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A general bilinear vector integral

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Since the time of the introduction of the Lebesgue integral, several types of extensions and generalizations have been studied. We shall be concerned with two such generalizations in the present paper.

The first extension is in the direction of integration when both the function to be integrated and the measure take values in a relatively general vector space[1]). This paper considers the case that there is a continuous bilinear "multiplication" defined on the product of the vector spaces in which the function and the measure take their values, the product lying in a (possibly different) vector space. The integral discussed here possesses many of the properties of the usual Lebesgue integral; in particular, we show that the well-known Vitali and Bounded Convergence theorems remain valid in this generality, while the natural extension of the Lebesgue Dominated Convergence theorem fails. The second extension is in the direction of replacing the usual requirement of countable additivity of the measure by the assumption of finite additivity. It was shown by Hildebrandt [20] and Fichtenholz and Kantorovitch [13] that this may be done for bounded functions, but some recent work of Dunford and Schwartz [12] demonstrates that it is also possible for unbounded functions, provided that almost everywhere convergence is replaced by convergence in measure.

The structure of the present paper is as follows: sections 1 and 2 introduce the basic terminology and elementary properties; section 3, the principal section, develops the general integral with respect to an additive set function. In section 4 the assumption of countable additivity is imposed and the main results of section 3 are recast in this light. Finally, in section 5 comparisons are made with other integrals. It is found that certain cases of the countably additive integral presented here reduce to (a) the Lebesgue integral, (b) the second Dunford [9] integral of vector functions with respect to a scalar measure (which includes the Bochner

[1]) Such integrals arise naturally in the definition of the concept of work, and in Ampère's law.

[5] integral and coincides with the Birkhoff [4] and Pettis [22] integrals for strongly-measurable functions), and (c) an integral introduced by Bartle, Dunford and Schwartz [3] for scalar functions with respect to a vector-valued measure.

The writer is pleased to acknowledge his debt to Professors Dunford and Schwartz for making their unpublished manuscript [12] available to him. The use of these notes has been an invaluable guide in the writing of this paper

1. Elementary notions. In the sequel we let X and Y denote two real or complex normed linear spaces. We assume that there is a bilinear mapping, which is denoted by juxtaposition, defined on $X\times Y$ with values in a Banach space Z, satisfying $|xy|\leqslant K|x||y|$ for some fixed positive number K. For example, (i) X and Y may be taken to be one and the same Banach algebra; (ii) one of the spaces X and Y may be a Banach space and the other its adjoint space; or (iii) one of the spaces X and Y may be a Banach space and the other the space of bounded linear operators on this space with values in a Banach space Z. We observe that the general case may be reduced to case (iii), but for reasons of symmetry we prefer to avoid doing so.

In the following S denotes an abstract set and $\mathfrak{S}$ a field of subsets of S, called the *measurable* subsets of S; hence $\mathfrak{S}$ is closed under finite unions, intersections and complements. By μ we signify an *additive* function on $\mathfrak{S}$ to Y: thus if $E,F\epsilon\mathfrak{S}$ and $E\cap F=\emptyset$, then $\mu(E\cup F)=\mu(E)+\mu(F)$. In section 4 we shall consider the additional properties when μ is countably additive, but for the present we assume only additivity.

The *semi-variation* of μ is the extended non-negative function $\|\mu\|$ whose value on a set E in $\mathfrak{S}$, denoted by $\|E\|$ or $\|\mu\|(E)$, is defined to be

$$\|E\| = \sup\Big|\sum x_i\mu(E_i)\Big|,$$

where the supremum is extended over all partitions of E into a finite number of disjoint sets $\{E_i\}\subset\mathfrak{S}$ and all finite collections of elements $\{x_i\}\subset X$ with $|x_i|\leqslant 1$. The *variation* of μ is the extended non-negative function $|\mu|$ whose value on a set E in $\mathfrak{S}$, denoted by $|E|$ or $|\mu|(E)$, is defined to be

$$|E| = \sup\sum|\mu(E_i)|,$$

where the supremum is taken over all partitions of E into a finite number of disjoint measurable sets.

The reader may readily verify that the semi-variation of μ is a monotone, subadditive function on $\mathfrak{S}$, and that the variation of μ is a monotone, additive function on $\mathfrak{S}$. It is evident that if E is in $\mathfrak{S}$, then $0\leqslant\|E\|\leqslant K|E|\leqslant+\infty$. If Y is the scalar field, then the finiteness of $\|E\|$ implies that of $|E|$, but for a general space it is possible that $\|E\|<\infty$ while $|E|=+\infty$. (For an example, see [15], p. 257). It is for this reason that we prefer to work with the semi-variation rather than the variation. If μ is countably additive, then the fact that $S\epsilon\mathfrak{S}$ leads to the conclusion that $\|S\|$ is finite, but this does not follow in the additive case.

It is technically convenient to extend the definition of $\|\mu\|$ and $|\mu|$ to arbitrary subsets of S. We do this as follows: if A is an arbitrary subset of S, then $\|A\|=\|\mu\|(A)$ is defined to be $\inf\{\|E\|: E\epsilon\mathfrak{S}, A\subset E\}$. We may extend $|\mu|$ similarly. It is easily seen that the extension of $\|\mu\|$ agrees with its former value on $\mathfrak{S}$ and is a monotone, subadditive function on the collection of all subsets of S.

We say that a subset A of S is a *μ-null set*[2]) if $\|A\|=0$, *i. e.* if for every $\varepsilon>0$ there is an E in $\mathfrak{S}$ such that $A\subset E$ and $\|E\|<\varepsilon$. We say that a proposition holds *μ-almost everywhere* if it holds outside of a null set.

A *μ-simple function* is a function $f: S\to X$ which assumes only a finite number of values $x_i, i=1,\ldots,n$, each non-zero value x_i being taken on a set E_i in $\mathfrak{S}$ with $\|E_i\|<\infty$. Such a function may be represented as a linear combination of characteristic functions; thus

$$(*)\qquad f=\sum_{i=1}^{n} x_i\chi_{E_i},\qquad E_i\epsilon\mathfrak{S},$$

where χ_E denotes the characteristic function of the set E. If f is a function and $M>0$, then f^M denotes the *M-truncation* of f and is defined by

$$f^M(s)\equiv\begin{cases} f(s), & \text{if}\quad |f(s)|<M,\\ \dfrac{f(s)}{|f(s)|}M, & \text{if}\quad |f(s)|\geqslant M.\end{cases}$$

It will be noted that f^M is a simple function if f is.

If f is a simple function with representation given by (*) and if E is in $\mathfrak{S}$, we define the *integral of f over E* to be

$$\int_E f(s)\mu(ds)\equiv\sum_{i=1}^{n} x_i\mu(E\cap E_i).$$

The reader will observe that the integral of a simple function is independent of the representation of the form (*) in its definition. We omit the proof of this statement and of the following lemma which will be used repeatedly:

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[2]) Subsequently, if confusion does not threaten, we will ordinarily omit explicit mention of the measure.

LEMMA 1. *Let μ be an additive function on $\mathfrak{S}$ to Y.*

(a) *For each fixed E in $\mathfrak{S}$, the integral over E is a linear mapping defined on the linear space of simple functions on S to X, and has values in Z.*

(b) *For each fixed simple function, the integral is an additive function on $\mathfrak{S}$.*

(c) *If f is a simple function and $|f(s)|<M$ for all s in $E\in\mathfrak{S}$, then*

$$\Big|\int_E f(s)\mu(ds)\Big| \leqslant M\|E\|.$$

2. Measurable functions. In extending the notion of integral to a larger class of functions we need the concepts of convergence in measure and of measurable function. A sequence $\{f_n\}$ of functions on S to X is said to *converge in μ-measure* to a function f if $\|(S,n,\varepsilon)\|\to 0$ as $n\to\infty$ for each $\varepsilon>0$, where we have put $(S,n,\varepsilon)\equiv\{s\epsilon S: |f_n(s)-f(s)|\geqslant\varepsilon\}$.

A similar definition can be given for a sequence of functions to be *fundamental in measure*. We say that a function is *μ-measurable*[3]) if it si the limit in measure of a sequence of simple functions. It is not difficult to show that the collection of all measurable functions on S to X is a linear space which is closed under the operation of convergence in measure of sequences. We also omit the demonstration that if f is a measurable function, then there exists a sequence $\{A_n\}$ of subsets of S with $\|A_n\|<\infty$ and such that f vanishes outside of $\bigcup_{n=1}^{\infty} A_n$.

A sequence $\{f_n\}$ of functions is said to be *μ-almost uniformly convergent* to a function f on S if for every $\varepsilon>0$ there is a subset A_ε of S such that $\|A_\varepsilon\|<\varepsilon$ and the convergence to f is uniform on $S-A_\varepsilon$.

LEMMA 2. *Let μ be an additive function on $\mathfrak{S}$ to Y.*

(a) *μ-almost uniform convergence implies convergence in μ-measure to the same function.*

(b) *μ-almost uniform convergence implies μ-almost everywhere convergence to the same function.*

The demonstration requires only trivial modifications in the usual proofs (cf. [18], p. 92, 89).

