

## A GENERAL CLASS OF IMPULSIVE EVOLUTION EQUATIONS

JINRONG WANG — MICHAL FEČKAN

---

**ABSTRACT.** One of the novelty of this paper is the study of a general class of impulsive differential equations, which is more reasonable to show dynamics of evolution processes in Pharmacotherapy. This fact reduces many difficulties in applying analysis methods and techniques in Bielecki's normed Banach spaces and thus makes the study of existence and uniqueness theorems interesting. The other novelties of this paper are new concepts of Ulam's type stability and Ulam–Hyers–Rassias stability results on compact and unbounded intervals.

### 1. Introduction

The dynamic of evolution processes in the real world is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. In general, these short-term perturbations are usually regarded as having acted instantaneously or appearing in the form of instantaneous impulses involving the corresponding differential equations. Many authors were devoted to study mild solutions to impulsive evolution equations with instantaneous impulses of the

---

2010 *Mathematics Subject Classification.* Primary: 35F55, 35R12; Secondary: 34G20, 37K45, 47D06.

*Key words and phrases.* Impulsive evolution equations, mild solutions, existence, stability.

The first author acknowledges the support by National Natural Science Foundation of China (11201091), Outstanding Scientific and Technological Innovation Talent Award of Education Department of Guizhou Province ([2014]240) and Doctor Project of Guizhou Normal College (13BS010). The second author acknowledges the support by Grants VEGA-MS 1/0071/14 and VEGA-SAV 2/0029/13 and by Slovak Research and Development Agency under the contract No. APVV-14-0378.

form

$$(1.1) \quad \begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in J' := J \setminus \{t_1, \dots, t_m\}, \quad J := [0, T], \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)), & k = 1, \dots, m, \end{cases}$$

where the linear unbounded operator  $A: D(A) \subseteq X \rightarrow X$  is the generator of a  $C_0$ -semigroup (analytic or compact)  $\{T(t), t \geq 0\}$  on a Banach space  $X$  with a norm  $\|\cdot\|$ ,  $f: J \times X \rightarrow X$  and  $I_k: X \rightarrow X$  and fixed impulsive time  $t_k$  satisfy  $0 = t_0 < t_1 < \dots, t_m < t_{m+1} = T$ , the symbols  $x(t_k^+) := \lim_{\varepsilon \rightarrow 0^+} x(t_k + \varepsilon)$  and  $x(t_k^-) := \lim_{\varepsilon \rightarrow 0^-} x(t_k + \varepsilon)$  represent the right and left limits of  $x(t)$  at  $t = t_k$ , respectively. Note that  $I_k$  in (1.1) is a sequences of instantaneously impulse operators and has been developed in physics, population dynamics, biotechnology, and so forth. For more details on differential equations with instantaneous impulses, one can see for instance the monographs [8], [10], [33], the works on not time variable impulses problem [4], [5], [12], [14]–[16], [26], [34], [36], [37] and time variable impulses problem [1]–[3], [17]–[19] and the references therein.

However, the action of instantaneous impulses seems do not describe some certain dynamics of evolution processes in Pharmacotherapy. Taking into consideration the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. Thus, we do not expect to use the model (1.1) to describe such process. In fact, the above situation is fallen in a new case of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval. To this end, Hernández and O'Regan [21] introduce a new class of impulsive evolution equations (with not instantaneous impulses) of the form

$$(1.2) \quad \begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}], \quad i = 0, \dots, m, \\ x(t) = g_i(t, x(t)), & t \in (t_i, s_i], \quad i = 1, \dots, m, \\ x(0) = x_0 \in X, \end{cases}$$

where  $A$  and  $f$  are the same as (1.1) and the fixed points  $s_i$  and  $t_i$  satisfy  $0 = t_0 = s_0 < t_1 < s_1 < t_2 < \dots < s_{m-1} < t_m < s_m < t_{m+1} = T$ , and  $g_i: [t_i, s_i] \times X \rightarrow X$  is continuous for all  $i = 1, \dots, m$ . Here  $g_i$  is regarded as continuous action process.

The concepts of mild solutions and classical solutions are introduced by Hernández and O'Regan [21] (see Definitions 2.1 and 2.2). Meanwhile, existence and uniqueness results of (1.2) are presented by using the theory of strongly continuous semigroup and compact semigroup via fixed point theorems (see Theorems 2.1 and 2.2, [21] and Theorems 2.1 and 2.2, [29]). Next, Pierri et al. [29] continue the work and development in [21] and study the existence and uniqueness of mild solutions to semilinear impulsive differential equations with not

instantaneous impulses in the fractional power space by virtue of the theory of analytic semigroup.

Next, we have a remark on the conditions in (1.2):

$$(1.3) \quad x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i], \quad i = 1, \dots, m,$$

where  $g_i \in C([t_i, s_i] \times X, X)$  and there are positive constants  $L_{g_i}$ ,  $i = 1, \dots, m$  such that

$$\|g_i(t, u_1) - g_i(t, u_2)\| \leq L_{g_i} \|u_1 - u_2\| \quad \text{for each } t \in [t_i, s_i] \text{ and all } u_1, u_2 \in X.$$

It follows from Theorems 2.1 and 2.2 in [21], [29], that a necessary condition  $\max\{L_{g_i} : i = 1, \dots, m\} < 1$  is considered. Then Banach fixed point theorem gives a unique  $z_i \in C([t_i, s_i], X)$  so that  $z = g_i(t, z)$  if and only if  $z = z_i(t)$ . So (1.3) is equivalent to

$$(1.4) \quad x(t) = z_i(t), \quad t \in (t_i, s_i], \quad i = 1, \dots, m,$$

which does not depend on the state  $x(\cdot)$ . Thus it is necessary to modify (1.3) and we recommend to consider the conditions

$$(1.5) \quad x(t) = g_i(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, \dots, m.$$

Comparing with (1.3), the new conditions (1.5) is a better and reasonable generalization of sudden impulses to not instantaneous ones.

Motivated by the above remark, we study impulsive evolution equations of the form

$$(1.6) \quad \begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}), \quad i = 0, \dots, m, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i = 1, \dots, m, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i], \quad i = 1, \dots, m. \end{cases}$$

Note that we consider in (1.6) that  $x \in C((t_k, t_{k+1}], X)$ ,  $k = 0, \dots, m$ , and there exist  $x(t_k^-)$ ,  $x(t_k^+)$ ,  $k = 1, \dots, m$  with  $x(t_k^-) = x(t_k)$ . Next this model is more suitable to show dynamics of evolution processes in Pharmacotherapy: the first equation denotes the health status of a patient; the second equation denotes the doctor takes some actions to test medicine for the patient practicably; the third equation denotes the testing medicine is valid for this patient and then begin to deal with the effect of patient for some time.

