A general characterization of the mean field limit for stochastic differential games

Daniel Lacker

ORFE, *Princeton University*

January 24, 2015

Section 1

Introduction

Mean field theories beyond physics

Mean field theory in a nutshell: Approximate statistical features of a n-particle system by a ∞-particle system.

Applications outside of physics:

- economics & finance (systemic risk, income distribution...)
- biology (flocking...)
- sociology (crowd dynamics, voter models...)
- electrical engineering (telecommunications...)

Mean field theories beyond physics

Mean field theory in a nutshell: Approximate statistical features of a n-particle system by a ∞-particle system.

Applications outside of physics:

- economics & finance (systemic risk, income distribution...)
- biology (flocking...)
- sociology (crowd dynamics, voter models...)
- electrical engineering (telecommunications...)

Why mean field game theory?

It replaces particles with rational agents. Laws of motion emerge in equilibrium and need not be prescribed exogenously.

Mean field theories beyond physics

Main novelties of MFG theory: continuous time (PDEs, SDEs) and rigorous connection to finite-population models

Most-studied so far: MFG analogs of McKean-Vlasov interacting diffusion models. $\rightsquigarrow$ Stochastic differential MFGs

Some recent literature: MFG analogs of

- Spin systems (Horst/Scheinkman)
- Stochastic coalescence (Duffie/Malamud/Manso)

Section 2

A prototypical MFG model

A model of systemic risk: Carmona/Fouque/Sun '13

Mean field model:

n banks with log-monetary reserves $(X_t^i)_{t\in[0,T]}$,

$$dX_t^i = a(\overline{X}_t - X_t^i)dt + \sigma\rho dW_t^i + \sigma\sqrt{1-\rho^2}dB_t,$$
$$\overline{X}_t = \frac{1}{n}\sum_{k=1}^{n} X_t^k$$

Rate of borrowing/lending between banks: $a > 0$

A model of systemic risk: Carmona/Fouque/Sun '13

Mean field model:
n banks with log-monetary reserves $(X_t^i)_{t\in[0,T]}$,

$$dX_t^i = a(\overline{X}_t - X_t^i)dt + \sigma\rho dW_t^i + \sigma\sqrt{1-\rho^2}dB_t,$$
$$\overline{X}_t = \frac{1}{n}\sum_{k=1}^{n} X_t^k$$

Rate of borrowing/lending between banks: $a > 0$

Goal: Find probabilities of systemic events of the form

$$\left\{\min_{0\le t\le T} \overline{X}_t \le D\right\}, \quad D = \text{ default level.}$$

A model of systemic risk: Carmona/Fouque/Sun '13

Mean field game model:
n banks with log-monetary reserves $(X^i_t)_{t\in[0,T]}$,

$$dX^i_t = \left[a(\overline{X}_t - X^i_t) + \alpha^i_t\right] dt + \sigma\rho dW^i_t + \sigma\sqrt{1-\rho^2}dB_t,$$
$$\overline{X}_t = \frac{1}{n}\sum_{k=1}^n X^k_t$$

Bank i chooses to borrow/lend from a central bank at rate α^i_t, to minimize some cost

$$\mathbb{E}\left[\int_0^T f(X^i_t, \overline{X}_t, \alpha^i_t)dt + g(X^i_T, \overline{X}_T)\right].$$

A model of systemic risk: Carmona/Fouque/Sun '13

Mean field game model:
n banks with log-monetary reserves $(X^i_t)_{t\in[0,T]}$,

$$dX^i_t = \left[a(\overline{X}_t - X^i_t) + \alpha^i_t\right] dt + \sigma\rho dW^i_t + \sigma\sqrt{1-\rho^2}dB_t,$$
$$\overline{X}_t = \frac{1}{n}\sum_{k=1}^n X^k_t$$

Bank i chooses to borrow/lend from a central bank at rate α^i_t, to minimize some cost

$$\mathbb{E}\left[\int_0^T f(X^i_t, \overline{X}_t, \alpha^i_t)dt + g(X^i_T, \overline{X}_T)\right].$$

Goal: Find systemic event probabilities in Nash equilibrium.

