A General Approach for the Stability Analysis of the Time-Domain Finite-Element Method for Electromagnetic Simulations

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***Abstract*—This paper presents a general approach for the stability analysis of the time-domain finite-element method (TDFEM) for electromagnetic simulations. Derived from the discrete system analysis, the approach determines the stability by analyzing the root-locus map of a characteristic equation and evaluating the spectral radius of the finite element system matrix. The approach is applicable to the TDFEM simulation involving dispersive media and to various temporal discretization schemes such as the central difference, forward difference, backward difference, and Newmark methods. It is shown that the stability of the TDFEM is determined by the material property and by the temporal and spatial discretization schemes. The proposed approach is applied to a variety of TDFEM schemes, which include: 1) time-domain finite-element modeling of dispersive media; 2) time-domain finite element-boundary integral method; 3) higher order TDFEM; and 4) orthogonal TDFEM. Numerical results demonstrate the validity of the proposed approach for stability analysis.**

***Index Terms*—Finite-element methods (FEM), numerical analysis, numerical stability, time-domain analysis, transient analysis.**

I. Introduction

In recent years, considerable attention has been devoted to time-domain numerical methods to simulate electromagnetic transients. The best known technique is the finite-difference time-domain (FDTD) method [1], and its stability has been investigated extensively [2]–[6]. Progress has also been made in the development of time-domain finite-element methods (TDFEMs) [7]. The TDFEM approaches developed so far can be grouped into two classes. One class of approaches directly solves Maxwell's equations, and operates in a leap-frog fashion similar to the FDTD method. These approaches are conditionally stable, and their stability can be analyzed by following the same lines of thought as in the FDTD [8], [9]. Another class of TDFEM approaches tackles the second-order vector wave equation, or the curl-curl equation, obtained by eliminating one of the field variables from Maxwell's equations [10]–[18]. These solvers can be formulated to be unconditionally stable [10]–[13] or conditionally stable [14]–[18]. In the unconditionally stable schemes, the time step is not constrained by a stability criterion. However, it is limited by the accuracy requirement and also by the spectral content of temporal signatures [11], [12]. These schemes are very useful for analyzing electromagnetic problems in which the electrical size of the finite elements varies by several orders of magnitude over the computational domain. In the conditionally stable schemes, although the time step is constrained by a stability criterion, these solvers usually yield better accuracy [12]. In addition, the TDFEM schemes can be constructed such that only the mass matrix is required to be solved at each time step [15], [16], [18], [19]. This feature permits the formulation of a purely explicit scheme, which eliminates a matrix solution at each time step [18], [19]. It also reduces the complexity of constructing preconditioners for the finite element (FE) system matrix when iterative solvers are utilized to solve the matrix equation. The stability of both the conditionally and unconditionally stable schemes has been addressed in a limited number of papers [11], [12]

In this paper, a general approach based on the discrete system analysis is developed for investigating the stability behavior of the TDFEMs for electromagnetic simulations involving dispersive media. It is shown that by tracing the roots of a characteristic equation in the complex z plane and by evaluating the spectral radius of the TDFEM matrix system, the stability condition can be determined. It is also shown that the proposed stability analysis is applicable to various temporal discretization schemes such as the central difference, forward difference, backward difference, and Newmark method. This approach is further applied to a variety of recently developed TDFEMs, which include 1) time-domain finite-element modeling of dispersive media [20], 2) the time-domain finite-element boundary-integral (FE-BI) method [21], [22], 3) higher order time-domain finite-element method [22], and 4) the orthogonal time-domain finite-element method [19]. Numerical results demonstrate its validity.

II. Stability Analysis

The stability analysis is an important issue in the numerical solution of initial-value problems. It has been studied extensively in the past for a variety of engineering problems [23], [24]. Here, we consider it for the TDFEM simulation of electromagnetic problems.

Consider the vector wave equation

$$\nabla \times {\mu_r}^{-1} \nabla \times \mathbf{E}(\mathbf{r}, t) + \mu_0 \epsilon_0 \partial_t^2 [\epsilon_r(t) \star \mathbf{E}(\mathbf{r}, t)] = 0 \qquad (1)$$

where $\epsilon_r(t)$ characterizes the dispersive effects of the medium, and $\star$ stands for the convolution. Assuming vanished tangential

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fields on the surface bounding the volume of interest, and using Galerkin's method, we obtain a weak-form solution

$$\mu_0\epsilon_0\partial_t^2\langle \mathbf{N}_i, \epsilon_r(t)\star \mathbf{E}(\mathbf{r},t)\rangle + {\mu_r}^{-1}\langle\nabla\times\mathbf{N}_i, \nabla\times\mathbf{E}(\mathbf{r},t)\rangle = 0 \quad (2)$$

where $\mathbf{N}_i(\mathbf{r})$ denotes the vector basis function. Expanding the electric field as

$$\mathbf{E}(\mathbf{r},t) = \sum_{j=1}^{N} u_j(t)\mathbf{N}_j(\mathbf{r}) \quad (3)$$

with N denoting the total number of expansion functions, and substituting (3) into (2), we obtain an ordinary differential equation

$$\mathbf{T}\frac{d^2}{dt^2}[\epsilon_r(t)\star u(t)] + \mathbf{S}u = 0 \quad (4)$$

in which

$$\begin{aligned}\mathbf{T}_{ij} &= \mu_0\epsilon_0\langle\mathbf{N}_i,\mathbf{N}_j\rangle\\ \mathbf{S}_{ij} &= \mu_r^{-1}\langle\nabla\times\mathbf{N}_i,\nabla\times\mathbf{N}_j\rangle.\end{aligned} \quad (5)$$

Here, for simplicity, the medium is assumed to be homogeneous throughout the computational domain. For the inhomogeneous case, the spatial variation of the permittivity can be taken into account in matrices $\mathbf{T}$ and $\mathbf{S}$, and the following approach for the stability analysis remains valid.