The rest of this paper is organized as follows. In Section 2, we give an existence and uniqueness result of (1.6) with  $x(0) = x_0 \in X$ . In Section 3, we introduce new four types of Ulam's type stability for differential equations with not instantaneous impulses (see Definitions 3.1–3.4). Ulam problem [35] has been attracted by many researchers, one can refer to the monographs of Cădariu [11], Hyers [22], [23], Jung [24], Rassias [30] and other mathematicians. For more recent contribution on such important fields, one can see the papers [6], [7], [13], [20], [25], [27], [31], [32], [37] and reference therein. We mainly present

the generalized Ulam–Hyers–Rassias stability results for the equation (1.6) on a compact interval.

Finally in Section 4, we extend our study to

$$(1.7) \quad \begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}), \quad i \in \mathbb{M}, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i \in \mathbb{M}, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i], \quad i \in \mathbb{M}, \end{cases}$$

where  $t \in \mathbb{R}_+ := [0, \infty)$ , the fixed points  $s_i$  and  $t_i$  satisfy  $t_i < s_i < t_{i+1}$  and either  $\mathbb{M} = \{1, \dots, m\}$  or  $\mathbb{M} = \mathbb{N}$  and then with  $\lim_{i \rightarrow \infty} t_i = \infty$ . We also set  $t_{m+1} = \infty$  for  $\mathbb{M} = \{1, \dots, m\}$ . Some extensions of Ulam–Hyers–Rassias stability for the case with infinite impulses are given. The existence and uniqueness result is also presented for this case.

### 2. An existence and uniqueness result

Set  $J := [0, T]$ . Throughout this paper, we need the Banach space

$$PC(J, X) := \{x : J \rightarrow X : x \in C((t_k, t_{k+1}], X), \quad k = 0, \dots, m$$

and there exist  $x(t_k^-)$  and  $x(t_k^+)$ ,  $k = 1, \dots, m$  with  $x(t_k^-) = x(t_k)$

endowed either with the Chebyshev  $PC$ -norm  $\|x\|_{PC} := \sup\{|x(t)| : t \in J\}$  or with the Bielecki  $PCB$ -norm  $\|x\|_{PCB} := \sup\{|x(t)|e^{-\Omega t} : t \in J\}$  for some  $\Omega \in \mathbb{R}$ .

We recall the following concepts of mild solutions of semilinear evolution equations with not instantaneous impulses.

DEFINITION 2.1 (see Definition 2.1, [21]). A function  $x \in PC(J, X)$  is called a mild solution of the problem

$$(2.1) \quad \begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}), \quad i = 0, \dots, m, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i = 1, \dots, m, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i], \quad i = 1, \dots, m, \\ x(0) = x_0 \in X, \end{cases}$$

if  $x(0) = x_0$  and

$$\begin{aligned} x(t) &= g_i(t, x(t_i^-)), & t \in (t_i, s_i], \quad i = 1, \dots, m; \\ x(t_i^+) &= g_i(t_i, x(t_i^-)), & i = 1, \dots, m, \\ x(t) &= T(t)x_0 + \int_0^t T(t-s)f(s, x(s)) \, ds, & t \in [0, t_1]; \\ x(t) &= T(t-s_i)g_i(s_i, x(t_i^-)) + \int_{s_i}^t T(t-s)f(s, x(s)) \, ds, & \\ & & t \in [s_i, t_{i+1}], \quad i = 1, \dots, m. \end{aligned}$$

Note that we consider  $x \in C((t_i, t_{i+1}], X)$ ,  $i = 0, \dots, m$ .

Cornering the existence and uniqueness of solutions to the problem (2.1), Hernández and O'Regan [21] initial obtain a interesting result under strong conditions via  $PC$ -norm. Here, we give another result under weak conditions via  $PCB$ -norm.

We introduce the following conditions:

(H<sub>0</sub>)  $A: D(A) \subseteq X \rightarrow X$  is the generator of a  $C_0$ -semigroup  $\{T(t), t \geq 0\}$  on a Banach space  $X$  with a norm  $\|\cdot\|$ . Then  $\|T(t)\| \leq Me^{\omega t}$  for some  $M \geq 1$  and  $\omega \in \mathbb{R}$  [28].

(H<sub>1</sub>)  $f \in C(J \times X, X)$ .

(H<sub>2</sub>) There exists a positive constant  $L_f$  such that

$$\|f(t, u_1) - f(t, u_2)\| \leq L_f \|u_1 - u_2\| \quad \text{for each } t \in J \text{ and all } u_1, u_2 \in X.$$

(H<sub>3</sub>)  $g_i \in C([t_i, s_i] \times X, X)$  and there are positive constants  $L_{g_i}$ ,  $i = 1, \dots, m$  such that

$$\|g_i(t, u_1) - g_i(t, u_2)\| \leq L_{g_i} \|u_1 - u_2\| \quad \text{for each } t \in [t_i, s_i] \text{ and all } u_1, u_2 \in X.$$

**THEOREM 2.2.** *Assume that (H<sub>0</sub>)–(H<sub>3</sub>) are satisfied. Then the equation (2.1) has a unique mild solution  $x \in PC(J, X)$ .*

**PROOF.** Consider a mapping  $F: PC(J, X) \rightarrow PC(J, X)$  defined by

$$(Fx)(0) = x_0;$$

$$(Fx)(t) = g_i \left( t, T(t - s_{i-1})g_{i-1}(s_{i-1}, x(t_{i-1}^-)) + \int_{s_{i-1}}^{t_i} T(t - s)f(s, x(s)) ds \right),$$

$$t \in (t_i, s_i], \quad i = 1, \dots, m;$$

$$(Fx)(t) = T(t)x_0 + \int_0^t T(t - s)f(s, x(s)) ds, \quad t \in [0, t_1];$$

$$(Fx)(t) = T(t - s_i)g_i(s_i, x(t_i^-)) + \int_{s_i}^t T(t - s)f(s, x(s)) ds,$$

$$t \in (s_i, t_{i+1}], \quad i = 1, \dots, m,$$

where we set  $g_0(t, x) := x_0$  and so  $L_{g_0} = 0$ . Obviously,  $F$  is well defined due to (H<sub>1</sub>).

Supposing  $\Omega > \omega$ , for any  $x, y \in PC(J, X)$  and  $t \in (s_i, t_{i+1}]$ ,  $i = 1, \dots, m$ , we have

$$\begin{aligned} & \| (Fx)(t) - (Fy)(t) \| \\ & \leq Me^{\omega(t-s_i)} L_{g_i} \|x(t_i^-) - y(t_i^-)\| + ML_f \int_{s_i}^t e^{\omega(t-s)} \|x(s) - y(s)\| ds \\ & \leq Me^{\omega(t-s_i) + \Omega t_i} L_{g_i} \|x - y\|_{PCB} + ML_f \int_{s_i}^t e^{\omega(t-s) + \Omega s} ds \|x - y\|_{PCB} \end{aligned}$$

$$\leq M \left( e^{\omega(t-s_i)+\Omega t_i} L_{g_i} + \frac{e^{\Omega t} L_f}{\Omega - \omega} \right) \|x - y\|_{PCB},$$

which implies that

$$\begin{aligned} e^{-\Omega t} \|(Fx)(t) - (Fy)(t)\| &\leq M \left( e^{\omega(t-s_i)+\Omega(t_i-t)} L_{g_i} + \frac{L_f}{\Omega - \omega} \right) \|x - y\|_{PCB} \\ &\leq M \left( e^{\Omega(t_i-s_i)} L_{g_i} + \frac{L_f}{\Omega - \omega} \right) \|x - y\|_{PCB}, \end{aligned}$$

for  $t \in (s_i, t_{i+1}]$ . Proceeding as above for  $t \in [0, t_1]$ , we obtain that

$$\begin{aligned} \|(Fx)(t) - (Fy)(t)\| &\leq ML_f \int_0^t e^{\omega(t-s)} \|x(s) - y(s)\| ds \\ &\leq ML_f \int_0^t e^{\omega(t-s)+\Omega s} ds \|x - y\|_{PCB} \leq \frac{ML_f e^{\Omega t}}{\Omega - \omega} \|x - y\|_{PCB}, \end{aligned}$$

which implies that

$$e^{-\Omega t} \|(Fx)(t) - (Fy)(t)\| \leq \frac{ML_f}{\Omega - \omega} \|x - y\|_{PCB}, \quad t \in [0, t_1].$$

Using the above estimates, similarly for  $t \in (t_i, s_i]$ ,  $i = 1, \dots, m$ , we derive

$$\begin{aligned} \|(Fx)(t) - (Fy)(t)\| &\leq ML_{g_i} \left( e^{\omega(t_i-s_{i-1})} L_{g_{i-1}} \|x(t_{i-1}^-) - y(t_{i-1}^-)\| \right. \\ &\quad \left. + L_f \int_{s_{i-1}}^{t_i} e^{\omega(t_i-s)} \|x(s) - y(s)\| ds \right) \\ &\leq ML_{g_i} \left( e^{\omega(t_i-s_{i-1})+\Omega t_{i-1}} L_{g_{i-1}} + \frac{e^{\Omega t_i} L_f}{\Omega - \omega} \right) \|x - y\|_{PCB}, \end{aligned}$$

which implies that

$$\begin{aligned} e^{-\Omega t} \|(Fx)(t) - (Fy)(t)\| &\leq ML_{g_i} \left( e^{\omega(t_i-s_{i-1})+\Omega(t_{i-1}-t_i)} L_{g_{i-1}} + \frac{L_f}{\Omega - \omega} \right) \|x - y\|_{PCB} \\ &\leq ML_{g_i} \left( e^{\Omega(t_{i-1}-s_{i-1})} L_{g_{i-1}} + \frac{L_f}{\Omega - \omega} \right) \|x - y\|_{PCB}, \end{aligned}$$

for  $t \in (t_i, s_i]$ ,  $i = 1, \dots, m$ . Summarizing the above estimates, we have that

$$\|Fx - Fy\|_{PCB} \leq L_F \|x - y\|_{PCB}$$

for

$$L_F := M \times \max_{1 \leq i \leq m} \left\{ \frac{L_f}{\Omega - \omega}, L_{g_i} \left( e^{\Omega(t_{i-1}-s_{i-1})} L_{g_{i-1}} + \frac{L_f}{\Omega - \omega} \right), e^{\Omega(t_i-s_i)} L_{g_i} + \frac{L_f}{\Omega - \omega} \right\}.$$

Obviously, one can choose a sufficient large  $\Omega > \omega$  such that  $L_F < 1$ , and so  $F$  is a contraction mapping. Then, one can derive the results immediately.  $\square$

**3. Concepts and results of Ulam’s type stability on a compact interval**

In this section, we introduce Ulam’s type stability concepts for the equation (1.6).

Let  $\varepsilon > 0$ ,  $\psi \geq 0$  and  $\varphi \in PC(J, \mathbb{R}_+)$  be nondecreasing. We consider the following inequalities

$$(3.1) \quad \begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varepsilon, & t \in (s_i, t_{i+1}), \quad i = 0, \dots, m, \\ \|y(t_i^+) - g_i(t_i, y(t_i^-))\| \leq \varepsilon, & i = 1, \dots, m, \\ \|y(t) - g_i(t, y(t_i^-))\| \leq \varepsilon, & t \in (t_i, s_i], \quad i = 1, \dots, m, \end{cases}$$

and

$$(3.2) \quad \begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varphi(t), & t \in (s_i, t_{i+1}), \quad i = 0, \dots, m, \\ \|y(t_i^+) - g_i(t_i, y(t_i^-))\| \leq \psi, & i = 1, \dots, m, \\ \|y(t) - g_i(t, y(t_i^-))\| \leq \psi, & t \in (t_i, s_i], \quad i = 1, \dots, m, \end{cases}$$

and

$$(3.3) \quad \begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varepsilon\varphi(t), & t \in (s_i, t_{i+1}), \quad i = 0, \dots, m, \\ \|y(t_i^+) - g_i(t_i, y(t_i^-))\| \leq \varepsilon\psi, & i = 1, \dots, m, \\ \|y(t) - g_i(t, y(t_i^-))\| \leq \varepsilon\psi, & t \in (t_i, s_i], \quad i = 1, \dots, m. \end{cases}$$

Now we set the vector space

$$Z := PC(J, X) \bigcap_{i=0}^m C^1((s_i, t_{i+1}), X) \bigcap_{i=0}^m C((s_i, t_{i+1}), D(A))$$

The following concepts are inspired by Wang et al. [37].

**DEFINITION 3.1.** The equation (1.6) is Ulam–Hyers stable if there exists a real number  $c_{f,m,g} > 0$  such that for each  $\varepsilon > 0$  and for each solution  $y \in Z$  of the inequality (3.1), there exists a mild solution  $x \in PC(J, X)$  of the equation (1.6) with

$$(3.4) \quad \|y(t) - x(t)\| \leq c_{f,m,g}\varepsilon, \quad t \in J.$$

**DEFINITION 3.2.** The equation (1.6) is generalized Ulam–Hyers stable if there exists  $\theta_{f,m,g} \in C(\mathbb{R}_+, \mathbb{R}_+)$ ,  $\theta_{f,m,g}(0) = 0$  such that for each solution  $y \in Z$  of the inequality (3.1), there exists a mild solution  $x \in PC(J, X)$  of the equation (1.6) with

$$(3.5) \quad \|y(t) - x(t)\| \leq \theta_{f,m,g}(\varepsilon), \quad t \in J.$$

**DEFINITION 3.3.** The equation (1.6) is Ulam–Hyers–Rassias stable with respect to  $(\varphi, \psi)$  if there exists  $c_{f,m,g,\varphi} > 0$  such that for each  $\varepsilon > 0$  and for each

solution  $y \in Z$  of the inequality (3.3), there exists a mild solution  $x \in PC(J, X)$  of the equation (1.6) with

$$(3.6) \quad \|y(t) - x(t)\| \leq c_{f,m,g,\varphi} \varepsilon (\varphi(t) + \psi), \quad t \in J.$$

DEFINITION 3.4. The equation (1.6) is generalized Ulam–Hyers–Rassias stable with respect to  $(\varphi, \psi)$  if there exists  $c_{f,m,g,\varphi} > 0$  such that for each solution  $y \in Z$  of the inequality (3.2) there exists a mild solution  $x \in PC(J, X)$  of the equation (1.6) with

$$(3.7) \quad \|y(t) - x(t)\| \leq c_{f,m,g,\varphi} (\varphi(t) + \psi), \quad t \in J.$$

REMARK 3.5. It is clear that: (a) Definition 3.1  $\Rightarrow$  Definition 3.2; (b) Definition 3.3  $\Rightarrow$  Definition 3.4; (c) Definition 3.3 for  $\varphi(t) = \psi = 1 \Rightarrow$  Definition 3.1.

REMARK 3.6. A function  $y \in Z$  is a solution of the inequality (3.3) if and only if there is  $G \in \bigcap_{i=0}^m C^1((s_i, t_{i+1}), X) \bigcap_{i=0}^m C((s_i, t_{i+1}), D(A))$  and  $g \in \bigcap_{i=1}^m C([t_i, s_i], X)$  (which depend on  $y$ ) such that:

- (a)  $\|G(t)\| \leq \varepsilon \varphi(t)$ ,  $t \in \bigcup_{i=0}^m (s_i, t_{i+1})$  and  $\|g(t)\| \leq \varepsilon \psi$ ,  $t \in \bigcup_{i=0}^m [t_i, s_i]$ ;
- (b)  $y'(t) = Ay(t) + f(t, y(t)) + G(t)$ ,  $t \in (s_i, t_{i+1})$ ,  $i = 0, \dots, m$ ;
- (c)  $y(t) = g_i(t, y(t_i^-)) + g(t)$ ,  $t \in (t_i, s_i]$ ,  $i = 1, \dots, m$ ;
- (d)  $y(t_i^+) = g_i(t_i, y(t_i^-)) + g(t_i)$ ,  $i = 1, \dots, m$ .

One can have similar remarks for the inequalities (3.1) and (3.2).

REMARK 3.7. If  $y \in Z$  is a solution of the inequality (3.3) then  $y$  is a solution of the following integral inequality

$$(3.8) \quad \left\{ \begin{array}{l} \|y(t) - g_i(t, y(t_i^-))\| \leq \varepsilon \psi, \quad t \in (t_i, s_i], \quad i = 1, \dots, m; \\ \left\| y(t) - T(t)y(0) - \int_0^t T(t-s)(s, y(s)) ds \right\| \\ \leq \varepsilon M \int_0^t e^{\omega(t-s)} \varphi(s) ds, \quad t \in [0, t_1]; \\ \|y(t_i^+) - g_i(t_i, y(t_i^-))\| \leq \varepsilon \psi, \quad i = 1, \dots, m; \\ \left\| y(t) - T(t-s_i)g_i(s_i, y(t_i^-)) - \int_{s_i}^t T(t-s)f(s, y(s)) ds \right\| \\ \leq \varepsilon M e^{\omega(t-s_i)} \psi + \varepsilon M \int_{s_i}^t e^{\omega(t-s)} \varphi(s) ds, \\ t \in [s_i, t_{i+1}], \quad i = 1, \dots, m. \end{array} \right.$$

In fact, by Remark 3.6 we get

$$(3.9) \quad \begin{cases} y'(t) = Ay(t) + f(t, y(t)) + G(t), & t \in (s_i, t_{i+1}), \quad i = 1, \dots, m; \\ y(t_i^+) = g_i(t_i, y(t_i^-)) + g(t_i), & i = 1, \dots, m; \\ y(t) = g_i(t, y(t_i^-)) + g(t), & t \in (t_i, s_i], \quad i = 1, \dots, m. \end{cases}$$



Clearly [28, p. 105], the solution  $y \in Z$  of the equation (3.9) is given by

$$\begin{aligned}
 y(t) &= g_i(t, y(t_i^-)) + g(t), & t \in (t_i, s_i], \quad i = 1, \dots, m; \\
 y(t) &= T(t)y(0) + \int_0^t T(t-s)(f(s, y(s)) + G(s)) ds, & t \in [0, t_1]; \\
 y(t) &= T(t-s_i)(g_i(s_i, y(t_i^-)) + g(t_i)) \\
 &\quad + \int_{s_i}^t T(t-s)(f(s, y(s)) + G(s)) ds, & t \in [s_i, t_{i+1}], \quad i = 1, \dots, m.
 \end{aligned}$$

For  $t \in [s_i, t_{i+1}]$ ,  $i = 0, \dots, m$ , we get

$$\begin{aligned}
 \left\| y(t) - T(t-s_i)g_i(s_i, y(t_i^-)) - \int_{s_i}^t T(t-s)f(s, y(s)) ds \right\| \\
 \leq M e^{\omega(t-s_i)} \|g(t_i)\| + \int_{s_i}^t e^{\omega(t-s)} \|G(s)\| ds \\
 \leq \varepsilon M e^{\omega(t-s_i)} \psi + \varepsilon M \int_{s_i}^t e^{\omega(t-s)} \varphi(s) ds.
 \end{aligned}$$

Proceeding as above, we derive that

$$\|y(t) - g_i(t, y(t_i^-))\| \leq \|g(t)\| \leq \varepsilon \psi, \quad t \in (t_j, s_j], \quad j = 1, \dots, m;$$

$$\begin{aligned}
 \left\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s, y(s)) ds \right\| &\leq M \int_0^t e^{\omega(t-s)} \|G(s)\| ds \\
 &\leq \varepsilon M \int_0^t e^{\omega(t-s)} \varphi(s) ds, \quad t \in [0, t_1].
 \end{aligned}$$

Similarly, one can give similar remarks for the solutions of the inequalities (3.2) and (3.1).

To discuss stability, we need the following additional assumption:

(H<sub>4</sub>) Let  $\varphi \in C(J, \mathbb{R}_+)$  be a nondecreasing function. There exists  $c_\varphi > 0$  such that

$$\int_0^t \varphi(s) ds \leq c_\varphi \varphi(t), \quad \text{for each } t \in J.$$

We need an impulsive Gronwall inequality which was given by Bainov and Simeonov (see Theorem 16.4, [9]).

LEMMA 3.8. *Let  $\mathbb{M}_0 := \mathbb{M} \cup \{0\}$  and the following inequality holds*

$$(3.10) \quad u(t) \leq a(t) + \int_0^t b(s)u(s) ds + \sum_{0 < t_k < t} \beta_k u(t_k^-), \quad t \geq 0,$$

where  $u, a, b \in PC(\mathbb{R}_+, \mathbb{R}_+)$ ,  $a$  is nondecreasing and  $b(t) > 0, \beta_k > 0, k \in \mathbb{M}$ . Then, for  $t \in \mathbb{R}_+$ , the following inequality is valid:

$$(3.11) \quad u(t) \leq a(t)(1 + \beta)^k \exp\left(\int_0^t b(s) ds\right), \quad t \in (t_k, t_{k+1}], \quad k \in \mathbb{M}_0,$$

where  $\beta = \sup_{k \in \mathbb{M}} \{\beta_k\}$ .

Now we are ready to state our main results in this section.

**THEOREM 3.9.** *Assume that (H<sub>1</sub>)–(H<sub>4</sub>) are satisfied. Then the equation (1.6) is Ulam–Hyers–Rassias stable with respect to  $(\varphi, \psi)$ .*

**PROOF.** Let  $y \in PC(J, D(A)) \cap C^1((s_i, t_{i+1}], X)$  be a solution of the inequality (3.3). Denote by  $x$  the unique mild solution of the impulsive Cauchy problem

$$(3.12) \quad \begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}), \quad i = 0, \dots, m, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i = 1, \dots, m, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i], \quad i = 1, \dots, m, \\ x(0) = y(0). \end{cases}$$

Then we get

$$x(t) = \begin{cases} g_i(t, x(t_j^-)), & t \in (t_j, s_j], \quad j = 1, \dots, m, \\ T(t)y(0) + \int_0^t T(t-s)f(s, x(s)) ds, & t \in [0, t_1], \\ T(t-s_i)g_i(s_i, x(t_i^-)) + \int_{s_i}^t T(t-s)f(s, x(s)) ds, & t \in (s_i, t_{i+1}], \quad i = 1, \dots, m. \end{cases}$$

Keep in mind of (3.8), for each  $t \in (s_i, t_{i+1}], i = 1, \dots, m$ , we have

$$\begin{aligned} & \left\| y(t) - T(t-s_i)g_i(s_i, y(t_i^-)) - \int_{s_i}^t T(t-s)f(s, y(s)) ds \right\| \\ & \leq \varepsilon M e^{|\omega|T} \left( \psi + \int_0^t \varphi(s) ds \right) \leq \varepsilon M e^{|\omega|T} (\psi + c_\varphi \varphi(t)), \end{aligned}$$

and for  $t \in (t_j, s_j], j = 1, \dots, m$ , we have

$$\|y(t) - g_i(t, y(t_i))\| \leq \varepsilon \psi,$$

and for  $t \in [0, t_1]$ , we have

$$\left\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s, y(s)) ds \right\| \leq \varepsilon M e^{|\omega|T} c_\varphi \varphi(t).$$

Hence, for each  $t \in (s_i, t_{i+1}], i = 1, \dots, m$ , we get

$$\begin{aligned} \|y(t) - x(t)\| & \leq \left\| y(t) - T(t-s_i)g_i(s_i, y(t_i^-)) - \int_{s_i}^t T(t-s)f(s, y(s)) ds \right\| \\ & \quad + M e^{|\omega|T} \|g_i(s_i, y(t_i^-)) - g_i(s_i, x(t_i^-))\| \\ & \quad + M e^{|\omega|T} \int_{s_i}^t \|f(s, y(s)) - f(s, x(s))\| ds \end{aligned}$$

$$\begin{aligned} &\leq M e^{|\omega|T} \left( \varepsilon(1 + c_\varphi)[\psi + \varphi(t)] + L_{g_i} \|y(t_i^-) - x(t_i^-)\| \right. \\ &\quad \left. + L_f \int_{s_i}^t \|y(s) - x(s)\| ds \right) \\ &\leq M e^{|\omega|T} \left( \varepsilon(1 + c_\varphi)[\psi + \varphi(t)] + L_f \int_0^t \|y(s) - x(s)\| ds \right. \\ &\quad \left. + \sum_{j=1}^i L_{g_j} \|y(t_j^-) - x(t_j^-)\| \right). \end{aligned}$$

Further, for  $t \in (t_j, s_j]$ ,  $j = 1, \dots, m$ , we have

$$\begin{aligned} \|y(t) - x(t)\| &\leq \|y(t) - g_i(t, y(t_i^-))\| + \|g_i(t, y(t_i^-)) - g_i(t, x(t_i^-))\| \\ &\leq \varepsilon\psi + \sum_{j=1}^i L_{g_j} \|y(t_j^-) - x(t_j^-)\| \\ &\leq M e^{|\omega|T} \left( \varepsilon(1 + c_\varphi)[\psi + \varphi(t)] \right. \\ &\quad \left. + L_f \int_0^t \|y(s) - x(s)\| ds + \sum_{j=1}^i L_{g_j} \|y(t_j^-) - x(t_j^-)\| \right). \end{aligned}$$

Next, for  $t \in [0, t_1]$ , we have

$$\begin{aligned} \|y(t) - x(t)\| &\leq M e^{|\omega|T} \left( \varepsilon c_\varphi \varphi(t) + L_f \int_0^t \|y(s) - x(s)\| ds \right) \\ &\leq M e^{|\omega|T} \left( \varepsilon(1 + c_\varphi)[\psi + \varphi(t)] + L_f \int_0^t \|y(s) - x(s)\| ds \right). \end{aligned}$$

Consequently, for  $t \in (t_i, t_{i+1}]$ , using Lemma 3.8, we derive that

$$\begin{aligned} \|y(t) - x(t)\| &\leq M e^{|\omega|T} (1 + c_\varphi) (1 + M e^{|\omega|T} L_g)^i e^{M e^{|\omega|T} L_f t} \varepsilon(\psi + \varphi(t)) \\ &\leq M e^{|\omega|T} (1 + c_\varphi) (1 + M e^{|\omega|T} L_g)^m e^{M e^{|\omega|T} L_f T} \varepsilon(\psi + \varphi(t)) \\ &:= c_{f,m,g,\varphi} \varepsilon(\psi + \varphi(t)), \end{aligned}$$

for  $L_g := \sup_{i \in \mathbb{M}} L_{g_i}$ , which implies that the equation (1.6) is Ulam–Hyers–Rassias stable with respect to  $(\varphi, \psi)$ . The proof is completed. □

Just repeating the procedure in Theorem 3.9, we establish without proof the following results.

**REMARK 3.10.** Under the assumptions of Theorem 3.9, we consider the equation (1.6) and the inequality (3.2). One can repeat the same process to verify that the equation (1.6) is generalized Ulam–Hyers–Rassias stable with respect to  $(\varphi, \psi)$ .

REMARK 3.11. Under the assumptions of Theorem 3.9, we consider the equation (1.6) and the inequality (3.1). One can repeat the same process to verify that the equation (1.6) is Ulam–Hyers stable.

To end this section, we give an example to illustrate our abstract results above.

EXAMPLE 3.12. We consider one-dimensional diffusion processes with not instantaneous changes of states. The example can explain either evolution of the temperature of the rod or the chemical concentration of the substance.

Consider the following impulsive partial differential equation

$$(3.13) \quad \begin{cases} \frac{\partial}{\partial t}x(t, y) = \frac{\partial^2}{\partial y^2}x(t, y), & y \in (0, 1), t \in [0, 1) \cup (2, 3], \\ \frac{\partial}{\partial y}x(t, 0) = \frac{\partial}{\partial y}x(t, 1) = 0, & t \in [0, 1) \cup (2, 3], \\ x(t, y) = \lambda x(1^-, y), & \lambda \in (-1, 1), t \in (1, 2], y \in (0, 1). \end{cases}$$

Hence  $J = [0, 3]$ ,  $0 = t_0 = s_0$ ,  $t_1 = 1$ ,  $s_1 = 2$  and  $T = 3$ . Let  $X = L^2(0, 1)$ . Define

$$Ax = \frac{\partial^2}{\partial y^2}x \quad \text{for } x \in D(A)$$

with

$$D(A) = \left\{ x \in X : \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \in X, x(0) = x(1) = 0 \right\}.$$

Then,  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $\{T(t), t \geq 0\}$  in  $X$ . Moreover,  $\|T(t)\| \leq 1$  for all  $t \geq 0$ . So  $M = 1$  and  $\omega = 0$ .

Denote  $x(t)(y) = x(t, y)$ ,  $f(t, x)(y) = 0$  and  $g_1(t, x)(y) = \lambda x(y)$ , then the problem (3.13) can be abstracted into

$$(3.14) \quad \begin{cases} x'(t) = Ax(t), & t \in (0, 1) \cup (2, 3), \\ x(t) = \lambda x(1^-), & t \in (1, 2]. \end{cases}$$

Clearly,  $(H_0)$ – $(H_3)$  are satisfied. Then the equation (3.14) with a given initial value has a unique solution  $x \in PC(J, X)$ . Moreover, we put  $\varphi(t) = e^t$  and  $\psi = 1$ . Then  $(H_4)$  holds if  $c_\varphi = 1$ . Thus, by Theorem 3.9, the equation (3.13) is Ulam–Hyers–Rassias stable with respect to  $(e^t, 1)$  on  $[0, 3]$  and  $c_{f, \psi, g, \varphi} = 2(1 + |\lambda|)$ .

#### 4. Extension stability results on the unbounded interval

In this section we consider the case  $J = \mathbb{R}_+$ . Then one can replace  $i = 1, \dots, m$  by  $i \in \mathbb{N}$  and  $i = 0, \dots, m$  by  $i \in \{0\} \cup \mathbb{N}$  in the inequalities (3.1)–(3.3) and (3.4)–(3.7), respectively. Then, one can rewrite four parallel stability

definitions like Definitions 3.1–3.4 when we take

$$Z := PC(J, X) \bigcap_{i \in \mathbb{M}_0} C^1((s_i, t_{i+1}), X) \bigcap_{i \in \mathbb{M}_0} C((s_i, t_{i+1}), D(A))$$

We state the following assumptions:

(H<sub>0</sub>')  $C_0$ -semigroup  $\{T(t), t \geq 0\}$  is exponentially stable, that is,  $\omega < 0$  in (H<sub>0</sub>).

(H<sub>1</sub>')  $f \in C([0, \infty) \times X, X)$ .

(H<sub>2</sub>') There exists positive constant  $L_f \in C(\mathbb{R}_+, (0, \infty))$  such that

$$\|f(t, u_1) - f(t, u_2)\| \leq L_f \|u_1 - u_2\|$$

for each  $t \in [0, \infty)$  and all  $u_1, u_2 \in X$ .

(H<sub>3</sub>')  $g_i \in C([t_i, s_i] \times X, X)$  and there are positive constants  $L_{g_i}, i \in \mathbb{N}$  such that

$$\|g_i(t, u_1) - g_i(t, u_2)\| \leq L_{g_i} \|u_1 - u_2\|$$

for each  $t \in [t_i, s_i]$  and all  $u_1, u_2 \in X$ .

(H<sub>4</sub>') There exists  $c_\varphi \geq 1$  such that

$$\int_{s_i}^t e^{\omega(s_i-s)} \varphi(s) ds \leq c_\varphi \varphi(t)$$

for any  $t \in [s_i, t_{i+1}]$  and  $i \in \mathbb{M}_0$ .

First we have the following extension of Theorem 2.2.

**THEOREM 4.1.** *Assume that (H<sub>0</sub>'), (H<sub>1</sub>')–(H<sub>3</sub>') are satisfied, then the equation (1.7) has a unique mild solution  $x \in PC([0, \infty), X)$  with  $x(0) = x_0 \in X$ .*

**PROOF.** Take  $T \in \mathbb{R}'_+ := \mathbb{R}_+ \setminus \mathbb{M}$ . Then by Theorem 2.2 the equation (1.7) has a unique mild solution  $x \in PC([0, T], X)$  with  $x(0) = x_0 \in X$ . Taking  $T \rightarrow \infty$ , the proof is finished. □

Now we pass to the stability problem.

**THEOREM 4.2.** *Assume that (H<sub>0</sub>')–(H<sub>4</sub>') are satisfied. Then the equation (1.7) is Ulam–Hyers–Rassias stable with respect to  $(\varphi, \psi)$  provided that*

$$\begin{aligned} L_g &:= \sup_{i \in \mathbb{M}} L_{g_i} < \infty, & \alpha &:= \inf_{i \in \mathbb{M}} (t_{i+1} - s_i) > 0, \\ \omega + ML_f &< 0, & \beta &:= ML_g e^{(\omega + ML_f)\alpha} < 1. \end{aligned}$$

**PROOF.** Let  $y \in Z$  be a solution of the inequality (3.3). Denote by  $x$  the unique mild solution of

$$(4.1) \quad \begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}), i \in \mathbb{M}_0, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i \in \mathbb{M}, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i], i \in \mathbb{M}, \\ x(0) = y(0). \end{cases}$$

Then we get

$$x(t) = \begin{cases} g_i(t, x(t_i^-)), & t \in (t_i, s_i], \quad i \in \mathbb{M}, \\ T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds, & t \in [0, t_1], \\ T(t-s_i)g_i(s_i, x(t_i^-)) + \int_{s_i}^t T(t-s)f(s, x(s))ds, & t \in (s_i, t_{i+1}], \quad i \in \mathbb{M}. \end{cases}$$

Here we note that we mean  $[a, \infty] = [a, \infty)$  to avoid confusion. By (3.8), for each  $t \in (s_i, t_{i+1}]$ ,  $i \in \mathbb{M}$ , we have

$$\begin{aligned} & \left\| y(t) - T(t-s_i)g_i(s_i, y(t_i^-)) - \int_{s_i}^t T(t-s)f(s, y(s))ds \right\| \\ & \leq \varepsilon M e^{\omega(t-s_i)}\psi + \varepsilon M \int_{s_i}^t e^{\omega(t-s)}\varphi(s)ds \leq \varepsilon M e^{\omega(t-s_i)}c_\varphi(\psi + \varphi(t)), \end{aligned}$$

and for  $t \in (t_i, s_i]$ ,  $i \in \mathbb{M}$ , we have  $\|y(t) - g_i(t, y(t_i^-))\| \leq \varepsilon\psi$ , and for  $t \in [0, t_1]$ , we have

$$\begin{aligned} \left\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s, y(s))ds \right\| & \leq \varepsilon M \int_0^t e^{\omega(t-s)}\varphi(s)ds \\ & \leq \varepsilon M c_\varphi e^{\omega t}\varphi(t). \end{aligned}$$

Hence, for each  $t \in (s_i, t_{i+1}]$ ,  $i \in \mathbb{M}$ , we get

$$\begin{aligned} \|y(t) - x(t)\| & \leq \left\| y(t) - T(t-s_i)g_i(s_i, y(t_i^-)) - \int_{s_i}^t T(t-s)f(s, y(s))ds \right\| \\ & \quad + M e^{\omega(t-s_i)}\|g_i(s_i, y(t_i^-)) - g_i(s_i, x(t_i^-))\| \\ & \quad + M \int_{s_i}^t e^{\omega(t-s)}\|f(s, y(s)) - f(s, x(s))\|ds \\ & \leq \varepsilon M c_\varphi e^{\omega(t-s_i)}(\psi + \varphi(t)) + M e^{\omega(t-s_i)}L_g\|y(t_i^-) - x(t_i^-)\| \\ & \quad + M L_f \int_{s_i}^t e^{\omega(t-s)}\|y(s) - x(s)\|ds, \end{aligned}$$

which implies for  $\bar{y}(t) := e^{-\omega t}y(t)$  and  $\bar{x}(t) := e^{-\omega t}x(t)$  that

$$\begin{aligned} \|\bar{y}(t) - \bar{x}(t)\| & \leq \varepsilon M c_\varphi e^{-\omega s_i}(\psi + \varphi(t)) \\ & \quad + M e^{\omega(t_i-s_i)}L_g\|\bar{y}(t_i^-) - \bar{x}(t_i^-)\| + M L_f \int_{s_i}^t \|\bar{y}(s) - \bar{x}(s)\|ds \end{aligned}$$

for  $t \in [s_i, t_{i+1}]$ ,  $i \in \mathbb{M}$ . By the Gronwall inequality (note  $\varphi(t)$  is nondecreasing), we derive

$$\|\bar{y}(t) - \bar{x}(t)\| \leq (\varepsilon M c_\varphi e^{-\omega s_i}(\psi + \varphi(t)) + M e^{\omega(t_i-s_i)}L_g\|\bar{y}(t_i^-) - \bar{x}(t_i^-)\|)e^{M L_f(t-s_i)},$$

which gives back

$$(4.2) \quad \|y(t) - x(t)\| \leq M e^{(\omega + M L_f)(t-s_i)}(\varepsilon c_\varphi(\psi + \varphi(t)) + L_g\|y(t_i^-) - x(t_i^-)\|)$$

for  $t \in [s_i, t_{i+1}]$ ,  $i \in \mathbb{M}$ , and in particular

$$\begin{aligned}
 (4.3) \quad & \|y(t_{i+1}^-) - x(t_{i+1}^-)\| \\
 & \leq M e^{(\omega + ML_f)(t_{i+1} - s_i)} (\varepsilon c_\varphi (\psi + \varphi(t_{i+1}^-)) + L_g \|y(t_i^-) - x(t_i^-)\|) \\
 & \leq M c_\varphi \varepsilon (\psi + \varphi(t_{i+1}^-)) + \beta \|y(t_i^-) - x(t_i^-)\|
 \end{aligned}$$

for  $i \in \mathbb{M}$  when  $t_{i+1} < \infty$ . Further, for  $t \in (t_i, s_i]$ ,  $i \in \mathbb{M}$ , we have

$$\begin{aligned}
 (4.4) \quad & \|y(t) - x(t)\| \leq \|y(t) - g_i(t, y(t_i^-))\| + \|g_i(t, y(t_i^-)) - g_i(t, x(t_i^-))\| \\
 & \leq \varepsilon \psi + L_g \|y(t_i^-) - x(t_i^-)\|.
 \end{aligned}$$

Moreover, for  $t \in [0, t_1]$ , we have

$$\begin{aligned}
 \|y(t) - x(t)\| & \leq \varepsilon M \int_0^t e^{\omega(t-s)} \varphi(s) ds + M \int_0^t e^{\omega(t-s)} L_f \|y(s) - x(s)\| ds \\
 & \leq \varepsilon M c_\varphi e^{\omega t} \varphi(t) + M L_f \int_0^t e^{\omega(t-s)} \|y(s) - x(s)\| ds
 \end{aligned}$$

which yields like above that

$$(4.5) \quad \|y(t) - x(t)\| \leq \varepsilon M c_\varphi e^{(\omega + ML_f)t} \varphi(t) \leq \varepsilon M c_\varphi \varphi(t)$$

for  $t \in [0, t_1]$ , and in particular

$$(4.6) \quad \|y(t_1^-) - x(t_1^-)\| \leq \varepsilon M c_\varphi \varphi(t_1^-).$$

Solving (4.3) and using (4.6) we derive

$$\begin{aligned}
 (4.7) \quad & \|y(t_i^-) - x(t_i^-)\| \leq M \varepsilon c_\varphi \sum_{j=2}^i (\psi + \varphi(t_j^-)) \beta^{i-j} + \beta^{i-1} \|y(t_1^-) - x(t_1^-)\| \\
 & \leq \frac{M \varepsilon c_\varphi (\psi + \varphi(t_i^-))}{1 - \beta} + \varepsilon M c_\varphi \varphi(t_i^-),
 \end{aligned}$$

since  $\varphi(\cdot)$  is nondecreasing.

Now let  $t \geq 0$ . Then either  $t \in [0, t_1]$  and (4.5) gives

$$(4.8) \quad \|y(t) - x(t)\| \leq \varepsilon M c_\varphi \varphi(t),$$

or  $t \in (t_i, t_{i+1}]$  for some  $i \in \mathbb{M}_0$ . Then either  $t \in (t_i, s_i]$  and then (4.4) and (4.7) give

$$\begin{aligned}
 (4.9) \quad & \|y(t) - x(t)\| \leq \varepsilon \psi + L_g \left( \frac{M \varepsilon c_\varphi (\psi + \varphi(t_i^-))}{1 - \beta} + \varepsilon M c_\varphi \varphi(t_i^-) \right) \\
 & \leq \varepsilon \left( \left( 1 + \frac{M L_g c_\varphi}{1 - \beta} \right) \psi + \left( \frac{M L_g c_\varphi}{1 - \beta} + M c_\varphi \right) \varphi(t) \right),
 \end{aligned}$$

or  $t \in (s_i, t_{i+1}]$  and then (4.2) and (4.7) give

$$\begin{aligned}
 (4.10) \quad \|y(t) - x(t)\| &\leq M e^{(\omega + ML_f)(t - s_i)} \left( \varepsilon c_\varphi (\psi + \varphi(t)) \right. \\
 &\quad \left. + L_g \left( \frac{M \varepsilon c_\varphi (\psi + \varphi(t_i^-))}{1 - \beta} + \varepsilon M c_\varphi \varphi(t_i^-) \right) \right) \\
 &\leq M \varepsilon c_\varphi \left( \left( 1 + \frac{L_g}{1 - \beta} \right) \psi + \left( 2 + \frac{L_g}{1 - \beta} \right) \varphi(t) \right).
 \end{aligned}$$

Using (4.8), (4.9) and (4.10) we have

$$\|y(t) - x(t)\| \leq c_{f,g,\varphi} \varepsilon (\psi + \varphi(t))$$

for any  $t \geq 0$ , where  $c_{f,g,\varphi} := M c_\varphi (2 + L_g / (1 - \beta))$ . Summarizing, we see that the equation (1.7) is Ulam–Hyers–Rassias stable with respect to  $(\varphi, \psi)$ .  $\square$

REMARK 4.3. If  $\gamma := \sup_{i \in \mathbb{M}_0} (t_{i+1} - s_i) < \infty$ , then assumption  $(H'_4)$  holds for any  $\varphi(t) = c e^{\omega' t}$ ,  $c > 0$  and  $\omega' > \omega$ . Indeed, we compute

$$\int_{s_i}^t e^{\omega(s_i - s)} \varphi(s) ds = c \int_{s_i}^t e^{\omega(s_i - s) + \omega' s} ds \leq c \frac{e^{\omega(s_i - t) + \omega' t}}{\omega' - \omega} \leq \frac{e^{-\omega \gamma}}{\omega' - \omega} \varphi(t)$$

for any  $t \in [s_i, t_{i+1}]$  and  $i \in \mathbb{M}_0$ , so  $c_\varphi = e^{-\omega \gamma} / (\omega' - \omega)$ . In particular, the constant function  $\varphi(t) = \varphi$  can be also used with  $\omega' = 0$ .

Finally, we give an example to illustrate our above results.

EXAMPLE 4.4. Consider

$$(4.11) \quad \begin{cases} \frac{\partial}{\partial t} x(t, y) = (\Delta_y - 3I)x(t, y), & y \in \Omega, \\ & t \in (2i + 1, 2(i + 1)], i \in \{0\} \cup \mathbb{N}, \\ \frac{\partial}{\partial y} x(t, y) = 0, & y \in \partial\Omega, t \geq 0, \\ x(t, y) = \sin i \cdot x(2i^-, y), & y \in \Omega, (2i, 2i + 1], i \in \mathbb{N}, \end{cases}$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain,  $\Delta_y$  is the Laplace operator in  $\mathbb{R}^2$ , and  $\partial\Omega \in C^2$ . Note now  $s_i = 2i + 1$  and  $t_i = 2i$ ,  $i \in \{0\} \cup \mathbb{N}$ . