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A General Central Limit Theorem for FKG Systems*

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Abstract. A central limit theorem is given which is applicable to (not necessarily monotonic) functions of random variables satisfying the FKG inequalities. One consequence is convergence of the block spin scaling limit for the magnetization and energy densities (jointly) to the infinite temperature fixed point of independent Gaussian blocks for a large class of Ising ferromagnets whenever the susceptibility is finite. Another consequence is a central limit theorem for the density of the *surface* of the infinite cluster in percolation models.

1. Introduction

For a translation invariant d-dimensional system of L_2 random variables (or random vectors), $\{X_k : k \in \mathbb{Z}^d\}$, we define for each $n = 1, 2, \ldots$, the block variables

$$X_k^n = n^{-d/2} \sum_{j \in B_k^n} (X_j - EX_j),$$

where B_k^n is a block of side length n located near nk,

$$B_k^n = \{j : nk_l \leqq j_l < n(k_l + 1) \text{ for } l = 1, \ldots, d\} = nk + B_0^n,$$

and E denotes expectation. In [N1] a central limit theorem for $\{X_k^n : k \in \mathbb{Z}^d\}$ as $n \to \infty$ was obtained under the additional assumptions that the X_k's obey the FKG inequalities [FKG] and $\sum_k \mathrm{Cov}(X_0, X_k) < \infty$.

In the context of a general Ising model, $\{\sigma_k : k \in \mathbb{Z}^d\}$ with energy density,

$$\mathscr{E}_k = -\sum_{j \in \mathbb{Z}^d} J(j-k)\sigma_j \sigma_k - h\sigma_k \quad \left(J(j) \geqq 0 \ \forall j \text{ and } \sum_{j \in \mathbb{Z}^d} J(j) < \infty\right),$$

and single site distribution $d\varrho(\sigma_j)$, the central limit theorem of [N1] implies convergence of the σ_j^n's to independent mean zero normal random variables of

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variance χ, providing

$$\chi \equiv \sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(\sigma_0, \sigma_k) \equiv \sum_{k\in\mathbb{Z}^d} E\sigma_0\sigma_k - E\sigma_0 E\sigma_k < \infty .$$

The results of [N1] do not yield any limit theorem for the block energy density variables $\{\mathscr{E}_k^n : k\in\mathbb{Z}^d\}$ (or for the bivariate vectors $\{(\sigma_k^n, \mathscr{E}_k^n)\}$) since the $\mathscr{E}_k$'s are not (coordinatewise) monotonic functions of the σ_j's and hence do not themselves satisfy the FKG inequalities. The central limit theorem of [IS] is applicable in principle to $\{\mathscr{E}_k^n\}$ (respectively to $\{(\sigma_k^n, \mathscr{E}_k^n)\}$) but its applicability requires detailed information (which is not generally available) concerning the zeros of the partition function in the inverse temperature variable β (respectively jointly in β and h).

In Sect. 2 of this paper, we combine methods from [L], [S], and [N1] to obtain a central limit theorem applicable to nonmonotonic functions of FKG variables. It is an immediate consequence of Theorem 3 and Proposition 4 of the next section that, at least for bounded spins (i.e. $d\varrho$ having compact support), the finiteness of the susceptibility χ implies convergence of $\{(\sigma_k^n, \mathscr{E}_k^n)\}$ as $n\to\infty$ to $\{Z_k = ({}_1Z_k, {}_2Z_k)\}$, where the Z_k's are independent mean zero normal random vectors with variance of ${}_1Z_k = \chi$, variance of ${}_2Z_k = \sum_j \mathrm{Cov}(\mathscr{E}_0, \mathscr{E}_j)$, and

$$\mathrm{Cov}({}_1Z_k, {}_2Z_k) = \sum_j \mathrm{Cov}(\sigma_0, \mathscr{E}_j) = \sum_j \mathrm{Cov}(\mathscr{E}_0, \sigma_j) .$$

The convergence of these sums is also a consequence of the finiteness of χ. In order to apply Theorem 3 to obtain such a result for unbounded spins, it seems necessary to assume something in addition to the finiteness of χ, such as (for example) the finiteness of $\sum_j \mathrm{Cov}((\sigma_0)^3, (\sigma_j)^3)$.

Another application of the results of the next section is to percolation models. Suppose Y_k, $k\in\mathbb{Z}^d$, are (zero- or one-valued) occupation random variables in some (independent or correlated) site percolation model, which are translation invariant and satisfy the FKG inequalities. Let U_k (respectively W_k) be the indicator function of the event that site k belongs to (respectively to the boundary of) an infinite cluster of occupied sites. The central limit theorem of [N1] is applicable to $\{U_k\}$ or to $\{\bar{U}_k \equiv U_k + W_k\}$ (for more details, see [NS]) but not to $\{W_k\}$ since W_k is not a monotonic function of the Y_j's. Theorem 3 of the next section however, is applicable to $\{W_k\}$ and to the bivariate system $\{(U_k, W_k)\}$ or $\{(U_k, \bar{U}_k)\}$); in particular if $\sum_j \mathrm{Cov}(U_0, U_j)$ and $\sum_j \mathrm{Cov}(\bar{U}_0, \bar{U}_j)$ are convergent, then $n^{-d/2}(M_n - nE(W_0))$ converges to a mean zero normal random variable with variance $= \sum_j \mathrm{Cov}(W_0, W_j)$, where M_n denotes the number of sites in the cube, $\{j : -n/2 \leqq j_l < n/2 \text{ for } l = 1, \ldots, d\}$, which belong to the boundary of an infinite cluster of occupied sites. A similar result would apply to sites in the "external boundary" of (the union of) infinite clusters (see [NS] for the definition).

For a fuller discussion of block limits and FKG inequalities, see [N1] and the references therein; for a survey of limit theorems and related results (including those of the next section) for positively and negatively dependent random variables in the context of $(d=1)$ sequences, see [N2]. For more information on percolation model central limit theorems, see [CG] and [NS].

2. A General Central Limit Theorem

Throughout this section, $\{Y_k : k \in \mathbb{Z}^d\}$ denotes a translation invariant system of (real) random variables and L_2 denotes the Hilbert space of complex-valued random variables which are measurable with respect to the σ-field generated by the Y_k's. We define D to be the L_2-closure of

$$\{f(Y_{j_1}, \ldots, Y_{j_m}) \in L_2 : m \geqq 1, \text{ each } j_i \in \mathbb{Z}^d, \text{ and } f \text{ is real and coordinatewise nondecreasing}\}.$$

We will assume throughout that $\{Y_k\}$ satisfies the FKG inequalities; i.e. $\mathrm{Cov}(U, V) \geqq 0$ for any $U, V \in D$.

For $V, V' \in L^2$, we write $V' \gg V$ if $V' - \mathrm{Re}(e^{i\alpha} V) \in D$ for all $\alpha \in \mathbb{R}$. Using the fact that D is a convex cone, we note that since $V' = [(V' - \mathrm{Re}(V)) + (V' - \mathrm{Re}(-V))]/2$, it follows first that $V' \gg V$ implies $V' \in D$, and second that $V' \gg V$ for real V if and only if both $V' + V$ and $V' - V$ are in D. Equivalently, $V' \gg V$ for real V if and only if there exist V^+, V^- in D such that $V = V^+ - V^-$ and $V' = V^+ + V^-$. The following proposition extends results of [L] and [S].

Proposition 1. *Suppose $U' \gg U$ and $V' \gg V$; then*

$$\mathrm{Cov}(U', V') \geqq \begin{cases} |\mathrm{Cov}(U, V)|, & \text{if } U \text{ or } V \text{ is real} \quad (1a) \\ |\mathrm{Cov}(U, V)|/2, & \text{otherwise}, \quad (1b) \end{cases}$$

and

$$\mathrm{Cov}(U', V') \geqq |\mathrm{Cov}(e^{iU}, e^{iV})|/2, \quad \text{if } U \text{ and } V \text{ are real}. \tag{2}$$

Proof. In proving (1a), we may assume U is real. Since then

$$|\mathrm{Cov}(U, V)| = \sup(\mathrm{Re}(e^{i\alpha}\,\mathrm{Cov}(U, V)) : \alpha \in \mathbb{R})$$

and $\mathrm{Re}(e^{i\alpha}\,\mathrm{Cov}(U, V)) = \mathrm{Cov}(U, \hat{V})$, where $\hat{V} = \mathrm{Re}(e^{i\alpha} V)$, it suffices to show that $\mathrm{Cov}(U, \hat{V}) \leqq \mathrm{Cov}(U', V')$; this follows from the identity,

$$\mathrm{Cov}(U', V') - \mathrm{Cov}(U, \hat{V}) = [\mathrm{Cov}(U' + U, V' - \hat{V}) + \mathrm{Cov}(U' - U, V' + \hat{V})]/2,$$

and the hypotheses that $U' \gg U$ and $V' \gg \hat{V}$. Inequality (1b) follows from (1a) and the bound

$$|\mathrm{Cov}(U, V)| = |\mathrm{Cov}(\mathrm{Re}\,U, V) + i\,\mathrm{Cov}(\mathrm{Im}\,U, V)| \leqq |\mathrm{Cov}(\mathrm{Re}\,U, V)| + |\mathrm{Cov}(\mathrm{Im}\,U, V)|.$$

Finally (2) follows from (1b) and the fact that

$$U' \gg \exp(iU) \tag{3}$$

and similarly for V'. To prove (3), we write $U = U^+ - U^-$, $U' = U^+ + U^-$ with $U^+, U^- \in D$, and then approximate $U^\pm$ by $f^\pm(Y_{j_1}, \ldots, Y_{j_m})$, where $f^\pm$ is (coordinatewise) nondecreasing; it suffices to show that $f' - \mathrm{Re}(\exp[i(\alpha + f)])$ is nondecreasing, where $f' = f^+ + f^-$, $f = f^+ - f^-$. To see this, denote by Δg the increment in the function $g(x_1, \ldots, x_m)$ when one or more of the x_i's is increased and observe that $|\Delta\,\mathrm{Re}(\exp[i(\alpha + f)])| \leqq |\Delta \exp[if]| \leqq |\Delta f| \leqq \Delta f'$. This completes the proof.

The following proposition extends Theorem 1 of [N1] from nondecreasing to arbitrary functions of the Y_j's at the cost of the factor, 2, in the right hand side of (4).

Proposition 2. *Suppose $U_1, \ldots, U_N$ are real and $U_l' \gg U_l$ for each l; then for any $r_1, \ldots, r_N \in \mathbb{R}$,*

$$\left|\phi^N(r_1, \ldots, r_N) - \prod_{l=1}^{N} \phi_l(r_l)\right| \leqq 2 \sum_{\substack{l,m=1 \\ l<m}}^{N} |r_l r_m| \operatorname{Cov}(U_l', U_m'), \tag{4}$$

where

$$\phi^N = E\left(\exp\left[i \sum_{l=1}^{N} r_l U_l\right]\right), \qquad \phi_l = E(\exp[i r_l U_l]).$$

Proof. Let

$$U = \sum_{l=1}^{N-1} r_l U_l, \qquad U' = \sum_{l=1}^{N-1} |r_l| U_l', \qquad V = r_N U_N, \qquad V' = |r_N| U_N'.$$

Then $U' \gg U$ and $V' \gg V$, so that we may apply (2) to obtain

$$\begin{aligned}\left|\phi^N - \prod_{l=1}^{N} \phi_l\right| &\leqq |\phi^N - \phi^{N-1}\phi_N| + \left|\phi^{N-1}\phi_N - \prod_{l=1}^{N} \phi_l\right| \\ &= |\operatorname{Cov}(e^{iU}, e^{iV})| + |\phi_N| \left|\phi^{N-1} - \prod_{l=1}^{N-1} \phi_l\right| \\ &\leqq 2 \operatorname{Cov}(U', V') + \left|\phi^{N-1} - \prod_{l=1}^{N-1} \phi_l\right|;\end{aligned}$$

(4) now follows by induction on N.

We denote by T_k the "shift by k" operator, defined initially by

$$T_k : f(Y_{j_1}, \ldots, Y_{j_m}) \to f(Y_{j_1+k}, \ldots, Y_{j_m+k})$$

and then extended by continuity to a unitary operator on all of L_2. For random vectors $X = ({}_1X, \ldots, {}_MX)$ and $X' = ({}_1X', \ldots, {}_MX')$, we write $X' \gg X$ if ${}_lX' \gg {}_lX$ for $l = 1, \ldots, M$. We note that since D is T_j-invariant, $X' \gg X$ implies $T_jX' \gg T_jX$.

Theorem 3. *Suppose $X_k = T_kX$, $X_k' = T_kX'$ for each $k \in \mathbb{Z}^d$, where X and X' are (real) random vectors with $X' \gg X$. If*

$$A_{il}' \equiv \sum_{j \in \mathbb{Z}^d} \operatorname{Cov}({}_iX_0', {}_lX_j') < \infty \quad \textit{for all} \quad i = l, \tag{5}$$

then $\{X_k^n : k \in \mathbb{Z}^d\}$ converges as $n \to \infty$ to $\{Z_k : k \in \mathbb{Z}^d\}$, where the Z_k's are independent mean zero normal random vectors with

$$\operatorname{Cov}({}_iZ_k, {}_lZ_k) = A_{il} \equiv \sum_{j \in \mathbb{Z}^d} \operatorname{Cov}({}_iX_0, {}_lX_j) \textit{ for all } i, l;$$

the convergence is in the sense of finite dimensional distributions, i.e.

$$\lim_{n \to \infty} E(g(X_{j_1}^n, \ldots, X_{j_m}^n)) = E(g(Z_{j_1}, \ldots, Z_{j_m}))$$

for any $m, j_1, \ldots, j_m$ and bounded continuous function g.

Proof. We first note that since the matrix A' is the limit of the (positive semidefinite) covariance matrix of X_0^n, it follows from (5) that $A'_{il} \leqq (A_{ii}A_{ll})^{1/2} < \infty$ for $i \neq l$. By essentially the same proof as for Theorem 2 of [N1] (but with Theorem 1 of [N1] replaced by Proposition 2 above) it follows that for real vectors $s_j, j \in \mathbb{Z}^d$,

$$W_j^n = s_j \cdot X_j^n \equiv \sum_{l=1}^{M} ({}_l s_j)({}_l X_j^n) \quad \text{and} \quad W_j'^n = \sum_{l=1}^{M} |{}_l s_j| ({}_l X_j'^n)$$

are such that

$$\lim_{n\to\infty} \operatorname{Cov}(W_j'^n, W_k'^n) = 0 \quad \forall j \neq k, \tag{7}$$

and

$$E(\exp[iW_j^n]) \to \exp[-(s_j \cdot As_j)/2]. \tag{8}$$

Now (7) together with Proposition 2 implies

$$\lim_{n\to\infty} \left[E\left(\exp\left[i \sum_{l=1}^{m} W_{j_l}^n\right]\right) - \prod_{l=1}^{m} E(\exp[iW_{j_l}^n]) \right] = 0. \tag{9}$$

Finally (8) and (9) (for arbitrary s_j's) imply by standard arguments the convergence of $\{X_k^n\}$ to $\{Z_k\}$.

We conclude with a proposition useful for verifying the hypotheses of Theorem 3 in the Ising model example discussed in Sect. 1; the proof uses an argument of [S].

Proposition 4. *Suppose the Y_k's are bounded (in absolute value) and $X = \sum_{j\in\mathbb{Z}^d} K_j Y_j Y_0$, where the K_j's are real with $\sum_j |K_j| < \infty$. Then there is a summable sequence K'_j such that*

$$X' \equiv \sum_j K'_j Y_j \geqslant X; \tag{10}$$

furthermore, the convergence of $\sum_j \operatorname{Cov}(Y_0, Y_j)$ implies that of $\sum_j \operatorname{Cov}(X'_0, X'_j)$, where $X'_j = T_j X'$.

Proof. Let L be the bound on $|Y_j|$, and define

$$h(y) = \begin{cases} -L, & \text{for } \quad y < -L, \\ y, & \text{for } -L \leqq y \leqq L, \\ L, & \text{for } \quad L < y. \end{cases}$$

Now $Y_j = h(Y_j)$, while $Ly_j + Ly_k - h(y_j) \cdot h(y_k)$ is a nondecreasing function of y_j and y_k; hence $LY_j + LY_0 \geqslant Y_j Y_0$ and (10) is valid with $K'_0 = LK_0 + L\sum_j K_j$ and $K'_j = LK_j$ for $j \neq 0$. The last statement of the proposition follows immediately from the identity,

$$\sum_j \operatorname{Cov}(X'_0, X'_j) = \sum_j \sum_{k_1} \sum_{k_2} K'_{k_1} K'_{k_2} \operatorname{Cov}(Y_{k_1}, Y_{k_2+j}) = \left(\sum_{j'} \operatorname{Cov}(Y_0, Y_{j'})\right)\left(\sum_k K'_k\right)^2.$$

Remark. A similar proposition applies to general random variables of the form

$$X=\sum_{m=1}^{\infty} \tilde{K}_{j_1\dots j_m} Y_{j_1} \dots Y_{j_m}, \quad \text{with} \sum_{m=1}^{\infty} \sum_{j_1,\dots,j_m} m|\tilde{K}_{j_1\dots j_m}| L^{m-1} < \infty .$$

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