Section 3

Stochastic differential mean field games

Stochastic differential games

Agents $i = 1, \dots, n$ have state process dynamics

$$dX_t^i = b(X_t^i, \bar{\mu}_t^n, \alpha_t^i)dt + \sigma dW_t^i + \sigma_0 dB_t,$$

$$\bar{\mu}_t^n := \frac{1}{n}\sum_{k=1}^n \delta_{X_t^k},$$

with $B, W^1, \dots, W^n$ independent, $(X_0^1, \dots, X_0^n)$ i.i.d.

Stochastic differential games

Agents $i = 1, \ldots, n$ have state process dynamics

$$dX_t^i = b(X_t^i, \bar{\mu}_t^n, \alpha_t^i)dt + \sigma dW_t^i + \sigma_0 dB_t,$$

$$\bar{\mu}_t^n := \frac{1}{n}\sum_{k=1}^n \delta_{X_t^k},$$

with $B, W^1, \ldots, W^n$ independent, $(X_0^1, \ldots, X_0^n)$ i.i.d.

Agent i chooses α^i to minimize

$$J_i^n(\alpha^1, \ldots, \alpha^n) := \mathbb{E}\left[\int_0^T f(X_t^i, \bar{\mu}_t^n, \alpha_t^i)dt + g(X_T^i, \bar{\mu}_T^n)\right].$$

Stochastic differential games

Agents $i = 1, \ldots, n$ have state process dynamics

$$dX_t^i = b(X_t^i, \bar{\mu}_t^n, \alpha_t^i)dt + \sigma dW_t^i + \sigma_0 dB_t,$$

$$\bar{\mu}_t^n := \frac{1}{n}\sum_{k=1}^n \delta_{X_t^k},$$

with $B, W^1, \ldots, W^n$ independent, $(X_0^1, \ldots, X_0^n)$ i.i.d.
Agent i chooses α^i to minimize

$$J_i^n(\alpha^1, \ldots, \alpha^n) := \mathbb{E}\left[\int_0^T f(X_t^i, \bar{\mu}_t^n, \alpha_t^i)dt + g(X_T^i, \bar{\mu}_T^n)\right].$$

Say $(\alpha^1, \ldots, \alpha^n)$ form an ϵ-Nash equilibrium if $\forall i = 1, \ldots, n$

$$J_i^n(\alpha^1, \ldots, \alpha^n) \le \epsilon + \inf_\beta J_i^n(\ldots, \alpha^{i-1}, \beta, \alpha^{i+1}, \ldots).$$

Mean field limit $n \to \infty$?

The problem

Given for each n an ϵ_n-Nash equilibrium $(\alpha^{n,1}, \ldots, \alpha^{n,n})$, with $\epsilon_n \to 0$, can we characterize the possible limits of $\bar{\mu}^n_t$?
Limiting behavior of a representative agent?

Mean field limit $n \to \infty$?

The problem

Given for each n an ϵ_n-Nash equilibrium $(\alpha^{n,1}, \ldots, \alpha^{n,n})$, with $\epsilon_n \to 0$, can we characterize the possible limits of $\bar{\mu}^n_t$?
Limiting behavior of a representative agent?

Previous results

Lasry/ Lions '06, Bardi '11, Feleqi '13, Gomes '13, Carmona/Fouque/Sun '13, Fischer '14

Mean field limit $n \to \infty$?

The problem

Given for each n an ϵ_n-Nash equilibrium $(\alpha^{n,1}, \dots, \alpha^{n,n})$, with $\epsilon_n \to 0$, can we characterize the possible limits of $\bar{\mu}_t^n$?
Limiting behavior of a representative agent?

Previous results

Lasry/ Lions '06, Bardi '11, Feleqi '13, Gomes '13, Carmona/Fouque/Sun '13, Fischer '14

A related, better-understood problem

Find a mean field game solution directly, and use it to construct an ϵ_n-Nash equilibrium for the n-player game.
See Huang/Malhamé/Caines '06 & many others.

Proposed mean field game limit, without common noise

Intuition and the existing literature suggest that $\bar{\mu}^n$ may converge to a mean field game (MFG) limit, a process μ satisfying:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right], \\ dX_t^\alpha & = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma dW_t, \\ & \end{cases}$$

Proposed mean field game limit, without common noise

Intuition and the existing literature suggest that $\bar{\mu}^n$ may converge to a mean field game (MFG) limit, a process μ satisfying:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right], \\ dX_t^\alpha & = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma dW_t, \\ \mu_t & = \mathrm{Law}(X_t^{\alpha^*}). \end{cases}$$

Proposed mean field game limit, with common noise

Intuition and the existing literature suggest that $\bar{\mu}^n$ may converge to a mean field game (MFG) limit, a process μ satisfying:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right], \\ dX_t^\alpha & = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma dW_t + \sigma_0 dB_t, \\ & \end{cases}$$

Proposed mean field game limit, with common noise

Intuition and the existing literature suggest that $\bar{\mu}^n$ may converge to a mean field game (MFG) limit, a process μ satisfying:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right], \\ dX_t^\alpha & = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma dW_t + \sigma_0 dB_t, \\ \mu_t & = \mathrm{Law}(X_t^{\alpha^*} \mid \mathcal{F}_t^B), \quad \mathcal{F}_t^B := \sigma(B_s : s \leq t). \end{cases}$$

Proposed mean field game limit, with common noise

Intuition and the existing literature suggest that $\bar{\mu}^n$ may converge to a mean field game (MFG) limit, a process μ satisfying:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right], \\ dX_t^\alpha & = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma dW_t + \sigma_0 dB_t, \\ \mu_t & = \mathrm{Law}(X_t^{\alpha^*} \mid \mathcal{F}_t^B), \quad \mathcal{F}_t^B := \sigma(B_s : s \leq t). \end{cases}$$

We call this a strong MFG solution, since μ_t is $\mathcal{F}_t^B$-adapted.

Proposed mean field game limit, with common noise

Intuition and the existing literature suggest that $\bar{\mu}^n$ may converge to a mean field game (MFG) limit, a process μ satisfying:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right], \\ dX_t^\alpha & = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma dW_t + \sigma_0 dB_t, \\ \mu_t & = \text{Law}(X_t^{\alpha^*} \mid \mathcal{F}_t^B), \quad \mathcal{F}_t^B := \sigma(B_s : s \le t). \end{cases}$$

We call this a strong MFG solution, since μ_t is $\mathcal{F}_t^B$-adapted. Without some kind of uniqueness (hard to come by!), we should expect only a weak solution:

$$\mu_t = \text{Law}(X_t^{\alpha^*} \mid \mathcal{F}_t^{\mu,B}), \text{ with } X_0, (\mu, B), W \text{ independent.}$$

Standing assumptions

Admissible controls for n-player game

Any $\mathcal{F}^n_t$-adapted process, where

$$\mathcal{F}^n_t \supset \sigma(X^1_0, \ldots, X^n_0, W^1_s, \ldots, W^n_s, B_s : s \leq t).$$

Technicalities

b, f, g continuous, control space $A \subset \mathbb{R}^k$ closed, b Lipschitz in (x, μ), growth assumptions...

Main results

Theorem (Mean field limit)

Given for each n an ϵ_n-Nash equilibrium with $\epsilon_n \to 0$, the sequence $(\bar{\mu}^n)_{n=1}^{\infty}$ is tight, and every limit is a weak MFG solution. Conversely, every weak MFG solution can be obtained as a limit in this way.

Main results

Theorem (Mean field limit)

Given for each n an ϵ_n-Nash equilibrium with $\epsilon_n \to 0$, the sequence $(\bar{\mu}^n)_{n=1}^{\infty}$ is tight, and every limit is a weak MFG solution. Conversely, every weak MFG solution can be obtained as a limit in this way.

Theorem (Existence, with R. Carmona & F. Delarue)

There exists a weak MFG solution.

Main results

Theorem (Mean field limit)

Given for each n an ϵ_n-Nash equilibrium with $\epsilon_n \to 0$, the sequence $(\bar{\mu}^n)_{n=1}^{\infty}$ is tight, and every limit is a weak MFG solution. Conversely, every weak MFG solution can be obtained as a limit in this way.

Theorem (Existence, with R. Carmona & F. Delarue)

There exists a weak MFG solution.

Theorem (Uniqueness, with R. Carmona & F. Delarue)

A Yamada-Watanabe-type theorem holds for MFGs, and under strong additional assumptions we have pathwise uniqueness.

Main results

Theorem (Mean field limit)

Given for each n an ϵ_n-Nash equilibrium with $\epsilon_n \to 0$, the sequence $(\bar{\mu}^n)_{n=1}^{\infty}$ is tight, and every limit is a weak MFG solution. Conversely, every weak MFG solution can be obtained as a limit in this way.