Applying the central difference discretization scheme to (4) and performing the Z transform [25], we obtain

$$(z-1)^2\epsilon_r(z)\mathbf{T}\tilde{u}(z) + \Delta t^2 z\mathbf{S}\tilde{u}(z) = 0 \quad (6)$$

which can be expressed as an eigenvalue problem

$$-(z-1)^2 z^{-1}\epsilon_r(z)\tilde{u}(z) = \Delta t^2\mathbf{T}^{-1}\mathbf{S}\tilde{u}(z) \quad (7)$$

where the value of $-(z-1)^2 z^{-1}\epsilon_r(z)$ corresponds to the eigenvalue of matrix $\Delta t^2\mathbf{T}^{-1}\mathbf{S}$. Denoting this eigenvalue as λ, obviously λ satisfies the following equation

$$(z-1)^2\epsilon_r(z) + \lambda z = 0 \quad (8)$$

which is termed a characteristic equation here, since it carries the characteristic information of the stability criterion. Clearly, the lower bound of λ is zero due to the property of matrices $\mathbf{T}$ and $\mathbf{S}$. The upper bound of λ, denoted as $\lambda_{\max}$, indicates a relation between the maximum time step and the spatial discretization, which has to be satisfied to ensure stability. If $\lambda_{\max}$ can reach inifinity, the scheme is unconditionally stable as the time step is irrelevant to the spatial discretization and can be chosen arbitrarily; otherwise, the scheme is conditionally stable. To determine $\lambda_{\max}$, we can trace the roots of the characteristic equation in the complex z plane. With λ increasing from zero to infinity, the roots of the characteristic equation change correspondingly. These roots are nothing but the poles of the linear system being investigated, which can be seen clearly from (7). Hence, when the roots leave the unit circle in the complex z plane [25], the instability occurs. The value of λ at this point yields the upper bound $\lambda_{\max}$. To ensure stability, all eigenvalues of the matrix system $\Delta t^2\mathbf{T}^{-1}\mathbf{S}$ should be smaller than $\lambda_{\max}$, which indicates that the maximum eigenvalue $\Delta t^2\rho(\mathbf{T}^{-1}\mathbf{S})$, where $\rho(\cdot)$ represents the spectral radius of $(\cdot)$, should be smaller than $\lambda_{\max}$. As a consequence, we deduce the following stability criterion:

$$\Delta t \le \frac{\sqrt{\lambda_{\max}}}{\sqrt{\rho(\mathbf{T}^{-1}\mathbf{S})}}. \quad (9)$$

Clearly, the above criterion implies that the stability of TDFEMs is determined by the spatial discretization, temporal discretization, as well as the material propoerty. Take the nondispersive medium, in which $\epsilon_r(z) = 1$, as an example. It can be shown that for λ greater than 4, the roots of (8) will go beyond the unit circle. Therefore, we conclude that $\lambda_{\max} = 4$. Hence, in a nondispersive medium, the stability condition can be written as

$$\Delta t \le \frac{2}{\sqrt{\rho(\mathbf{T}^{-1}\mathbf{S})}} \quad (10)$$

which agrees with that deduced in [16], [18]. It should be noted that in a dispersive medium, $\lambda_{\max}$ is Δt dependent since $\epsilon_r(z)$ is Δt dependent. Hence, to determine the maximally allowed time step, we can start from an initial guess of Δt, find $\lambda_{\max}$, verify whether (9) is satisfied, modify Δt correspondingly, and then repeat this procedure until the optimum value of $\lambda_{\max}$ is identified. For the determination of the spectral radius of matrix $\mathbf{T}^{-1}\mathbf{S}$, any numerical techniques designed for large sparse generalized eigenvalue problems are suitable for use. The recently developed implicitly restarted Arnoldi method [26], [27] is a good choice due to its reliability and its efficiency in both CPU time and memory requirement.

Although, the stability analysis derived above is based on the central difference scheme in time, it is also applicable to the backward difference [15], forward difference, Newmark method [12], [13], three-point recurrence scheme [11] (which is a subset of the Newmark method), etc. For instance, if the backward difference is used to discretize (4), (6) will become

$$(z-1)^2\epsilon_r(z)\mathbf{T}\tilde{u}(z) + \Delta t^2 z^2\mathbf{S}\tilde{u}(z) = 0 \quad (11)$$

which yields the characteristic equation

$$(z-1)^2\epsilon_r(z) + \lambda z^2 = 0. \quad (12)$$

In free space, the roots of (12) can be easily found as

$$z = \frac{2\pm\sqrt{-4\lambda}}{2(1+\lambda)} \quad (13)$$

whose magnitude is $1/\sqrt{1+\lambda}$. These roots never go beyond the unit circle in the complex z plane because the eigenvalue λ is always nonnegative. Hence, the stability of the backward difference scheme is unconditionally guaranteed, which agrees with that deduced in [15].

If the forward difference is used to discretize (4), (12) becomes

$$(z-1)^2\epsilon_r(z) + \lambda = 0. \quad (14)$$

Evidently, in the free space and dynamic case, the roots of the above equation are always outside the unit circle. Hence, the forward difference will result in the definite instability as stated in [15]. If the Newmark method is used to discretize (4) with $\gamma = 0.5$, we obtain

$$\begin{aligned}\mathbf{T}[\epsilon_r(t) \star u(t)]^{n+1} &+ \beta \Delta t^2 \mathbf{S} u^{n+1} \\ &= 2\mathbf{T}[\epsilon_r(t) \star u(t)]^n - [(1-2\beta)\Delta t^2 \mathbf{S}]u^n \\ &\quad - \mathbf{T}[\epsilon_r(t) \star u(t)]^{n-1} - \beta \Delta t^2 \mathbf{S} u^{n-1}. \end{aligned} \tag{15}$$

After performing the Z transform on (15), we have

$$(z-1)^2 \epsilon_r(z) \mathbf{T} \tilde{u}(z) + \Delta t^2 [\beta z^2 + (1-2\beta) z + \beta] \mathbf{S} \tilde{u}(z) = 0 \tag{16}$$

which yields the characteristic equation

$$(z-1)^2 \epsilon_r(z) + \lambda \left[\beta z^2 + (1-2\beta) z + \beta\right] = 0. \tag{17}$$

In free space, the roots of (17) can be evaluated as

$$z = \frac{2 - \lambda(1-2\beta) \pm \sqrt{(1-4\beta)\lambda^2 - 4\lambda}}{2(1+\lambda\beta)}. \tag{18}$$

Obviously, when $\beta \geq 0.25$, the magnitude of these roots is locked at one. Hence, the Newmark method with $\beta \geq 0.25$ and $\gamma = 0.5$ is unconditionally stable, which agrees with that deduced in [11]–[13] although, here we used a much simpler approach to arrive at the conclusion. Besides its simplicity, this approach is general and applicable to the stability analysis involving any kind of $\epsilon_r(z)$.

As shown in (6), (11), and (16), for dispersive media $\epsilon_r(z)$ is involved, which changes the stability behavior of TDFEM schemes. The introduction of an absorbing boundary condition in the TDFEM procedures may also have an impact on the stability. In addition, the use of different kinds of vector basis functions changes the property of the matrix $\mathbf{T}^{-1}\mathbf{S}$, and hence, the stability of the entire numerical scheme. These issues are addressed in detail in Sections III–VI. Throughout these sections, the temporal discretization scheme is assumed to be central difference, although the proposed approach also applies to the other discretization schemes.

III. TDFEM Modeling of Dispersive Media

For any time-domain based numerical method to accurately perform wide-band electromagnetic simulations, one has to incorporate the effect of medium dispersion in its formulation. Over the past decade, several approaches have been proposed for the FDTD method [28]–[31]. Little work has been reported on the dispersion modeling in the TDFEM. Recently, a general formulation has been developed for the TDFEM modeling of electromagnetic fields in a general dispersive medium [20]. In Sections III-A–C, we take the lossy medium, plasma, and the Debye medium as examples to analyze the stability of TDFEM procedures in dispersive media.

A. Lossy Medium

For a lossy medium characterized by conductivity σ, assuming vanished tangential electric or magnetic fields on the truncating outer boundary, the finite-element discretization will yield the following ordinary differential equation

$$\mathbf{T}\frac{d^2 u}{dt^2} + \mathbf{R}\frac{du}{dt} + \mathbf{S}u = 0 \tag{19}$$

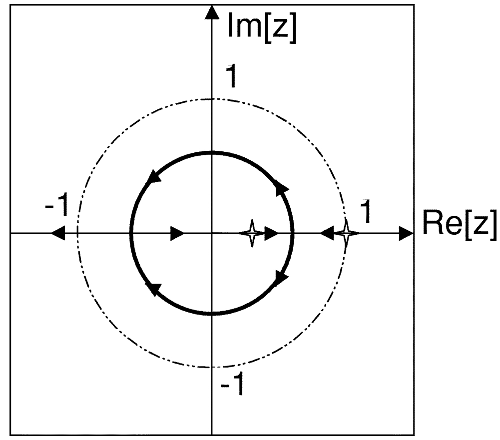


Fig. 1. Root-locus map of a lossy medium.

where $\mathbf{T}$, $\mathbf{R}$, and $\mathbf{S}$ denote matrices whose elements are given by

$$\begin{aligned}\mathbf{T}_{ij} &= \mu_0 \epsilon_0 \langle \mathbf{N}_i, \mathbf{N}_j \rangle \\ \mathbf{S}_{ij} &= \mu_r^{-1} \langle \nabla \times \mathbf{N}_i, \nabla \times \mathbf{N}_j \rangle \\ \mathbf{R}_{ij} &= \mu_0 \sigma \langle \mathbf{N}_i, \mathbf{N}_j \rangle. \end{aligned} \tag{20}$$

In the case that the central difference scheme is used, the permittivity $\epsilon_r(z)$ can be derived as

$$\epsilon_r(z) = 1 + \frac{\sigma \Delta t}{2\epsilon_0} \frac{z+1}{z-1}. \tag{21}$$

Substituting $\epsilon_r(z)$ into (8), we have

$$\left[1 + \frac{\sigma \Delta t}{2\epsilon_0} \frac{z+1}{z-1}\right] (z-1)^2 = -\lambda z. \tag{22}$$

Without loss of generality, assuming $\sigma \Delta t \epsilon_0^{-1} = 1$, we can draw a root-locus map on the complex z plane as shown in Fig. 1. When $\lambda = 0$, there are two roots in (22): one is located at $z = 1/3$, and the other at $z = 1$. As λ increases, the two roots move along the real axis in opposite directions until they encounter each other at $z = 1/\sqrt{3}$. After overlapping, the roots split; one follows the upper circle in the complex z plane, and the other traces the lower one until they encounter for the second time at $z = -1/\sqrt{3}$. Next, with the continual increase of λ, the roots split again. One goes toward the negative infinity, and the other goes toward zero. It is evident that the instability occurs exactly at $z = -1$, where the first root leaves the unit circle. The value of λ at this point is identified as $\lambda_{\max}$, which can be calculated as $\lambda_{\max} = 4$. Since in many cases, the instability is exactly produced at $z = -1$, $\lambda_{\max}$ can be obtained immediately by replacing z in (8) with -1. Fig. 1 also shows that in a lossy medium, the wave is attenuated, since the radius of the circle is less than one. However, the stability threshold is intact compared to the case of free space, although we expect that the lossy medium delays the occurrence of instability if the stability criterion is not satisfied.

To demonstrate the validity of the above stability analysis, the problem of scattering from an empty box of dimension

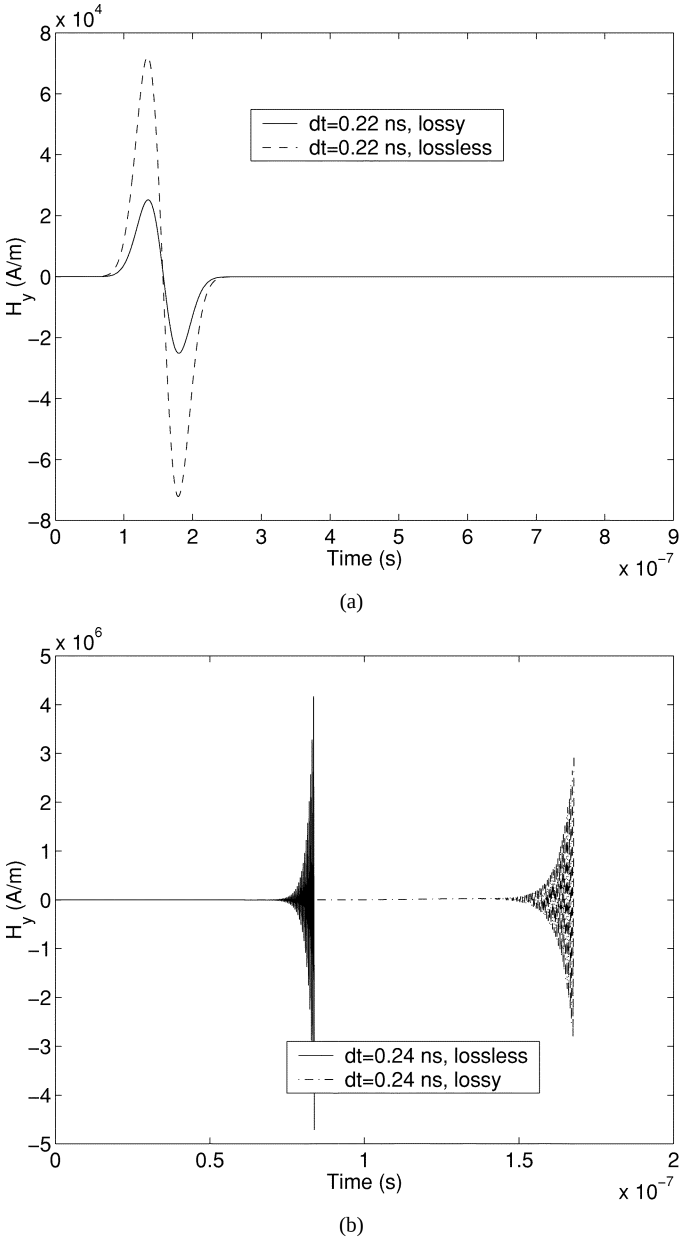


Fig. 2. Scattering from an empty box filled with lossless or lossy medium. (a) H_y generated by using $\Delta t = 0.22$ ns. (b) H_y generated by using $\Delta t = 0.24$ ns.

$1.0 \times 0.5 \times 0.75$ m^3 (a dielectric volume with $\epsilon_r = \mu_r = 1$) is considered. The box is illuminated by a Neumann pulse

$$\mathbf{E}^{\mathrm{inc}}(\mathbf{r}, t) = \hat{\mathbf{E}} \left\{ 2[t - t_0 - c^{-1}\hat{\mathbf{k}} \cdot (\mathbf{r} - \mathbf{r}_0)]\tilde{\tau}^{-2} \times \exp\left\{ \frac{-[t - t_0 - c^{-1}\hat{\mathbf{k}} \cdot (\mathbf{r} - \mathbf{r}_0)]^2}{\tilde{\tau}^2} \right\} \right\} \quad (23)$$

with the pulse parameters defined as $\hat{\mathbf{k}} = -\hat{\mathbf{x}}$, $\hat{\mathbf{E}} = \hat{\mathbf{z}}$, $t_0 = 156$ ns, $\mathbf{r}_0 = 0.5\hat{\mathbf{x}} + 0.5\hat{\mathbf{y}} + 0.375\hat{\mathbf{z}}$ m, and $\tilde{\tau} = 31.5$ ns. The box is filled by the lossy medium characterized by $\sigma = 0.04$ s/m. The computational domain is discretized into 40 tetrahedra, resulting in 83 unknowns. It is truncated by the Dirichlet type of boundary condition enforced at the outer boundary. The zeroth-order Whitney edge elements are utilized to describe the unknown fields. By an eigenvalue analysis of matrix $\mathbf{T}^{-1}\mathbf{S}$, it is found that the spectral radius is equal to 7.1×10^{19}. Hence, from (10), it is determined that the time step should satisfy $\Delta t \leq 0.23$ ns to guarantee stability. As analyzed, this constraint on the time step should be valid for both lossy and lossless cases. These conclusions are validated by numerical simulations. Fig. 2 shows the y component of the magnetic field calculated by using two different time steps. It is evident that when the time step is chosen to be 0.22 ns, the numerical simulation can be kept stable. When the time step is increased to 0.24 ns, instability occurs, which agrees very well with our theoretical prediction. Also, we observe that the introduction of loss does delay the time at which the instability happens.

B. Plasma

For plasma that is characterized by plasma frequency ω_p and damping frequency ν_c, discarding the contribution from sources, the finite-element discretization yields the following ordinary differential equation:

$$\mathbf{T}\frac{d^2u}{dt^2} + \mathbf{S}u + \mathbf{Z}\psi = 0 \quad (24)$$

where $\mathbf{T}$, $\mathbf{S}$, and $\mathbf{Z}$ are square matrices given by

$$\begin{aligned} \mathbf{T}_{ij} &= \mu_0\epsilon_0\langle \mathbf{N}_i, \mathbf{N}_j\rangle \\ \mathbf{S}_{ij} &= \mu_r^{-1}\langle \nabla \times \mathbf{N}_i, \nabla \times \mathbf{N}_j\rangle + \mu_0\epsilon_0\omega_p^2\langle \mathbf{N}_i, \mathbf{N}_j\rangle \\ \mathbf{Z}_{ij} &= -\mu_0\epsilon_0\omega_p^2\nu_c\langle \mathbf{N}_i, \mathbf{N}_j\rangle \end{aligned} \quad (25)$$

and ψ is a vector whose elements can be evaluated from

$$\psi_i(t) = e^{-\nu_c t}\bar{u}(t) \star u_i(t) \quad (26)$$

in which $\bar{u}(t)$ stands for a unit step function.

In the case that the central difference is used to discretize (24), $\epsilon_r(z)$ can be readily derived as

$$\epsilon_r(z) = 1 + \frac{\Delta t^2\omega_p^2 z}{(z-1)^2} - \frac{0.5\omega_p^2\nu_c\Delta t^3(z + e^{-\nu_c\Delta t})}{(z - e^{-\nu_c\Delta t})(z-1)^2}. \quad (27)$$

By substituting $\epsilon_r(z)$ into (8), and tracing the roots in the complex z plane, the allowed maximum eigenvalue $\lambda_{\max}$, and thereby also the constraint on the time step, can be identified.

The example considered here is a metallic sphere coated with plasma. The metallic sphere has a radius of 0.8 m and the coating has a thickness of 0.2 m. This coated sphere is illuminated by an x polarized incident Neumann pulse, defined by $\hat{\mathbf{k}} = \hat{\mathbf{z}}$, $\hat{\mathbf{E}} = \hat{\mathbf{x}}$, $t_0 = 165$ ns, $\mathbf{r}_0 = -1.2\hat{\mathbf{z}}$ m, and $\tilde{\tau} = 33.3$ ns. The computational region is discretized into 1956 tetrahedra, yielding 2704 unknowns. The plasma frequency ω_p and damping frequency ν_c are chosen to be 50 Mrad/s. An eigenvalue analysis reveals that the spectral radius of matrix $\mathbf{T}^{-1}\mathbf{S}$ is equal to 3.11×10^{20}. Substituting (27) into (8), we find $\lambda_{\max}$ to be slightly less than 4. Hence, from (9) we deduce the stability condition $\Delta t \leq 0.113$ ns. Fig. 3 shows the calculated electric field at $\mathbf{r} = 0.1\hat{\mathbf{y}} - 1.18\hat{\mathbf{z}}$ m using $\Delta t = 0.093$ ns and $\Delta t = 0.12$ ns, respectively. It is evident that the numerical result agrees very well with the theoretical stability analysis.

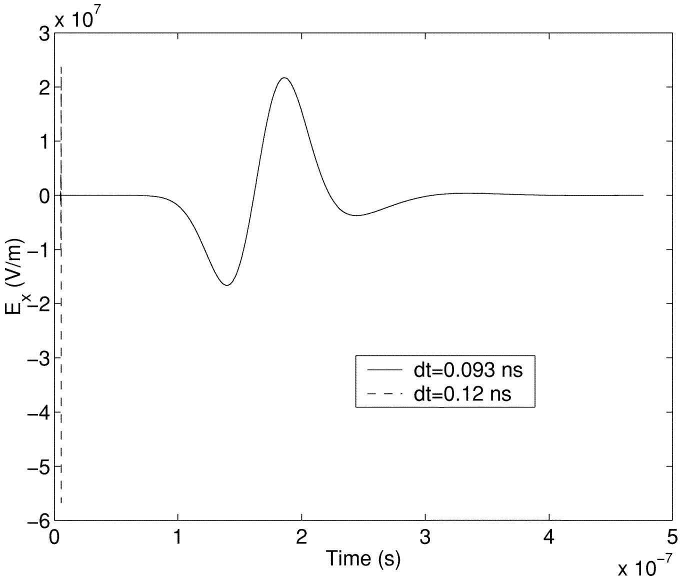


Fig. 3. Scattering from a metallic sphere coated with plasma ($\omega_p = \nu_c = 50$ Mrad/s).

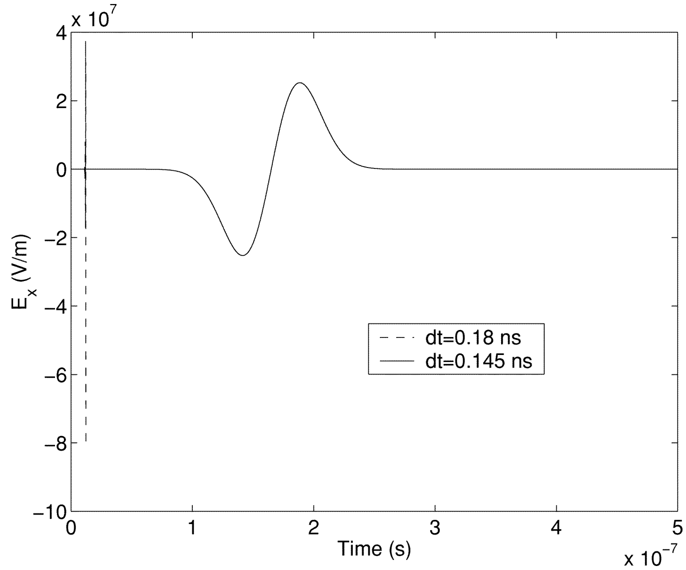


Fig. 4. Scattering from a metallic sphere coated with a Debye medium ($\epsilon_s = 4.0$, $\epsilon_\infty = 1.0$, and $\tau = 0.15$ ns.

C. Debye Medium

For a Debye medium that is characterized by relaxation time τ, relative dielectric constant ϵ_s at dc and ϵ_∞ at infinite frequency, the TDFEM solution yields the following ordinary differential equation:

$$\mathbf{T}\frac{d^2u}{dt^2} + \mathbf{R}\frac{du}{dt} + \mathbf{S}u + \mathbf{Z}\psi = 0 \tag{28}$$

in which matrices $\mathbf{T}$, $\mathbf{R}$, $\mathbf{S}$, and $\mathbf{Z}$ are given by

$$\begin{aligned} \mathbf{T}_{ij} =& \mu_0\epsilon_0\epsilon_\infty\langle \mathbf{N}_i, \mathbf{N}_j\rangle \\ \mathbf{S}_{ij} =& \mu_r^{-1}\langle \nabla\times\mathbf{N}_i, \nabla\times\mathbf{N}_j\rangle \\ & -\tau^{-2}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\langle \mathbf{N}_i, \mathbf{N}_j\rangle \\ \mathbf{R}_{ij} =& \tau^{-1}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\langle \mathbf{N}_i, \mathbf{N}_j\rangle \\ \mathbf{Z}_{ij} =& \tau^{-3}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\langle \mathbf{N}_i, \mathbf{N}_j\rangle \end{aligned} \tag{29}$$

and vector ψ can be evaluated from

$$\psi_i(t) = e^{-t/\tau}\bar{u}(t) \star u_i(t). \tag{30}$$

By discretizing (28) in the time domain using the central difference and performing the Z transform, we obtain the permittivity as seen in (31) at the bottom of the page.

Assuming $\epsilon_s = 4.0, \epsilon_\infty = 1.0$, and $\tau = 0.15$ ns, we simulated the same metallic sphere described in the preceding section, except that now the plasma coating is replaced by a layer of the Debye medium. By a stability analysis, the allowed maximum eigenvalue $\lambda_{\max}$ is identified as 8.93, resulting in a maximum time step 0.169 ns. Obviously, compared to free space, the Debye medium allows for a larger time step for the TDFEM numerical simulation. Fig. 4 shows the calculated electric field at $\mathbf{r} = 0.1\hat{\mathbf{y}}$–$1.18\hat{\mathbf{z}}$ m. The first-order absorbing boundary condition is placed at the dielectric interface to truncate the computational domain. As can be seen clearly from Fig. 4, the numerical simulation is stable at $\Delta t = 0.145$ ns. However, when Δt goes up beyond 0.17 ns, the numerical simulation becomes unstable, which agrees well with the theoretical prediction.

IV. Time-Domain FE-BI Method

The hybrid FE-BI method (see [32] and references therein), is a powerful numerical technique for solving open-region electromagnetic scattering problems. Although this approach has been thoroughly studied within the context of frequency domain solvers, its time-domain version is developed only very recently [21], [22]. This method employs the boundary integral representation to accurately truncate the computational domain and the multilevel plane-wave time-domain algorithm to efficiently evaluate the boundary integrals (BIs). Here, we discuss its stability.

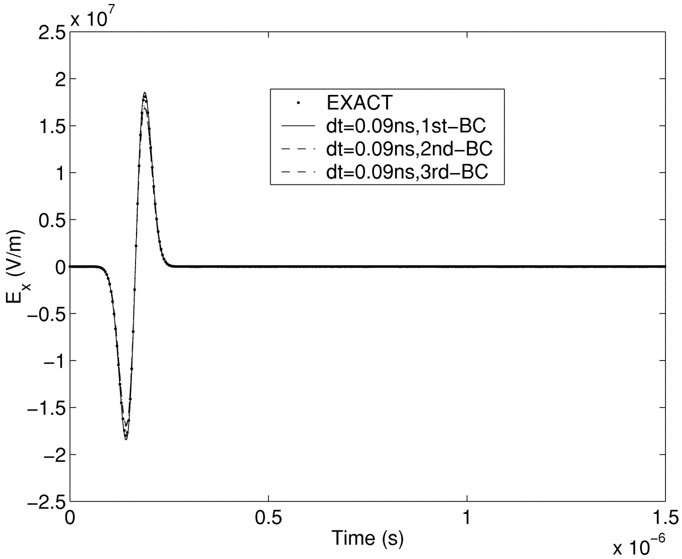
The time-domain FE-BI discretization of an open-region electromagnetic problem renders the following ordinary differential equation:

$$\mathbf{T}\frac{d^2u}{dt^2} + \mathbf{Q}\frac{du}{dt} + \mathbf{S}u + w = 0 \tag{32}$$

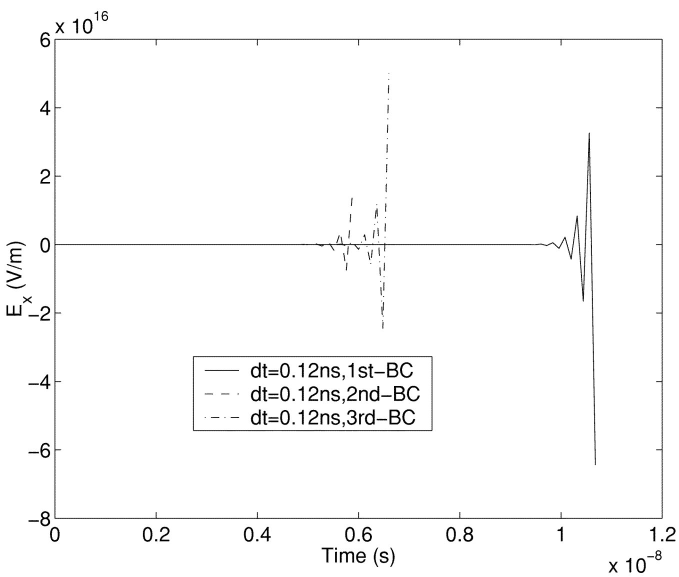
where

$$\begin{aligned} \mathbf{T}_{ij} =& \mu_0\epsilon\langle \mathbf{N}_i, \mathbf{N}_j\rangle \\ \mathbf{S}_{ij} =& \mu_r^{-1}\langle \nabla\times\mathbf{N}_i, \nabla\times\mathbf{N}_j\rangle \\ \mathbf{Q}_{ij} =& c^{-1}\langle \hat{n}\times\mathbf{N}_i, \hat{n}\times\mathbf{N}_j\rangle_{S_o} \\ w_i =& \langle \mathbf{N}_i, \mathbf{U}\rangle_{S_o}. \end{aligned} \tag{33}$$

$$\epsilon_r(z) = \epsilon_\infty + \Delta t\tau^{-1}(\epsilon_s-\epsilon_\infty) \times \frac{0.5(z^2-1)(z-e^{-\Delta t/\tau}) - \Delta t\tau^{-1}z(z-e^{-\Delta t/\tau}) + 0.5\Delta t^2\tau^{-2}z(z+e^{-\Delta t/\tau})}{(z-1)^2(z-e^{-\Delta t/\tau})\epsilon_\infty}. \tag{31}$$



(a)



(b)

Fig. 5. Scattering from a conducting sphere. (a) E_x calculated by using $\Delta t = 0.09$ ns. (b) E_x calculated by using $\Delta t = 0.12$ ns.

In (33), $\mathbf{U}$ is formulated as an impedance boundary condition [22]

$$\hat{n} \times [\mu_r^{-1}\nabla \times \mathbf{E}(\mathbf{r},t)] + c^{-1}\hat{n} \times \partial_t[\hat{n} \times \mathbf{E}(\mathbf{r},t)] = \mathbf{U} \quad \mathbf{r} \in S_o \quad (34)$$

which is evaluated using BIs. The introduction of $\mathbf{U}$ yields an efficient scheme to hybridize finite elements with boundary integrals. This scheme not only preserves the sparsity of the FE system matrix but also generates solutions devoid of spurious modes. These spurious modes are likely to be supported by alternative global boundary condition implementations such as those based on evaluating only the tangential electric or tangential magnetic field at the truncating outer boundary [33].

Clearly, the contribution of $\mathbf{U}$ in (32) is equivalent to the excitation source, which should be discarded in the stability analysis. The introduction of the first temporal derivative is

TABLE I
SPECTRAL RADIUS AND MAXIMUM TIME STEP GENERATED BY THE FIRST-KIND, SECOND-KIND, AND IMPEDANCE BOUNDARY CONDITIONS

Boundary Condition	1st-kind	2nd-kind	impedance
$\rho(\mathbf{T}^{-1}\mathbf{S})$	3.11×10^{20}	3.21×10^{20}	3.21×10^{20}
$\Delta t_{\max}$	0.113 ns	0.112 ns	0.112 ns

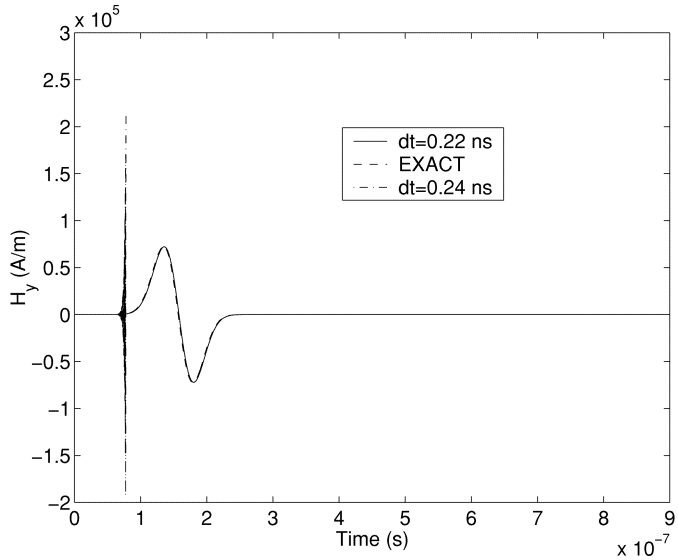
TABLE II
SPECTRAL RADIUS AND MAXIMUM TIME STEP FOR THE ZEROTH-, FIRST-, AND SECOND–ORDER VECTOR ELEMENTS

Order	0th	1st	2nd
$\rho(\mathbf{T}^{-1}\mathbf{S})$	7.1×10^{19}	3.04×10^{20}	8.75×10^{20}
$\Delta t_{\max}$ (ns)	0.23	0.115	0.068

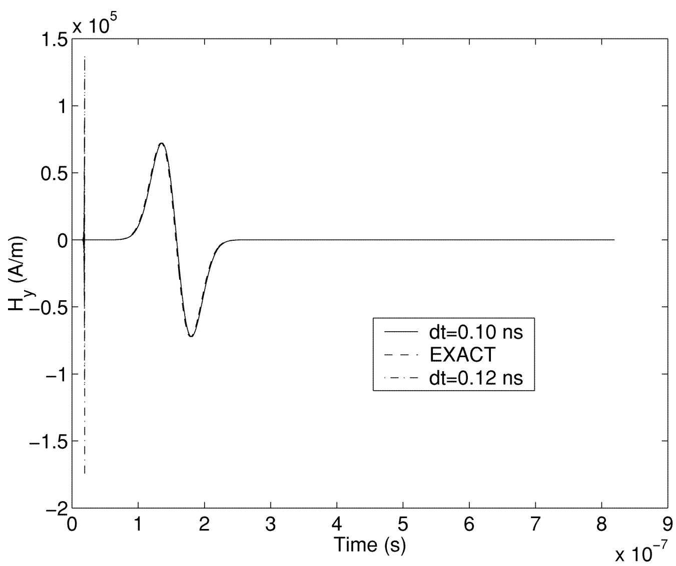
equivalent to the introduction of loss. Consequently, the stability of the time-domain FE-BI procedure should not be affected according to the analysis in Section III-A, except that the instability can be delayed to occur when the stability criterion is not satisfied.

For the purpose of validation, we consider the problem of scattering from a conducting sphere. The problem is set up similarly to that for the stability analysis in plasma and Debye medium, except that now the sphere is not coated. For comparison, the time-domain FE-BI solution is also formulated using the first- and second-kind boundary conditions, for which we evaluate only the tangential electric or magnetic field at the truncating outer boundary. Table I gives the spectral radius of the matrix $\mathbf{T}^{-1}\mathbf{S}$ as well as the resulting maximum time step. The spectral radius of matrix $\mathbf{T}^{-1}\mathbf{S}$ generated by the first-kind boundary condition is slightly different from that generated by the other two boundary conditions. This is due to the explicit enforcement of the boundary condition on the tangential electric field, and hence, the reduction of unknowns. Fig. 5 shows the calculated x component of the electric field observed at point $\mathbf{r} = 0.1\hat{\mathbf{y}}–1.18\hat{\mathbf{z}}$ m. Obviously, when the time step is 0.09 ns, any of these three boundary conditions can produce stable results. However, when the time step is increased to 0.12 ns, none of these three boundary conditions can make the simulation stable. This agrees with the theoretical analysis shown in Table I. Besides, it can be observed that using the impedance boundary condition, instability does occur at a later time compared to the case using the second-kind boundary condition.

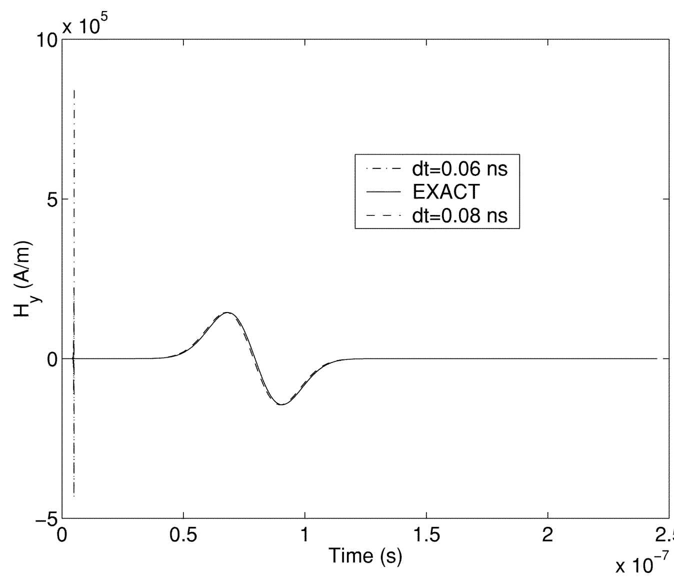
We should note that the hybrid FE-BI schemes based on the first- and second-kind boundary conditions suffer from the problem of interior resonances, as discussed in detail in [21] and [22]. This problem is caused by the improper formulation of the boundary integral equations. Although, it can also lead to instability in the time-domain solution, the nature of the problem is fundamentally different from the one dealt with in this paper. In the example discussed in this section, the incident spectrum does not include any interior resonance frequency. Hence, the hybrid FE-BI schemes based on the first- and second-kind boundary conditions can also yield accurate and stable solutions.



(a)



(a)



(b)

Fig. 6. Scattering from an empty box. (a) H_y generated by using the zeroth-order basis functions. (b) H_y generated by using the first-order basis functions. (c) H_y generated by using the second-order basis functions.

V. Higher Order TDFEM

In our recently developed higher order TDFEM schemes [22], higher order basis functions are utilized to accurately model the unknown fields. It is shown that the efficiency and accuracy can be enhanced greatly by the use of higher order basis functions. However, the stability of the resulting TDFEM is affected because the property of matrix $\mathbf{T}^{-1}\mathbf{S}$ is changed. Hence, a theoretical analysis of stability is important for the correct use of higher order basis functions.

We consider the scattering from an empty box as depicted in Section III-A. This box is subdivided into 40 tetrahedra, which results in 83 unknowns using the zeroth-order elements, 366 unknowns using the first-order elements, and 969 unknowns using the second-order elements. From an eigenvalue analysis, the spectral radius of matriz $\mathbf{T}^{-1}\mathbf{S}$ is calculated, and the results are given in Table II together with the maximum time steps. It can be seen that the property of the FE system matrix deteriorates rapidly when the higher order basis functions are used, resulting in a smaller time step to ensure stability. Fig. 6 shows the calculated y component of the magnetic field at point $\mathbf{r} = 0.35\hat{\mathbf{x}} + 0.41\hat{\mathbf{y}} + 0.1\hat{\mathbf{z}}$ m generated by using the zeroth-, first-, and second-order basis functions, respectively. Evidently, the numerical results agree very well with the theoretical stability analysis.

Considering the decreased time step, one is likely to question the efficiency of higher order schemes. To clarify this point, we reconsider the above problem by discretizing the box into 40 second-order tetrahedra, 118 first-order tetrahedra, and 672 zeroth-order tetrahedra, respectively, so that the three discretizations yield a similar number of unknowns. We then calculate the maximally allowed time step and find that they are about the same, as can be seen in Table III. Considering its higher order accuracy, the higher order TDFEM scheme is

TABLE III
NO. OF UNKNOWNS AND MAXIMUM TIME STEP FOR THE ZEROTH-, FIRST-, AND SECOND-ORDER VECTOR ELEMENTS

Order	0th	1st	2nd
Unknowns	923	972	969
$\Delta t_{\max}$ (ns)	0.067	0.09	0.068

more efficient than the lower-order schemes, as numerically demonstrated in [22].

VI. ORTHOGONAL TDFEM

Due to its ability to handle unstructured meshes and its capacity to impose continuity conditions across material interfaces, the TDFEM is a powerful numerical method for analyzing electromagnetic problems involving complex geometries and inhomogeneous media. However, the TDFEM does not enjoy widespread popularity when compared to the FDTD method. One of the major reasons is that most of the present TDFEMs require a matrix equation to be solved at each time step. This problem can be eliminated by constructing a set of orthogonal vector basis functions that yield a diagonal mass matrix [18]. Recently, a set of three-dimensional (3-D) orthogonal vector basis functions has been developed for the TDFEM solution of vector wave equations [19]. Here, the stability of the resulting TDFEM solution is analyzed. We consider the problem of scattering from an empty box as analyzed in Section III-A. The box is illuminated by a z-polarized Neumann pulse defined by $t_0 = 51.99$ ns and $\tilde{\tau} = 10.5$ ns. The computational domain is discretized into 40 tetrahedra, resulting in 300 unknowns. The eigenvalue analysis of matrix $\mathbf{T}^{-1}\mathbf{S}$ shows that its spectral radius is equal to 2.91×10^{20}. Thus, it is found from (9) that the time step should be less than 0.12 ns to guarantee stability.

Fig. 7 gives the z component of the electric field at $\mathbf{r} = 0.83\hat{\mathbf{x}} + 0.42\hat{\mathbf{y}} + 0.56\hat{\mathbf{z}}$ m generated by the orthogonal TDFEM using $\Delta t = 0.11$ ns and $\Delta t = 0.13$ ns, respectively. It is clear that the numerical experiments are in agreement with the proposed stability analysis.

VII. CONCLUSION

In this paper, a general approach was developed for the stability analysis of time-domain finite-element numerical schemes for electromagnetic simulations. This approach took the dispersion of material into consideration, and was not constrained by the temporal discretization scheme. It was shown that by identifying $\lambda_{\max}$ from the root-locus map of a characteristic equation and by obtaining the spectral radius of the finite-element matrix system, the stability condition can be determined. The successful application to a variety of recently developed TDFEMs, which include (1) time-domain finite-element modeling of dispersive media, (2) time-domain finite element-boundary integral method, (3) higher order TDFEM, and (4) orthogonal TDFEM, validates the proposed approach for stability analysis.

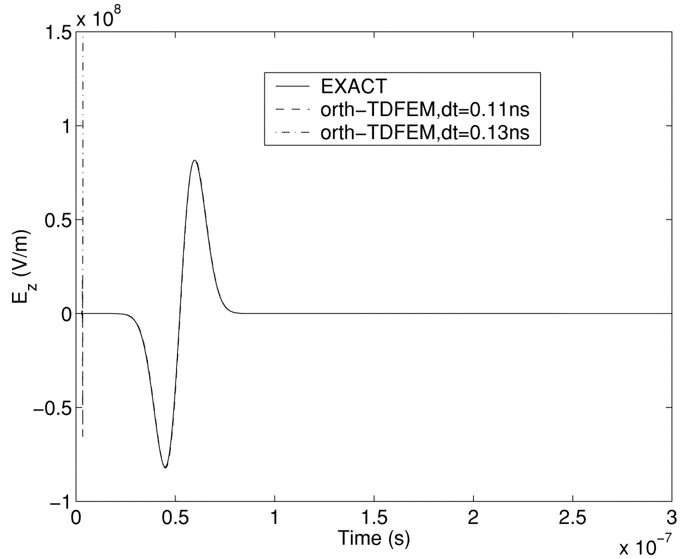


Fig. 7. Scattering from an empty box simulated using the orthogonal TDFEM.

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