equiv 2[e_{2i} \otimes a^2 - d(e_{2i+2} \otimes ab)] \bmod 4.$$

Thus, in $H(W(p, Z_p)^{(n)} \otimes_\pi K^p)$, if $\bar{a}$ is the mod p reduction of a, then $\beta\{e_{2i} \otimes \bar{a}^p\} = \{e_{2i-1} \otimes \bar{a}^p\}$ in case (a) and $\beta\{e_{2i+1} \otimes \bar{a}^2\} = \{e_{2i} \otimes \bar{a}^2\}$ in case (b). Since θ is the mod p reduction of the map $\theta: W(p, Z)^{(n)} \otimes_\pi K^p \longrightarrow K$, $\beta\theta_* = \theta_*\beta$, and the result follows.

Of course, (i) and (ii) imply that the P_s and βP_s are natural homomorphisms (except, if $n < \infty$, for the last operation). A check of constants gives the following corollary of part (iii).

<u>Corollary 2.4.</u> Let $(K, \theta) \in \mathcal{C}(p, n)$. Then $P_q(x) = x^p$ if $p = 2$ and $x \in H_q(K)$ or if $p > 2$ and $x \in H_{2q}(K)$. If (K, θ) is unital, then $P_s(e) = 0$ for $s \neq 0$.

The implications of (iv) and (v) for the P_s and βP_s are clear if $p > 2$. If $p = 2$, we have the following corollary of (v).

<u>Corollary 2.5.</u> If $(K, \theta) \in \mathcal{C}(2, \infty)$ is reduced mod 2, then $\beta P_{s+1} = sP_s$.

The following result is the external Cartan formula.

<u>Proposition 2.6.</u> Let (K, θ) and (L, θ') be objects of $\mathcal{C}(p, n)$. Let $x \in H_q(K)$ and $y \in H_r(L)$. Consider $x \otimes y \in H(K) \otimes H(L) = H(K \otimes L)$.

(i) If $p = 2$, then $D_i(x \otimes y) = \sum_{j+k=i} D_j(x) \otimes D_k(y)$ for $i \leq n$.

(ii) If $p > 2$, then $D_{2i}(x \otimes y) = (-1)^{mqr} \sum_{j+k=i} D_{2j}(x) \otimes D_{2k}(y)$ for $2i \leq n$, and

$$D_{2i+1}(x \otimes y) = (-1)^{mqr} \sum_{j+k=i} (D_{2j+1}(x) \otimes D_{2k}(y) + (-1)^q D_{2j}(x) \otimes D_{2k+1}(y)) \quad \text{for } 2i+1 \leq n.$$

<u>Proof.</u> By Lemmas 1.1 and 1.3, we may work in $W^{(n)} \otimes_\pi [H(K) \otimes H(L)]^p$. Since π operates trivially on $(x \otimes y)^p$, we may compute $\theta_*(e_i \otimes (x \otimes y)^p)$ by means of the induced coproduct on $W^{(n)} \otimes_\pi Z_p$, as given in (2) of Definition 1.2. The result follows by direct calculation from

$$\theta_*(e_i \otimes (x \otimes y)^p) = (\theta_* \otimes \theta'_*)(1 \otimes T \otimes 1)(\psi \otimes U)(e_i \otimes (x \otimes y)^p).$$

A trivial verification of constants, together with part (iv) of Proposition 2.3, yields the following corollary.

Corollary 2.7. Let (K, θ) and (L, θ') be objects of $\mathcal{C}(p, n)$. Let $x \in H_q(K)$ and $y \in H_r(L)$. Then $P_s(x \otimes y) = \sum_{i+j=s} P_i(x) \otimes P_j(y)$ and, if $p > 2$, $\beta P_{s+1}(x \otimes y) = \sum_{i+j=s} (\beta P_{i+1}(x) \otimes P_j(y) + (-1)^q P_i(x) \otimes \beta P_{j+1}(y))$.

Of course, if (K, θ) is a Cartan object in $\mathcal{C}(p, n)$, then the corollary and the naturality of the operations imply that the P_s and, if $p > 2$, the βP_s on $H(K)$ satisfy the internal Cartan formulas

$$P_s(xy) = \sum_{i+j=s} P_i(x)P_j(y) \quad \text{and}$$

(1)

$$\beta P_{s+1}(xy) = \sum_{i+j=s} (\beta P_{i+1}(x)P_j(y) + (-1)^{\deg x} P_i(x)\beta P_{j+1}(y)) .$$

3. Chain level operations, suspension, and spectral sequences

In this section, we define chain level Steenrod operations and use them to prove that the homology operations commute with suspension. The chain level operations can also be used to study the behavior of Steenrod operations in spectral sequences and, in particular, we shall prove a general version of the Kudo transgression theorem.

Theorem 3.1. Let $(K, \theta) \in \mathcal{C}(p, \infty)$. Then there exist functions $P_s: K_q \longrightarrow K_{q+s}$ if $p = 2$ and $P_s: K_q \longrightarrow K_{q+2s(p-1)}$ and $\beta P_s: K_q \longrightarrow K_{q+2s(p-1)-1}$ if $p > 2$ which satisfy the following properties.

(i) $dP_s = P_s d$ and $d\beta P_s = -\beta P_s d$

(ii) If a is a cycle which represents $x \in H(K)$, then $P_s(a)$ and $\beta P_s(a)$ are cycles which represent $P_s(x)$ and $\beta P_s(x)$.

(iii) If $f: (K, \theta) \longrightarrow (K', \theta')$ is a morphism in $\mathcal{P}(p, \infty)$, so that $f\theta = \theta'(1 \otimes f^p)$, then $fP_s = P_s f$ and $f\beta P_s = \beta P_s f$.

Proof. Let $a \in K_q$ and write $b = d(a) \in K_{q-1}$. In the case $p = 2$, define

(1) $P_s(a) = \theta(c)$, where $c = e_{s-q+1} \otimes b \otimes a + e_{s-q} \otimes a \otimes a \in W \otimes_\pi K^2$.

The verification of (i), (ii), and (iii) is trivial. Thus assume that $p > 2$. Let (a, b) denote the subcomplex of K with basis a and b, so that $(a, b)^p \subset K^p$. Define $s: (a, b) \longrightarrow (a, b)$, of degree one, by $s(a) = 0$ and $s(b) = a$. Then $ds + sd = 1$ on (a, b). Let $S = 1^{p-1} \otimes s$ on $(a, b)^p$. Then $dS + Sd = 1$ on $(a, b)^p$ and S is given explicitly by $S(ea) = 0$ and $S(eb) = (-1)^{\deg e} ea$ for $e \in (a, b)^{p-1}$. Define $t_i \in (a,b)^p$ for $0 \leq i \leq p$ by the inductive formula

(2) $t_o = b^p$; $t_1 = b^{p-1}a$; $t_{2k} = S(\alpha^{-1} t_{2k-1} - t_{2k-1})$; $t_{2k+1} = S(Nt_{2k})$.

Since $dS + Sd = 1$, an easy calculation demonstrates that

(3) $d(t_1) = t_o$; $d(t_{2k}) = (\alpha^{-1} - 1)t_{2k-1}$ and $d(t_{2k+1}) = Nt_{2k}$, $1 \leq k \leq m$.

A straightforward induction, which uses the explicit formula for S, yields

(4) $t_{2k} = \sum_I (-1)^{kq}(k-1)!\, b^{i_1}a^2b^{i_2}a^2 \cdots b^{i_k}a^2$, $1 \le k \le m$, summed over all

k-tuples $I = (i_1, \ldots, i_k)$ such that $\sum i_j = p-2k$; and

(5) $t_{2k+1} = \sum_I (-1)^{kq}\, k!\, b^{i_1}a^2 \ldots b^{i_k}a^2b^{i_{k+1}}a$, $0 \le k \le m$, summed over all

$(k+1)$-tuples $I = (i_1, \ldots, i_{k+1})$ such that $\sum i_j = p-1-2k$.

In particular, $t_p = t_{2m+1} = (-1)^{mq} m!\, a^p$ (since each $i_j = 0$). Now let $j = (2s-q+1)(p-1)$ and define chains c and c' in $W \otimes_\pi K^p$ by the following formulas (where, by convention, $e_i = 0$ if $i < 0$):

(6) $$c = \sum_{k=0}^{m} (-1)^k e_{j-2k} \otimes t_{2k+1} - \sum_{k=1}^{m} (-1)^k e_{j+1-2k} \otimes (\alpha^{-1}-1)^{p-2} t_{2k} ;$$

(7) $$c' = \sum_{k=0}^{m} (-1)^k e_{j-1-2k} \otimes t_{2k+1} + \sum_{k=1}^{m} (-1)^k e_{j-2k} \otimes t_{2k} .$$

Then an easy computation, which uses Definition 1.2 and (3), gives

(8) $$d(c) = e_j \otimes b^p \text{ and } d(c') = -e_{j-1} \otimes b^p \quad (j = (2s-q+1)(p-1))$$

In calculating $d(c)$, the salient observations are that $Nt_p = 0$, that $\alpha e_i \otimes t = e_i \otimes \alpha^{-1} t$ for $t \in K^p$ by the very definition of a tensor product, and that $(\alpha^{-1}-1)^{p-1} = N$ in $Z_p\pi$. Finally, define

(9) $$P_s(a) = (-1)^s \nu(q-1)\theta(c) \text{ and } \beta P_s(a) = (-1)^s \nu(q-1)\theta(c') .$$

If a is a cycle, $b = 0$, then $t_i = 0$ for $i < p$ and $t_p = (-1)^{mq} m!\, a^p$, hence

(10) $$c = (-1)^{m(q+1)} m!\, e_{(2s-q)(p-1)} \otimes a^p \text{ and } c' = (-1)^{m(q+1)} m!\, e_{(2s-q)(p-1)-1} \otimes a^p.$$

It is easy to verify that $\nu(q) = (-1)^{m(q+1)} m!\, \nu(q-1)$ and now (ii) is obvious from (9) and (10) and (i) follows from (8), (9), and (10), applied to the cycle $b \in K_{q-1}$. Part (iii) is immediate from (9).

The remaining results of this section are corollaries of the theorem and its proof. We first define and study a very general notion of suspension.

Definition. Let $f: K' \longrightarrow K$ and $g: K \longrightarrow K''$ be morphisms of Λ-complexes such that $gf = 0$. Define σ: Ker $f_* \longrightarrow$ Coker g_* by the formula $\sigma\{b'\} = \{g(a)\}$, where b' represents $\{b'\} \in$ Ker f_* and $d(a) = f(b')$ in K. It is trivial to verify that σ is well-defined, and we call σ the suspension.

We can now prove that the P_s commute with suspension. We remark that if $n = \infty$, the hypotheses of the next theorem simplify to the requirement that f and g be morphisms in $\mathcal{P}(p, \infty)$ such that $gf = 0$. For $n < \infty$, the stated hypotheses arise in practice in the study of iterated loop spaces.

Theorem 3.3. Let $(K', \theta') \in \mathcal{C}(p, n+1)$ and let $(K'', \theta'') \in \mathcal{C}(p, n)$. Let K be a Z_p-complex and let $f: K' \longrightarrow K$ and $g: K \longrightarrow K''$ be morphisms of complexes such that $gf = 0$. Define a subcomplex $\tilde{K}$ of $W^{(n+1)} \otimes K^p$ by

$$\tilde{K} = W^{(n+1)} \otimes f(K')^p + \overline{W}^{(n+1)} \otimes f(K')^{p-1} \otimes K + W^{(n)} \otimes K^p,$$

where $\overline{W}^{(n+1)} = W^{(n)} \oplus Z_p e_{n+1}$ (that is, $e_{n+1} \in \overline{W}^{(n+1)}$ but $\alpha^i e_{n+1} \notin \overline{W}^{(n+1)}$ for $i \leq i < p$). Suppose given a π-morphism $\theta: \tilde{K} \longrightarrow K$ (where, by convention, π does not act on $e_{n+1} \otimes f(K')^{p-1} \otimes K$) such that the following diagram is commutative:

$$\begin{array}{ccccc}
W^{(n+1)} \otimes (K')^p & \xrightarrow{1 \otimes f^p} & \tilde{K} & \xrightarrow{1 \otimes g^p} & W^{(n)} \otimes (K'')^p \\
\downarrow \theta' & & \downarrow \theta & & \downarrow \theta'' \\
K' & \xrightarrow{f} & K & \xrightarrow{g} & K''
\end{array}$$

(Here $gf = 0$ ensures that $(1 \otimes g^p)(\tilde{K}) \subset W^{(n)} \otimes (K'')^p$.) Observe that Ker f_* is closed under the P_s and βP_s and that there are well-defined induced P_s on Coker g_*. Let $x \in$ Ker f_*. Then $\sigma P_s(x) = P_s \sigma(x)$ and $\sigma \beta P_s(x) = -\beta P_s \sigma(x)$ whenever $P_s(x)$ and $\beta P_s(x)$ are defined.

Proof. Let $\deg(x) = q-1$ and let $b' \in K'$ represent x. Let $b = f(b')$ and let $d(a) = b$ in K, so that $g(a)$ represents $\sigma(x)$. The hypothesis guarantees that if s is such that $P_s(x)$ or $\beta P_s(x)$ is defined, then the chain level operation $P_s(a)$ or $\beta P_s(a)$ constructed in the previous proof is also defined. Of course, this is clear if $n = \infty$; if $n < \infty$, we need only verify that all elements involved in the definition of $P_s(a)$ or $\beta P_s(a)$ are present in $\tilde{K}$. For example, if $p = 2$, the last operation $P_s(x)$ occurs for $s = q+n$ and then $P_s(a) = \theta(c)$, where $c = e_{n+1} \otimes b \otimes a + e_n \otimes a \otimes a$, and c is indeed in $\tilde{K}$. Now our diagram ensures that $fP_s(b') = P_s f(b')$, hence $fP_s(b') = dP_s(a)$, and that $gP_s(a) = P_s g(a)$. $\sigma P_s(x) = P_s \sigma(x)$ follows from the definition of σ, and the proof that $\sigma\beta P^s(x) = -\beta P_s \sigma(x)$ is equally simple.

Note that if $p > 2$ and all objects are reduced mod p, then $\sigma\beta = -\beta\sigma$, which is consistent with the theorem. The theorem implies that $\sigma(x^p) = 0$ and that $\sigma\beta P_s(x) = 0$ if $p > 2$ and $\deg(x) = 2s-1$; if (K'', θ'') is reduced mod p, the latter statement becomes $\beta\sigma(x)^p = 0$. The operation $\beta P_s(x)$, $\deg(x) = 2s - 1$, plays a special role in many applications; the following very useful technical result about about this operation is known as the Kudo transgression theorem. It applies to the Dyer-Lashof operations in the homology Serre spectral sequence of the path-space fibration $\Omega^n X \longrightarrow P\Omega^{n-1} X \longrightarrow \Omega^{n-1} X$, to the Steenrod operations in the cohomology Serre spectral sequence of a fibration $F \longrightarrow E \longrightarrow B$ (with $K' \longrightarrow K \longrightarrow K''$ being $C^*(B) \longrightarrow C^*(E) \longrightarrow C^*(F)$, graded by subscripts) and to the spectral sequence of Adams [1, p. 210] for cocommutative Hopf algebras.

Theorem 3.4. Assume, in addition to the hypotheses of Theorem 3.3, that K has an increasing fitration $\{F_i K\}$, that $H_o(K') = Z_p = H_o(K'')$, and that there is a morphism of complexes $\pi: K \otimes f(K') \longrightarrow K$ such that either

(i) K', K, and K'' are positively graded, $F_i K = 0$ if $i < 0$, $F_i K_i = K$ if $i \geq 0$, $f(K') \subset F_o K$, $\pi(F_i K \otimes f(K')) \subset F_i K$, and f and g induce isomorphisms $E^2 f: H_j(K') \longrightarrow E^2_{oj} K$ and $E^2 g: K \longrightarrow H_i(K'')$ and π induces a morphism $E^2\pi: E^2_{ij} K \otimes E^2_{ok} K \longrightarrow E^2_{i,j+k} K$ such that the composite morphism $E^2\pi[(E^2 g)^{-1} \otimes E^2 f]: H_i(K'') \otimes H_j(K') \longrightarrow E^2_{ij} K$ is an isomorphism; or

(ii) K', K, and K'' are negatively graded, $F_iK = K$ if $i \geq 0$, $F_{i-1}K_i = 0$ if $i \leq 0$, $f(K'_i) \subset F_iK$, $\pi(F_iK \otimes f(K'_j)) \subset F_{i+j}K$, and f and g induce isomorphisms $E^2f: H_i(K') \longrightarrow E^2_{i,o}K$ and $E^2_g: E^2_{oj}K \longrightarrow H_j(K'')$ and π induces a morphism $E^2\pi: E^2_{ij}K \otimes E^2_{ko}K \longrightarrow E^2_{i+k,j}K$ such that the composite morphism $E^2\pi[(E^2g)^{-1} \otimes E^2f]: H_j(K'') \otimes H_i(K') \longrightarrow E^2_{ij}K$ is an isomorphism.

Let τ be the transgression, $\tau = d_t: E^t_{to}K \longrightarrow E^t_{0,t-1}K$ in (i) and $\tau = d_{1-t}: E^{1-t}_{ot}K \longrightarrow E^{1-t}_{t-1,o}K$ $(t < 0)$ in (ii). Then τ is the inverse additive relation to σ, and if $y \in H_q(K'')$ transgresses to $x \in H_{q-1}(K')$, then $P_s(y)$ and if $p > 2$, $\beta P_s(y)$ transgresses to $P_s(x)$ and $-\beta P_s(x)$, whenever the operations are defined. Moreover, if $p > 2$ and $q = 2s$, then $y^{p-1} \otimes x$ transgresses to $-\beta P_s(x)$ (that is, $d_{q(p-1)}(y^{p-1} \otimes x) = -\beta P_s(x)$ in case (i) and $d_{1-q(p-1)}(y^{p-1} \otimes x) = -\beta P_s(x)$ in case (ii) provided that

(iii) if $a_j \in F_{i_j}K$, then $\theta(e_k \otimes a_1 \otimes \ldots \otimes a_p) \in F_iK$, $i = \sum i_j + k$, and

(iv) The restriction of θ to $e_o \otimes K^{p-1} \otimes f(K')$ induces a morphism $E^2\theta: (E^2_{*o}K)^{p-1} \otimes E^2_{o*}K \longrightarrow E^2K$ in (i) and $E^2\theta: (E^2_{o*}K)^{p-1} \otimes E^2_{*o}K \rightarrow E^2K$ in (ii) such that $E^2\theta = E^2\pi[(E^2g)^{-1}\phi(E^2g)^{p-1} \otimes 1]$, where $\phi: H(K'')^{p-1} \longrightarrow H(K'')$ is the iterated product.

Proof. Let $y \in H_q(K'')$. By the definition of the differentials in the spectral sequence of a filtered complex, $\tau(y)$ is defined if and only if y is represented by $g(a)$ for some $a \in K_q$ such that $d(a) = f(b')$ for some cycle $b' \in K'_{q-1}$, and then $x = \tau(y) = \{b'\}$. Thus the first statement follows from the properties of the chains $P_s(a)$ and $\beta P_s(a)$. For the second statement, consider $\beta P_s(a)$, $q = 2s$. Since $d\beta P_s(a) = -\beta P_s f(b')$, a and b' as above, it suffices to prove that $\beta P_s(a)$ represents $y^{p-1} \otimes x$ in E^2K. $\beta P_s(a) = -m!\,\theta(c')$ by (9) of the proof of Theorem 3.1 and the observation that $\nu(q-1) = \nu(2s-1) = (-1)^{s-1}m!$. In the definition (7) of c', the term with $k = m$ in the first sum involves e_{-1} (since

$q = 2s$ implies $j = p-1$) and is therefore zero. The term with $k = m$ in the second sum is $(-1)^m e_o \otimes t_{p-1}$ where, by (4), $t_{p-1} = (m-1)! \sum_{i=0}^{m-1} a^{2i} b a^{2(m-i)}$, $b = d(a)$. It is easy to see that

$$\sum_{i=0}^{m-1} a^{2i} b a^{2(m-i)} = P(\alpha) a^{p-1} b \ , \quad \text{where } P(\alpha) = \sum_{i=1}^{m} \alpha^{2i-1} \ ,$$

and direct calculation shows that $P(\alpha) = m + Q(\alpha)$, in $Z_p\pi$, where $Q(\alpha) = \sum_{j=1}^{m} j(\alpha^{2j} + \alpha^{2j+1})$. Let $c'' = (-1)^m (m-1)! \, e_1 \otimes Q(\alpha) a^{p-1} \otimes b \in W \otimes K^p$. Then $c' - d(c'') = (-1)^m m! \, e_o \otimes a^{p-1} b$ plus a linear combination of terms $e_i \otimes g$ such that g has $i+1$ factors b and $p-i-1$ factors a. Condition (iii) ensures that each $\theta(e_i \otimes g)$ has lower filtration than does $\theta(e_o \otimes a^{p-1} b)$ and condition (iv) ensures that $\theta(e_o \otimes a^{p-1} b)$ represents $y^{p-1} \otimes x \in E^2 K$. Since $\beta P_s(a)$ and $-m! \, \theta(c' - d(c'')) = \theta(e_o \otimes a^{p-1} b)$ represent the same element of $E^2 K$, the proof is complete.

The following proposition gives a general prescription for the study of Steenrod operations in spectral sequences; it will be useful in the study of the cohomology of the Steenrod algebra in [18]. In the applications, the determination of the function f is often quite difficult and depends on how the given θ was constructed.

Proposition 3.5. Let (K, θ) be an increasingly filtered object of $\mathcal{C}(p, \infty)$ Suppose given a function $f(i, j, k)$ such that

(i) If $a_t \in F_{i_t} K_{i_t + j_t}$, where $\sum i_t = i$ and $\sum j_i = j$, then

$$\theta(e_k \otimes a_1 \otimes \dots \otimes a_p) \in F_{f(i,j,k)} K_{i+j+k} \, ;$$

(ii) $f(i, j, k) > f(i-r, j+r-1, k+1)$, $r \geq 1$; and

(iii) $f(i, j, k) \geq r + f(i-pr, j+p(r-1), k+p-1)$, $r \geq 1$.

Let $y \in E^r_{ij} K$. Then there exist elements $P_s(y) \in E^t_{k\ell} K$ and, if $p > 2$, $\beta P_s(y) \in E^{t'}_{k'\ell'}$ such that $d^t P_s(y) = P_s d^r(y)$ and $d^{t'} \beta P_s(y) = -\beta P_s d^r(y)$, where

(iv) If $p = 2$, then $k = f(2i, 2j, s-i-j)$, $\ell = i+j+s-k$, and

$t = k - f(2i - 2r, 2j + 2(r-1), s+1-i-j)$,

(v) If $p > 2$, then $k = f(pi, pj, (2s-i-j)(p-1))$, $\ell = i+j+2s(p-1)-k$, and

$t = k - f(pi - pr, pj + p(r-1), (2s+1-i-j)(p-1))$.

(vi) If $p > 2$, then $k' = f(pi, pj, (2s-i-j)(p-1)-1)$, $\ell' = i+j+2s(p-1)-1-k$, and

$t' = k' - f(pi - pr, pj + p(r-1), (2s+1-i-j)(p-1)-1)$.

Proof. Let $a \in F_i K_{i+j}$ represent y and let $b = d(a) \in F_{i-r} K_{i+j-1}$. Consider the chain $P_s(a)$ constructed in Theorem 3.1. By (ii), all summands of $P_s(a)$ other than that involving $e_n \otimes a^p$ (for the appropriate n) have lower filtration than k and, by (i), $\theta(e_n \otimes a^p)$ has filtration k. Since $dP_s(b)$, where $P_s(b) \in F_{k-t} K$ by (i) and $k - t \geq r$ by (iii), the statement about $P_s(y)$ follows. The proof for $\beta P_s(y)$ is similar.

4. The Adem relations

We here show that the Adem relations are valid for the Steenrod operations in $H(K)$ if $(K, \theta) \in \mathcal{C}(p, \infty)$ satisfies certain hypotheses. The general algebraic context is distinctly advantageous in the proof. We are able to exploit a trick (Lemma 4.3) used by Adem to prove the classical Adem relations, and this trick would not be available in a categorical approach to Steenrod operations since it depends on the usage of objects of $\mathcal{C}(p, \infty)$ which are not present in many categories of interest, such as infinite loop spaces and cocommutative Hopf algebras.

We require some notations and definitions before we can proceed to the proof.

Let Σ_{p^2} act as permutations on $\{(i,j) \mid 1 \leq i \leq p,\ 1 \leq j \leq p\}$. Embed π in Σ_{p^2} by letting $\alpha(i,j) = (i+1, j)$. Define $\alpha_i \in \Sigma_{p^2}$, $1 \leq i \leq p$, by

$\alpha_i(i,j) = (i,j+1)$ and $\alpha_i(k,j) = (k,j)$ for $k \neq i$, and let $\beta = \alpha_1 \cdots \alpha_p$ so that $\beta(i,j) = (i,j+1)$. Then

(1) $\alpha\alpha_i = \alpha_{i+1}\alpha$; $\alpha_i\alpha_j = \alpha_j\alpha_i$; and $\alpha\beta = \beta\alpha$.

Let α_i generate π_i and β generate ν, so that π_i and ν are cyclic of order p. Let $\sigma = \pi\nu$ and let τ be generated by the α_i and α. Then $\sigma \subset \tau$, τ is a p-Sylow subgroup of Σ_{p^2}, and τ is a split extension of $\pi_1 \cdots \pi_p$ by π.

Let $W_1 = W$ and $W_2 = W$ regarded, respectively, as π-free and ν-free resolutions of Z_p. Let ν operate trivially on W_1, let π operate trivially on W_2, and let σ operate diagonally on $W_1 \otimes W_2$. Then $W_1 \otimes W_2$ is a σ-free resolution of Z_p.

If M is any ν-module, let τ operate on M^p by letting α operate by cyclic permutation and by letting α_i operate on the i-th factor M as does β. Let α_i operate trivially on W_1. Then τ operates on W_1 and we let τ operate diagonally on $W_1 \otimes M^p$. In particular, $W_1 \otimes W_2^p$ is then a τ-free resolution of Z_p.

Let K be any Z_p-complex. We let Σ_{p^2} operate on K^{p^2} by permutations with the (i, j)-th factor K being the j-th factor K in the i-th factor K^p of $K^{p^2} = (K^p)^p$. We let ν operate in the standard fashion on $W_2 \otimes K^p$ (β acting as cyclic permutation on K^p). By the previous paragraph, this fixes an operation of τ on $W_1 \otimes (W_2 \otimes K^p)^p$.

Let Y be any Σ_{p^2}-free resolution of Z_p and let $w: W_1 \otimes W_2^p \longrightarrow Y$ be any τ-morphism over Z_p. w exists since Y is acyclic, and any two choices of w are τ-homotopic.

With these notations, we have the following definition.

<u>Definition 4.1.</u> Let $(K, \theta) \in \mathcal{C}(p,n)$. We say that (K,θ) is an Adem object if there exists a Σ_{p^2}-morphism $\xi: Y^{(n)} \otimes K^{p^2} \longrightarrow K$ such that the following diagram is τ-homotopy commutative:

$$\begin{array}{ccccc}
(W_1 \otimes W_2^{\,p})^{(n)} \otimes K^{p^2} & \xrightarrow{\;w \otimes 1\;} & Y^{(n)} \otimes K^{p^2} & & \\
\Big\downarrow 1 \otimes U & & & \searrow \xi & \\
 & & & & K \\
 & & & \nearrow \theta & \\
W_1^{(n)} \otimes (W_2^{(n)} \otimes K^p)^p & \xrightarrow{\;1 \otimes \theta^p\;} & W_1^{(n)} \otimes K^p & &
\end{array}$$

Here U is the evident shuffle map, and is clearly a τ-morphism (Σ_{p^2} acts trivially on K and α_i acts trivially on $W_1^{(n)} \otimes K^p$).

For clarity, we only treat the case $n = \infty$ below. The relations obtained will be valid for operations on $H_q(K)$, (K, θ) an Adem object of $\mathcal{C}(p, n)$, provided that n is sufficiently large relative to q.

We first show that the tensor product of Adem objects is an Adem object and then use this fact to show that any relations valid on $H_{q_i}(K)$ for all Adem objects (K, θ) and suitable q_i will necessarily be valid on $H_q(K)$ for arbitrary q.

Lemma 4.2. If (K, θ) and (L, θ') are Adem objects of $\mathcal{C}(p, \infty)$, then $(K \otimes L, \tilde{\theta})$ is an Adem object of $\mathcal{C}(p, \infty)$.

Proof. $\tilde{\theta}$ is as defined in Definitions 2.1. By hypothesis, we are given ξ and ξ' such that (K, ξ) and (K, η') are objects of $\mathcal{C}(\Sigma_{p^2}, \infty, Z_p)$, hence we may define $\tilde{\xi}$ as in Definitions 2.1 so that $(K \otimes L, \tilde{\xi}) \in \mathcal{C}(\Sigma_{p^2}, \infty, Z_p)$. We must show that the diagram of Definition 4.1, for $K \otimes L$, is τ-homotopy commutative, and this follows easily from a simple chase of a large diagram and the definition of $\tilde{\theta}$ and $\tilde{\xi}$. The crucial point is the observation that since $W_1 \otimes W_2^{\,p}$ is τ-free and $Y \otimes Y$ is acyclic, the following diagram is τ-homotopy commutative where V is the evident shuffle and $\psi : Y \longrightarrow Y \otimes Y$ is any given Σ_{p^2}-coproduct:

$$\begin{array}{ccccc} W_1 \otimes W_2^p & \xrightarrow{\psi \otimes \psi^p} & W_1 \otimes W_1 \otimes (W_2 \otimes W_2)^p & \xrightarrow{V} & W_1 \otimes W_2^p \otimes W_1 \otimes W_2^p \\ \downarrow w & & & & \downarrow w \otimes w \\ Y & & \xrightarrow{\psi} & & Y \otimes Y \end{array}$$

Let F_p denote the free associative algebra generated by $\{P_s | s \in Z\}$ and, if $p > 2$, $\{\beta P_s | s \in Z\}$. Let $J_p \subset F_p$ denote the two-sided ideal consisting of all elements $a \in F_p$ such that $ax = 0$ for all $x \in H(K)$ and all Adem objects $(K, \theta) \in \mathcal{C}(p, \infty)$. Let $B_p = F_p / J_p$. B_p is a universal Steenrod algebra. Both the classical Steenrod algebra and the Dyer-Lashof algebra [17] are quotients of B_p.

Lemma 4.3. Let $a \in F_p$. Let $\{q_i | i \geq 0\}$ be a strictly decreasing sequence of integers. Suppose that $ax = 0$ for all $x \in H_{q_i}(K)$, $i \geq 0$, and all Adem objects $(K, \theta) \in \mathcal{C}(p, \infty)$. Then $a \in J_p$.

Proof. Let K be an Adem object in $\mathcal{C}(p, \infty)$ and let $x \in H_q(K)$. We must prove that $ax = 0$. Choose $r < 0$ such that $q + r = q_i$ for some i. There exists an Adem object $(L_r, \theta_r) \in \mathcal{C}(p, \infty)$ and a class $y \in H_r(L_r)$ such that $P_o(y) = y$, $P_s(y) = 0$ for $s \neq 0$, and $\beta P_s(y) = 0$ for all s. Such an object can easily be constructed explicitly, but it is quicker to appeal to the results of section 8, which show that the singular cochains of a $(-r)$-sphere, graded by non-positive subscripts, provide such an object. Now $(K \otimes L_r, \tilde{\theta})$ is an Adem object of $\mathcal{C}(p,\infty)$ by the previous lemma. By the external Cartan formula, Corollary 2.7, $a(x \otimes y) = ax \otimes y$. Since $x \otimes y \in H_{q_i}(K \otimes L_r)$, $a(x \otimes y) = 0$ and therefore $ax = 0$, as was to be shown.

The Adem relations will be proven by chosing the diagram of Definition 4.1, and we shall need some information about the homology of σ, τ, and Σ_{p^2}. Let $\phi: W_1 \otimes W_2 \longrightarrow W_1 \otimes W_2^p$ be a σ-morphism over Z_p. Define $\gamma \in \Sigma_{p^2}$ by $\gamma(i, j) = (j, i)$. Observe that $\gamma^2 = 1$ and $\gamma\alpha = \beta\gamma$. For $q \in Z$, conjugation by γ

gives a commutative diagram

$$\begin{array}{ccccc} H_*(\sigma; Z_p(q)) & \xrightarrow{\phi_*} & H_*(\tau; Z_p(q)) & \xrightarrow{w_*} & H_*(\Sigma_{p^2}; Z_p(q)) \\ \downarrow \gamma_* & & & & \downarrow 1 \\ H_*(\sigma; Z_p(q)) & \xrightarrow{\phi_*} & H_*(\tau; Z_p(q)) & \xrightarrow{w_*} & H_*(\Sigma_{p^2}; Z_p(q)) \end{array}$$

Thus $w_*(\phi_* - \phi_*\gamma_*) = 0$. The following lemmas compute γ_* and ϕ_*. Note that $H_*(\tau; Z_p(q)) = H_*(\tau; Z_p)$ since τ contains only even permutations.

Lemma 4.4. γ_* is given on $H_*(\sigma; Z_p(q))$ by $\gamma_*(e_i \otimes e_j) = (-1)^{ij+mq} e_j \otimes e_i$.

Proof. Define $\gamma_\# : W_1 \otimes W_2 \longrightarrow W_1 \otimes W_2$ by the formula

$$\gamma_\#(\alpha^k e_i \otimes \beta^\ell e_j) = (-1)^{ij} \alpha^\ell e_j \otimes \beta^k e_i .$$

Then $d\gamma_\# = \gamma_\# d$ and $\gamma_\#(\mu x) = (\gamma\mu\gamma^{-1})\gamma_\#(x)$ for $\mu \in \sigma$ and $x \in W_1 \otimes W_2$. Thus $\gamma_\# \otimes \gamma: (W_1 \otimes W_2) \otimes_\sigma Z_p(q) \longrightarrow (W_1 \otimes W_2) \otimes_\sigma Z_p(q)$ induces γ_*. Since the sign of γ is $(-1)^m$, $\gamma \cdot 1 = (-1)^{mq}$ in $Z_p(q)$, and the result follows.

Before computing ϕ_*, we fix notations concerning binomial coefficients.

Notations 4.5. Let i and j be integers. Define $(i,j) = (i+j)!/i!j!$ if $i \geq 0$ and $j \geq 0$ $(0! = 1)$ and define $(i,j) = 0$ if $i < 0$ or $j < 0$. Recall that if $i \geq 0$ and $j \geq 0$ have p-adic expansions $i = \sum a_k p^k$ and $j = \sum b_k p^k$, then $(i,j) \equiv \prod_k (a_k, b_k) \bmod p$. Clearly $(a_k, b_k) \not\equiv 0 \bmod p$ if and only if $a_k + b_k < p$, hence $(i,j) \not\equiv 0 \bmod p$ if and only if $\sum (a_k + b_k)p^k$ is the p-adic expansion of $i+j$.

Lemma 4.6. $H_*(\tau; Z_p) = H_*(\pi; H_*(\nu; Z_p)^p)$ and $\phi_*: H_*(\sigma; Z_p) \longrightarrow H_*(\tau; Z_p)$ is given by the following formulas (with sums taken over the integers).

(i) If $p = 2$, $\phi_*(e_r \otimes e_s) = \sum_k (k, s-2k) e_{r+2k-s} \otimes e^2_{s-k}$; and

(ii) If $p > 2$, $\phi_*(e_r \otimes e_s) = \sum_k (-1)^k \nu(s)(k, [s/2] - pk)\, e_{r+(2pk-s)(p-1)} \otimes e^p_{s-2k(p-1)}$

$$-\delta(r)\delta(s-1) \sum_k (-1)^k \nu(s-1)(k, [^{s-1}/2] - pk) e_{r+p+(2pk-s)(p-1)} \otimes e^p_{s-2k(p-1)-1},$$

where $\nu(2j+\varepsilon) = (-1)^j(m!)^{\varepsilon}$ and $\delta(2j+\varepsilon) = \varepsilon$, j any integer, $\varepsilon = 0$ or 1 .

Proof. Let $\overline{W}_2 = W_2 \otimes_\nu Z_p = H_*(\nu; Z_p)$. By the definition of the action of τ on $W_1 \otimes W_2^p$, we have that $(W_1 \otimes W_2^p) \otimes_\tau Z_p = W_1 \otimes_\pi \overline{W}_2^p$ as a Z_p-complex, and the first part follows. Of course, $H_*(\tau; Z_p)$ is now computed by Lemma 1.3. ϕ_* could be computed directly, but it is simpler to use topology. Let $K(Z_p, 1) = E/\nu$, where ν operates properly on the acyclic space E, so that, by [14, IV 11], $C_*(E) = Z_p\nu \otimes C_*(E/\nu)$, with Z_p coefficients. Let $D: E \longrightarrow E^p$ be the iterated diagonal. Then $1 \otimes D: W_1 \otimes C_*(E) \longrightarrow W_1 \otimes C_*(E^p)$ is a σ-morphism, and we shall obtain a σ-morphism $\Phi: W_1 \otimes C_*(E^p) \longrightarrow W_1 \otimes C_*(E)^p$ in Lemma 7.1. Let $d = \Phi(1 \otimes D)$ and let $f: W_2 \longrightarrow C_*(E)$ be a ν-morphism over Z_p. Since $W_1 \otimes W_2$ is σ-free and $W_1 \otimes C_*(E)^p$ is acyclic (and τ-free, with the evident τ-action), the following diagram is σ-homotopy commutative.

$$\begin{array}{ccc} W_1 \otimes W_2 & \xrightarrow{\quad\phi\quad} & W_1 \otimes W_2^p \\ \Big\downarrow{\scriptstyle 1 \otimes f} & & \Big\downarrow{\scriptstyle 1 \otimes f^p} \\ W_1 \otimes C_*(E) & \xrightarrow{\quad d \quad} & W_1 \otimes C_*(E)^p \end{array}$$

Therefore $\phi_* = d_*: H_*(\sigma; Z_p) \longrightarrow H_*(\tau; Z_p)$, and this can clearly be computed from the quotient map $d: W_1 \otimes_\pi C_*(E/\nu) \longrightarrow W_1 \otimes_\pi C_*(E/\nu)^p$. We shall prove the following formulas in Proposition 9.1.

(a) If $p = 2$, $d_*(e_r \otimes e_s) = \sum_k e_{r+2k-s} \otimes P_*^k(e_s)^2$; and

(b) If $p > 2$, $d_*(e_r \otimes e_s) = \sum_k (-1)^k \nu(s) e_{r+(2pk-s)(p-1)} \otimes P_*^k(e_s)^p$

$$- \delta(r) \sum_k (-1)^k \nu(s-1) e_{r+p+(2pk-s)(p-1)} \otimes P_*^k \beta(e_s)^p .$$

Here the P_*^k are the duals to the Steenrod operations in $H^*(K(Z_p, 1); Z_p) = H^*(\nu; Z_p)$. The latter operations satisfy $P^0 = 1$ and the internal Cartan formula, hence, by (1.2), if w_t is dual to e_t, then

(c) If $p = 2$, $P^k(w_t) = (k, t-k)w_{k+t}$, hence $P_*^k(e_s) = (k, s-2k)e_{s-k}$; and

(d) If $p > 2$, $P^k(w_t) = (k, [t/2] - k)w_{t+2k(p-1)}$, hence

$$P_*^k(e_s) = (k, [s/2] - pk)e_{s-2k(p-1)} \,.$$

Combining (a) and (c), we obtain (i). Combining (b), (d), and $\beta(e_i) = \delta(i-1)e_{i-1}$, we obtain (ii).

<u>Theorem 4.7.</u> The following relations among the P_s and βP_s are valid on all homology classes of all Adem objects in $\mathcal{C}(p, \infty)$.

(i) If $p = 2$ and $a > 2b$, $P_a P_b = \sum_i (2i-a, a-b-i-1)P_{a+b-i}P_i$

(ii) If $p > 2$ and $a > pb$, $P_a P_b = \sum_i (-1)^{a+i}(pi - a, a-(p-1)b-i-1)P_{a+b-i}P_i$

and $\beta P_a P_b = \sum_i (-1)^{a+i}(pi-a, a-(p-1)b-i-1)\beta P_{a+b-i}P_i$

(iii) If $p > 2$ and $a \geq pb$, $P_a \beta P_b = \sum_i (-1)^{a+i}(pi-a, a - (p-1)b-i)\beta P_{a+b-i}P_i$

$$- \sum_i (-1)^{a+i}(pi - a-1, a-(p-1)b-i)P_{a+b-i}\beta P_i$$

and

$$\beta P_a \beta P_b = - \sum_i (-1)^{a+i}(pi - a-1, a - (p-1)b-i)\beta P_{a+b-i}\beta P_i$$

<u>Proof.</u> Note first that the second relations of (ii) and (iii) are implied by the first for objects which are reduced mod p, but are logically independent in our general setting. Let (K, θ) be an Adem object in $\mathcal{C}(p, \infty)$ and let $x \in H_q(K)$. Definition 4.1 implies that we have a Z_p-homotopy commutative diagram

$$\begin{array}{ccccc}
(W_1 \otimes W_2^{\,p}) \otimes_\tau K^{p^2}(q) & \xrightarrow{w \otimes 1} & Y \otimes_{\Sigma_{p^2}} K^{p^2}(q) & \searrow^{\xi} & \\
\Big\downarrow {\scriptstyle 1 \otimes u} & & & & K(q) \\
W_1 \otimes_\pi (W_2 \otimes K^p)^p(q) & \xrightarrow{1 \otimes \theta^p} & W_1 \otimes_\pi K^p(q) & \nearrow_{\theta} &
\end{array}$$

Since x^{p^2} is Σ_{p^2} invariant in $K^{p^2}(q)$, we have, for all r and s,

(a) $\xi_*(w \otimes 1)_*(e_r \otimes e_s^{\,p} \otimes x^{p^2}) = \xi_*(w_*(e_r \otimes e_s^{\,p}) \otimes x^{p^2})$.

In the other direction, U introduces the sign $(-1)^{mqs}$ and we have

(b) $\theta_*(1\otimes\theta^p)_*(1\otimes U)_*(e_r\otimes e_s^p\otimes x^{p^2}) = (-1)^{mqs}D_rD_s(x)$.

Since $w_*\emptyset_* = w_*\emptyset_*\gamma_*$, Lemma 4.4 gives the formula

(c) $w_*\emptyset_*(e_r\otimes e_s) = (-1)^{rs+mq}w_*\emptyset_*(e_s\otimes e_r)$.

Combining formulas (a) and (c), we obtain the formula

(d) $\xi_*(w\otimes 1)_*(\emptyset_*(e_r\otimes e_s)\otimes x^{p^2}) = (-1)^{rs+mq}\xi_*(w\otimes 1)_*(\emptyset_*(e_s\otimes e_r)\otimes x^{p^2})$.

In view of (b), (d) gives relations on iterated operations, and these relations are explicit since $\emptyset_*$ is known. We prove the three parts of the theorem successively In all parts, the statements about binomial coefficients are verified by writing out the p-adic expansions of the relevant integers and appealing to the remarks in Notations 4.5.

(i) By (b) and (d), Lemma 4.6 implies the formula

(e) $\sum_k (k, s-2k)D_{r+2k-s}D_{s-k}(x) = \sum_\ell (\ell, r-2\ell)D_{s+2\ell-r}D_{r-\ell}(x)$.

Formula (e) is valid for all r and s, and we set $r = a-2q$ and $s = b-q$ for our fixed $a > 2b$. If we then change variables to $j = b-k$ and $i = a-q-\ell$ and apply Definition 2.2, we obtain

(f) $\sum_j (b-j, 2j-b-q)P_{a+b-j}P_j(x) = \sum_i (a-q-i, 2i-a)P_{a+b-i}P_i(x)$.

The condition $a > 2b$ guarantees that the same terms do not appear with non-zero coefficients on both sides of (f). Now suppose that $q = b-2^t+1$ for some $t > 0$. Then, if $j \neq b$,

$$(b-j, 2j-b-q) = (b-j, 2^t-1-2(b-j)) = 0.$$

On the right side of (f), $P_{a+b-i}P_i(x) = 0$ unless $2^t-1 = b-q \geq 2i-a$, while if $2^t > 2i-a$, then

$$(a-q-i, 2i-a) = (2i-a, a-b-i-1+2^t) = (2i-a, a-b-i-1).$$

Thus (f) reduces to the desired relation (i) when $q = b-2^t+1$ for some $t > 0$. By Lemma 4.3, it follows that (i) holds for all q.

(ii) Observe that $\beta w_* = w_* \beta$ and that, by the proof of part (v) of Proposition 2.3, the Bockstein operation β in $H_*(\tau; Z_p)$ is given by $\beta(e_k \otimes e_\ell^p) = \beta(e_k) \otimes e_\ell^p$, where $\beta(e_k) = \delta(k-1)e_{k-1}$. Now since (c) holds with $\emptyset_*$ replaced by $\beta\emptyset_*$, so does (d); that is,

(d') $$\xi_*(w \otimes 1)_*(\beta\emptyset_*(e_r \otimes e_s) \otimes x^{p^2}) = (-1)^{rs+mq} \xi_*(w \otimes 1)_*(\beta\emptyset_*(e_s \otimes e_r) \otimes x^{p^2}).$$

Replace r and s by 2r and 2s in (d) and (d') and let $\varepsilon = 0$ or 1; then, by (b) and Lemma 4.6, (d) and (d') imply the following formula for $\varepsilon = 0$ and $\varepsilon = 1$, respectively.

(g) $$\sum_k (-1)^k \nu(2s)(k, s-pk) D_{2r+(2pk-2s)(p-1)-\varepsilon} D_{2s-2k(p-1)}(x)$$
$$= \sum_\ell (-1)^{\ell+mq} \nu(2r)(\ell, r-p\ell) D_{2s+(2p\ell-2r)(p-1)-\varepsilon} D_{2r-2\ell(p-1)}(x).$$

In (g), set $r = a(p-1)-pqm$ and $s = b(p-1)-qm$ and change variables to $j = b-k$ and $i = a-mq-\ell$. Let $\beta^o P_s = P_s$ and $\beta^1 P_s = \beta P_s$, by abuse; then, by Definitions 2.2 and a check of constants, we obtain

(h) $$\sum_j (-1)^{b+j}(b-j, pj-b-mq)\beta^\varepsilon P_{a+b-j} P_j(x)$$
$$= \sum_i (-1)^{a+i}(a-mq-i, pi-a)\beta^\varepsilon P_{a+b-i} P_i(x) .$$

Again, $a > pb$ ensures that the same terms do not appear on both sides of (h). Now suppose that $q = 2b - 2(1+p+\ldots+p^{t-1})$, $t > 0$. Then $(b-j, pj-b-mq) = 0$ unless $j = b$ and, on the other side of (h), $\beta^\varepsilon P_{a+b-i} P_i(x) = 0$ unless $1+\ldots+p^{t-1} = b-q/2 \geq pi-a$, when $p^t > pi-a$ implies

$$(a-mq-i, pi-a) = (pi-a, a-(p-1)b+p^t-1-i) = (pi-a, a-(p-1)b-i-1).$$

Thus (h) reduces to the desired relations (ii) when $q = 2b-2(1+\ldots+p^{t-1})$ for some $t > 0$. By Lemma 4.3, it follows that (ii) holds for all q.

(iii) Replace r and s by 2r and 2s-1 in (d) and (d'); then, by (b) and Lemma 4.6, (d) and (d') imply the following formula for $\varepsilon = 0$ and $\varepsilon = 1$, respectively.

(i) $$\sum_k (-1)^{k+mq}\,\nu(2s-1)(k,s-1-pk)D_{2r+(2pk-2s+1)(p-1)-\varepsilon}D_{2s-1-2k(p-1)}(x)$$
$$= (1-\varepsilon)\sum_{\ell}(-1)^{\ell+mq}\nu(2r)(\ell,r-p\ell)D_{2s-1+(2p\ell-2r)(p-1)}D_{2r-2\ell(p-1)}(x)$$
$$-\sum_{\ell}(-1)^{\ell}\,\nu(2r-1)(\ell,r-1-p\ell)D_{2s+(2p\ell-2r+1)(p-1)-\varepsilon}D_{2r-1-2\ell(p-1)}(x).$$

In (i), set $r = a(p-1)-pqm$ and $s = b(p-1)-qm$ and change variables to $j = b-k$ and $i = a-mq-\ell$. By Definitions 2. 2, we obtain

(j) $$\sum_j (-1)^{b+j}(b-j,pj-b-mq-1)\beta^{\varepsilon}P_{a+b-j}\beta P_j(x)$$
$$= (1-\varepsilon)\sum_i (-1)^{a+i}(a-mq-i,pi-a)\beta P_{a+b-i}P_i(x)$$
$$-\sum_i (-1)^{a+i}(a-mq-i,pi-a-1)\beta^{\varepsilon}P_{a+b-i}\beta P_i(x)\ .$$

Again, $a \geq pb$ ensures that the same terms do not occur on both sides of the equation. Now suppose that $q = 2b-2p^t$, $t > 0$. Then $(b-j,pj-b-mq-1) = 0$ unless $j = b$. On the other side of (j), $\beta P_{a+b-i}P_i(x) = 0$ unless $p^t > pi-a$, when

$$(a-mq-i,pi-a) = (pi-a,a-(p-1)b-i+(p-1)p^t) = (pi-a,a-(p-1)b-i),$$

and $\beta^{\varepsilon}P_{a+b-i}\beta P_i(x) = 0$ unless $p^t > pi-a-1$, when

$$(a-mq-i,pi-a-1) = (pi-a-1,a-(p-1)b-i+(p-1)p^t) = (pi-a-1,a-(p-1)b-i)\ .$$

Thus (j) reduces to the desired relations (iii) when $q = 2b-2p^t$ for some $t < 0$, and Lemma 4. 3 implies that (iii) holds for all q.

Remark 4. 8. It should be observed, for use in section 9, that the relations (f), (h), and (j) derived in the proof above are valid for arbitrary integers a and b (without the restrictions $a > pb$ or $a \geq pb$). Indeed, these conditions on a and b were only required in order to obtain disjoint non-trivial terms on the two sides of the cited equations.

5. Reindexing for cohomology.

We have geared our discussion to homology, but the reformulation appropriate to cohomology is obtained by a minor and standard change of notation. Thus let K be a Z_p-complex Z-graded by superscripts, with d of degree plus one. If we regrade K by $K_{-q} = K^q$, then the theory of the previous sections applies. Equivalently, we can regrade W by non-positive superscripts and reformulate the theory. Obviously, this in no way changes the proofs. Let $(K, \theta) \in \mathcal{C}(p, \infty)$, with K and W graded by superscripts, and let $x \in H^q(K)$. Then $D_i(x) = \theta_*(e^{-i} \otimes x^p) \in H^{pq-i}(K)$, $i \geq 0$, and we may define $P^s(x) = P_{-s}(x)$ and, if $p > 2$, $\beta P^s(x) = \beta P_{-s}(x)$. Explicitly, $P^s(x)$ and $\beta P^s(x)$ are defined by the formulas

(1) If $p = 2$, $P^s(x) = D_{q-s}(x) \in H^{q+s}(K)$, where $D_i = 0$ for $i < 0$; and

(2) If $p > 2$, $P^s(x) = (-1)^s \nu(-q) D_{(q-2s)(p-1)}(x) \in H^{q+2s(p-1)}(K)$ and

$\beta P^s(x) = (-1)^s \nu(-q) D_{(q-2s)(p-1)-1}(x) \in H^{q+2s(p-1)+1}(K)$, where

$D_i = 0$ for $i < 0$ and if $q = 2j - \varepsilon$, $\varepsilon = 0$ or 1, then $\nu(-q) = (-1)^j (m!)^{\varepsilon}$.

Of course, if $p = 2$, we should write $P^s = Sq^s$ in order to conform to standard notations, but we prefer to retain the notation P^s. In this way, the Cartan formula and Adem relations are formally the same in the cases $p = 2$ and $p > 2$.

The P^s and βP^s are natural homomorphisms and are defined for all integers s. If $(K, \theta) \in \mathcal{C}(p, \infty)$ and $x \in H^q(K)$, then

(3) If $p = 2$, $P^s(x) = 0$ if $s > q$ and $P^q(x) = x^2$; and

(4) If $p > 2$, $P^s(x) = 0$ if $2s > q$, $\beta P^s(x) = 0$ if $2s \geq q$, and $P^s(x) = x^p$ if $2s = q$.

Note that we do not claim that $P^s(x) = 0$ if $s < 0$ or that $P^0 = 1$; these formulas are not true in general. If (K, θ) is unital, then $P^s(e) = 0$ for $s \neq 0$. If (K, θ) is reduced mod p, then

(5) $\beta P^{s-1} = sP^s$ if $p = 2$ and βP^s is the composition of P^s and the Bockstein β if $p > 2$.

The external Cartan formula now reads

(6) $$P^s(x \otimes y) = \sum_{i+j=s} P^i(x) \otimes P^j(y) \text{ and, if } p > 2,$$

$$\beta P^{s+1}(x \otimes y) = \sum_{i+j=s} (\beta P^{i+1}(x) \otimes P^j(y) + (-1)^{\deg x} P^i(x) \otimes \beta P^{j+1}(y)).$$

We have $\sigma P^s = P^s \sigma$ and $\sigma \beta P^s = -\beta P^s \sigma$ and of course the Kudo transgression theorem takes on a more familiar form with grading by superscripts in case (ii). The Adem relations, reformulated in terms of the P^s, take on the form given in the following corollary.

Corollary 5.1. The following relations among the P^s and βP^s are valid on all cohomology classes of all Adem objects in $\mathcal{C}(p, \infty)$

(i) If $p \geq 2$, $a < pb$, and $\varepsilon = 0$ or 1 if $p > 2$, $\varepsilon = 0$ if $p = 2$, then

$$\beta^{\varepsilon} P^a P^b = \sum_i (-1)^{a+i}(a - pi, (p-1)b - a+i-1)\beta^{\varepsilon} P^{a+b-i} P^i$$

(ii) If $p > 2$, $a \leq pb$, and $\varepsilon = 0$ or 1, then

$$\beta^{\varepsilon} P^a \beta P^b = (1-\varepsilon) \sum_i (-1)^{a+i}(a - pi, (p-1)b - a+i-1)\beta P^{a+b-i} P^i$$
$$- \sum_i (-1)^{a+i}(a-pi-1, (p-1)b-a+i)\beta^{\varepsilon} P^{a+b-i} \beta P^i$$

(where, by abuse of notation, $\beta^0 P^s = P^s$ and $\beta^1 P^s = \beta P^s$).

While the two forms of the Adem relations given in Theorem 4.7 and the corollary are completely equivalent, they work out quite differently in practice. The relations of Theorem 4.7 apply to positive complexes, in homology, with $a, b \geq 0$; but $a, b \geq 0$ in Theorem 4.7 corresponds to $a, b \leq 0$ in the corollary, which is designed for use in cohomology with $a, b \geq 0$. For this reason, the Dyer-Lashof algebra [17], which operates on the homology of infinite loop spaces, is a very different algebraic object than the classical Steenrod algebra.

6. Cup-i products, Browder operations, and higher Bocksteins.

We here discuss $\cup_i$-products and certain homology operations of two variables, which were first studied by Browder in [4]; these operations occur in the presence of a $\cup_n$-product and the absence of a $\cup_{n+1}$-product and are central to the study of the homology of $(n+1)$-fold loop spaces. We shall also obtain a very useful result, Proposition 6.8, on higher Bockstein operations. In section 10, we shall show that this result suffices to give a complete computation of the mod p cohomology Bockstein spectral sequence of $K(\pi, n)$ for any Abelian group π and any prime p.

Throughout this section, Λ is a commutative ring, π is the cyclic group of order 2 with generator α, and W is the canonical $\Lambda\pi$-free resolution of Λ. Let $\Delta_i = \alpha + (-1)^i \in \Lambda\pi$, so that $d(e_i) = \Delta_i e_{i-1}$ for $i \geq 1$. If $(K, \theta) \in \mathcal{C}(\pi, n, \Lambda)$, then we may assume that the restriction of θ to $e_o \otimes K \otimes K$ agrees with the given product on K by (i) of Definition 2.1.

<u>Definition 6.1.</u> Let $(K, \theta) \in \mathcal{C}(\pi, n, \Lambda)$ and let $x \in K_q$ and $y \in K_r$. For $0 \leq i \leq n$, define $x \cup_i y = (-1)^{\frac{1}{2}i(i+1)} \theta(e_i \otimes x \otimes y)$. Then $\cup_o$ is the product on K and if $i > 0$, then $\cup_i : K \otimes K \longrightarrow K$ is a chain homotopy of degree i from $\cup_{i-1}$ to $(-1)^{i-1} \cup_{i-1} \cdot \alpha$; that is,

(i) $$d(x \cup_i y) = (-1)^i d(x) \cup_i y + (-1)^{i+q} x \cup_i d(y) + x \cup_{i-1} y + (-1)^{i+qr} y \cup_{i-1} x.$$

If $\Lambda = Z_2$ and $x \in K_q$ is a cycle, then $P_{i+q}\{x\} = D_i\{x\} = \{x \cup_i x\}$, which, in cohomology, was Steenrod's first definition [25] of the squares. We now define the Browder operations for $(K, \theta) \in \mathcal{C}(\pi, n, \Lambda)$.

<u>Definition 6.2.</u> Let $(K, \theta) \in \mathcal{C}(\pi, n, \Lambda)$, $n < \infty$, and let $x \in H_q(K)$ and $y \in H_r(K)$. Observe that if a and b are representative cycles for x and y, then $\Delta_{n+1} e_n \otimes a \otimes b$ is a cycle in $W^{(n)} \otimes K^2$ whose homology class $\Delta_{n+1} e_n \otimes x \otimes y$ depends only on x and y. Define $\lambda_n(x, y) \in H_{q+r+n}(K)$ by

$\lambda_n(x, y) = (-1)^{nq+1}\theta_*(\Delta_{n+1}e_n \otimes x \otimes y)$. Note that we have chosen not to pass to equivariant homology; of course, we can do so, and, in $W^{(n)} \otimes_\pi K^2$,

$$(-1)^{nq+1}\Delta_{n+1}e_n \otimes a \otimes b = (-1)^{nq+n}e_n \otimes a \otimes b - (-1)^{nq+qr}e_n \otimes b \otimes a.$$

Thus $\lambda_n(x, y)$ is represented by $(-1)^{nq+n+\frac{1}{2}n(n+1)}(a \cup_n b - (-1)^{n+qr} b \cup_n a)$.

The following proposition contains many of the elementary properties of the λ_n; its proof is immediate from the definition.

Proposition 6.3. Let $(K, \theta) \in \mathcal{C}(\pi, n, \Lambda)$, $n < \infty$, and consider $\lambda_n : H_q(K) \times H_r(K) \longrightarrow H_{q+r+n}(K)$.

(i) λ_n induces a homomorphism $\lambda_n : H_q(K) \otimes H_r(K) \longrightarrow H_{q+r+n}(K)$

(ii) If $f: K \longrightarrow K'$ is a morphism in $\mathcal{C}(\pi, n, \Lambda)$, then $\lambda_n(f_* \otimes f_*) = f_*\lambda_n$

(iii) If θ is the restriction to $W^{(n)} \otimes K^2$ of $\theta' : W^{(n+1)} \otimes K^2 \longrightarrow K$, then $\lambda_n = 0$

(iv) If $n = 0$, then $\lambda_o(x, y) = xy - (-1)^{qr}yx$

(v) If (K, θ) is unital and the restriction of θ to $W^{(n)} \otimes (e \otimes K + K \otimes e)$ is homotopic to $\varepsilon \otimes \phi$, ϕ the product, then $\lambda_n(x, e) = 0 = \lambda_n(e, y)$.

(vi) $\lambda_n(x, y) = (-1)^{qr+1+n(q+r+1)}\lambda_n(y, x)$ and, if $2 = 0$ in Λ, $\lambda_n(x, x) = 0$

(Note that the first part implies $2\lambda_n(x, x) = 0$ if $n+q$ is even.)

The λ_n satisfy the following analog of the external Cartan formula.

Proposition 6.4. Let (K, θ) and (L, θ') be in $\mathcal{C}(\pi, n, \Lambda)$, $n < \infty$ and Λ a field. Let $x \in H_q(K)$, $x' \in H_r(K)$, $y \in H_s(L)$ and $y' \in H_t(L)$. Then

$$\lambda_n(x \otimes y, x' \otimes y') = (-1)^{r(s+n)}xx' \otimes \lambda_n(y, y') + (-1)^{s(r+t+n)}\lambda_n(x, x') \otimes y'y.$$

Proof. Let a, a', b, b' represent x, x', y, y' respectively. Let $c = (-1)^{n(q+s)+1}\tilde{\theta}(\Delta_{n+1}e_n \otimes a \otimes b \otimes a' \otimes b')$, so that c represents $\lambda_n(x\otimes y, x' \otimes y')$. By (1) of Definition 1.2,

$\psi(e_n) = \sum_{j=0}^{n} e_j \otimes \alpha^j e_{n-j}$ in W, and the definition of $\tilde{\theta}$ shows that

$$c = \sum_{j=0}^{n} (-1)^{rs+n(1+r+s)-j(q+r)} \theta(e_j \otimes a \otimes a') \otimes \theta'(\alpha^j e_{n-j} \otimes b \otimes b')$$
$$- \sum_{j=0}^{n} (-1)^{rs+n(r+s)-j(q+r)} \theta(\alpha e_j \otimes a \otimes a') \otimes \theta'(\alpha^{j+1} e_{n-j} \otimes b \otimes b')$$

Let $e = \sum_{j=1}^{n} (-1)^{rs+n(r+s)-(j+1)(q+r)} \theta(e_j \otimes a \otimes a') \otimes \theta'(\alpha^j e_{n+1-j} \otimes b \otimes b')$. Then a straightforward calculation demonstrates that

$$c + d(e) = (-1)^{r(s+n)+sn+n} \theta(e_o \otimes a \otimes a') \otimes \theta'(\Delta_{n+1} e_n \otimes b \otimes b')$$
$$+ (-1)^{s(r+n)+qn+n} \theta(\Delta_{n+1} e_n \otimes a \otimes a') \otimes \theta'(\alpha^{n+1} e_o \otimes y \otimes y') .$$

Since L is homotopy commutative for $n > 0$, $\theta'(\alpha^{n+1} e_o \otimes b \otimes b')$ represents $(-1)^{st} y'y$ for any n, and the result follows.

We next prove that the λ_n commute with suspension.

<u>Proposition 6.5.</u> Let $(K', \theta') \in \mathcal{C}(\pi, n+1, \Lambda)$ and $(K'', \theta'') \in \mathcal{C}(\pi, n, \Lambda)$. Let K be a Λ-complex and let $f: K' \longrightarrow K$ and $g: K \longrightarrow K''$ be morphisms of complexes such that $gf = 0$. Define

$$\tilde{K} = W^{(n+1)} \otimes f(K')^2 + \overline{W}^{(n+1)} \otimes f(K') \otimes K + W^{(n)} \otimes K^2 ,$$

where $\overline{W}^{(n+1)} = W^{(n)} + \Lambda e_{n+1}$ $(\alpha e_{n+1} \notin \overline{W}^{(n+1)})$. Suppose given a π-morphism $\theta: \tilde{K} \longrightarrow K$ such that the following diagram is commutative:

$$\begin{array}{ccccc} W^{(n+1)} \otimes K' \otimes K' & \xrightarrow{1 \otimes f \otimes f} & \tilde{K} & \xrightarrow{1 \otimes g \otimes g} & W^{(n)} \otimes K'' \otimes K'' \\ \downarrow \theta' & & \downarrow \theta & & \downarrow \theta'' \\ K' & \xrightarrow{f} & K & \xrightarrow{g} & K'' \end{array}$$

Let $x, y \in \operatorname{Ker} f_*$. Then $\sigma\lambda_{n+1}(x, y) = \lambda_n(\sigma x, \sigma y) \in \operatorname{Coker} g_*$.

<u>Proof.</u> Let $a' \in K'_q$ and $b' \in K'_r$ represent x and y respectively. Let $a = f(a')$ and $b = f(b')$ and choose $u \in K_{q+1}$ and $v \in K_{r+1}$ such that $d(u) = a$

and $d(v) = b$. Define $c \in \tilde{K}$ by

$$c = (-1)^q \Delta_{n+1} e_n \otimes u \otimes v - (-1)^{n+q} e_{n+1} \otimes a \otimes v - (-1)^{r(q+1)} e_{n+1} \otimes b \otimes u.$$

Then a straightforward calculation demonstrates that

$$d(c) = e_{n+1} \otimes a \otimes b + (-1)^{n+qr} e_{n+1} \otimes b \otimes a = (-1)^n \Delta_{n+2} e_{n+1} \otimes a \otimes b$$

Thus $(-1)^{(n+1)q+1} f\theta'(\Delta_{n+2} e_{n+1} \otimes a' \otimes b') = (-1)^{(n+1)(q+1)} d\theta(c)$ and

$$(-1)^{(n+1)(q+1)} g\theta(c) = (-1)^{n(q+1)+1} \theta''(\Delta_{n+1} e_n \otimes g(u) \otimes g(v)),$$

by our commutative diagram, and this proves the result.

The analog for the Browder operations of the Adem relations is the following Jacobi identity: let $x \in H_q(K)$, $y \in H_r(L)$, and $z \in H_s(K)$; then, under appropriate hypotheses,

$$(-1)^{(q+n)(s+n)} \lambda_n(x, \lambda_n(y, z)) + (-1)^{(r+n)(q+n)} \lambda_n(y, \lambda_n(z, x)) + (-1)^{(s+n)(r+n)} \lambda_n(z, \lambda_n(x, y)) = 0,$$

and, if $3 = 0$ in Λ and $q+n$ is odd, $\lambda_n(x, \lambda_n(x, x)) = 0$. We omit the proof as an easier geometric argument can be obtained for the homology of $(n+1)$-fold loop spaces. This identity, and the identity of (vi) of Proposition 6.3, lead to a notion of λ_n-algebra which generalizes that of Lie algebra (or λ_o-algebra). There is also a notion of restricted λ_n-algebra which is important for the applications. In the case $\Lambda = Z_2$, the restriction is already present in our algebraic context; it is the last Steenrod operation for an object $(K, \theta) \in \mathcal{C}(2, n)$. The following addendum to Proposition 2.3 gives some properties of this operation that are needed in the study of $(n+1)$-fold loop spaces.

Proposition 6.5. Let $(K, \theta) \in \mathcal{C}(2, n)$. Let $\xi_n = P_{q+n}: H_q(K) \longrightarrow H_{2q+n}(K)$. Then

(i) $\xi_n(x+y) = \xi_n(x) + \xi_n(y) + \lambda_n(x, y)$, and

(ii) $\beta\xi_n(x) = (q+n-1)P_{q+n-1}(x) + \lambda_n(x, \beta x)$ if (K, θ) is reduced mod 2

Proof. For (i), if a and b represent x and y, then, in K^2, $(a+b)^2 = a^2 + b^2 + \Delta_{n+1}ab$, and the error term $\Delta_{n+1}e_n \otimes a \otimes b$ yields the stated deviation from additivity of ξ_n. Part (ii) follows from a glance at the proof of (v) of Proposition 2.3.

We now relate the λ_n to the Bockstein operations on H(K) when $(K, \theta) \in \mathcal{C}(\pi, n, Z_p)$ is reduced mod p. In contrast to the Steenrod operations, the higher Bocksteins are all of interest.

Proposition 6.7. Let $(K, \theta) \in \mathcal{C}(\pi, n, Z_p)$ be reduced mod p. Let $x, y \in H(K)$, $\deg(x) = q$. Assume that $\beta_r(x)$ and $\beta_r(y)$ are defined. Then $\beta_r\lambda_n(x, y)$ is defined and, modulo indeterminacy,

$$\beta_r\lambda_n(x, y) = \lambda_n(\beta_r x, y) + (-1)^{n+q}\lambda_n(x, \beta_r y).$$

Proof. Let $(K, \theta) = (\tilde{K} \otimes Z_p, \tilde{\theta} \otimes Z_p)$. Let $a, b \in K$ be such that their mod p reductions $\bar{a}, \bar{b} \in K$ represent x and y. We may assume that $d(a) = p^r a'$ and $d(b) = p^r b'$; the mod p reduction $\bar{a}'$ and $\bar{b}'$ of a' and b' represent $\beta_r(x)$ and $\beta_r(y)$. In $W^{(n)} \otimes \tilde{K}^2$, $d(\Delta_{n+1}e_n \otimes a \otimes b) = (-1)^n p^r \Delta_{n+1}e_n \otimes (a' \otimes b + (-1)^q a \otimes b')$. By reduction mod p and a check of signs, this implies the result.

Surprisingly, the following fundamental result appears not to be in the literature, although it is presumably well-known. It allows complete calculation of the mod p homology Bockstein spectral sequence of $QX = \varinjlim \Omega^n S^n X$ for any space X and, as we shall show later, the mod p cohomology Bockstein spectral sequence of $K(\pi, n)$. Together with the previous result, it also suffices for the computation of the mod p homology Bockstein spectral sequence of $\Omega^n S^n X$, $n \geq 1$.

Proposition 6.8. Let K be a Z-graded associative differential ring which is flat as a Z-module. Let K have a $\cup_1$-product such that

(a) $d(a \cup_1 b) = -d(a) \cup_1 b - (-1)^{\deg a} a \cup_1 d(b) + ab - (-1)^{\deg a \deg b} ba$

and, for case (ii), such that the Hirsch formula (b) holds

(b) $ab \cup_1 c = (-1)^{\deg a} a(b \cup_1 c) + (-1)^{\deg b \deg c}(a \cup_1 c)b$.

Let β_r denote the r-th mod p Bockstein on $H(K \otimes Z_p)$, $\beta_1 = \beta$. Let $y \in H_{2q}(K \otimes Z_p)$ and assume that $\beta_{r-1}(y)$ is defined, $r \geq 2$. Then $\beta_r(y^p)$ is defined and, modulo indeterminacy,

(i) If $p = 2$ and $r = 2$, $\beta_2(y^2) = \beta(y)y + P_{2q}\beta(y)$

(ii) If $p > 2$ and $r = 2$, $\beta_2(y^p) = \beta(y)y^{p-1} + \sum_{j=1}^{m} j\lambda_1(\beta(y)y^{j-1}, \beta(y)y^{p-j-1})$

(iii) If $p \geq 2$ and $r \geq 3$, $\beta_r(y^p) = \beta_{r-1}(y)y^{p-1}$.

<u>Proof.</u> Let $b \in K_{2q}$ be such that its mod p reduction $\bar{b}$ represents y. We may assume that $d(b) = p^{r-1}a$, and then a is a cycle whose mod p reduction $\bar{a}$ represents $\beta_{r-1}(y)$. Clearly we have

$$d(b^p) = p^{r-1} \sum_{i=1}^{p} b^{i-1}ab^{p-i}, \quad \text{and}$$

$$d(ab^{p-i} \cup_1 b^{i-1}) \equiv ab^{p-1} - b^{i-1}ab^{p-i} \mod p^{r-1}, \quad 2 \leq i \leq p .$$

Therefore $d(b^p + p^{r-1} \sum_{i=2}^{p} ab^{p-i} \cup_1 b^{i-1}) \equiv p^r ab^{p-1} \mod p^{2r-2}$.

If $r \geq 3$, then $2r-2 > r$ and part (iii) follows. Thus let $r = 2$; we must now take into account the terms arising from $d(b) = pa$ in $d(ab^{p-i} \cup_1 b^{i-1})$. If $p = 2$, then

$$d(b^2 + 2a \cup_1 b) = 4ab + 4a \cup_1 a.$$

Since the mod 2 reduction of $a \cup_1 a$ represents $P_{2q}\beta(y)$, this proves (i). Thus assume that $p > 2$. Then

$$d(b^p + p \sum_{i=2}^{p} ab^{p-i} \cup_1 b^{i-1}) = p^2 ab^{p-1} + p^2 c + p^2 c' , \quad \text{where}$$

$$c = \sum_{i=2}^{p-1} \sum_{j=1}^{p-i} ab^{j-1}ab^{p-i-j} \cup_1 b^{i-1} \quad \text{and} \quad c' = \sum_{1 \leq j < i \leq p} ab^{p-i} \cup_1 b^{j-1}ab^{i-j-1} .$$

By the Hirsch formula, and a separate reindexing of the two resulting sums, we find that

$$c = \sum_{2 \leq j < i \leq p} [(ab^{i-j-1} \cup_1 b^{j-1})ab^{p-i} - ab^{p-i}(ab^{i-j-1} \cup_1 b^{j-1})].$$

Therefore if $e = \sum_{2 \leq j < i \leq p} ab^{p-i} \cup_1 (ab^{i-j-1} \cup_1 b^{j-1})$, then

$$d(e) \equiv -c + \sum_{2 \leq j < i \leq p} ab^{p-i} \cup_1 (ab^{i-2} - b^{j-1}ab^{i-j-1}) \bmod p .$$

Comparing $d(e)$ to c', we easily find that

$$d(b^p + p\sum_{i=2}^{p} ab^{p-i} \cup_1 b^{i-1} + p^2 e) \equiv p^2 ab^{p-1} + p^2 \sum_{i=2}^{p} (i-1)ab^{p-i} \cup_1 ab^{i-2}$$

$$\equiv p^2 ab^{p-1} + p^2 \sum_{j=1}^{m} j(ab^{p-j-1} \cup_1 ab^{j-1} - ab^{j-1} \cup_1 ab^{p-j-1}) \bmod p^3 .$$

By Definition 6.2, this implies (ii) and so completes the proof.

Of course, the terms involving λ_1 in (ii) are zero if K admits a $\cup_2$-product. The general result is needed for second loop spaces. The Hirsch formula is valid for the cochains of a space [8, 16], for the chains of a second loop space [10], and for the dual of the bar construction of a cocommutative Hopf algebra [16]. In connection with this formula, we make the following remarks.

Remarks 6.9. Let K be an associative differential Z_p-algebra, $p > 2$, with a $\cup_1$-product which satisfies the Hirsch formula. Define $\langle\ \rangle^p: H_{2s-1}(K) \longrightarrow H_{2sp-2}(K)$ as follows. Let a_1 represent $x \in H_{2s-1}(K)$ and define $a_i = \frac{1}{i} a_{i-1} \cup_1 a_1$ for $2 \leq i < p$. Then $d(a_i) = \sum_{j=1}^{i-1} a_j a_{i-j}$ and $\tilde{a} = \sum_{j=1}^{p-1} a_j a_{p-j}$ is a cycle. A computation demonstrates that if $\{a_i' \mid 1 \leq i < p\}$ is any set of elements of K such that a_1' represents x and $d(a_i') = a_j' a_{i-j}'$ for $2 \leq i < p$, then $\tilde{a}' = \sum_{j=1}^{p-1} a_j' a_{p-j}'$ is homologous to $\tilde{a}$. Thus the class of $\tilde{a}$ depends only on x, and we define $\langle x\rangle^p = \{\tilde{a}\}$. In the applications, it is often the case that if $(K, \theta) \in \mathcal{C}(p, p-2)$, then $\langle x\rangle^p = -\beta P_s(x)$. Kraines [11] has proven this result for the cohomology of spaces, where it reads

$\langle x\rangle^p = -\beta P^s(x)$ for $x \in H^{2s+1}(K)$, and Kochman [10] has proven it for the homology of iterated loop spaces. A general proof within our algebraic context should be possible, but appears to be difficult.

7. The category of simplicial Λ-modules

We here develop some machinery that will allow us to apply the theory of the previous sections to a large simplicial category $\mathcal{SA}$. We shall specialize to specific categories of interest in the next section. We assume familiarity with the basic definitions of the theory of simplicial objects and of acyclic models (see, e.g., [15, §1,2,28,29]). Let Λ be a commutative ring, and let $\mathcal{A}$, $\mathcal{CA}$ and $\mathcal{SA}$ denote the categories of (ungraded) Λ-modules, positively graded Λ-complexes, and simplicial Λ-modules. Let $C: \mathcal{SA} \to \mathcal{CA}$ be the normalized chain complex functor (for $K \in \mathcal{SA}$, $C(K)$ is the quotient of K, regarded as a chain complex with $d = \sum (-1)^i d_i$, by the subcomplex generated by the degenerate simplices). Define $H_*(K) = H(C(K))$ and $H^*(K) = H(C^*(K))$, where $C^*(K) = \mathrm{Hom}_\Lambda(C(K), \Lambda)$ is given the differential $\delta(f)(k) = (-1)^{q+1} f(dk)$ for $f \in C^q(K)$ and $k \in C_{q+1}(K)$. The following key lemma is based on ideas of Dold [5].

Lemma 7.1. Let π be a subgroup of Σ_r and let W be a $\Lambda\pi$-free resolution of Λ such that $W_o = \Lambda\pi$ with $\Lambda\pi$-generator e_o. Let $K_1, \ldots, K_r \in \mathcal{SA}$; then there exists a morphism of Λ-complexes

$$\Phi : W \otimes C(K_1 \otimes \cdots \otimes K_r) \longrightarrow W \otimes C(K_1) \otimes \cdots \otimes C(K_r)$$

which is natural in the K_i and satisfies the following properties:

(i) For $\sigma \in \pi$, the following diagram is commutative:

$$\begin{array}{ccc} W \otimes C(K_1 \times \dots \times K_r) & \xrightarrow{\Phi} & W \otimes C(K_1) \otimes \dots \otimes C(K_r) \\ \downarrow \sigma & & \downarrow \sigma \\ W \otimes C(K_{\sigma(1)} \times \dots \times K_{\sigma(r)}) & \xrightarrow{\Phi} & W \otimes C(K_{\sigma(1)}) \otimes \dots \otimes C(K_{\sigma(r)}). \end{array}$$

(ii) Φ is the identity homomorphism on $W \otimes C_o(K_1 \times \dots \times K_r)$.

(iii) $\Phi(e_o \otimes k_1 \otimes \dots \otimes k_r) = e_o \otimes \xi(k_1 \otimes \dots \otimes k_r)$ if $k_i \in K_i$ is a j-simplex, where $\xi : C(K_1 \times \dots \times K_r) \longrightarrow C(K_1) \otimes \dots \otimes C(K_r)$ is the Alexander-Whitney map.

(iv) $\Phi(W \otimes C_j(K_1 \times \dots \times K_r)) \subset \sum_{k \leq rj} W \otimes [C(K_1) \otimes \dots \otimes C(K_r)]_k$.

Moreover, any two such Φ are naturally equivariantly homotopic

Proof. Since $(K \times L)_j = K_j \otimes L_j$, formulas (ii) and (iii) make sense. Write $A_j = C_j(K_1 \times \dots \times K_r)$ and $B_j = [C(K_1) \otimes \dots \otimes C(K_r)]_j$. We construct Φ on $W_i \otimes A_j$ by induction on i and for fixed i by induction on j. Formula (ii) defines Φ for $j = 0$ and all i and formulas (i) and (iii) define Φ for $i = 0$ and all j. Thus let $i \geq 1$ and $j \geq 1$ and assume that Φ is defined for $i' < i$ and for the given i and $j' < j$. Choose a $\Lambda\pi$-basis $\{w_k\}$ for W_i. It suffices to define Φ on $w \otimes x$ for $w \in \{w_k\}$ and $x \in A_j$, since Φ can then be uniquely extended to all of $W_i \otimes A_j$ by (i). Let $\Lambda\Delta[j]$ denote the free simplicial Λ-module generated by the standard simplicial j-simplex [15, p. 14]. Then the functor $w \otimes A_j$ is represented by the r-fold Cartesian product $\Lambda\Delta[j]^r$, and $W \otimes B(\Lambda\Delta[j]^r)$ is acyclic. Therefore $\Phi(w \otimes \Delta_j \otimes \dots \otimes \Delta_j)$ can be defined by choosing a chain whose boundary is $\Phi d(w \otimes \Delta_j \otimes \dots \otimes \Delta_j)$, and Φ can be carried over to arbitrary $w \otimes k_1 \otimes \dots \otimes k_r$ by representability. Now (i), (ii), and (iii) are clearly satisfied and (iv) follows from the fact that $C_k(\Lambda\Delta[j]) = 0$ for $k > j$. The proof that Φ is unique up to natural equivariant homotopy is equally simple.

Remarks 7.2. Define $\Psi: W \otimes C(K_1) \otimes \ldots \otimes C(K_r) \longrightarrow W \otimes C(K_1 \times \ldots \times K_r)$ by $\Psi = 1 \otimes \eta$ where $\eta: C(K_1) \otimes \ldots \otimes C(K_r) \longrightarrow C(K_1 \times \ldots \times K_r)$ is the shuffle map. Since η, unlike ξ, is commutative, Ψ is equivariant. By an easy acyclic models proof, $\Phi\Psi$ and $\Psi\Phi$ are equivariantly homotopic to the respective identity maps.

We shall only be interested in the case $K_1 = \ldots = K_r$; here $\Phi: W \otimes C(K^r) \longrightarrow W \otimes C(K)^r$ is a natural morphism of $\Lambda\pi$-complexes. The general case was required in the proof in order to have the representability of the functors A_j. Starting with objects of the following category $\mathcal{DA}$, we shall use Φ to obtain diagonal approximations and so to pass to the category $\mathcal{P}(\pi, \infty, \Lambda) \subset \mathcal{C}(\pi, \infty, \Lambda)$ defined in Definitions 2.1.

Definitions 7.3. Let $\mathcal{DA}$ denote the following category. The objects of $\mathcal{DA}$ are pairs (K, D) where $K \in \mathcal{SA}$ and $D: K \longrightarrow K \times K$ is a morphism in $\mathcal{SA}$ such that $(D \times 1) = (1 \times D)D$ and $tD = D$, where $t(x \otimes y) = y \otimes x$. The morphisms $f: (K, D) \longrightarrow (K', D')$ in $\mathcal{DA}$ are those morphisms $f: K \longrightarrow K'$ in $\mathcal{SA}$ such that $(f \times f)D = D'f$.

Each $K \in \mathcal{SA}$ admits the natural diagonal $D(k) = k \otimes k$, and $\mathcal{SA}$ is thereby embedded as a full subcategory of $\mathcal{DA}$. However, an object $K \in \mathcal{SA}$ may admit other interesting diagonals. For example, if K is a simplicial cocommutative coassociative Λ-coalgebra, then the coproduct $\psi: K \longrightarrow K \times K$ is a permissible diagonal; that is, $(K, \psi) \in \mathcal{DA}$. The following remarks will be of use in the study of relative and reduced cohomology.

Remarks 7.4. (i) If $L \subset K$ in $\mathcal{SA}$, define $H_*(K, L) = H(C(K/L))$ and $H^*(K, L) = H(C^*(K/L))$. If $(K, D) \in \mathcal{DA}$ and $D(L)$ is contained in $L \times K + K \times L$, then K/L admits the diagonal $\overline{D}$ induced from the composite $K \xrightarrow{D} K \times K \xrightarrow{\pi \times \pi} K/L \times K/L$, where $\pi: K \longrightarrow K/L$ is the projection, and then π is a morphism in $\mathcal{DA}$.

(ii) Let $\tilde{\Lambda} = \Lambda\Delta[0] \in \mathcal{SA}$; thus $\tilde{\Lambda}_n = \Lambda$ for $n \geq 0$, each d_i and s_i is the identity, and $C(\tilde{\Lambda}) = C_o(\tilde{\Lambda}) = \Lambda$. Give $\tilde{\Lambda}$ the natural diagonal. We say that $(K, D) \in \mathcal{DA}$ is unital if we are given a monomorphism $\nu : \tilde{\Lambda} \longrightarrow K$ in $\mathcal{DA}$ and an epimorphism $\varepsilon : K \longrightarrow \tilde{\Lambda}$ in $\mathcal{SA}$ such that $\varepsilon\nu = 1$ and $(\varepsilon \times 1)D = (1 \times \varepsilon)D$ (where $\tilde{\Lambda} \times K = K = K \times \tilde{\Lambda}$). If (K, D) is unital and $IK = \operatorname{Ker} \varepsilon$, then $K = \nu(\tilde{\Lambda}) \oplus IK$ and for $k \in IK$, $D(k) = k \otimes \nu(1) + \nu(1) \otimes k + \overline{D}(k)$, where $\overline{D}(k) \in IK \times IK$. Clearly, $(IK, \overline{D})$ is isomorphic to $(K/\nu(\tilde{\Lambda}), \overline{D})$ in $\mathcal{DA}$.

If $(K, D) \in \mathcal{DA}$, then $C^*(K)$ is an associative differential Λ-algebra, with cup product defined as the composite

(1) $\quad \cup : C^*(K) \otimes C^*(K) \xrightarrow{\alpha} [C(K) \otimes C(K)]^* \xrightarrow{\xi^*} C^*(K \times K) \xrightarrow{D^*} C^*(K)$.

Here α is the natural map, $\alpha(x \otimes y)(k \otimes \ell) = (-1)^{\deg y \deg k}\, x(k) y(\ell)$, and ξ is the Alexander-Whitney map. If (K, D) is unital, then $C^*(K)$ is unital (via ν^*) and augmented (via ε^*).

We now define a functor $\Gamma : \mathcal{DA} \longrightarrow \mathcal{P}(\pi, \infty, \Lambda)$ and then show how to use Γ to apply our general theory to $H^*(K)$ for $(K, D) \in \mathcal{DA}$ in the case $\Lambda = Z_p$.

<u>Definitions 7.5.</u> Let $(K, D) \in \mathcal{DA}$ and write D for the iterated diagonal $K \longrightarrow K^r$. Let $\pi \subset \Sigma_r$ and let W be a $\Lambda\pi$-free resolution of Λ with $W_o = \Lambda\pi$. Define $\Delta : W \otimes C(K) \longrightarrow C(K)^r$ to be the composite

(2) $\quad \Delta : W \otimes C(K) \xrightarrow{1 \otimes D} W \otimes C(K^r) \xrightarrow{\Phi} W \otimes C(K)^r \xrightarrow{\varepsilon \otimes 1} C(K)^r$

Let $\alpha : C^*(K)^r \longrightarrow [C(K)^r]^*$ be the natural map and define a $\Lambda\pi$-morphism $\theta : W \otimes C^*(K)^r \longrightarrow C^*(K)$ by the formula

(3) $\quad \theta(w \otimes x)(k) = (-1)^{\deg w \deg x}\, \alpha(x)(\Delta(w \otimes k)), \; w \in W, \; x \in C^*(K)^r, \; k \in C(K)$.

Since θ may be defined for $\pi = \Sigma_r$ and then factored through $j : W \longrightarrow V$ as in Definition 2.1, and the resulting composite is $\Lambda\pi$-homotopic to the original θ defined in terms of W, θ satisfies condition (ii) of Definition 2.1. By Lemma 7.1, formula (3) specializes to give

(4) $\quad \theta(e_o \otimes x) = D^* \xi^* \alpha(x)$ for any $x \in C^*(K)^r$, and

(5) $\theta(w \otimes x) = \mathcal{E}(w) D^* \alpha(x)$ if $x \in C^o(K)^r$ and $w \in W$.

By (1) and (4), θ satisfies condition (i) of Definition 2.1. Since θ is natural on morphisms in $\mathcal{DA}$, we thus obtain a contravariant functor $\Gamma: \mathcal{DA} \to \mathcal{P}(\pi, \infty, \Lambda)$ by letting $\Gamma(K, D) = (C^*(K), \theta)$ on objects and $\Gamma(f) = C^*(f)$ on morphisms. By (5), if (K, D) is unital in $\mathcal{DA}$ then $\Gamma(K, D)$ is unital in $\mathcal{P}(\pi, \infty, \Lambda)$. If $\Lambda = Z_p$, π is cyclic of order p, and $(K, D) = (\tilde{K} \otimes Z_p, \tilde{D} \otimes Z_p)$ where $\tilde{K}$ is a Z-free simplicial Z-module, we agree to choose θ for K to be the mod p reduction of θ for $\tilde{K}$; then $\Gamma(K, D)$ is reduced mod p (since $C(\tilde{K})$ is Z-free and therefore $C^*(\tilde{K})$ is Z-flat, as required by Definition 2.1).

Observe that, by Definition 6.1, we now have $\cup_i$-products in $C^*(K)$ for any $(K, D) \in \mathcal{DA}$. When $\Lambda = Z_p$, the results of Proposition 2.3 will clearly apply to the Steenrod operations P^s defined on the cohomology of objects $(K, D) \in \mathcal{DA}$. If (K, D) and (L, D') are objects of $\mathcal{DA}$, then $K \times L$ admits the diagonal $\tilde{D} = (1 \times t \times 1)(D \times D')$; if D and D' are the natural diagonals, then so is $\tilde{D}$. Thus $(C^*(K \times L), \theta)$ is defined in $\mathcal{C}(\pi, \infty, \Lambda)$. The following lemma compares $(C^*(K \times L), \theta)$ to $(C^*(K) \otimes C^*(L), \tilde{\theta})$ and will imply the applicability of the external Cartan formula to $H^*(K \times L)$ when $\Lambda = Z_p$.

<u>Lemma 7.6.</u> For any objects (K, D) and (L, D') in $\mathcal{DA}$, the following diagram is $\Lambda\pi$-homotopy commutative

$$\begin{array}{ccc} W \otimes C^*(K \times L)^r & \xrightarrow{\quad\theta\quad} & C^*(K \times L) \\ {\scriptstyle 1 \otimes (\xi^*)^r} \uparrow \downarrow {\scriptstyle 1 \otimes (\eta^*)^r} & & {\scriptstyle \xi^*} \uparrow \downarrow {\scriptstyle \eta^*} \\ W \otimes [C^*(K) \otimes C^*(L)]^r & \xrightarrow{\quad\tilde{\theta}\quad} & C^*(K) \otimes C^*(L) \end{array}$$

That is, η^* and ξ^* are morphisms in the category $\mathcal{C}(\pi, \infty, \Lambda)$.

<u>Proof.</u> By the definitions of θ and $\tilde{\theta} = (\theta \otimes \theta)(1 \otimes T \otimes 1)(\psi \otimes U)$, it suffices to show that the following diagram is $\Lambda\pi$-homotopy commutative:

$$\begin{array}{ccc} W\otimes C(K\times L) & \xrightarrow{\Delta = (\varepsilon\otimes 1)\Phi(1\otimes\tilde{D})} & C(K\times L)^r \\ {\scriptstyle 1\otimes\eta}\uparrow\;\downarrow{\scriptstyle 1\otimes\xi} & & {\scriptstyle \eta^r}\uparrow\;\downarrow{\scriptstyle \xi^r} \\ W\otimes C(K)\otimes C(L) & \xrightarrow{U(\Delta\otimes\Delta)(1\otimes T\otimes 1)(\psi\otimes 1\otimes 1)} & [C(K)\otimes C(L)]^r \end{array}$$

Since $(1\otimes D\otimes 1\otimes D')(1\otimes T\otimes 1)(\psi\otimes 1\otimes 1) = (1\otimes T\otimes 1)(\psi\otimes 1\otimes 1)(1\otimes D\otimes D')$, if we let $\emptyset = (\varepsilon\otimes 1)\Phi$ and let $u\colon K^r\times L^r \to (K\times L)$ be the evident shuffle, so that $\tilde{D} = u(D\times D')$, then this diagram becomes

$$\begin{array}{ccccccc} W\otimes C(K\times L) & \xrightarrow{1\otimes D\times D'} & W\otimes C(K^r\times L^r) & \xrightarrow{1\otimes u} & W\otimes C([K\times L]^r) & \xrightarrow{\emptyset} & C(K\times L)^r \\ {\scriptstyle 1\otimes\eta}\uparrow\;\downarrow{\scriptstyle 1\otimes\xi} & & {\scriptstyle 1\otimes\eta}\uparrow\;\downarrow{\scriptstyle 1\otimes\xi} & & & & {\scriptstyle \eta^r}\uparrow\;\downarrow{\scriptstyle \xi^r} \\ W\otimes C(K)\otimes C(L) & \xrightarrow{1\otimes D\otimes D'} & W\otimes C(K^r)\otimes C(L^r) & & \xrightarrow{U(\emptyset\otimes\emptyset)(1\otimes T\otimes 1)(\psi\otimes 1\otimes 1)} & & [C(K)\otimes C(L)]^r \end{array}$$

The left-hand square commutes by the naturality of η and ξ. Since the diagonals are not involved inthe right-hand square, we can prove that it commutes up to $\Lambda\pi$-homotopy by an acyclic models argument, with K^r and L^r replaced by $K_1\times\ldots\times K_r$ and $L_1\times\ldots\times L_r$ so as to have domains given by representable functors for fixed $w\in W$. On zero simplices, the diagram commutes for any $w\in W$ and on $e_o\in W$, as the simplices vary, the diagram is Λ-homotopy commutative by a standard acyclic models argument. This starts the inductive construction of the desired homotopies, and the proof is completed precisely as was the proof of Lemma 7.1.

Corollary 7.7. If $(K,D)\in \mathcal{DA}$, then $\Gamma(K,D)$ is a Cartan object of $\mathcal{C}(\pi,\infty,\Lambda)$.

Proof. Since $D\colon K\to K\times K$ is commutative and associative, it is a morphism in $\mathcal{DA}$. Therefore the cup product (1) is a morphism in $\mathcal{C}(\pi,\infty,\Lambda)$, as required.

Lemma 7.8. If $(K, D) \in \mathcal{PA}$, $\Lambda = Z_p$, then $\Gamma(K, D)$ is an Adem object of $\mathcal{C}(p, \infty)$.

Proof. In the notations of Definition 4.1 (with $Y_o = Z_p \Sigma_{p^2}$), it suffices to prove that the following diagram is τ-homotopy commutative:

$$\begin{array}{ccccc} W_1 \otimes W_2{}^p \otimes C^*(K)^{p^2} & \xrightarrow{w \otimes 1} & Y \otimes C^*(K)^{p^2} & \xrightarrow{\theta} & C^*(K) \\ \downarrow{\scriptstyle 1 \otimes U} & & & & \| \\ W_1 \otimes (W_2 \otimes C^*(K)^p)^p & \xrightarrow{1 \otimes \theta^p} & W_1 \otimes C^*(K)^p & \xrightarrow{\theta} & C^*(K) \end{array}$$

All maps θ are as defined in Definition 7.5; by dualization, it suffices to prove that the following diagram is τ-homotopy commutative

$$\begin{array}{ccccc} W_1 \otimes W_2{}^p \otimes C(K) & \xrightarrow{w \otimes 1} & Y \otimes C(K) & \xrightarrow{\Delta} & C(K)^{p^2} \\ \downarrow{\scriptstyle T \otimes 1} & & & & \uparrow{\scriptstyle \Delta^p} \\ W_2{}^p \otimes W_1 \otimes C(K) & \xrightarrow{1 \otimes \Delta} & W_2{}^p \otimes C(K)^p & \xrightarrow{U} & (W_2 \otimes C(K))^p \end{array}$$

Let $\phi = (\mathcal{E} \otimes 1)\Phi$ and define $\alpha = \phi(w \otimes 1): W_1 \otimes W_2{}^p \otimes C(K^{p^2}) \longrightarrow C(K)^{p^2}$. Since $\Delta = \phi(1 \otimes D)$, $\Delta(w \otimes 1) = \alpha(1 \otimes 1 \otimes D)$, $D: C(K) \longrightarrow C(K^{p^2})$. By the naturality of ϕ, the following diagram is commutative:

$$\begin{array}{ccccccc} W_1 \otimes W_2{}^p \otimes C(K) & \xrightarrow{T \otimes D} & W_2{}^p \otimes W_1 \otimes C(K^p) & \xrightarrow{1 \otimes \phi} & W_2{}^p \otimes C(K)^p & \xrightarrow{U} & (W_2 \otimes C(K))^p \\ \downarrow{\scriptstyle 1 \otimes 1 \otimes D} & & \downarrow{\scriptstyle 1 \otimes 1 \otimes D} & & \downarrow{\scriptstyle 1 \otimes D^p} & & \downarrow{\scriptstyle (1 \otimes D)^p} \\ W_1 \otimes W_2{}^p \otimes C(K^{p^2}) & \xrightarrow{T \otimes 1} & W_2{}^p \otimes W_1 \otimes C(K^{p^2}) & \xrightarrow{1 \otimes \phi} & W_2{}^p \otimes C(K^p)^p & \xrightarrow{U} & (W_2 \otimes C(K^p))^p \end{array}$$

Let $\beta = \phi^p U(1 \times \phi)(T \times 1): W_1 \times W_2{}^p \times C(K^{p^2}) \longrightarrow C(K)^{p^2}$. By the diagram above, $\Delta^p U(1 \times \Delta)(T \times 1) = \beta(1 \times 1 \times D)$. Thus it suffices to prove that α and β are τ-homotopic. Since α and β do not involve the diagonal, this can easily be shown by acyclic models precisely as in our previous proofs.

The following theorem summarizes properties of the Steenrod operations that are valid for arbitrary objects of the category $\mathcal{DA}$, $\Lambda = Z_p$. Of course, we use the notations of section 5 since we are dealing with cohomology.

Theorem 7.9. Let $(K,D) \in \mathcal{DA}$, $\Lambda = Z_p$. Then there exist natural homomorphisms P^s and, if $p > 2$, βP^s defined on each $H^q(K)$; degree $(P^s) = s$ if $p = 2$ and $\deg(\beta^{\varepsilon} P^s) = 2s(p-1)+\varepsilon$, $\varepsilon = 0$ or 1, if $p > 2$. These cohomology operations on $\mathcal{DA}$ satisfy the properties

(i) $\beta^{\varepsilon} P^s = 0$ if $s < 0$ or if $p = 2$ $(\varepsilon = 0)$ and $s > q$ or if $p > 2$ and $2s+\varepsilon > q$.

(ii) $P^s(x) = x^2$ if $p = 2$ and $s = q$; $P^s(x) = x^p$ if $p > 2$ and $2s = q$

(iii) If $(K,D) = (\tilde{K} \otimes Z_p, \tilde{D} \otimes Z_p)$, where $\tilde{K}$ is Z-free, then $\beta P^{s-1} = sP^s$ if $p = 2$ and βP^s is the composition of β and P^s if $p > 2$.

(iv) $P^s = \sum P^i \otimes P^{s-i}$ and $\beta P^s = \sum(\beta P^i \otimes P^{s-i} + P^i \otimes \beta P^{s-i})$ on $H^*(K \times L)$; the internal Cartan formula is satisfied in $H^*(K)$

(v) If $f: K' \to K$ and $g: K \to K''$ are morphisms in $\mathcal{DA}$ such that $gf = 0$, then $\sigma\beta^{\varepsilon} P^s = (-1)^{\varepsilon} \beta^{\varepsilon} P^s \sigma$, where $\sigma: H^q(K'') \to H^{q-1}(K')$ is the suspension associated with $C^*(K'') \to C^*(K) \to C^*(K')$.

(vi) If $L \subset K$ and $D(L) \subset L \times L$, then $\delta\beta^{\varepsilon} P^s = (-1)^{\varepsilon} \beta^{\varepsilon} P^s \delta$ where $\delta: H^q(L) \to H^{q+1}(K,L)$ is the connecting homomorphism.

(vii) The $\beta^{\varepsilon} P^s$ satisfy the Adem relations as stated in Corollary 5.1.

Proof. For (i), we must prove that $\beta^{\varepsilon} P^s = 0$ for $s < 0$ (the rest is the convention $e_i = 0$ for $i < 0$); by formulas (5.1) and (5.2), it suffices to show that $D_i(x) = 0$ for $i > (p-1)q$, $\deg(x) = q$. By (3) of Definition 7.5, it suffices to show that $\Delta(e_i \otimes k) = 0$ for $k \in C_{pq-i}(K)$. Now $\Delta = (\varepsilon \otimes 1)\Phi(1 \otimes D)$ and, by (iv) of Lemma 7.1, if $i > (p-1)q$, then

$$\Phi(e_i \otimes D(k)) \in \sum_{j < pq} W_{pq-j} \otimes [C(K)]^p_j \subset \operatorname{Ker}(\varepsilon \otimes 1).$$

(ii) and (iii) follow from Proposition 2.3; (iv) follows from Corollary 2.7, Lemma 7.6, and Corollary 7.7; (v) and (vi) follow from Theorem 3.3, noting for

(vi) that the suspension associated with $C^*(K/L) \longrightarrow C^*(K) \longrightarrow C^*(L)$ is the inverse additive relation to the connecting homomorphism δ; (vii) follows from Theorem 4.7 and Lemma 7.8.

By Theorem 3.4, the Kudo transgression theorem applies to appropriate spectral sequences involving objects of $\mathcal{BA}$ and, under the hypotheses of (iii) of the theorem, Proposition 6.8 applies to compute the higher Bocksteins on p-th powers of elements of $H^*(K)$, $(K, D) \in \mathcal{BA}$. In the next section, we shall show how to compoute P^o for arbitrary objects $(K, D) \in \mathcal{BA}$ and shall give non-trivial examples to show that $P^o \neq 1$ in general.

8. Simplicial sets and simplicial restricted Lie algebras

We shall here obtain the Steenrod operations on the cohomology of topological spaces, simplicial sets, and simplicial restricted Lie algebras, and shall consider the evaluation of P^o on $H^*(K)$ for any $(K, D) \in \mathcal{BA}$, $\Lambda = Z_p$,

Let $\mathcal{S}$ denote the category of simplicial sets. For $K \in \mathcal{S}$, let $\tilde{K}$ denote the free simplicial Abelian group generated by K. Let Λ be a commutative ring and define a functor $A: \mathcal{S} \longrightarrow \mathcal{BA}$ by letting $A(K) = \tilde{K} \otimes \Lambda$ with its natural diagonal D; here $\mathcal{BA}$ is as defined in Definition 7.3 and D is induced from the diagonal $k \longrightarrow (k, k)$ on K. Composing A with Γ of Definition 7.5, we obtain a functor $\Gamma A: \mathcal{S} \longrightarrow \mathcal{P}(\pi, \infty, \Lambda)$ for any $\pi \subset \Sigma_r$. Let $\mathcal{J}$ denote the category of topological spaces and let $S: \mathcal{J} \longrightarrow \mathcal{S}$ be the total singular complex functor. Then $\Gamma AS: \mathcal{J} \longrightarrow \mathcal{P}(\pi, \infty, \Lambda)$ is defined. If (K, L) is a simplicial pair, define $A(K, L) = \tilde{K}/\tilde{L} \otimes \Lambda$. Then ΓA is defined on the category $\mathcal{S}_2$ of simplicial pairs and ΓAS is defined on the catetogy $\mathcal{J}_2$ of topological pairs. Since the normalized cochains with coefficients in Λ of a simplicial pair (K, L) and of a topological pair (X, Y) may be defined as $C^*(K, L) = C^*(\tilde{K}/\tilde{L} \otimes \Lambda)$ and $C^*(X, Y) = C^*(SX, SY)$, the results of the previous section apply to the cohomology of simplicial and topological pairs.

For the remainder of this section, we take $\Lambda = Z_p$ and we let π be cyclic of order p. Via ΓA: $\mathcal{S}_2 \to \mathcal{P}(p, \infty)$, we have Steenrod operations P^s, $s \geq 0$, on $H^*(K, L)$ for all $(K, L) \in \mathcal{S}_2$, hence on $H^*(X, Y)$ for all $(X, Y) \in \mathcal{T}_2$. Of course, if $p = 2$, P^s is usually denoted by Sq^s. Theorem 7.9 gives all of the standard properties of the P^s except $P^o = 1$. We now show that $P^o = 1$ follows from the previously obtained properties of the P^s.

Proposition 8.1. P^o is the identity operation and, if $p = 2$, P^1 is the Bockstein operation on the cohomology of simplicial (or topological) pairs.

Proof. Since $\beta P^o = P^1$ if $p = 2$, it suffices to prove that $P^o = 1$. If $(K, L) \in \mathcal{S}_2$, L non-empty, then $H^*(K, L) = H^*(K/L, P)$, where P is a point complex. Thus it suffices to prove that $P^o(x) = x$ for $x \in \tilde{H}^n(K) = H^n(K, P)$, since the result for L empty will follow trivially. If $K^{(n)}$ is the n-skeleton of K, then $\tilde{H}^n(K) \to \tilde{H}^n(K^{(n)})$ is a monomorphism, and we may thus assume that $K = K^{(n)}$. Then, by the Hopf Theorem [24, p. 431], there exists $f: K \to S^n$ such that $f^*(i_n^*) = x$, where $i_n^* \in \tilde{H}^n(S^n)$ is the fundamental class of the simplicial n-sphere. It therefore suffices to prove that $P^o(i_n^*) = i_n^*$. Now for any K, the suspension isomorphism $S^*: H^{q+1}(SK) \to \tilde{H}^q(K)$ may be defined as the composite $H^{q+1}(SK) \to H^{q+1}(CK, K) \longrightarrow \tilde{H}^q(K)$, where CK is the simplicial cone of K, hence S^* commutes with the P^s. Since $S^*(i_n^*) = i_{n-1}^*$ for $n \geq 1$ and $P^o(i_o^*) = (i_o^*)^p = i_o^*$ (where i_o^* generates $\tilde{H}(S^o) = Z_p$), this proves the result.

We now use the fact that $P^o = 1$ on the cohomology of simplicial sets to show how to compute P^o on $H^*(K)$ for any object $(K, D) \in \mathcal{DA}$. In fact, we have the following addendum to Lemma 7.1 when W is the canonical $Z_p\pi$-free resolution of Z_p.

Lemma 8.2. Let $K_i \in \mathcal{A}$, $1 \leq i \leq p$, and let k_i be a q-simplex of K_i. Then, for any $\Phi: W \otimes C(K_1 \times \ldots \times K_p) \to W \otimes C(K_1) \otimes \ldots \otimes C(K_p)$ which satisfies

satisfies the conclusions of Lemma 7.1,

$$(\varepsilon\otimes 1)\Phi(e_{q(p-1)}\otimes k_1\otimes\ldots\otimes k_p) = (-1)^{mq}\nu(-q)^{-1}k_1\otimes\ldots\otimes k_p,$$

where $\nu(-q) = 1$ if $p = 2$ and $\nu(-2j+\varepsilon) = (-1)^j(m!)^{\varepsilon}$, $\varepsilon = 0$ or 1, if $p > 2$.

Proof. Let Δ_q be the fundamental q-simplex of $\Lambda\Delta[q]$. Since Δ_q is a Z_p-basis for $C_q(\Lambda\Delta[q])$, we clearly have that

(i) $\Phi(e_{q(p-1)}\otimes\Delta_q\otimes\ldots\otimes\Delta_q) \equiv \gamma e_o\otimes\Delta_q\otimes\ldots\otimes\Delta_q$ mod $\mathrm{Ker}(\varepsilon\otimes 1)$, $\gamma\in Z_p\pi$.

By the naturality of Φ (or by the proof of Lemma 7.1), (i) implies

(ii) $(\varepsilon\times 1)\Phi(e_{q(p-1)}\otimes k_1\otimes\ldots\otimes k_p) = \varepsilon(\gamma)k_1\otimes\ldots\otimes k_p$ for any $k_i\in(K_i)_q$.

To evaluate $\varepsilon(\gamma)$, let $i_q\in C_q(S^q)$ represent the fundamental class of $H_q(S^q)$; we may take $S^q = \Delta[q]/\dot{\Delta}[q]$ so that i_q is a basis for $C_q(S^q)$ and $i_q^*\in C^q(S^q)$ is well-defined. By (ii) and $D(i_q) = i_q\otimes\ldots\otimes i_q$, Definition 7.5 gives

(iii) $\theta(e_{q(p-1)}\otimes i_q^{*p})(i_q) = \alpha(i_q^{*p})[(\varepsilon\otimes 1)\Phi(e_{q(p-1)}\otimes i_q^p)] = (-1)^{mq}\varepsilon(\gamma)$.

Since $P^o\{i_q^*\} = \nu(-q)D_{q(p-1)}\{i_q^*\} = i_q^*$, $\nu(-q)\theta(e_{q(p-1)}\otimes(i_q^{*p}) = i_q^*$. Thus $(-1)^{mq}\nu(-q)\varepsilon(\gamma) = 1$ and the result is proven.

Corollary 8.3. Let $(K,D)\in\mathcal{SA}$. Write $D(k) = \sum k^{(1)}\otimes\ldots\otimes k^{(p)}\in C_q(K^p)$ for $k\in C_q(K)$, and regard each $k^{(i)}$ as an element of $C_q(K)$. Let $x\in C^q(K)$ be a cocycle. Then $P^o\{x\}$ is represented by that cocycle $y\in C^q(K)$ such that $y(k) = \sum x(k^{(1)})\cdots x(k^{(p)})\in Z_p$ for each $k\in C_q(K)$. In particular, if $D(k) = N\ell\in K^p$ for each $k\in K$, where $N = \sum\alpha^i\in Z_p\pi$, then $P^o = 0$ on $H^*(K)$.

Proof. By formulas (5.1) and (5.2), $y = \nu(-q)\theta(e_{q(p-1)}\otimes x^p)$ represents $P^o\{x\}$, and the result follows by an easy computation from Definition 7.5 and the lemma.

We now give a useful application of the theory for which the Steenrod operations satisfy the results of Theorem 7.9 and $P^o = 0$. Let $\mathcal{L}$ and $\mathcal{H}$ denote the categories of restricted Lie algebras and of primitively generated Hopf algebras over Z_p. Let $F: \mathcal{A} \to \mathcal{L}$ denote the free restricted Lie algebra functor, let $V: \mathcal{L} \to \mathcal{H}$ denote the universal enveloping algebra functor, and let $P: \mathcal{H} \to \mathcal{L}$ denote the functor which assigns to $H \in \mathcal{H}$ its restricted Lie algebra of primitive elements. By a result of Milnor and Moore [19, Theorem 6.11], $PV(L) = L$ for $L \in \mathcal{L}$ and $VPH = H$ for $H \in \mathcal{H}$. By Theorems of Witt and Friedrich [9, Theorems 7 and 9, p.168-170], extended to restricted Lie algebras, if $K \in \mathcal{A}$ and $T(K)$ is the tensor algebra of K, then $V(FK) = T(K)$ in $\mathcal{H}$ and $FK = PT(K)$ in $\mathcal{L}$. These statements clearly remain valid for the categories $s\mathcal{A}$, $s\mathcal{L}$ and $s\mathcal{H}$ of simplicial objects in $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{H}$ [see 15, Definition 2.1]. We shall need the following algebraic lemma.

Lemma 8.4. Let L be a restricted Lie algebra and let $IV(L) = \text{Ker}\, \varepsilon$, $\varepsilon: V(L) \to Z_p$, be its augmentation ideal. Let $\psi: V(L) \longrightarrow V(L)^p = V(L) \otimes \ldots \otimes V(L)$ denote the iterated coproduct. Then, for each $x \in IV(L)$, there exists $y \in V(L)^p$ such that $\psi(x) = Ny$.

Proof. Let $\pi: FK \to L$ represent L as a quotient of a free restricted Lie algebra. Then $V(\pi) = V(FK) \to V(L)$ is an epimorphism of Hopf algebras, and we may assume that $L = FK$. Clearly we may also assume that K is a finite dimensional Z_p-module. Since $T(K)$ admits a grading under which it is connected, [19, Proposition 4.20] implies that the p-th power operation ξ is zero on the augmentation ideal of the dual Hopf algebra $T(K)^*$. The cocommutativity of $T(K)$ implies that, for $x \in IT(K)$, $\psi(x)$ can be written in the form $\psi(x) = Ny + \sum z_i \otimes \ldots \otimes z_i$ in $T(K)^p$. By the triviality of ξ on $IT(K)^*$, each $z_i = 0$ and the result follows.

We now sketch a definitional framework for the study of homotopy invariants of simplicial restricted Lie algebras. Define a category $s\mathcal{L}_2$ as follows. The objects of $s\mathcal{L}_2$ are pairs (L, M) such that $L \in s\mathcal{L}$ and M is a restricted Lie ideal of L and the morphisms $f: (L, M) \to (L', M')$ in $s\mathcal{L}_2$ are morphisms

$f: L \longrightarrow L'$ in $\mathcal{SL}$ such that $f(M) \subset M'$. Two such morphisms, f and g, are said to be Lie homotopic if there exist morphisms of restricted Lie algebras $h_i: L_q \longrightarrow L'_{q+1}$, $0 \leq i \leq q$, such that $h_i(M_q) \subset M'_{q+1}$ and the identities (i) - (iii) of [15, Definition 5.1] are satisfied. Define the homotopy, homology, and cohomology groups of $(L, M) \in \mathcal{SL}_2$ by

(1) $\pi_*(L, M) = H_*(L/M)$ and

(2) $H_*(L, M) = H_*(IV(L/M))$ and $H^*(L, M) = H^*(IV(L/M))$

The homology and cohomology groups on the right sides of these equations are as defined at the start of section 7, with L/M and $IV(L/M)$ regarded as simplicial Z_p-modules. The argument of [15, Proposition 5.3] shows that Lie homotopic morphisms in $\mathcal{SL}_2$ induce the same morphisms on homotopy, homology, and cohomology. By [15, Theorem 22.1], $\pi_*(L, M)$ and $H_*(L, M)$ are, respectively, the homotopy groups of L/M and of $IV(L/M)$ regarded as simplicial sets. The Hurewicz homomorphism $h: \pi_*(L, M) \longrightarrow H_*(L, M)$ may thus be defined as the map induced on homotopy from the inclusion $L/M \longrightarrow IV(L/M)$. Since $IV(L/M) = IV(L)/IV(M)$, we have natural long exact homotopy, homology, and cohomology sequences on pairs $(L, M) \in \mathcal{SL}_2$, and h defines a natural transformation of long exact sequences. Note that $H_*(L, M)$ is the augmentation ideal of the Hopf algebra $H_*(V(L/M))$ if $H_*(V(L/M))$ is of finite type. Consider $FAS^n = F(\tilde{S}^n \otimes Z_p)$, where S^n is the simplicial n-sphere. It can be proven that $\pi_n(L)$ is the Z_p-module of Lie homotopy equivalence classes of morphisms $FAS^n \longrightarrow L$ for $L \in \mathcal{SL}$ and that $H_*(FAS^n) \cong \tilde{H}_*(\Omega S^{n+1})$ is the augmentation ideal of the free commutative algebra on one primitive generator of degree n.

Our theory immediately yields Steenrod operations on $H^*(L, M)$.

<u>Theorem 8.5.</u> There exist natural homomorphisms P^s and, if $p > 2$, βP^s defined on $H^*(L, M)$ for $(L, M) \in \mathcal{SL}_2$. These operations satisfy the con-

clusions of Theorem 7.9 (except that the hypothesis of (iii) is not satisfied in general) and, in addition, the operation P^o is identically zero.

Proof. We may regard $C_*(L,M) = C(IV(L/M))$ as an object of $\mathcal{DA}$, with diagonal $\overline{D} = \overline{\psi}$, the reduced coproduct as defined in Remarks 7.4(ii). Thus Theorem 7.9 applies directly, and $P^o = 0$ follows from the previous lemma and corollary.

In [22], Priddy has given a different definition of $H_*(L)$ and $H^*(L)$. Let $\overline{W}$ be the functor from simplicial Z_p-algebras to $\mathcal{SA}$ defined by Moore [20]. If A is a simplicial Z_p-algebra, then $\overline{W}_o(A) = Z_p$ and $\overline{W}_q(A) = A_{q-1} \otimes \ldots \otimes A_o$, $q > 0$, as Z_p-modules. The face and degeneracy operators are as defined in [15, p.87]. For $L \in \mathcal{SL}$, $\overline{W}V(L)$ is a simplicial cocommutative coalgebra with coproduct ψ given by

$$\psi(a_{q-1} \otimes \ldots \otimes a_o) = \sum (a'_{q-1} \otimes \ldots \otimes a'_o) \otimes (a''_{a-1} \otimes \ldots \otimes a'_o),$$

where $a_i \in V_i(L)$ satisfies $\psi(a_i) = \sum a'_i \otimes a''_i$. Priddy defines $H_*(L) = H_*(I\overline{W}V(L))$ and $H^*(L) = H^*(I\overline{W}V(L))$, where $I\overline{W}V(L)$ is regarded as a simplicial Z_p-module. For spectra, Priddy's definition and ours clearly differ only by a shift of degree; with his definition, $H^*(FAS^n) \cong \tilde{H}^*(S^{n+1}) = Z_p$. By Definition 7.3 and Remarks 7.4(ii), $(C(I\overline{W}V(L), \overline{\psi}) \in \mathcal{DA}$ and therefore Priddy's $H^*(L)$ alos admits Steenrod operations which satisfy the conclusions of Theorem 7.9 (except, in general, for (iii)) and $P^o = 0$.

9. The dual homology operations; Nishida's theorem

For applications to loop spaces and to obtain a result used in the proof of the Adem relations, we shall discuss the homology operations P^s_* whose duals are the Steenrod operations on the mod p cohomology of a space X. Of course, $H^*(X) = H_*(X)^* = \mathrm{Hom}_{Z_p}(H_*(X), Z_p)$ and, if $H_*(X)$ is of finite type, $H_*(X) = H^*(X)^*$. Define P^s_* on $H_*(X)$ by $P^s = (P^s_*)^*$; P^s_* is clearly well-defined if $H_*(X)$ is of finite type, and either a direct limit argument or the next proposition imply that P_*^s is well-defined in general. P_*^s lowers degrees by s if $p = 2$ and by $2s(p-1)$ if $p > 2$. Our results on the P^s immediately yield the dual results for the P_*^s. We shall write the operations P_*^s on the left; the order of composition in the dual Adem relations must thus be reversed (that is, $H_*(X)$ is a left module over the opposite algebra of the Steenrod algebra). The following proposition was used in the proof of Lemma 4.6. Formula (2) of the proof is essentially Steenrod's definition [30] of the D_i.

Proposition 9.1. Let X be a space and let $d = \Phi(1 \otimes D): W \otimes_\pi C_*(X) \longrightarrow W \otimes_\pi C_*(X)^p$. Consider $d_*: H_*(\pi; H_*(X)) \longrightarrow H_*(\pi; H_*(X)^p)$. Let $x \in H_s(X)$. Then

(i) If $p = 2$, $d_*(e_r \otimes x) = \sum_k e_{r+2k-s} \otimes P_*^k(x) \otimes P_*^k(x)$.

(ii) If $p > 2$,
$$d_*(e_r \otimes x) = \nu(s) \sum_k (-1)^k e_{r+(2pk-s)(p-1)} \otimes P_*^k(x)^p - \delta(r)\, \nu(s-1) \sum_k (-1)^k e_{r+p+(2pk-s)(p-1)} \otimes P_*^k \beta(x)^p.$$

where $\nu(2j+\varepsilon) = (-1)^j (m!)^\varepsilon$ and $\delta(2j+\varepsilon) = \varepsilon$, $\varepsilon = 0$ or 1.

Proof. We may assume that $H_*(X)$ is of finite type. We shall compute $d^*: H^*(\pi; H^*(X)^p) \longrightarrow H^*(\pi) \otimes H^*(X)$ and then dualize. In the notations of Lemma 1.3, $H_*(X)^p \cong A \oplus Z_p\pi \otimes B$ as a π-module, and $H_*(\pi; H_*(X)^p) \cong H_*(\pi) \otimes A \oplus B$. It follows that $H^*(\pi; H^*(X)^p) \cong H^*(\pi) \otimes A^* \oplus B^*$. We claim first that $d^*(B^*) = 0$. To see this, we make explicit the isomorphism from B^* to the homology of $(W \otimes_\pi Z_p\pi \otimes B)^*$. For $y \in B^*$, define $\tilde{y} \in (W \otimes_\pi Z_p\pi \otimes B)^*$ by

$$\tilde{y}(w \otimes \alpha^i \otimes b) = \varepsilon(w) y(b) \quad \text{for } w \in W,\ 0 \le i < p,\ b \in B.$$

Then y is a cocylce and $y \longrightarrow \tilde{y}$ induces the desired isomorphism. Define $\nu: Z_p\pi \longrightarrow Z_p$ by $\nu(1) = 1$ and $\nu(\alpha^i) = 0$, $1 \le i < p$, and define $\bar{y} \in (W \otimes Z_p\pi \otimes B)^*$ for $y \in B^*$ by

$$\bar{y}\,(w \otimes \alpha^i \otimes b) = \varepsilon(w)\, \nu(\alpha^i) y(b) \quad \text{for } w \in W,\ 0 \le i < p,\ b \in B.$$

Clearly $\tilde{y}(w\otimes\alpha^i\otimes b) = \bar{y}(N[w\otimes\alpha^i\otimes b])$. Therefore

$$d^*(\tilde{y})(w\otimes x) = \tilde{y}d_*(w\otimes x) = \bar{y}\,d_*(Nw\otimes x) = 0$$

for $w \in W$ and $x \in H_*(X)$ since $Nw\otimes x$ is a boundary in $W\otimes H_*(X)$ (because $d(e_{2i}) = Ne_{2i-1}$ and $d(T^{p-2}e_{2i+1}) = Ne_{2i}$ in W). This proves that $d^*(B^*) = 0$.

We next compute d^* on $H^*(\pi)\otimes A^*$. Let $y \in H^q(X)$ and $x \in H_{pq-i}(X)$. By Definition 7.5, we have the formula

(1) $\quad D_i(y)(x) = \theta_*(e_i\otimes y^p)(x) = (-1)^{iq}d^*(\varepsilon\otimes 1)^*(y^p)(e_i\otimes x)$

(where the isomorphism α from the tensor product of duals to the dual of tensor products has been omitted from the notation).

Let w_i be dual to e_i. Then $(\varepsilon\otimes 1)^*(y^p) = w_o\otimes y^p$ and therefore

$$d^*(w_o\otimes y^p)(e_i\otimes x) = (-1)^{iq}D_i(y)(x).$$

For any $z \in H^{pq-i}(X)$, the sign in the definition of α gives

$$(w_i\otimes z)(e_i\otimes x) = (-1)^{i(pq-i)}z(x).$$

Comparing these formulas, we see that

(2) $$d^*(w_o\otimes y^p) = \sum (-1)^i w_i\otimes D_i(y) \ .$$

To compute $d^*(w_j\otimes y^p)$ for $j > 0$, observe that if $\rho : W \longrightarrow \overline{W} = Z_p\otimes_\pi W$ is the natural epimorphism, then we have the commutative diagram:

$$\begin{array}{ccc}
W\otimes_\pi H_*(X) & \xrightarrow{\quad d_*\quad} & W\otimes_\pi H_*(X)^p \\
\psi\otimes 1 \downarrow & & \downarrow \psi\otimes 1 \\
(W\otimes W)\otimes_\pi H_*(X) & \xrightarrow{\quad 1\otimes d_*\quad} & (W\otimes W)\otimes_\pi H_*(X)^p \\
\rho\otimes 1\otimes 1 \downarrow & & \downarrow \rho\otimes 1\otimes 1 \\
\overline{W}\otimes(W\otimes_\pi H_*(X)) & \xrightarrow{\quad 1\otimes d_*\quad} & \overline{W}\otimes(W\otimes_\pi H_*(X)^p)
\end{array}$$

(The upper rectangle requires an easy acyclic models argument.) Dually, d^* is a morphism of $H^*(\pi)$-modules. Now $w_j(w_o\otimes y^p) = w_j\otimes y^p$ and, in $H^*(\pi)$, $w_jw_i = w_{i+j}$ if $p = 2$ or either i or j is even and $w_jw_i = 0$ if $p > 2$ and i and j are odd by formula (1.2). Therefore (2) implies the formulas

(3) if $p = 2$, $d^*(w_j \otimes y^p) = \sum_i w_{i+j} \otimes D_i(y)$ and

(4) if $p > 2$, $d^*(w_j \otimes y^p) = \sum_i w_{2i+j} \otimes D_{2i}(y) - \delta(j+1) \sum w_{2i+1+j} \otimes D_{2i+1}(y)$.

By formulas (5.1) and (5.2) and a reindexing, (3) and (4) become

(5) if $p = 2$, $d^*(w_j \otimes y^p) = \sum_k w_{j+q-k} \otimes P^k(y)$ and

(6) if $p > 2$, $d^*(w_j \otimes y^p) = \nu(-q)^{-1} \sum_k (-1)^k w_{j+(q-2k)(p-1)} \otimes P^k(y)$

$$- \delta(j+1)\, \nu(-q)^{-1} \sum_k (-1)^k w_{j+(q-2k)(p-1)-1} \otimes \beta P^k(y).$$

We now dualize. $d_*(H_*(\pi) \otimes H_*(X)) \subset H_*(\pi) \otimes A$ since $d^*(B^*) = 0$. For $x \in H_s(X)$, we may therefore write

$$d_*(e_r \otimes x) = \sum e_{r+s-pq} \otimes E_{qr}(x)^p, \quad E_{qr}(x) \in H_q(X).$$

Let $y \in H^q(X)$. Using the Kronecker pairing $<\ ,\ >$, we have

(7) $<w_{r+s-pq} \otimes y^p, d_*(e_r \otimes x)> = (-1)^{(r+s-q+m)q} <y, E_{qr}(x)>$.

Since $<P^k(y), x> = <y, P^k_*(x)>$, (5) implies that if $p = 2$, then

(8) $<d^*(w_{r+s-2q} \otimes y^2), e_r \otimes x> = <w_r \otimes P^{s-q}(y), e_r \otimes x> = <y, P_*^{s-q}(x)>$.

Thus $E_{qr}(x) = P_*^{s-q}(x)$ if $p = 2$ and, with $k = s-q$, this implies (i). Now assume that $p > 2$. By (6), $d^*(w_{r+s-pq} \otimes y^p)$ has a summand involving w_r only if $q = s-2k(p-1)-\varepsilon$, $k \geq 0$ and $\varepsilon = 0$ or 1, hence $E_{qr}(x) = 0$ for other values of q. For $q = s-2k(p-1)$, (6) gives

(9) $<d^*(w_{r+(2pk-s)(p-1)} \otimes y^p), e_r \otimes x> = \nu(-q)^{-1}(-1)^{k+rq} <y, P_*^k(x)>$.

By (7) and (9), $E_{qr}(x) = (-1)^{k+mq}\nu(-q)^{-1}P_*^k(x)$ if $q = s-2k(p-1)$; since $(-1)^{mq}\, \nu(-q)^{-1} = \nu(q) = \nu(s)$, this yields the first sum of (ii). Observe next that $<\beta y, x> = (-1)^{q+1} <y, \beta x>$ by the chain and cochain definitions of the Bockstein and the sign convention $\delta(f) = (-1)^{\deg f+1} fd$ used in defining $C^*(X)$.

Now for $q = s-2k(p-1)-1$, (6) gives

(10) $\quad \langle d^*(w_{r+p+(2pk-s)(p-1)} \otimes y^p), e_r \otimes x\rangle = \delta(r)\nu(-q)^{-1}(-1)^{k+r(q+1)+q}\langle y, P_*^k\beta(x)\rangle$

By (7) and (10), $E_{qr}(x) = (-1)^{k+r+mq}\nu(-q)^{-1}\delta(r)P_*^k\beta(x)$ if $q = s-2k(p-1)-1$; since $\delta(r) = 0$ if r is even and $(-1)^{mq}\nu(-q)^{-1} = \nu(q) = \nu(s-1)$, this yields the second sum of (ii) and so completes the proof.

Remark 9.2. The proof above used no properties specific to topological spaces and so applies to compute

$$d_* = \Phi_*(1\otimes D)_*: H_*(\pi; H_*(K)) \longrightarrow H_*(\pi; H_*(K)^p)$$

in terms of P_*^s and βP_*^s (where βP_*^s is defined by $(\beta P_*^s)^* = -\beta P^s$ if no Bockstein is present) for arbitrary objects $(K, D) \in \mathcal{PA}$.

We now give a new proof of a result due to Nishida [21], which is essential to the computation of Steenrod operations in iterated loop spaces. Let $K(Z_p, 1) = E/\pi$ where π operates properly on the acyclic space E; by [14, IV 11], $C_*(E) = Z_p\pi \otimes C_*(E/\pi)$. Let $\sigma: E \longrightarrow E/\pi$ be the projection and let $f: W \longrightarrow C_*(E)$ be a π-morphism over Z_p. If $\overline{W} = Z_p \otimes_\pi W$, then f induces $\overline{f}: \overline{W} \longrightarrow C_*(E/\pi)$, and $\overline{f}\rho$ is homotopic to σf, $\rho: W \longrightarrow \overline{W}$. By Remarks 7.2, if η is the shuffle map, then we have the following homotopy commutative diagram for any space X:

$$\begin{array}{ccccc}
W\otimes_\pi C_*(X) & \xrightarrow{f\otimes 1} & C_*(E)\otimes_\pi C_*(X) & \xrightarrow{\eta} & C_*(E\times_\pi X) \\
\downarrow d & & \downarrow 1\otimes D & & \downarrow 1\times D \\
W\otimes_\pi C_*(X)^p & \xrightarrow{f\otimes\eta} & C_*(E)\otimes_\pi C_*(X^p) & \xrightarrow{\eta} & C_*(E\times_\pi X^p) \\
\downarrow \psi\otimes 1 & & \downarrow D\otimes 1 & & \downarrow D\times 1 \\
W\otimes W\otimes_\pi C_*(X)^p & \xrightarrow{\eta(f\otimes f)\otimes\eta} & C_*(E\times E)\otimes_\pi C_*(X^p) & \xrightarrow{\eta} & C_*(E\times E\times_\pi X^p) \\
\downarrow \rho\otimes 1\otimes 1 & & \downarrow \sigma\times 1\otimes 1 & & \downarrow \sigma\times 1\times 1 \\
\overline{W}\otimes W\otimes_\pi C_*(X)^p & \xrightarrow{\eta(\overline{f}\otimes f)\otimes\eta} & C_*(E/\pi\times E)\otimes_\pi C_*(X^p) & \xrightarrow{\eta} & C_*(E/\pi\times E\times_\pi X^p)
\end{array}$$

Let $\mu_* = (\rho\otimes 1\otimes 1)_*(\psi\otimes 1)_*: H_*(\pi; H_*(X)^p) \longrightarrow H_*(\pi)\otimes H_*(\pi; H_*(X)^p)$. The horizontal arrows are homology isomorphisms and we therefore have Steenrod operations P_*^s on $H_*(\pi; H_*(X))$, $H_*(\pi; H_*(X)^p)$, and $H_*(\pi)\otimes H_*(\pi; H_*(X)^p)$ such that d_*

and μ_* commute with the P_*^s. The following theorem uses d_* and μ_* to evaluate the P_*^s on $H_*(\pi; H_*(X)^p)$. Our result differs from Nishida's by a sign; the reason for this is that our formulas (2) and (6) in the proof above differ from the corresponding formulas in [30, p. 103 and p. 119]. We were pedantic about signs in the preceding proof because of this disagreement. We shall need the following identity on binomial coefficients in the proof of the theorem.

Lemma 9.3. $\sum_i (i, a-i)(n-i, i+b-n) = (n, a+b-n)$ for $a \geq 0$, $b \geq 0$, and $n \geq 0$.

Proof. The result is obvious if $b = 0$, when $i = n$ gives the only non-zero summand on the left. Using $(c-1, d) + (c, d-1) = (c, d)$, we find that the result for the triples $(a, b-1, n)$ and $(a, b-1, n-1)$ implies the result for the triple (a, b, n).

Theorem 9.4. Let X be a space, $x \in H_q(X)$. Then, in $H_*(\pi; H_*(X)^p)$,

(i) If $p = 2$, $P_*^s(e_r \otimes x^2) = \sum_i (s-2i, r+q-2s+2i) e_{r-s+2i} \otimes P_*^i(x)^2$.

(ii) If $p > 2$, $P_*^s(e_r \otimes x^p) = \sum (s-pi, [\frac{r}{2}] + qm - ps + pi) e_{r+2(pi-s)(p-1)} \otimes P_*^i(x)^p$

$$+ \delta(r)\alpha(q) \sum_i (s-pi-1, [\tfrac{r+1}{2}] + qm - ps + pi) e_{r+p+2(pi-s)(p-1)} \otimes P_*^i \beta(x)^p$$

where $\alpha(q) = \nu(q)^{-1}\nu(q-1) = -(-1)^{mq} m!$ and $\delta(2j + \varepsilon) = \varepsilon$, $\varepsilon = 0$ or 1.

Proof. We assume that $s > 0$, since the result is trivial for $s = 0$. If $r = 0$, then $e_o \otimes x^p$ is in the image of $H_*(E \times X^p) \longrightarrow H_*(E \times_\pi X^p)$. In $H_*(E \times X^p)$, $P_*^s(e_o \otimes x^p) = \sum e_o \otimes P_*^{i_1}(x) \otimes \ldots \otimes P_*^{i_p}(x)$ summed over all p-tuples $(i_1, \ldots, i_p)$ such that $\sum i_p = s$. The sum of all terms with any $i_j \neq i_k$ lies in $e_o \otimes NH_*(X)^p$ and is thus zero in $H_*(E \times_\pi X^p)$. Therefore $P_*^s(e_o \otimes x^p) = 0$ unless $s = pt$, when $P_*^s(e_o \otimes x^p) = e_o \otimes P_*^t(x)^p$, which is in agreement with (i) and (ii). Recall that, by Definition 1.2 and the proof of Lemma 4.6, we have the following relations in $H_*(\pi)$.

(a) If $p = 2$, $P_*^i(e_j) = (i, j-2i)e_{j-i}$ and $\psi(e_r) = \sum e_j \otimes e_{r-j}$

(b) If $p > 2$, $P^i_*(e_j) = (i, [j/2] - pi)e_{j-2i(p-1)}$ and $\psi(e_r) = \sum \delta(r,j)e_j \otimes e_{r-j}$ where $\delta(r,j) = 1$ unless r is even and j is odd, when $\delta(r,j) = 0$.

If $q = 0$, then $d_*(e_r \otimes x) = e_r \otimes x^p$ and therefore

$$P^s_*(e_r \otimes x^p) = d_*(P^s_*(e_r) \otimes x) = P^s_*(e_r) \otimes x^p ;$$

by (a) and (b), the result holds in this case. We now proceed by induction on q and for fixed q by induction on r. Thus assume the result for $q' < q$ and for our fixed q and $r' < r$. Let $z = P^s_*(e_r \otimes x)$ and let z' denote the right side of the equation to be proven. Write $z - z' = \sum e_i \otimes y_i \in H_*(\pi; H_*(X)^p)$. We shall first prove that $\mu_*(z-z') = e_o \otimes (z-z')$; this will imply that $y_i = 0$ for all $i > 0$ since if i is maximal such that $y_i \neq 0$, then $e_i \otimes e_o \otimes y_i$ clearly occurs as a non-zero summand of $\mu_*(z-z')$. We shall then prove that $y_o = 0$ by explicit computation and so complete the proof. We give the details separately in the cases $p = 2$ and $p > 2$.

(i) $p = 2$. Since $\mu_*(z) = P^s_*\mu_*(e_r \otimes x^2)$, we find by (a), the Cartan formula in $H_*(\pi) \otimes H_*(\pi; H_*(X)^2)$, and the induction hypothesis on r, that

(c) $$\mu_*(z) = P^s_*\Big(\sum_j e_j \otimes e_{r-j} \otimes x^2\Big) = \sum_{i,j} (i, j-2i)e_{j-i} \otimes P^{s-i}_*(e_{r-j} \otimes x^2), \text{ where}$$

$$P^{s-i}_*(e_{r-j} \otimes x^2) = \sum_k (s-i-2k, r-j+q-2s+2i+2k)e_{r-j-s+i+2k} \otimes P^k_*(x)^2 \text{ if } j > 0.$$

The terms with $j = i > 0$ are zero since $(i, -i) = 0$. Applying the lemma to those (i,j) such that $j-i = \ell > 0$, with $a = \ell$, $b = r-\ell+q-s$, and $n = s-2k$, we see that (c) reduces to the formula

(d) $$\mu_*(z) = e_o \otimes z + \sum_{k,\ell>0} (s-2k, r+q-2s+2k)e_\ell \otimes e_{r-s+2k-\ell} \otimes P^k_*(x)^2 .$$

A glance at the right side of (i) shows that $\mu_*(z-z') = e_o \otimes (z-z')$, hence $y_i = 0$ for $i > 0$. To compute y_o, observe that $P^o = 1$ and Proposition 9.1 imply

(e) $$e_r \otimes x^2 = d_*(e_{r+q} \otimes x) + \sum_{k>0} e_{r+2k} \otimes P^k_*(x)^2 .$$

$P_*^s d_* = d_* P_*^s$, the Cartan formula on $H_*(\pi) \otimes H_*(X)$, and Proposition 9.1 evaluate $P_*^s d_*(e_{r+q} \otimes x)$, and the induction hypothesis on q evaluates $P_*^s(e_{r+2k} \otimes P_*^k(x)^2)$ for $k > 0$. Carrying out these computations, we find that (e) implies the formula

$$\text{(f)} \qquad z = \sum_{k,\ell} (s-\ell, r+q-2s+2\ell) e_{r-s+2k+2\ell} \otimes P_*^k P_*^\ell (x)^2 + \sum_{k>0,\ell} (s-2\ell, r+q+k-2s+2\ell) e_{r-s+2k+2\ell} \otimes P_*^\ell P_*^k (x)^2 .$$

In principle, (f) must imply (i) directly, but our argument with μ_* shows that we need only consider those terms involving e_o, with $2(k+\ell) = s-r$. Let $t = k+\ell-s$ and $c = q-k-\ell$; then these terms become

$$\text{(g)} \qquad \sum_{k\geq 0} (k-t, t+c-2k) e_o \otimes P_*^k P_*^{s+t-k}(x)^2 + \sum_{\ell < s+t} (c+\ell-s, s-2\ell) e_o \otimes P_*^\ell P_*^{s+t-\ell}(x)^2.$$

By formula (f) of the proof of Theorem 4.7, rephrased as in section 5 and dualized (with the order of composition reversed under dualization since we are writing the operations P_*^s on the left), and by Remarks 4.8, (g) would be zero if $\ell = s+t$ were allowed in the second sum; thus (g) reduces to

$$\text{(h)} \qquad (c+t, -s-2t) e_o \otimes P_*^{s+t}(x)^2 = (q-s, r) e_o \otimes P_*^{1/2(s-r)}(x)^2 .$$

Since (h) is equal to the summand of z' involving e_o, it follows that $y_o = 0$.

(ii) $p > 2$. For brevity of notation, write $d = 2(p-1)$. As in the case $p = 2$, we find by (b) and induction on r that

$$\text{(i)} \qquad \mu_*(z) = \sum_{ij} \delta(r,j)(i, [j/2]-pi) e_{j-di} \otimes P_*^{s-i}(e_{r-j} \otimes x^p), \text{ where, if } j > 0,$$

$$P_*^{s-i}(e_{r-j} \otimes x^p) = \sum_k (s-i-pk, [\tfrac{r-j}{2}] + qm-ps+pi+pk) e_{r-j+d(pk-s+i)} \otimes P_*^k(x)^p$$
$$+ \delta(r-j)\alpha(q) \sum_k (s-i-pk-1, [\tfrac{r-j+1}{2}] + qm-ps+pi+pk) e_{r-j+p+d(pk-s+i)} \otimes P_*^k \beta(x)^p$$

The terms with $j = di > 0$ are zero. By the lemma, applied to those (i, j) such that $j-di = \ell > 0$, with $a = [\ell/2]$, $b = [\frac{r-\ell}{2}] + qm-s(p-1)$, and $n = s-pk$ for the first sum and $a = [\ell/2]$, $b = [\frac{r-\ell-1}{2}] + qm - s(p-1)$, and $n = s-pk-1$ for the second sum, (i) reduces to

$$\text{(j)} \quad \mu_*(z) = e_o \otimes z + \sum_{k,\ell>0} \delta(r,\ell)(s-pk, [\tfrac{r}{2}]+qm-ps+pk)e_\ell \otimes e_{r+d(pk-s)-\ell} \otimes P_*^k(x)^p$$

$$+ \sum_{k,\ell>0} \delta(r,\ell)\delta(r-\ell)\,\alpha(q)(s-pk-1, [\tfrac{r+1}{2}]+qm-ps+pk)e_\ell \otimes e_{r+p+d(pk-s)-\ell} \otimes P_*^k\beta(x)^p .$$

Now $\delta(r,\ell)\delta(r-\ell) = \delta(r)\delta(r+1,\ell)$, and it follows from a glance at (ii) that $\mu_*(z-z') = e_o \otimes (z-z')$. Thus $y_i = 0$ for $i > 0$. To compute y_o, observe that Proposition 9.1 implies that

$$\text{(k)} \quad e_r \otimes x^p = \nu(q)^{-1} d_*(e_{r+q(p-1)} \otimes x) - \sum_{k>0} (-1)^k e_{r+dpk} \otimes P_*^k(x)^p$$
$$+ \delta(r)\alpha(q) \sum_k (-1)^k e_{r+p+dpk} \otimes P_*^k\beta(x)^p .$$

Precisely as in the case $p = 2$, we can compute P_*^s on the right side of (k); carrying out this computation, we find

$$(\ell) \quad z = \sum_{k,\ell} (-1)^k (s-\ell, [r/2]+qm-ps+p\ell)e_{r+d(pk+p\ell-s)} \otimes P_*^k P_*^\ell(x)^p$$

$$- \delta(r)\alpha(q) \sum_{k,\ell} (-1)^k (s-\ell, [r/2]+qm-ps+p\ell)e_{r+p+d(pk+p\ell-s)} \otimes P_*^k \beta P_*^\ell(x)^p$$

$$- \sum_{k>0,\ell} (-1)^k (s-p\ell, [r/2]+k(p-1)+qm-ps+p\ell)e_{r+d(pk+p\ell-s)} \otimes P_*^\ell P_*^k(x)^p$$

$$- \delta(r)\alpha(q) \sum_{k>0,\ell} (-1)^k (s-p\ell-1, [\tfrac{r+1}{2}] + k(p-1)+qm-ps+p\ell)e_{r+p+d(pk+p\ell-s)} \otimes P_*^\ell \beta P_*^k(x)^p$$

$$+ \delta(r)\alpha(q) \sum_{k,\ell} (-1)^k (s-p\ell, [\tfrac{r+1}{2}]+k(p-1)+qm-ps+p\ell)e_{r+p+d(pk+p\ell-s)} \otimes P_*^\ell P_*^k \beta(x)^p .$$

Consider the first and third sums, with $r+d(pk+p\ell-s) = 0$. Let $t = k+\ell-s$ and $c = q-d(k+\ell)$. Then these two sums become

$$\text{(m)} \quad \sum_k (-1)^k (k-t, t+mc-pk)e_o \otimes P_*^k P_*^{s+t-k}(x)^p$$
$$- \sum_{\ell<s+t} (-1)^{s+t+\ell} (\ell+mc-s, s-p\ell)e_o \otimes P_*^\ell P_*^{s+t-\ell}(x)^p .$$

Consider the remaining sums of (ℓ), with $r+p+d(pk+p\ell-s) = 0$; r is odd, hence $\delta(r) = 1$. Let t be as above and let $c' = c-1$. Then these three sums become

$$\text{(n)}\qquad -\alpha(q)\sum_{k}(-1)^{k}(k-t,t+mc'-pk-1)e_{o}\otimes P_{*}^{k}\beta P_{*}^{\ell}(x)^{p}$$

$$-\alpha(q)\sum_{\ell<s+t}(-1)^{s+t+\ell}(\ell+mc'-s,s-p\ell-1)e_{o}\otimes P_{*}^{\ell}\beta P_{*}^{s+t-\ell}(x)^{p}$$

$$+\alpha(q)\sum_{\ell}(-1)^{s+t+\ell}(\ell+mc'-s,s-p\ell)e_{o}\otimes P_{*}^{\ell}P_{*}^{s+t-\ell}\beta(x)^{p}\,.$$

By formulas (h) and (j) of the proof of Theorem 4.7 (with $\varepsilon = 0$), rephrased as in section 5 and dualized, we see that (m) and (n) would be zero if $\ell = s+t$ were allowed in the second sums. Therefore, by an easy verification, (m) and (n) reduce to the following expressions, where $i = k+\ell = s+t$.

(o) $\quad (s-pi, \frac{r}{2} + qm - ps+pi)e_o \otimes P_*^i(x)^p$ with $dpi = ds-r$, and

(p) $\quad \alpha(q)(s-pi-1, \frac{r+1}{2} + qm - ps + pi)e_o \otimes P_*^i\beta(x)^p$ with $dpi = ds-r-p$.

Clearly (o) is equal to the summand of z' involving e_o in its first sum and (p) is equal to the summand of z' involving e_o in its second sum. Thus $y_o = 0$ and the proof is complete.

10. The cohomology of $K(\pi, n)$ and the axiomatization of the P^s

We recall the structure of $H^*(K(\pi,n); Z_p) = H^*(\pi, n, Z_p)$ and compute completely the mod p cohomology Bockstein spectral sequence of $K(\pi, n)$ in this section. We also show (as should be well-known) that Serre's proof [23] of the axiomatization of the Sq^i using $K(Z_2, n)$ can be simply modified so as to apply in the case of odd primes. We shall consider only the cyclic groups $\pi = Z_{p^t}$, $1 \leq t \leq \infty$, where, by convention, $Z_{p^\infty} = Z$. We first fix conventions on admissible monomials relative to t.

Notations 10.1. (a) $p = 2$. For $I = (s_1, \ldots, s_k)$, we say that I is admissible if $s_i \geq 2s_{i+1}$ and $s_k \geq 1$. The length, degree, and excess of I are defined by $\ell(I) = k$, $d(I) = \sum s_j$, and, if $I = (s, J)$, $e(I) = s - d(J)$. Define

$P_t^I = P^{s_1} \ldots P^{s_{k-1}} P_t^{s_k}$, where $P_t^s = P^s$ if $s \geq 2$, $P_t^1 = \beta_t$ if $t < \infty$, and $P_\infty^1 = 0$; thus, if $t = \infty$, we agree that admissibility requires $s_k \geq 2$. The empty sequence I is admissible, with length, degree, and excess zero, and I determines the identity operation.

(b) $p > 2$. For $I = (\varepsilon_1, s_1, \ldots, \varepsilon_k, s_k, \varepsilon_{k+1})$, $\varepsilon_i = 0$ or 1, we say that I is admissible if $s_i \geq ps_{i+1} + \varepsilon_{i+1}$ and $s_k \geq 1$ or if $k = 0$, when $I = (\varepsilon)$. Define $\ell(I) = k$, $d(I) = \sum \varepsilon_j + \sum 2s_j(p-1)$, and, if $I = (\varepsilon, s, J)$, $e(I) = 2s + \varepsilon - d(J)$. Define $P_t^I = \beta^{\varepsilon_1} P^{s_1} \ldots \beta^{\varepsilon_k} P^{s_k} \beta_t^{\varepsilon_{k+1}}$, where $\beta_t^o = 1$ for all t, $\beta_t^1 = \beta_t$ for $t < \infty$, and $\beta_\infty^1 = 0$; thus, if $t = \infty$, we agree that admissibility requires $\varepsilon_{k+1} = 0$.

We now give a quick calculation of $H^*(Z_{p^t}, n, Z_p)$.

<u>Lemma 10. 2.</u> $H^*(Z, 1, Z_p) = E(i_1)$ and $H^*(Z, 2, Z_p) = P(i_2)$. If $t < \infty$, then $H^*(Z_{2^t}, 1, Z_2) = P(i_1)$, with $\beta_t(i_1) = i_1^2$, and $H^*(Z_{p^t}, 1, Z_p) = E(i_1) \otimes P(\beta_t(i_1))$ if $p > 2$.

<u>Proof.</u> $K(Z, 1) = S^1$ and $K(Z, 2) = CP^\infty$, so the first statement is obvious. For the second statement, $H^*(Z_{p^t}, 1, Z_p) = H^*(Z_{p^t}; Z_p)$, and we can define a ΛZ_{p^t}-free resolution of Λ, with coproduct, precisely as in Definition 1.2 (with p there replaced by p^t) for any commutative ring Λ. The result follows by an easy computation.

<u>Theorem 10. 3.</u> If $n \geq 2$ (or if $n = 1$ and either $p > 2$ or $t < \infty$), then $H^*(Z_{p^t}, n, Z_p)$ is the free commutative algebra generated by the following set: $\{P_t^I i_n \mid$ I is admissible and $e(I) < n$ or, if $p > 2$, $e(I) = n$ and ${}_1 = 1\}$. Moreover, $H^*(Z_{p^t}, n, Z_p)$ is a primitively generated Hopf algebra.

<u>Proof.</u> The lemma gives the result for $t < \infty$ and $n = 1$ and for $t = \infty$ and $n = 2$. Assume the result for n-1. Of course, the Serre spectral sequence $\{E_r\}$ of $K(Z_{p^t}, n-1) \longrightarrow E \longrightarrow K(Z_{p^t}, n)$, E acyclic, satisfies

$$E_2 = H^*(Z_{p^t}, n, Z_p) \otimes H^*(Z_{p^t}, n-1, Z_p) \text{ and } E_\infty = Z_p .$$

First, let $p = 2$; then, regarding squares as Steenrod operations, we see that we may rewrite the polynomial algebra $H^*(Z_{2^t}, n-1, Z_2)$, additively, as the exterior algebra $E(S)$, where

$$S = \{ P_t^I i_{n-1} \mid I \text{ is admissible and } e(I) < n \}.$$

By Theorem 3.4, $P_t^I i_{n-1}$ transgresses to $P_t^I i_n$. Define an abstract spectral sequence of differential algebras, $\{E'_r\}$, by letting $E'_2 = P(\tau S) \otimes E(S)$, where $\tau(S)$ is a copy of S with degrees augmented by one, and by requiring $s \in S$ to transgress to $\tau s \in \tau(S)$. Clearly $E'_\infty = Z_2$. Define a morphism of spectral sequences $f_r: E'_r \longrightarrow E_r$ by $f_2 = g \otimes 1$, where $g: P(\tau S) \longrightarrow H^*(Z_{2^t}, n, Z_2)$ is the morphism of algebras defined on generators by $g(\tau P_t^I i_{n-1}) = P_t^I i_n$; clearly commutation with the differentials determines f_r for $r > 2$. Since f_2^{o*} and f_∞ are isomorphisms, $f_2^{*o} = g$ is an isomorphism by the comparison theorem [14, p. 355].

Now let $p > 2$. We may rewrite $H^*(Z_{p^t}, n, Z_p)$, additively, as $E(S) \otimes Q(T)$, where Q denotes a truncated polynomial algebra ($x^p = 0$ for $x \in T$) and where

$$S = \{P_t^I i_{n-1} \mid I \text{ is admissible, } e(I) < n-1, \ d(I) + n \text{ even}\},$$

$$T = \{P_t^I i_{n-1} \mid I \text{ is admissible, } e(I) < n, \ d(I)+n \text{ odd}\}.$$

(Note that $e(I) \equiv d(I) \bmod 2$, hence $d(I)+n$ even and $e(I) = n-1$ is impossible.)

By Theorem 3.4, $P_t^I i_{n-1}$ transgresses to $(-1)^{d(I)} P_t^I i_n$ and, if $d(I) + n = 2q+1$, $P_t^I i_n \otimes (P_t^I i_{n-1})^{p-1}$ transgresses to $(-1)^n \beta P^q P_t^I i_n$. Define an abstract spectral sequence of differential algebras, $\{E'_r\}$, as follows. Let

$$E'_2 = [P(\tau S) \otimes E(\tau) \otimes P(\mu T)] \otimes [E(S) \otimes Q(T)]$$

(the bracketed expressions are the base and fibre, respectively). Here τS and τT are copies of S and T with degrees augmented by one and μT is a copy of T with degrees multiplied by p and then augmented by two. The differentials in $\{E'_r\}$ are specified by requiring $s \in S$ to transgress to $\tau s \in \tau S$, $t \in T$ to transgress to $\tau t \in \tau T$, and $\tau t \otimes t^{p-1}$ to transgress to $\mu t \in \mu T$. An easy computation demonstrates that $E'_\infty = Z_p$. Define a morphism of spectral sequences

$f_r : E'_r \longrightarrow E_r$ by $f_2 = g \otimes 1$, where $g: P(\tau S) \otimes E(\tau T) \otimes P(\mu T) \longrightarrow H^*(Z_{p^t}, n, Z_p)$ is the morphism of algebras defined on generators by $g(\tau P_t^I i_{n-1}) = (-1)^{d(I)} P_t^I i_n$ and $g(\mu P_t^I i_{n-1}) = -\beta P^q P_t^I i_n$ if $d(I) + n = 2q+1$. As in the case $p = 2$, the f_r for $r > 2$ are determined by commutation with the differentials, and g is an isomorphism by the comparison theorem. The last statement follows since, by the external Cartan formula, if X is an H-space and $x \in H^*(X)$ satisfies $\psi(x) = \sum x' \otimes x''$, then

$$\psi P^s(x) = \sum_{i+j=s} \sum P^i(x') \otimes P^j(x'') \text{ and } \psi\beta(x) = \sum(\beta(x') \otimes x'' + (-1)^{\deg x'} x' \otimes \beta(x'')).$$

Thus, since i_n and $\beta_t(i_n)$ are primitive, so are all of the $P_t^I i_n$.

We can now compute the mod p cohomology Bockstein spectral sequence $\{E_r\}$ of $K(Z_{p^t}, n)$. Recall that $\{E_r\}$ is a spectral sequence of differential algebras such that $E_1 = H^*(p^t, n, Z_p)$ and E_{r+1} is the homology of E_r with respect to β_r for $r \geq 1$. Since $H^*(p^t, n, Z)$ is a direct sum of cyclic groups with one generator of order p^r for each basis element of $\text{Im}(\beta_r) \subset E_r$ and one generator of infinite order for each basis element of E_∞, the integral cohomology of $K(Z_{p^t}, n)$ is completely determined, additively, by $\{E_r\}$. If $t < \infty$ and $n = 1$, Lemma 10, 2 implies that $E_1 = E_t$ and $E_{t+1} = E_\infty = Z_p$, hence we need only consider the case $n \geq 2$.

<u>Theorem 10.4.</u> Let $n \geq 2$. Define a subset S of the set of generators for $E_1 = H^*(Z_{p^t}, n, Z_p)$ given in Theorem 10.3 by

(a) If $p = 2$, $S = \{P_t^I i_n \mid s_1$ and $d(I) + n$ are even and $\ell(I) > 0\}$.

(b) If $p > 2$, $S = \{P_t^I i_n \mid \varepsilon_1 = 0$, $d(I) + n$ is even, and $\ell(I) > 0\}$.

For $y \in S$, define $z(y) = \beta(y)y + P^{2q}\beta(y)$ if $p = 2$ and degree $(y) = 2q$ and define $z(y) = \beta(y)y^{p-1}$ if $p > 2$. Define an algebra $A_r(n, t)$ by

(c) $A_r(2n, \infty) = P\{i_n\}$ and $A_r(2n+1, \infty) = E\{i_n\}$.

(d) $A_r(2n, t) = P\{i_{2n}\} \otimes E\{\beta_t(2n)\}$ if $r \leq t < \infty$ and

$$A_r(2n,t) = P\{i_{2n}^{p^{r-t}}\} \otimes E\{z(i_{2n})i_{2n}^{p^{r-t}-p}\} \text{ if } r > t,$$

where $z(i_{2n}) = \beta(i_{2n})i_{2n} + P^{2n}\beta(i_{2n})$ if $p = 2$ and $t = 1$

and $z(i_{2n}) = \beta_t(i_{2n})i_{2n}^{p-1}$ if either $p > 2$ or $t > 1$

(e) $A_r(2n+1,t) = E\{i_{2n+1}\} \otimes P\{\beta_t(i_{2n+1})\}$ if $r \leq t < \infty$ and

$A_r(2n+1,t) = Z_p$ if $r > t$.

Then, if $r \geq 1$, $E_{r+1} = P\{y^{p^r} \mid y \in S\} \otimes E\{z(y)y^{p^r - p} \mid y \in S\} \otimes A_{r+1}(n,t)$, $\beta_{r+1}(y^{p^r}) = z(y)y^{p^r - p}$ for $y \in S$, and $\beta_{r+t}(i_{2n}^{p^r}) = z(i_{2n})i_{2n}^{p^r - p}$.

Proof. We first compute E_2 separately in the cases $p = 2$ and $p > 2$. Let $p = 2$ and define subsets T and U of the set of generators of E_1 by

$T = \{P_t^I i_n \mid s_1$ is even, $d(I)+n$ is odd, $e(I) < n-1$, and $\ell(I) > 0\}$

$U = \{P_t^I i_n \mid d(I)+n$ is odd, $e(I) = n-1$, and $\ell(I) > 0\}$.

Recall that $\beta P^{s-1} = sP^s$ and observe that if $P_t^I i_n \in U$, then $I = (2q, J)$, where $d(J) + n = 2q+1$, and $\beta P_t^I i_n = (P_t^J i_n)^2$. Let C be the (additive) subcomplex of E_1 which is the tensor product of the following collections of subcomplexes:

(i) $P\{\beta(y)\} \otimes E\{y\}$ for $y \in T$, and

(ii) $P\{z^2\} \otimes E\{y\}$ for $y = P^{2q}z \in U$, $\deg(z) = 2q+1$.

Let IC be the positive degree elements of C. Then $H(IC) = 0$ under β, and therefore E_2 is isomorphic to $H(E_1/IC)$. If $y \in T \cup U$, then $P^{2q}y \in U$, $\deg y = 2q+1$, and therefore C is actually a subalgebra of E_1 and E_1/IC is a quotient differential algebra of E_1. It is easy to see that

$$E_1/IC = P\{y \mid y \in S\} \otimes E\{\beta(y) \mid y \in S\} \otimes A_1'(n,t),$$

where $A_1'(n,t)$ is the quotient of the polynomial algebra generated by i_n and, if $t < \infty$, $\beta_t(i_n)$ by the ideal generated by i_n^2 if n is odd or by $\beta_t(i_n)^2$ if n is even. Therefore

$$H(E_1/IC) = P\{y^2 \mid y \in S\} \otimes E\{\beta(y)y \mid y \in S\} \otimes A_2'(n,t),$$

where $A_2'(n,t) = A_2(n,t)$ unless n is even and $t = 1$, when $A_2'(n,t) = P\{i_n^2\} \otimes E\{\beta(i_n)i_n\}$. For $y \in S$ or $y = i_n$ if n is even and $t = 1$, $z(y) = \beta(y)y + P^{2q}\beta(y)$, $\deg y = 2q$, is a cycle in E_1 which projects to the cycle $\beta(y)y$ in E_1/IC. Since $z(y)^2$ bounds in E_1, it follows that E_2 has the stated form if $p = 2$. Next, let $p > 2$ and define a subset T of the set of generators of E_1 by

$$T = \{P_t^I i_n \mid \varepsilon_1 = 0,\ d(I) + n \text{ is odd, and } \ell(I) > 0\}.$$

Then, as a differential algebra, E_1 breaks up into the tensor product of the following collection of subalgebras:

(iii) $P\{y\} \otimes E\{\beta(y)\}$ for $y \in S$

(iv) $E\{y\} \otimes P\{\beta(y)\}$ for $y \in T$

(v) The free commutative algebra generated by i_n and, if $t < \infty$, $\beta_t(i_n)$.

The algebras in (iii) have homology $P\{y^p\} \otimes E\{z(y)\}$, those of (iv) are acyclic, and that of (v) has homology $A_2(n,t)$, hence E_2 is as stated. Now assume that E_{r+1} is as stated, $r \geq 1$ and any p. Then Proposition 6.8 computes β_{r+1} and E_{r+1} breaks up into the tensor product of $A_{r+1}(n,t)$ with subalgebras of the form $P\{x\} \otimes E\{\beta_{r+1}(x)\}$, where $x = y^{p^r}$, $y \in S$. This proves the result.

Finally, we prove the axiomatization of the P^s on topological spaces. Recall first that the Cartan formula and $P^o = 1$ imply that the P^s commute with suspension [28, 30] and that we have shown in Proposition 8.1 that $P^o = 1$ is implied by the commutation of P^o with S^* and the fact that P^o is the p-th power on a zero dimensional class. Thus the axioms we choose (for convenience of proof) are in fact redundant.

Theorem 10.5. There exists a unique family $\{P^s \mid s \geq 0\}$ of natural homomorphisms $H^*(X; Z_p) \to H^*(X; Z_p)$ such that $\deg(P^s) = s$ if $p = 2$, $\deg(P^s) = 2s(p-1)$ if $p > 2$, and

(i) P^o is the identity homomorphism

(ii) $P^s(x) = x^p$ if $p = 2$ and $s = \deg x$ or $p > 2$ and $2s = \deg x$

(iii) $P^s(x) = 0$ if $p = 2$ and $s > \deg x$ or $p > 2$ and $2s > \deg x$

(iv) $P^s(x \otimes y) = \sum_{i+j=s} P^i(x) \otimes P^j(y)$ for $x \otimes y \in H^*(X \times Y)$

(v) $\sigma^* P^s = P^s \sigma^*$, where σ^* is the **suspension** of a fibration.

Proof. Suppose given $\{R^s \mid s \geq 0\}$ which also satisfy the axioms. If $x \in H^n(X, Z_p)$, then $x = f^*(i_n)$ for some $f: X \to K(Z_p, n)$, hence it suffices to prove that $P^s(i_n) = R^s(i_n)$. The result is obvious from (i), (ii), and (iii) if $n = 1$ or if $p > 2$ and $n = 2$. Assume that $P^s(i_{n-1}) = R^s(i_{n-1})$ for all s and consider $y = P^s(i_n) - R^s(i_n)$, $0 < s < n$ if $p = 2$ and $0 < 2s < n$ if $p > 2$. By (v), $\sigma^*(y) = 0$, where

$$\sigma^*: H^*(Z_p, n, Z_p) \to H^*(Z_p, n-1, Z_p).$$

If $p = 2$, σ^* is an isomorphism in degrees less than $2n$ and therefore $y = 0$. Let $p > 2$. As shown in the proof of Theorem 10.3, (i) and (iv) imply that both $P^s(i_n)$ and $R^s(i_n)$ are primitive. By Theorem 10.3, we see that

$$\{P^I i_n \mid I \text{ admissible, } e(I) \leq n\}$$

is a basis for the primitive elements of $H^*(Z_p, n, Z_p)$. The only elements of this set which are in $\operatorname{Ker} \sigma^*$ are p-th powers and elements of the form $\beta P^q(y)$, $\deg y = 2q+1$, which have degree $2pq + 2$. If n is odd, then y has odd degree and is thus zero. If n is even, then all primitive elements in $\operatorname{Ker} \sigma^*$ have degree at least pn, which is greater than the degree of y, and again $y = 0$.

11. Cocommutative Hopf algebras

In this section, we consider the following category $\mathcal{C}$. The objects of $\mathcal{C}$ are triples $C = (E, A, F)$ where A is a (Z-graded) cocommutative Hopf algebra over a commutative ring Λ, E is a right and F is a left (Z-graded) cocommutative A-coalgebra. Thus E and F are A-modules and cocommutative coalgebras (not necessarily unital or augmented), and their coproducts ψ are morphisms of A-modules. We say that C is unital if E and F are unital and augmented and their units and augmentations are morphisms of A-modules. A morphism $\gamma: C \longrightarrow C'$ in $\mathcal{C}$ is a triple $\gamma = (\alpha, \lambda, \beta)$, where $\lambda: A \longrightarrow A'$ is a morphism of Hopf algebras and $\alpha: E \longrightarrow E'$ and $\beta: F \longrightarrow F'$ are λ-equivariant morphisms of coalgebras; thus $\alpha(ea) = \alpha(e)\lambda(a)$ and $\beta(af) = \lambda(a)\beta(f)$ for $e \in E$, $a \in A$, and $f \in F$. We say that γ is unital if α and β are morphisms of unital augmented coalgebras. For C and C' in $\mathcal{C}$, define $C \otimes C' = (E \otimes E', A \otimes A', F \otimes F') \in \mathcal{C}$ and observe that

$$\psi = (\psi, \psi, \psi): C = (E, A, F) \longrightarrow (E \otimes E,\ A \otimes A,\ F \otimes F) = C \otimes C$$

is a morphism in $\mathcal{C}$; clearly ψ is unital if C is unital. Define homology and cohomology functors on the category $\mathcal{C}$ by

(1) $\quad H_{st}(C) = \mathrm{Tor}_{st}^{(A,\Lambda)}(E, F) \quad$ and $\quad H^{st}(C) = \mathrm{Ext}_{(A,\Lambda)}^{st}(E, F^*)$.

We shall define and study Steenrod operations on $H^*(C)$ when $\Lambda = Z_p$. The results here generalize work of Liulevicius [13].

In the following definitions, we recall the description of $H^*(C)$, with its product, in terms of the bar construction.

Definitions 11.1. For $C = (E, A, F) \in \mathcal{C}$, let $\overline{C} = (A, A, F) \in \mathcal{C}$. Let JA be the cokernel of the unit $\Lambda \longrightarrow A$. Define the bar construction $B(C)$ as follows. $B(C) = E \otimes T(JA) \otimes F$ as a Λ-module, where $T(JA)$ is the tensor algebra on JA. Write elements of $B(C)$ in the form $e[a_1|\dots|a_s]f$; such an element has homological degree s, internal degree $t = \deg e + \sum \deg a_i + \deg f$, and total degree s+t. Define $\varepsilon: B(C) \longrightarrow E \otimes_A F$ and $d: B_{s,*}(C) \longrightarrow B_{s-1,*}(C)$ by

(2) $\varepsilon(e[\ ]f) = e \otimes f, \quad \varepsilon(e[a_1|\dots|a_s]f) = 0,$ and

(3)
$$d(e[a_1|\dots|a_s]f) = -\bar{e}a_1[a_2|\dots|a_s]f - \sum_{i=1}^{s-1} \bar{e}[\bar{a}_1|\dots|\bar{a}_{i-1}|\bar{a}_i a_{i+1}|a_{i+2}|\dots|a_s]f - \bar{e}[\bar{a}_1|\dots|\bar{a}_{s-1}]a_s f \ , \quad \text{where } \bar{x} = (-1)^{1+\deg x}x \ .$$

If $E = A$, then d is a morphism of left A-modules and $dS + Sd = 1 - \sigma\varepsilon$, where $\sigma: F \longrightarrow B(\overline{C})$ and $S: B_{s,*}(\overline{C}) \longrightarrow B_{s+1,*}(\overline{C})$ are defined by the formulas

(4) $\sigma(f) = [\ ]f$ and $S(a[a_1|\dots|a_s]f) = [a|a_1|\dots|a_s]f$.

Clearly $d = 1 \otimes_A d$ on $B(C) = E \otimes_A B(\overline{C})$. By adjoint associativity,

$$\mathrm{Hom}_A(B(\overline{C}), E^*) \cong B^*(C) \cong \mathrm{Hom}_A(B(E,A,A), F^*).$$

Therefore (1) admits the equivalent reformulation

(5) $H_*(C) = H(B(C))$ and $H^*(C) = H(B^*(C)) = \mathrm{Ext}_{(A,\Lambda)}(F, E^*)$.

Definitions 11.2. Let C and C' be objects of $\mathcal{C}$. Define the Alexander-Whitney map $\xi: B(C \otimes C') \longrightarrow B(C) \otimes B(C')$ and the shuffle map $\eta: B(C) \otimes B(C') \longrightarrow B(C \otimes C')$ by the formulas

(6)
$$\xi(e \otimes e'[a_1 \otimes a_1'|\dots|a_s \otimes a_s']f \otimes f') = \sum_{k=0}^{s} (-1)^{\mu(k)} e[a_1|\dots|a_k]a_{k+1}\dots a_s f \otimes e'a_1'\dots a_k'[a_{k+1}'|\dots|a_s']f',$$

where $\mu(k) = \deg e'(k + \deg a_1 \dots a_s f) + \sum_{i=1}^{k} \deg a_i' \,(k - i + \deg a_{i+1} \dots a_s f)$
$$+ \sum_{j=k+1}^{s} (1 + \deg a_j') \deg a_{j+1} \dots a_s f \quad , \quad \text{and}$$

(7)
$$\eta(e[a_1|\dots|a_s]f \otimes e'[a_{s+1}|\dots|a_{s+t}]f') = \sum_{\pi} (-1)^{\nu(\pi)} e \otimes e'[a_{\pi(1)}|\dots|a_{\pi(s+t)}]f \otimes f' \ ,$$

where $a_i \in A$ if $i \leq s$, $a_i \in A'$ if $i > s$, the sum is taken over all (s,t)-shuffles π (see [15, p. 17]), and

$$\nu(\pi) = \sum_{\pi(i) > \pi(s+j)} (1 + \deg a_i)(1 + \deg a_{s+j}).$$

The unnormalized bar construction $E \otimes T(A) \otimes F$ admits a structure of simplicial graded Λ-module under which ξ and η are in fact the classical normalized Alexander-Whitney and shuffle maps. Define $D = \xi B(\psi): B(C) \longrightarrow B(C) \otimes B(C)$. Then D gives $B(C)$ a structure of coassociative coalgebra; if C is unital, then $B(C)$ is unital and augmented. If $E = A$, then D coincides with the morphism of left A-modules defined inductively by

(8) $\qquad D([\ ]f) = \sum [\ ]f' \otimes [\ ]f''$ if $\psi(f) = \sum f' \otimes f''$, and

(9) $\qquad DS = SD$, where $S = S \otimes 1 + \sigma\varepsilon \otimes S$ on $B(\overline{C}) \otimes B(\overline{C})$.

Clearly D on $B(C) = E \otimes_A B(\overline{C})$ is the composite

$$E \otimes_A B(\overline{C}) \xrightarrow{\psi \otimes D} E \otimes E \otimes_A B(\overline{C}) \otimes B(\overline{C}) \xrightarrow{1 \otimes T \otimes 1} E \otimes_A B(\overline{C}) \otimes E \otimes_A B(\overline{C}).$$

We define the cup product on $B^*(C)$ to be the composite

(10) $\qquad \cup : B^*(C) \otimes B^*(C) \xrightarrow{\alpha} [B(C) \otimes B(C)]^* \xrightarrow{D^*} B^*(C).$

We have the following analog of Lemma 7.1; a more precise analog (in terms of ξ) could also be proven, and an alternative proof by semi-simplicial rather than homological techniques is available.

Lemma 11.3. Let π be a subgroup of Σ_r and let W be a $\Lambda\pi$-free resolution of Λ such that $W_o = \Lambda\pi$ with $\Lambda\pi$ generator e_o. Let $C \in \mathfrak{C}$. Bigrade $W \otimes B(C)$ by $[W \otimes B(C)]_{st} = \sum_{i+j=s} W_i \otimes B_{jt}(C)$. Then there exists a morphism of bigraded $\Lambda\pi$-complexes $\Delta: W \otimes B(C) \longrightarrow B(C)^r$ which is natural in C and satisfies the following properties:

(i) $\quad \Delta(w \otimes b) = 0$ if $b \in B_{o,*}(C)$ and $w \in W_i$ for $i > 0$

(ii) $\quad \Delta(e_o \otimes b) = D(b)$ if $b \in B(C)$, where D is the iterated coproduct

(iii) $\quad$ If $E = A$, then Δ is a morphism of left A-modules, where A operates on $W \otimes B(C)$ by $a(w \otimes b) = (-1)^{\deg w \deg a}\, w \otimes ab$.

(iv) $\Delta(W_i \otimes B_{st}(C)) = 0$ if $i > (r-1)s$.

Moreover, any two such Δ are naturally $\Lambda\pi$-homotopic.

Proof. Observe first that the cocommutativity of A ensures that (iii) is compatible with the π-equivariance of Δ. Observe next that it suffices to prove the result when $E = A$, since we can then define Δ on $W \otimes B(C) = E \otimes_A W \otimes B(\overline{C})$ to be the composite

$$E \otimes_A W \otimes B(\overline{C}) \xrightarrow{\psi \otimes \Delta} E^r \otimes_A B(\overline{C})^r \xrightarrow{U} [E \otimes_A B(\overline{C})]^r$$

where U is the evident shuffle. We define Δ on $W_i \otimes B_{st}(\overline{C})$ by induction on i and for fixed i by induction on s. Formula (i) defines Δ for $s = 0$ and all $i > 0$ and formula (ii) and π-equivariance defines Δ for $i = 0$ and all s. Let $i \geq 1$ and $s \geq 1$ and assume that Δ has been defined for $i' < i$ and for our given i and $s' < s$. Let $\{w_k\}$ be a $\Lambda\pi$-basis for W_i. By (iii) and π-equivariance, it suffices to define $\Delta(w \otimes S(y))$ for $w \in \{w_k\}$ and $y \in B_{s-1,*}(\overline{C})$. Let $S = \sum_{i=1}^{r} (\sigma\varepsilon)^i \otimes S \otimes 1^{r-i-1}$ on $B(\overline{C})^r$. Then $dS + Sd = 1 - (\sigma\varepsilon)^r$. We define

(v) $\qquad \Delta(w \otimes S(y)) = (-1)^{\deg w} S\Delta(w \otimes y) + S\Delta(d(w) \otimes S(y))$.

Observe that (v) is equivalent to (ii) on $w = e_o$ and that (v) is well-defined by the induction hypothesis. To verify that $d\Delta = \Delta d$, write (v) in the form $\Delta(1 \otimes S) = S\Delta(1 \otimes 1 + d \otimes S)$. Then:

$$\begin{aligned}
d\Delta(1 \otimes S) &= dS\Delta(1 \otimes 1 + d \otimes S) = (1 - Sd)\Delta(1 \otimes 1 + d \otimes S)\\
&= [\Delta - S\Delta(d \otimes 1 + 1 \otimes d)](1 \otimes 1 + d \otimes S)\\
&= \Delta + \Delta(d \otimes S) - S\Delta(d \otimes 1) - S\Delta(1 \otimes d) + S\Delta(d \otimes dS)\\
&= \Delta + \Delta(d \otimes S) - S\Delta(d \otimes 1) - S\Delta(1 \otimes d) + S\Delta(d \otimes 1) - S\Delta(d \otimes Sd)\\
&= \Delta + \Delta(d \otimes S) - \Delta(1 \otimes dS) = \Delta + \Delta(d \otimes S) - \Delta + \Delta(1 \otimes dS)\\
&= \Delta(d \otimes 1 + 1 \otimes d)(1 \otimes S)
\end{aligned}$$

(where no terms involving $\sigma\varepsilon$ are relevant by (i) and an easy verification). Thus (i), (ii), (iii), and (v), together with π-equivariance, provide an explicit construction of a natural morphism Δ of $\Lambda\pi$-complexes. To see that (iv) holds, observe that if $w \in \{w_k\} \subset W_i$ and $y = a_1[a_2|\dots|a_s]f$, then $\Delta(w \otimes S(y))$ is a linear combination in $B(\overline{C})^r$ of terms involving precisely the factors $a_i^{(j)}$ and $f^{(j)}$ in the $B(\overline{C})$, where $\psi(a_i) = \sum a_i^{(1)} \otimes \dots \otimes a_i^{(r)}$ and $\psi(f) = \sum f^{(1)} \otimes \dots \otimes f^{(r)}$ give the iterated coproducts. Thus no summand of $\Delta(w \otimes S(y))$ can have homological degree greater than rs. Since $\Delta(w \otimes S(y))$ has homological degree $i+s > rs$ if $i > (r-1)s$, this proves (iv). The uniqueness of Δ up to $\Lambda\pi$-homotopy follows easily by use of the contracting homotopy S on $B(\overline{C})^r$.

We now pass to the category $\mathcal{P}(\pi, \infty, \Lambda)$ of Defitions 2.1.

<u>Definition 11.4.</u> Let $C \in \mathcal{C}$. Let $\alpha: B^*(C)^r \longrightarrow [B(C)^r]^*$ be the natural map and define a $\Lambda\pi$-morphism $\theta: W \otimes B^*(C) \longrightarrow B^*(C)^r$ by the formula

(11) $\theta(w \otimes x)(k) = (-1)^{\deg w \deg x}\alpha(x)\Delta(w \otimes k), \quad w \in W, \quad x \in B^*(C)^r, \ k \in B(C).$

Since θ may be defined for $\pi = \Sigma_r$ and then factored through $j: W \longrightarrow V$ as in Definition 2.1, and the resulting composite is naturally $\Lambda\pi$-homotopic to the original θ defined in terms of W, θ satisfies condition (ii) of Definition 2.1. By the lemma, formula (11) specializes to give

(12) $\theta(e_0 \otimes x) = D^*\alpha(x)$ for any $x \in B^*(C)^r$ and

(13) $\theta(w \otimes x) = \varepsilon(w)D^*\alpha(x)$ for any $x \in B^{0,*}(C)^r$ and $w \in W$.

By (10) and (12), θ satisfies condition (i) of Definition 2.1. Since θ is natural on morphisms in $\mathcal{C}$, we thus obtain a functor $\Gamma: \mathcal{C} \longrightarrow \mathcal{P}(\pi, \infty, \Lambda)$ by setting $\Gamma(C) = (B^*(C), \theta)$ on objects and $\Gamma(\gamma) = B^*(\gamma)$ on morphisms. By (13), if C is unital in $\mathcal{C}$, then $\Gamma(C)$ is unital in $\mathcal{P}(\pi, \infty, \Lambda)$. If $\Lambda = Z_p$, π is cyclic of order p, and $C = \tilde{C} \otimes Z_p$ where $\tilde{C}$ is Z-free (that is, $\tilde{E}, \tilde{A}$, and $\tilde{F}$ are Z-free), then we agree to choose θ for C to be the mod p reduction of θ for C; $\Gamma(C)$ will thus be reduced mod p. Note that if $x \in B^*(C)$ has bidegree (s,t), then $\theta(w \otimes x)$ has bidegree $(s - \deg w, t)$. $W \otimes B^*(C)^r$ and $B^*(C)$ should be thought of as regarded by total degree in defining the functor Γ.

Observe that, by Definition 6.1, we now have $\cup_i$-products in $B^*(C)$ for any $C \in \mathcal{C}$. When $\Lambda = Z_p$, the results of Proposition 2.3 will clearly apply to the Steenrod operations in $H^*(C)$, and the following lemmas will imply the applicability of the external Cartan formula and the Adem relations.

Lemma 11.5. For any objects C and C' in $\mathcal{C}$, the following diagram is $\Lambda\pi$-homotopy commutative.

$$\begin{array}{ccc} W \otimes B^*(C \otimes C')^r & \xrightarrow{\theta} & B^*(C \otimes C') \\ {\scriptstyle 1\otimes(\xi^*)^r}\uparrow \downarrow{\scriptstyle 1\otimes(\eta^*)^r} & & {\scriptstyle \xi^*}\uparrow\downarrow{\scriptstyle \eta^*} \\ W \otimes [B^*(C) \otimes B^*(C')]^r & \xrightarrow{\tilde{\theta}} & B^*(C) \otimes B^*(C') \end{array}$$

Proof. It suffices to prove the $\Lambda\pi$-homotopy commutativity of the diagram

$$\begin{array}{ccc} W \otimes B(C) \otimes B(C') & \xrightarrow{U(\Delta\otimes\Delta)(1\otimes T\otimes 1)(\psi\otimes 1\otimes 1)} & [B(C) \otimes B(C')]^r \\ {\scriptstyle 1\otimes\xi}\uparrow \downarrow{\scriptstyle 1\otimes\eta} & & {\scriptstyle \xi^r}\uparrow\downarrow{\scriptstyle \eta^r} \\ W \otimes B(C \otimes C') & \xrightarrow{\Delta} & B(C \otimes C')^r \end{array}$$

and this diagram need only be studied with C and C' replaced by $\overline{C}$ and $\overline{C}'$. Since $B(\overline{C} \otimes \overline{C}')^r$ and $[B(\overline{C}) \otimes B(\overline{C}')]^r$ have obvious contracting homotopies, the result follows by an easy double induction like that in the proof of Lemma 11.3.

Corollary 11.6. If $C \in \mathcal{C}$, then $\Gamma(C)$ is a Cartan object of $\mathcal{C}(\pi, \infty, \Lambda)$

Lemma 11.7. If $C \in \mathcal{C}$, $\Lambda = Z_p$, then $\Gamma(C)$ is an Adem object of $\mathcal{C}(p, \infty)$.

Proof. Precisely as in the proof of Lemma 7.8, it suffices to prove the τ-homotopy commutativity of the following diagram:

$$\begin{array}{ccccc}
W_1 \otimes W_2^{\,p} \otimes B(C) & \xrightarrow{w \otimes 1} & Y \otimes B(C) & \xrightarrow{\quad \Delta \quad} & B(C)^{p^2} \\
\big\downarrow {\scriptstyle T \otimes 1} & & & & \big\uparrow {\scriptstyle \Delta^p} \\
W_2^{\,p} \otimes W_1 \otimes B(C) & \xrightarrow{1 \otimes \Delta} & W_2^{\,p} \otimes B(C)^p & \xrightarrow{\quad U \quad} & (W_2 \otimes B(C))^p
\end{array}$$

We need only consider this diagram with C replaced by $\overline{C}$, and, since $B(\overline{C})^{p^2}$ has a contracting homotopy, the result then holds by another easy double induction.

The following theorem summarizes the properties of the P^s and βP^s on $H^*(C)$ for $C \in \mathcal{C}$, $\Lambda = Z_p$. We shall be very precise as to grading since there is considerable confusion on this point in the literature. We are thinking of $H^*(C)$ as regraded by total degree in applying our general theory. An alternative formulation that is sometimes convenient will be given after the theorem.

Theorem 11.8. Let $C \in \mathcal{C}$, $\Lambda = Z_p$. Then there exist natural homomorphisms P^i and, if $p > 2$, βP^i defined on $H^*(C)$, with

(a) $P^i : H^{s,t}(C) \longrightarrow H^{s+i-t,\,2t}(C)$ if $p = 2$;

(b) $P^i : H^{st}(C) \longrightarrow H^{s+(2i-t)(p-1),\,pt}(C)$ and

$\beta P^i : H^{st}(C) \longrightarrow H^{s+1+(2i-t)(p-1),\,pt}(C)$ if $p > 2$.

These operations satisfy the following properties:

(i) $P^i = 0$ if $p = 2$ and either $i < t$ or $i > s+t$

$P^i = 0$ if $p > 2$ and either $2i < t$ or $2i > s+t$

$\beta P^i = 0$ if $p > 2$ and either $2i < t$ or $2i \geq s+t$

(ii) $P^i(x) = x^p$ if $p = 2$ and $i = s+t$ or if $p > 2$ and $2i = s+t$

(iii) If $C = \tilde{C} \otimes Z_p$, where $\tilde{C}$ is Z-free, then $\beta P^{i-1} = iP^i$ if $p = 2$ and βP^i is the composition of β and P^i if $p > 2$.

(iv) $P^j = \sum P^i \otimes P^{j-i}$ and $\beta P^j = \sum (\beta P^i \otimes P^{j-i} + P^i \otimes \beta P^{j-i})$ or $H^*(C \otimes C')$; the internal Cartan formula is satisfied in $H^*(C)$.

(v) If $\gamma: C' \longrightarrow C$ and $\phi: C \longrightarrow C''$ are unital morphisms in $\mathcal{C}$ such that $\phi\gamma = 0$ on the cokernels of the units, then $\sigma P^i = P^i\sigma$ and $\sigma\beta P^i = -\beta P^i\sigma$, where $\sigma: H^{st}(C'') \longrightarrow H^{s-1,t}(C')$ is the suspension.

(vi) The P^i and βP^i satisfy the Adem relations as stated in Corollary 5.1.

Proof. If $x \in H^{st}(C)$, then $D_i(x) = \theta_*(e_i \otimes x^p) \in H^{ps-i,pt}(C)$. The P^i and βP^i are defined by formulas (5.1) and (5.2), with x having its total degree $q = s+t$; thus (a) and (b) are valid. The vanishing of $P^i(x)$ for $i < t$ if $p = 2$ and of $\beta^\varepsilon P^i(x)$ for $2i < t$ if $p > 2$ follows from part (iv) of Lemma 11.3. The remainder of the theorem follows immediately from our general theory and the previous lemmas. For (v), note that the composite $B^*(C'') \xrightarrow{B^*(\phi)} B^*(C) \xrightarrow{B^*(\gamma)} B^*(C')$ is zero on the kernel of the augmentation $B^*(C'') \longrightarrow Z_p$. An alternative formulation of (v) in the non-unital case can easily be obtained.

In addition to (v), the Kudo transgression theorem, Theorem 3.4, applies to appropriate spectral sequences involving objects of $\mathcal{C}$. The hypothesis of (iii) is seldom satisfied in practice, and βP^i is generally an independent operation having nothing to do with any Bockstein. There is an alternative definition of the operations, which amounts to the following regrading of our operations . Define

(c) $\tilde{P}^i = Sq^i = P^{i+t}: H^{st}(C) \longrightarrow H^{s+i,2t}(C)$ if $p = 2$;

(d) $\tilde{P}^i = P^{i+t}: H^{s,2t}(C) \longrightarrow H^{s+2i(p-1),2pt}(C)$ and
$\beta\tilde{P}^i = \beta P^{i+t}: H^{s,2t}(C) \longrightarrow H^{s+2i(p-1)+1,2pt}(C)$ if $p > 2$.

This regrading is reasonable if $p = 2$, but has the effect of eliminating all operations on $H^{st}(C)$ for t odd if $p > 2$; of course, these operations are non-trivial since, if s and t are odd, the p-th power operation on $H^{st}(C)$ is non-trivial in general. The results of the theorem can easily be transcribed for the $\tilde{P}^i$ and $\beta\tilde{P}^i$; for example, the Adem relations are still correct precisely as stated but with all P^i and βP^i replaced by $\tilde{P}^i$ and $\beta\tilde{P}^i$. The motivation for the reindexing is just the desire to make $\tilde{P}^o$ the first non-trivial operation. This operation is of particular importance in the applications, and we now evaluate it.

Definition 11.9. Let $C = (E, A, F) \in \mathcal{C}$, where E, A, and F are positively graded and of finite type. Then $B^*(C)$ may be identified with $E^* \otimes T(IA^*) \otimes F^*$. Define $\lambda : B^{st}(C) \longrightarrow B^{s,pt}(C)$ by

$$\lambda(\varepsilon[\alpha_1|\dots|\alpha_s]\phi) = \varepsilon^p[\alpha_1{}^p|\dots|\alpha_s{}^p]\phi^p .$$

Then λ commutes with the differential and induces $\lambda_* : H^{st}(C) \longrightarrow H^{s,pt}(C)$. Of course, if $p > 2$, then $\lambda(\varepsilon[\alpha_1|\dots|\alpha_s]\phi) = 0$ if ε, α_i, or ϕ has odd degree and thus $\lambda_* = 0$ if t is odd.

Proposition 11.10. Let $C = (E, A, F) \in \mathcal{C}$, where E, A, and F are positively graded and of finite type. Let $x \in H^{s,t}(C)$ where t is even if $p > 2$. Then $\tilde{P}^0(x) = \lambda_*(x)$.

Proof. Let $y = e[a_1|\dots|a_s]f \in B_{s,pt}(C)$. A straightforward, but tedious, calculation demonstrates that

$$\Delta(e_{s(p-1)} \otimes y) = \sum (-1)^{ms}\, \nu(-s)^{-1}(e'[a_1'|\dots|a_s']f') + Nz,$$

where the sum is taken over the symmetric summands $e' \otimes \dots \otimes e'$, $a_i' \otimes \dots \otimes a_i'$, and $f' \otimes \dots \otimes f'$ of the iterated coproducts. (A moment's reflection on the case $C = (Z_p, Z_pG, Z_p)$, where G is a group, and a glance at Lemma 8.2 should convince the reader of the plausibility of this statement.) The result now follows easily from the definitions.

Remarks 11.11. If $p > 2$, then it can be shown by a tedious calculation that $-\beta\tilde{P}^0(x) = \langle x \rangle^p$ for $x \in H^{1,2t}(C)$, where $C = (Z_p, A, Z_p) \in \mathcal{C}$ and $\langle x \rangle^p$ is as defined in Remarks 6.9. It is possible that $-\beta\tilde{P}^s(x) = \langle x \rangle^p$ for $x \in H^{2s+1,2t}(C)$ and any $C \in \mathcal{C}$, but this appears to be difficult to prove.

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