Theorem (Existence, with R. Carmona & F. Delarue)

There exists a weak MFG solution.

Theorem (Uniqueness, with R. Carmona & F. Delarue)

A Yamada-Watanabe-type theorem holds for MFGs, and under strong additional assumptions we have pathwise uniqueness.

Theorem (Existence, with K. Webster)

"Translation invariant" MFGs admit strong solutions.

Section 4

A surprise in the case of no common noise

Interacting particle system without common noise

Particles $i = 1, \dots, n$ have dynamics

$$dX_t^i = b(X_t^i, \bar{\mu}_t^n)dt + \sigma dW_t^i,$$
$$\bar{\mu}_t^n := \frac{1}{n}\sum_{k=1}^{n} \delta_{X_t^k},$$

with $W^1, \dots, W^n$ independent, $(X_0^1, \dots, X_0^n)$ i.i.d.

Mean field limit $n \to \infty$

Theorem

1. $\bar{\mu}^n$ *are tight in* $C([0,T];\mathcal{P}(\mathbb{R}))$.

Mean field limit $n \to \infty$

Theorem

1. *$\bar{\mu}^n$ are tight in $C([0,T];\mathcal{P}(\mathbb{R}))$.*
2. *Every weak limit μ is such that a.e. realization $\nu \in C([0,T];\mathcal{P}(\mathbb{R}))$ satisfies the McKean-Vlasov (MV) equation:*

$$\begin{cases} dX_t = b(X_t,\nu_t)dt + \sigma dW_t, \\ \nu_t = Law(X_t). \end{cases}$$

See: Oelschläger '84, Gärtner '88.

McKean-Vlasov equations (MFG without control)

Strong McKean-Vlasov solution: A deterministic μ s.t.:

$$\begin{cases} dX_t & = b(X_t, \mu_t)dt + \sigma dW_t, \\ \mu_t & = \text{Law}(X_t). \end{cases}$$

McKean-Vlasov equations (MFG without control)

Strong McKean-Vlasov solution: A deterministic μ s.t.:

$$\begin{cases} dX_t & = b(X_t, \mu_t)dt + \sigma dW_t, \\ \mu_t & = \mathrm{Law}(X_t). \end{cases}$$

Weak McKean-Vlasov solution: A stochastic μ s.t.:

$$\begin{cases} dX_t & = b(X_t, \mu_t)dt + \sigma dW_t, \\ \mu_t & = \mathrm{Law}(X_t \mid \mathcal{F}_t^{\mu}), \quad \text{with } X_0,\ W,\ \mu \text{ independent.} \end{cases}$$

McKean-Vlasov equations (MFG without control)

Strong McKean-Vlasov solution: A deterministic μ s.t.:

$$\begin{cases} dX_t &= b(X_t, \mu_t)dt + \sigma dW_t, \\ \mu_t &= \mathrm{Law}(X_t). \end{cases}$$

Weak McKean-Vlasov solution: A stochastic μ s.t.:

$$\begin{cases} dX_t &= b(X_t, \mu_t)dt + \sigma dW_t, \\ \mu_t &= \mathrm{Law}(X_t \mid \mathcal{F}^\mu_t), \quad \text{with } X_0, \ W, \ \mu \text{ independent}. \end{cases}$$

Theorem

A random measure μ is a weak solution if and only if it is concentrated on the set of strong solutions, that is a.e. realization is a strong solution.

MFG solutions without common noise

Strong MFG solution: A deterministic μ s.t.:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right], \\ dX_t^\alpha & = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma dW_t, \\ \mu_t & = \mathrm{Law}(X_t^{\alpha^*}). \end{cases}$$

MFG solutions without common noise

Strong MFG solution: A deterministic μ s.t.:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right], \\ dX_t^\alpha & = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma dW_t, \\ \mu_t & = \mathrm{Law}(X_t^{\alpha^*}). \end{cases}$$

Weak MFG solution: A stochastic μ s.t.:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right], \\ dX_t^\alpha & = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma dW_t, \\ \mu_t & = \mathrm{Law}(X_t^{\alpha^*} \mid \mathcal{F}_t^\mu), \quad \text{with } X_0,\ W,\ \mu \text{ independent.} \end{cases}$$

Weak vs strong MFG solutions

Until now, the MFG literature only considered strong solutions:

A natural question:

Are weak MFG solutions concentrated on the set of strong MFG solutions? In other words, is a.e. realization of a weak MFG solution a strong MFG solution?