3. The general integral. We are now prepared to introduce the general integral and show that it possesses at least some of the properties usually associated with a Lebesgue theory of integration. Throughout this section μ is an additive function on $\mathfrak{S}$ to Y.

[3]) This notion of measurability is somewhat more restrictive than that employed by some authors, but it is sufficient for the purposes of integration.

Definition 1. A function f on S to X is said to be *μ-integrable over S* if there is a sequence $\{f_n\}$ of simple functions on S to X satisfying the conditions:

(i) the sequence $\{f_n\}$ converges in measure to f;

(ii) the sequence $\{\lambda_n\}$ of indefinite integrals

$$\lambda_n(E)\equiv\int_E f_n(s)\mu(ds),\qquad E\epsilon\mathfrak{S},$$

has the property that given any $\varepsilon>0$ there is a $\delta>0$ such that if E is in $\mathfrak{S}$ and $\|E\|<\delta$, then $|\lambda_n(E)|<\varepsilon$, $n=1,2,\ldots$;

(iii) the sequence $\{\lambda_n\}$ has the property that given any $\varepsilon>0$ there is a set E_ε in $\mathfrak{S}$ with $\|E_\varepsilon\|<\infty$ and such that if G is in $\mathfrak{S}$ and $G\subset S-E_\varepsilon$, then $|\lambda_n(G)|<\varepsilon$, $n=1,2,\ldots$

We remark that condition (iii) is trivially satisfied in case $\|S\|<\infty$, but otherwise it is important. Condition (ii) is frequently described by saying that the sequence $\{\lambda_n\}$ is *uniformly absolutely continuous* with respect to $\|\mu\|$, and (iii) by saying that $\{\lambda_n\}$ is *equicontinuous* with respect to $\|\mu\|$. In the case that X and Y are the spaces of scalars a definition of this sort has been employed by Riesz [26] and Graves [17] to lead to a simple and rapid development of integration theory. Hildebrandt ([21], p. 117) reports that he employed a similar approach in an unpublished paper [19] dealing with the integration of vector-valued functions with respect to a scalar measure.

THEOREM 1. *If f is integrable over S in the sense of definition 1, then for each E in $\mathfrak{S}$ the limit of the indefinite integrals exists in the norm of Z. This limit is denoted by $\lambda(E)$ or by*

$$\int_E f(s)\mu(ds),$$

and is called the value of the indefinite integral λ at E, or the integral of over the set E. In addition, the limit

$$\lambda(E)=\lim_{n\to\infty}\lambda_n(E)$$

exists in the norm of Z uniformly for E in $\mathfrak{S}$.

Proof. Let $\varepsilon>0$ be given and take δ as in condition (ii) and E^ε with $\|E_\varepsilon\|\neq 0$ as in (iii). By (i) there exists an integer N_ε such that if m and n are any fixed integers larger than N_ε, then there is a set $F\epsilon\mathfrak{S}$ with $\|F\|<\delta$ such that if $s\bar{\epsilon}F$ then $|f_m(s)-f_n(s)|<\varepsilon/\|E_\varepsilon\|$.

Now if $E \epsilon \mathfrak{S}$ is arbitrary, then we have from the above and Lemma 1 that

$$|\lambda_m(E)-\lambda_n(E)| \leqslant |\lambda_m(E\cap F)|+|\lambda_n(E\cap F)|+$$
$$+|\lambda_m[E-(F\cup E_\varepsilon)]|+|\lambda_n[E-(F\cup E_\varepsilon)]|+$$
$$+|\lambda_m[(E-F)\cap E_\varepsilon]-\lambda_n[(E-F)\cap E_\varepsilon]|$$
$$<\varepsilon+\varepsilon+\varepsilon+\varepsilon+\Big|\int_{(E-F)\cap E_\varepsilon}\{f_m(s)-f_n(s)\}\mu(ds)\Big|<4\varepsilon+(\varepsilon/\|E_\varepsilon\|)\|(E-F)\cap E_\varepsilon\|<5\varepsilon.$$

This proves the existence and the uniformity of the limit.

The fact that the integral is independent of the sequence of simple functions used to define it is readily shown and will be omitted. It will also be clear what is meant for a function to be integrable over a subset E in $\mathfrak{S}$.

THEOREM 2. (a) *If E is in $\mathfrak{S}$, the set of functions integrable over E is a linear space and the integral over E is a linear mapping of this space into Z.*

(b) *If f is integrable over S, the integral of f is an additive function on the field $\mathfrak{S}$.*

(c) *If f is integrable over S, then*

$$\lim_{\|E\|\to 0}\int_E f(s)\mu(ds)=0.$$

(d) *If f is integrable over S, then given any $\varepsilon>0$ there is a set E_ε in $\mathfrak{S}$ such that if G is in $\mathfrak{S}$ and $G\subset S-E_\varepsilon$ then*

$$\Big|\int_G f(s)\mu(ds)\Big|<\varepsilon.$$

We omit a detailed proof of this theorem. Properties (a) and (b) require Lemma 1 and properties (c) and (d) are consequences od definition 1 (ii) and (iii) and the uniformity of the limit established in Theorem 1.

We say that a function f on S to X is *μ-essentially bounded* on a subset A of S if

$$\inf_N \sup_{s\epsilon A-N}|f(s)|<\infty,$$

where the infimum is taken over all null sets N. We write $\operatorname{ess\,sup}_{s\epsilon A}|f(s)|$ for this number.

THEOREM 3. *An essentially bounded measurable function is integrable over any set E in $\mathfrak{S}$ with $\|E\|<\infty$.*

Proof. Let

$$M_1\equiv\operatorname{ess\,sup}_{s\epsilon E}|f(s)|,$$

and suppose that the sequence $\{f_n\}$ of simple functions converges in measure to f. Let $M\equiv 2+M_1$, then there exists a null set B such that $\{s\epsilon E\colon |f(s)|\geqslant M_1+1\}\subset B$, and hence $\{s\epsilon E\colon |f_n(s)|\geqslant M\}\subset B\cup(E,n,1)$. Therefore, for the sequence of truncated simple functions $\{f_n^M\}$, we have

$$\{s\epsilon E\colon|f_n^M(s)-f(s)|\geqslant 2\varepsilon\}\subset\{s\epsilon E\colon|f_n^M(s)-f_n(s)|\geqslant\varepsilon\}\cup(E,n,\varepsilon)$$
$$\subset B\cup(E,n,1)\cup(E,n,\varepsilon),$$

and so $\{f_n^M\}$ converges in measure to f on E. Hence we may and do assume that the sequence $\{f_n\}$ is uniformly bounded; from Lemma 1(c) it follows that definition 1(ii) is satisfied. Since (iii) is automatic for $\|E\|<\infty$, we conclude that f is integrable over E.

THEOREM 4. *If f is measurable and essentially bounded on a set E in $\mathfrak{S}$ with $\|E\|<\infty$, then*

$$\Big|\int_E f(s)\mu(ds)\Big|\leqslant\{\operatorname{ess\,sup}_{s\epsilon E}|f(s)|\}\,\|E\|.$$

This was given for a simple function in Lemma 1 (c). The general case follows from a slight refinement of the argument in the first part of the preceding theorem and from theorem 1.

We now prove a theorem which, in the case of scalars, is essentially due to G. Vitali. It derives its importance from the fact that it is a key to the interchange of limits and integration.

THEOREM 5 (VITALI CONVERGENCE THEOREM). *Let f be a function on S to X and let $\{f_n\}$ be a sequence of integrable functions which are such that*

(i) *the sequence $\{f_n\}$ converges in measure to f;*

(ii) *the sequence of indefinite integrals is uniformly absolutely continuous with respect to $\|\mu\|$;*

(iii) *the indefinite integrals are equicontinuous with respect to $\|\mu\|$.*

Then it follows that f is an integrable function and that

$$\int_E f(s)\mu(ds)=\lim_{n\to\infty}\int_E f_n(s)\mu(ds),\qquad E\epsilon\mathfrak{S}.$$

Furthermore, the limit is uniform for E in $\mathfrak{S}$.

Proof. We shall first prove the integrability of f. Since f_n possesses this property, definition 1 implies that there is a simple function g_n such that $\|\{s\epsilon S: |f_n(s)-g_n(s)|\geqslant 2^{-n}\}\|<2^{-n}$. From the uniformity of the limit in theorem 1, we may also suppose that g_n is selected such that

(*) $$|\lambda_n(E)-\lambda_n'(E)|<2^{-n}, \quad E\epsilon\mathfrak{S},$$

where we have put λ_n and λ_n' for the indefinite integrals of f_n and g_n respectively. Suppose that such a simple function g_n has been chosen for $n=1,2,\ldots$ Since

$$\{s\epsilon S: |g_n(s)-f(s)|\geqslant 2\varepsilon\}\subset\{s\epsilon S: |g_n(s)-f_n(s)|\geqslant\varepsilon\}\cup\{s\epsilon S: |f_n(s)-f(s)|\geqslant\varepsilon\},$$

it follows that $\{g_n\}$ converges in measure to f. Also, since

(**) $$|\lambda_n'(E)|<|\lambda_n(E)|+2^{-n}, \quad E\epsilon\mathfrak{S},$$

it follows that condition (ii) of definition 1 is satisfied for the sequence $\{g_n\}$. Condition (iii) of that definition for the sequence $\{g_n\}$ is a consequence of hypothesis (iii), the inequality (**) and the fact that a finite number of simple functions vanish outside of a set of finite $\|\mu\|$-measure. Hence f is integrable. We conclude from theorem 1 that

$$\int_E f(s)\mu(ds)=\lim_{n\to\infty}\lambda_n'(E),$$

and that this convergence is uniform on $\mathfrak{S}$. Applying (*), the statement is proved.

In the cases when either X or Y is the scalar field, the integrability of a function g implies that of the function $|g(\cdot)|$. Further, in these cases, one may ordinarily use the Vitali Convergence theorem to derive a result generalizing the Lebesgue Dominated Convergence theorem. In the case at hand, these remarks are not true, in general, as we shall show.

Examples. (a) Let $X=Y=$real Euclidean three-space, with the usual inner product, and let δ_1,δ_2 and δ_3 be the unit coordinate vectors. Let λ be Lebesgue measure on $S=[0,1]$ and let $\mu(E)=\lambda(E)\delta_1$. It is easy to see that the function $g(s)\equiv s^{-1}\delta_2$ is μ-integrable and its integral over any measurable set is zero. However, the function $|g(\cdot)|$ is not integrable.

(b) Let S,X,Y,μ, and g be as in (a). If $f_n(s)\equiv s^{-1+(1/n)}\delta_1$, then $\{f_n\}$ is a sequence of integrable functions with $|f_n(s)|\leqslant|g(s)|$ and such that $\{f_n\}$ converges almost everywhere and in measure to $f_0(s)\equiv s^{-1}\delta_1$. Since f_0 is not integrable, the usual formulation of the Lebesgue Dominated Convergence theorem does not hold, in general.

(c) Let S,X,Y, and μ be as in (a). Let $h_n(s)\equiv 1\delta_2$ and $h_0(s)\equiv 1\delta_3$, so each h_n is integrable and

$$\int_E h_0(s)\mu(ds)=\lim_{n\to\infty}\int_E h_n(s)\mu(ds)=0$$

uniformly for E in $\mathfrak{S}$. Condititions (ii) and (iii) of the Vitali Convergence theorem are satisfied, but $\{h_n\}$ does not converge at any point to h_0 and not in measure. Thus the converse of the Vitali theorem fails.

Despite (b), a slight alteration in statement renders valid a form of the Dominated Convergence theorem.

THEOREM 6 (DOMINATED CONVERGENCE THEOREM). *Let $\{f_n\}$ be a sequence of integrable functions on S to X which converges in measure to a function f. If there exists an integrable function g such that if E is in $\mathfrak{S}$ and $n=1,2,\ldots$, then*

$$\Big|\int_E f_n(s)\mu(ds)\Big|\leqslant\Big|\int_E g(s)\mu(ds)\Big|,$$

then we may conclude that f is integrable on S and

$$\int_E f(s)\mu(ds)=\lim_{n\to\infty}\int_E f_n(s)\mu(ds), \quad E\epsilon\mathfrak{S}.$$

Proof. It follows from theorem 2(c) that condition (ii) of theorem 5 is satisfied, and from 2(d) that (iii) is. Therefore the Vitali theorem may be applied.

A weaker, but somewhat more convenient, result follows:

THEOREM 7 (BOUNDED CONVERGENCE THEOREM). *Let $\{f_n\}$ be a sequence of integrable functions on S to X which converges in measure to f. If $|f_n(s)|\leqslant M$ for almost all $s\epsilon S$, then f is integrable over any set E in $\mathfrak{S}$ with $\|E\|<\infty$ and*

$$\int_E f(s)\mu(ds)=\lim_{n\to\infty}\int_E f_n(s)\mu(ds).$$

In the theorem just stated and in the next one, the requirement that $\|E\|<\infty$ cannot be dropped, as is easily seen.

THEOREM 8. *Let $\{f_n\}$ be a sequence of integrable functions which converge almost uniformly to f. Then f is integrable over any set E in $\mathfrak{S}$ with $\|E\|<\infty$ and*

$$\int_E f(s)\mu(ds)=\lim_{n\to\infty}\int_E f_n(s)\mu(ds).$$

This is an immediate consequence of theorem 5 once it is observed that lemma 2(a) implies that condition (i) of that theorem is satisfied and that the almost uniform convergence implies condition (ii).

4. The countably additive case. In the preceding section a theory of integration was constructed under the assumption that the measure μ was merely finitely additive. In this section we suppose that $\mathfrak{S}$ is a σ-field and that μ is *countably additive* in the sense that if $\{E_n\}$ is a disjoint sequence in $\mathfrak{S}$, then

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n),$$

where the series converges unconditionally in the norm of Y. Under this restriction we can prove that if $f: S \to X$ is integrable, then the indefinite integral of f is a countably additive set function on $\mathfrak{S}$ to Z. If f is a simple function this conclusion follows immediately from the countable additivity of μ; in the general case it follows from the definition of the integral and the uniformity of the convergence established in theorem 1.

Ordinarily the countable additivity of the measure insures that the requirement of convergence in measure — made frequently throughout section 3 — can be replaced by almost everywhere convergence. We have not been able to demonstrate this without further restrictions. Fortunately these restrictions are frequently automatic.

Definition 2. We say that a countably additive measure μ defined on a σ-field $\mathfrak{S}$ to Y has the **-property* (with respect to X) if there is a non-negative finite-valued countably additive measure ν on $\mathfrak{S}$ such that $\nu(E) \to 0$ if and only if $\|E\| = \|\mu\|(E) \to 0$. When we are assuming this we will mark the theorems with an asterisk, and employ terms such as the "*-integral".

The *-property is available under a variety of circumstances: (a) if $y^*\mu$ is a finite countably additive scalar-valued measure for each $y^* \in Y^*$, it is seen in [3] that μ has the *-property with respect to the scalar field, (b) if μ is scalar-valued, or (c) if μ has a finite variation in the sense of section 1, then μ has the *-property with respect to any Banach space X. It is not difficult to show from a theorem of Saks [27], that if μ has the *-property, then $\|S\| < \infty$. In addition, $\|\mu\|$ is countably subadditive on subsets of S. We shall use these two facts freely.

(*) LEMMA 3. (a) *If a sequence $\{f_n\}$ of functions on S to X converges in μ-measure to a function f, then some subsequence converges μ-almost uniformly to f.*

(b) *If a sequence $\{f_n\}$ converges μ-almost everywhere to f, then it converges μ-almost uniformly to f.*

Proof. Statement (a) is proved precisely as in the usual case (cf. [18], p. 93). To prove (b), let $\varepsilon > 0$ and take $E_0 \in \mathfrak{S}$ such that $\|E_0\| < \varepsilon$ and $f_n(s) \to f(s)$ for all $s \bar{\epsilon} E_0$. Let $\delta = \delta(\varepsilon) > 0$ be such that if $E \in \mathfrak{S}$ and $\nu(E) < \delta$ then $\|E\| < \varepsilon$. By the standard proof of the theorem of Egoroff (cf. [18], p. 88) which is valid for vector-valued functions, we conclude that there is a set $E_1 \in \mathfrak{S}$ with $\nu(E_1) < \delta$ such that $\{f_n\}$ converges to f uniformly on $S - (E_0 \cup E_1)$. But $\|E_0 \cup E_1\| < 2\varepsilon$, and so $\{f_n\}$ converges μ-almost uniformly to f.

Thus we conclude that if μ has the *-property, then a function is measurable if and only if it is the limit almost everywhere of a sequence of simple functions. Further, the family of measurable functions is closed under the operation of almost everywhere convergence of sequences. We now show that integrability takes a simpler, though equivalent, form.

(*) THEOREM 9. *A function f on S to X is integrable if and only if there exists a sequence $\{f_n\}$ of simple functions such that*

(i) *the sequence $\{f_n\}$ converges to f almost everywhere;*

(ii) *the sequence $\{\lambda_n\}$ of indefinite integrals converges in the norm of Z for each E in $\mathfrak{S}$.*

Proof. Let f satisfy the hypotheses of definition 1. Lemmas 3(a) and 2(b) imply that some subsequence converges almost everywhere, and from theorem 1 we conclude that the corresponding subsequence of indefinite integrals converges for each $E \in \mathfrak{S}$. Conversely, if $\{f_n\}$ satisfies the present hypotheses, then by lemmas 3(b) and 2(a), the sequence $\{f_n\}$ converges in measure. Since $\|S\| < \infty$, it suffices to establish condition (ii) of definition 1. To do this we use Lemma 1(c) and definition 2 to observe that for each $n = 1, 2, \ldots$, we have

$$(+) \qquad \lim_{\nu(E) \to 0} \lambda_n(E) = 0.$$

Furthermore, by hypothesis (ii) we have that the function λ on $\mathfrak{S}$ to Z defined by

$$\lambda(E) = \lim_{n \to \infty} \lambda_n(E),$$

exists for each E in $\mathfrak{S}$. It follows from the well-known Vitali-Hahn-Saks theorem (cf. [27]), which is valid for countably additive functions with values in a Banach space, that the convergence in (+) is uniform in n. Thus definition 1(ii) is verified.

Because of its importance, we explicitly restate theorem 5 in a form appropriate for measures with the *-property.

(*) THEOREM 10. *If $\{f_n\}$ is a sequence of integrable functions which are such that*

(i) *the sequence $\{f_n\}$ converges almost everywhere to f;*

(ii) *given* $\varepsilon>0$ *there is a* $\delta>0$ *such that if* E *is in* $\mathfrak{S}$ *and* $\|E\|<\delta$, *then*

$$\Big|\int_E f_n(s)\mu(ds)\Big|<\varepsilon, \qquad n=1,2,\ldots;$$

then we may conclude that f *is integrable on* S *and*

$$\int_E f(s)\mu(ds)=\lim_{n\to\infty}\int_E f_n(s)\mu(ds),$$

uniformly for E *in* $\mathfrak{S}$.

Corresponding replacement of almost everywhere convergence by convergence in measure is possible in theorems 6 and 7.

5. Relations with other integrals. Since there are already many abstract integrals, it is proper that we indicate the connection between them and the integrals in sections 3 and 4. For a very readable account of abstract integration the reader should consult Hildebrandt [21]. In most cases studied in the past, μ is countably additive, so unless explicit mention to the contrary is indicated, we shall assume this condition. Also, for simplicity, we shall consider only the case of finite measure.

Scalar functions; scalar measure

If X, Y and Z are all the field of scalars, then it is clear that if a function is Lebesgue-integrable, then it is *-integrable, and conversely. Hence the *-integral reduces to the Lebesgue integral in this case.

Independently, Hildebrandt [20] and Fichtenholz and Kantorovitch [13] employed an integral for bounded functions with respect to a measure with finite variation which was assumed to be finitely additive. It is readily verified that the integral of section 3, with X, Y, and Z taken as the scalars, includes this integral.

Vector functions; scalar measure

The first abstract integral was studied by Graves [16] and was of the Riemann type. However, it is not subsumed in our discussion, since a Graves-integrable function need not be almost separably-valued ([11], p. 166)[4]). If it is, then it is *-integrable to the same value.

Probably the most frequently-used abstract integral is that introduced by Bochner [5], and also studied by Dunford [8] and Hildebrandt [19] (cf. [21], p. 117). It may be seen that any function which is Bochner-integrable is *-integrable to the same value. This follows very readily from [21] (p. 117-118) or from theorem 9 and the definition employed by Dunford [8]. However, there are functions which are *-integrable but not Bochner-integrable ([4], p. 377).

Dunford [9] introduced an integral (the second Dunford integral) by declaring a function f on S to X to be integrable with respect to a finite positive measure μ if it satisfies the hypothesis of theorem 9. Thus the *-integral contains this Dunford integral, and therefore ([21], p. 123) it coincides for measurable functions with the integrals of Birkhoff [4] and of Gelfand [14] and Pettis [22]. Birkhoff's integral is more general than the *-integral, however, since it also integrates certain multiply-valued functions. The Gelfand-Pettis integral is more general than the *-integral in that it does not require the integrable function to be essentially contained in a separable manifold. The convergence theorems presented here compare favorably with those in [4] and [22]. Again, the Phillips integral [23] includes this case of the *-integral, since it is defined for functions with values in a locally convex topological linear space and includes both the Birkhoff and Gelfand-Pettis integrals.

The only extensive development of a finitely additive integral in the spirit presented here that is known to the writer is due to Dunford and Schwartz [12]. While the approach is different, it may be seen that if μ is a scalar measure, the integral of section 3 contains the Dunford-Schwartz integral.

Scalar functions; vector measure

Integrals of the Riemann type which allow one to integrate scalar-valued continuous functions with respect to a vector-valued measure have been treated by Dunford ([11], p. 312). In the generality treated in [11], they are subsumed here.

Alexiewicz [1] employed an integral of a bounded function with respect to a finitely additive measure with values in an F-space. The discussion is close to the treatment in Fichtenholz and Kantorovitch [13] and the Banach space case of this integral is included in the results of section 3.

A Lebesgue-type theory of integration for unbounded functions with respect to a countably additive measure was presented by Bartle, Dunford and Schwartz [3]. It follows from theorem 9 that the *-integral contains this theory.

[4]) It may be seen that if μ is countably additive, then any function integrable in the sense of section 3 is essentially separably-valued.

Vector functions; vector measure

The first integral of this sort known to the writer is due to Gowurin [15]. It is of the Riemann type and the discussion is almost entirely limited to bounded functions. A convergence theorem along the lines of theorem 8 is presented. Section 3 contains and extends the Gowurin theory. Similar Riemann integrals and their extensions were defined and employed by Bochner and Taylor [6] — they were only incidentally concerned with the development of an integration theory, however.

The first Lebesgue theory in the bilinear case was presented by Price [24] and is along the lines of the Birkhoff integral. Insofar as it permits the integration of multiply-valued functions it is more general than the *-integral. In the Price integral, $X=Z$ and the measure μ is a function on a σ-field to the space of bounded linear operators in the space X, and such that (a) if $E\subset E_0 \epsilon \mathfrak{S}$ and $\mu(E_0)=0$, then $\mu(E)=0$; (b) if $\mu(E)\neq 0$, then $\mu(E)$ has a bounded inverse; and (c) μ is countably additive. The portion of Price's paper most closely related to the present one is Part IV ([24], p. 25-34); here he shows that bounded measurable functions are integrable in his sense and obtains a bounded convergence theorem of the same sort as theorem 7. Unbounded functions are integrated (cf. [24], p. 32-34) only when μ has finite variation, and here Price obtains a dominated convergence theorem. It is seen, then, that the results of this part of [24] are contained in what we have done.

We now turn to the remarkable Rickart integral [25]; we are concerned primarily with his bilinear integral ([25], p. 511-519). This integral is more general than ours in that it permits the integration of multiply-valued functions in a locally convex linear topological space. On the other hand it is countably additive and requires ([25], p. 518) that if $E\subset E_0 \epsilon S$ and $x\mu(E_0)=0$ for all $x \epsilon X$, then $x\mu(E)=0$, $x\epsilon X$. Nevertheless, Rickart obtains theorems related to theorems 3 and 10. Direct comparisons between the Rickart integral and the one presented here are difficult due to the radically different nature of these integrals. It is clear, however, that neither definitely contains the other.

In [7], Day treated the case where $X=Z$ and μ is defined on a σ-field $\mathfrak{S}$ with values in the space of bounded operators in X. In much of [7] (p. 596-603), it is assumed that μ is countably additive, but in a sense appropriate to the strong operator topology rather than in the uniform operator topology as in section 4. The possibility of integrating bounded measurable functions is shown, as is a theorem of the bounded convergence type. In addition, the permutability of the integral and a bounded linear operator is discussed. Except for the last result, section 3 extends these results to some extent.

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Zum distributiven Gesetz der reellen Zahlen

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Im Jahre 1952 habe ich im Zusammenhange mit einem Problem aus der Algebra der geometrischen Objekte (vgl. [4]) die Frage gestellt, welche (möglichst schwache) Voraussetzungen über die Funktionen f und g in der Gleichung des distributiven Gesetzes

$$g[f(x,y),z]=f[g(x,z),g(y,z)] \tag{1}$$

hinreichend sind, um die Folgerung zu ziehen, daß f und g einen Automorphismus inbezug auf die Addition und Multiplikation im Bereiche der reellen Zahlen bilden.

Die gestellte Frage habe ich beantwortet. Die Lösung habe ich im Jahre 1953 Herrn J. Łoś schriftlich mitgeteilt und im Jahre 1954 habe ich sie in polnischer Sprache veröffentlicht [2].

Im Jahre 1953 ist eine Arbeit von Herrn M. Hosszú [3] erschienen, die unter gewissen Regularitätsannahmen über die Funktionen f und g eine allgemeine Lösung der Gleichung (1) gibt.

Da mein Satz inzwischen eine Anwendung in der Wahrscheinlichkeitsrechnung gefunden hat [1], da zweitens die Zeitschrift, in welcher mein Ergebnis erschienen ist, schwer zugänglich ist, und da letztens es gelungen ist eine der Voraussetzungen meines Satzes abzuschwächen, habe ich mich entschlossen den Satz nochmals zu publizieren.

SATZ. *Wenn die Funktionen $f(x,y)$ und $g(x,y)$ folgende Voraussetzungen erfüllen:*

I. *sie sind reell und in der ganzen Ebene definiert;*

II. *sie gehören auf der ganzen Ebene der Klasse C_1 an (d. h. sie besitzen stetige Ableitungen erster Ordnung)*[1];

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[1]) Meine ursprüngliche Voraussetzung II war etwas stärker; sie lautete, daß die Funktion g mit den ersten Ableitungen $\partial g/\partial x$, $\partial g/\partial y$ und außerdem mit der zweiten Ableitung $\partial^2 g/\partial x\partial y$ (auf einer gewissen Geraden) ausgestattet ist. Meiner Schülerin U. Stono-Wróbel ist es gelungen diese Voraussetzung abzuschwächen. Dadurch habe ich die Beweismethode (in bezug auf die frühere) in zwei Punkten umändern müssen.