Weak vs strong MFG solutions

Until now, the MFG literature only considered strong solutions:

A natural question:

Are weak MFG solutions concentrated on the set of strong MFG solutions? In other words, is a.e. realization of a weak MFG solution a strong MFG solution?

Answer

NO.

Weak vs strong MFG solutions

Until now, the MFG literature only considered strong solutions:

A natural question:

Are weak MFG solutions concentrated on the set of strong MFG solutions? In other words, is a.e. realization of a weak MFG solution a strong MFG solution?

Answer

NO.

Conclusion

Strong solutions are not enough to describe mean field limits.

Weak vs strong MFG solutions

Until now, the MFG literature only considered strong solutions:

A natural question:

Are weak MFG solutions concentrated on the set of strong MFG solutions? In other words, is a.e. realization of a weak MFG solution a strong MFG solution?

Answer

NO.

Conclusion

Strong solutions are not enough to describe mean field limits.

The obstruction

When μ is deterministic, the control α can anticipate μ.

A resolution

A sufficient condition

For each deterministic $\mu = (\mu_t)_{t\in[0,T]}$, find an optimal control $\alpha^*[\mu] = (\alpha^*[\mu]_t)_{t\in[0,T]}$. Suppose

$$\alpha^*[\mu]_t = \alpha^*[\mu_{\cdot\wedge t}]_t, \qquad \text{for all } t, \mu.$$

Then every weak solution is concentrated on the set of strong solutions.

Open problem

For a family of optimal control problems parametrized by paths $(\mu_t)_{t\in[0,T]}$, under what conditions is the dependence of the optimal control on the parameter adapted?

Section 5

MFG limit proof outline

Interacting particle system with common noise

Particles $i = 1, \ldots, n$ have dynamics

$$dX_t^i = b(X_t^i, \bar{\mu}_t^n)dt + \sigma dW_t^i + \sigma_0 dB_t,$$
$$\bar{\mu}_t^n := \frac{1}{n}\sum_{k=1}^n \delta_{X_t^k},$$

with $B, W^1, \ldots, W^n$ independent, $(X_0^1, \ldots, X_0^n)$ i.i.d.

Mean field limit $n \to \infty$, an unorthodox approach

Theorem

1. $(\bar{\mu}^n, B, W^1, X^1)$ *are tight in* $C([0,T]; \mathcal{P}(\mathbb{R}) \times \mathbb{R}^3)$.

Mean field limit $n \to \infty$, an unorthodox approach

Theorem

1. $(\bar{\mu}^n, B, W^1, X^1)$ *are tight in* $C([0,T]; \mathcal{P}(\mathbb{R}) \times \mathbb{R}^3)$.
2. *Every weak limit* (μ, B, W, X) *solves the conditional McKean-Vlasov (CMV) equation:*

$$\begin{cases} dX_t &= b(X_t, \mu_t)dt + \sigma dW_t + \sigma_0 dB_t, \\ \mu_t &= Law(X \mid \mathcal{F}_t^{\mu,B}), \;\; \text{with } X_0, (\mu, B), W \text{ independent.} \end{cases}$$

Mean field limit $n \to \infty$, an unorthodox approach

Theorem

1. $(\bar{\mu}^n, B, W^1, X^1)$ *are tight in* $C([0,T]; \mathcal{P}(\mathbb{R}) \times \mathbb{R}^3)$.
2. *Every weak limit* (μ, B, W, X) *solves the conditional McKean-Vlasov (CMV) equation:*

$$\begin{cases} dX_t &= b(X_t, \mu_t)dt + \sigma dW_t + \sigma_0 dB_t, \\ \mu_t &= Law(X \mid \mathcal{F}_t^{\mu,B}), \ \textit{with } X_0, (\mu, B), W \ \textit{independent}. \end{cases}$$

This approach keeps track of a representative particle and thus adapts well to the MFG setting.

Proving the MFG limit

Theorem

Given for each n an ϵ_n-Nash equilibrium $(\alpha^{n,1}, \ldots, \alpha^{n,n})$ with $\epsilon_n \to 0$, the sequence $(\bar{\mu}^n)_{n=1}^{\infty}$ is tight, and every limit is a weak MFG solution.

Proof outline

1. Deal with lack of exchangeability.
2. Control the controls.
3. Prove tightness.
4. Check dynamics and fixed point condition at limit.
5. Prove optimality of limits.

A pipe dream

If $\alpha_t^{n,i} = \hat{\alpha}(t, X_t^i, \bar{\mu}_t^n)$ for some nice function $\hat{\alpha}$, $\forall 1 \leq i \leq n$, then reduce to the particle system case.

Step 1: Exchangeability

Naive idea

Study the joint law of $(\bar{\mu}^n, B, W^1, \alpha^{n,1}, X^1)$.

Step 1: Exchangeability

Naive idea

Study the joint law of $(\bar{\mu}^n, B, W^1, \alpha^{n,1}, X^1)$.

Problem

No reason to expect $(\alpha^{n,1}, \ldots, \alpha^{n,n})$ or $(X^1, \ldots, X^n)$ to be exchangeable.

Step 1: Exchangeability

Naive idea

Study the joint law of $(\bar{\mu}^n, B, W^1, \alpha^{n,1}, X^1)$.

Problem

No reason to expect $(\alpha^{n,1}, \ldots, \alpha^{n,n})$ or $(X^1, \ldots, X^n)$ to be exchangeable.

Solution

Randomly select representative agent. Study the laws

$$Q_n = \frac{1}{n} \sum_{i=1}^{n} \mathrm{Law}\left(\bar{\mu}^n, B, W^i, X^i, \alpha^{n,i}\right)$$
$$= \mathrm{Law}\left(\bar{\mu}^n, B, W^U, X^U, \alpha^{n,U}\right),$$

where $U \sim \mathrm{Uniform}\{1, \ldots, n\}$ is independent of everything.

Step 2: Control the controls

Problem

Find a good space for the controls, $\alpha^{n,i}$. Compactness is difficult in $L^0([0,T];A)$, with topology of convergence in measure.

Step 2: Control the controls

Problem

Find a good space for the controls, $\alpha^{n,i}$. Compactness is difficult in $L^0([0,T];A)$, with topology of convergence in measure.

Solution

Use relaxed controls,

$$\mathcal{V} := \text{weak closure}\left\{dt\delta_{\alpha(t)}(da) : \alpha \in L^0([0,T];A)\right\} \cong \left(L^0([0,T];\mathcal{P}(A)),\ \tau_{\text{relaxed}}\right).$$

Drift with a relaxed control Λ is $\int_A b(X_t,\mu_t,a)\Lambda_t(da)$.

Step 2: Control the controls

Problem

Find a good space for the controls, $\alpha^{n,i}$. Compactness is difficult in $L^0([0,T];A)$, with topology of convergence in measure.

Solution

Use relaxed controls,

$$\begin{aligned}\mathcal{V} &:= \text{weak closure}\left\{dt\delta_{\alpha(t)}(da) : \alpha \in L^0([0,T];A)\right\} \\ &\cong \left(L^0([0,T];\mathcal{P}(A)),\ \tau_{\text{relaxed}}\right).\end{aligned}$$

Drift with a relaxed control Λ is $\int_A b(X_t, \mu_t, a)\Lambda_t(da)$.

An extreme case

Suppose $g \equiv f \equiv 0$. Then any strategies $(\alpha^{n,1}, \ldots, \alpha^{n,n})$ are Nash, and any relaxed control can arise in the limit.

Step 5: Optimality

Problem

What is the right class of admissible (relaxed) controls Λ for the MFG?

Natural but bad choice #1

Require Λ adapted to the filtration $\mathcal{F}_t^{X_0,\mu,B,W}$ generated by (X_0, μ, B, W), the given sources of randomness for the control problems. This class is too small, and does not necessarily contain our limit.

Step 5: Optimality

Problem

What is the right class of admissible (relaxed) controls Λ for the MFG?

Natural but bad choice #1

Require Λ adapted to the filtration $\mathcal{F}_t^{X_0,\mu,B,W}$ generated by (X_0, μ, B, W), the given sources of randomness for the control problems. This class is too small, and does not necessarily contain our limit.

Natural but bad choice #2

Require that B and W remain Wiener processes with respect to the filtration generated by $(X_0, \mu, B, W, \Lambda)$. This class is too large, and our limit may not be optimal in this class.

Step 5: Optimality

The right choice

Require Λ to be compatible, meaning that $\mathcal{F}^{\Lambda}_t$ is conditionally independent of $\mathcal{F}^{X_0,\mu,B,W}_T$ given $\mathcal{F}^{X_0,\mu,B,W}_t$, for each t.

Step 5: Optimality

The right choice

Require Λ to be compatible, meaning that $\mathcal{F}_t^{\Lambda}$ is conditionally independent of $\mathcal{F}_T^{X_0,\mu,B,W}$ given $\mathcal{F}_t^{X_0,\mu,B,W}$, for each t.

Lemma

Under any weak limit, the relaxed control Λ is compatible.

Step 5: Optimality

The right choice

Require Λ to be compatible, meaning that $\mathcal{F}_t^{\Lambda}$ is conditionally independent of $\mathcal{F}_T^{X_0,\mu,B,W}$ given $\mathcal{F}_t^{X_0,\mu,B,W}$, for each t.

Lemma

Under any weak limit, the relaxed control Λ is compatible.

Lemma

A relaxed control Λ is compatible if and only if there exists a sequence of $\mathcal{F}_t^{X_0,\mu,B,W}$-adapted strict controls $\hat{\alpha}_k = \hat{\alpha}_k(t, X_0, \mu, B, W)$, continuous in μ, such that

$$(X_0, \mu, B, W, \hat{\alpha}_k(t, X_0, \mu, B, W)) \Rightarrow (X_0, \mu, B, W, \Lambda).$$

Step 5: Optimality

Fix a weak limit (μ, B, W, Λ, X). Show Λ optimal among compatible controls:

1. Consider first a $\mathcal{F}_t^{X_0,\mu,B,W}$-adapted strict control $\hat{\alpha}(t, X_0, \mu, B, W)$, with $\hat{\alpha}$ continuous in μ.
2. Construct an admissible strategy for the n-player game via $\beta_t^{n,i} = \hat{\alpha}(t, X_0^i, \bar{\mu}^n, B, W^i)$.
3. By ϵ_n-Nash property in n-player game, $\alpha^{n,i}$ is nearly superior to $\beta^{n,i}$ for agent i.
4. Passing the inequality to the limit (using continuity of $\hat{\alpha}$ in μ), Λ is superior to $\hat{\alpha}(t, X_0, \mu, B, W)$.
5. Conclude by approximating general compatible controls by such $\hat{\alpha}(t, X_0, \mu, B, W)$.

Section 6

Refinements

Strict controls

We are more interested in MFG solutions with strict controls, meaning $\Lambda_t = \delta_{\alpha_t}$ for some A-valued process α.

Theorem

Suppose for each (x, μ) the set

$$\{(b(x, \mu, a), z) : a \in A, \ z \geq f(x, \mu, a)\}$$

is convex. Then for every weak MFG solution there exists another weak MFG solution with strict control with the same $Law(\mu, B, W, X)$.

Strong controls

We are even more interested in MFG solutions with strong controls, meaning $\Lambda_t = \delta_{\alpha_t}$ for some $\mathcal{F}_t^{X_0,\mu,B,W}$-progressive A-valued process α.

Theorem

Suppose b is affine in (x,a), f is strictly convex in (x,a), and g is convex in x. Then every weak MFG solution necessarily has strong control.

$\Rightarrow$ Can state MFG limit theorem without reference to relaxed controls or compatibility

Uniqueness

We are even more interested in strong MFG solutions, meaning the control is strong and also η is B-measurable, so

$$\mu_t = \mathrm{Law}(X_t \in \cdot \mid \mathcal{F}_t^B).$$

Theorem

Suppose $b = b(x, a)$ is affine in (x, a) and independent of the mean field, f is strictly convex in (x, a), g is convex in x, $f = f_1(t, x, \mu) + f_2(t, x, a)$, and monotonicity holds: $\forall \mu, \nu$,

$$\int \left[f_1(t, x, \mu) - f_1(t, x, \nu) + g(x, \mu) - g(x, \nu)\right](\mu - \nu)(dx) \geq 0.$$

Then "pathwise uniqueness" holds, and the unique weak MFG solution is strong. In particular, for every sequence of ϵ_n-Nash equilibria converges to the unique MFG solution.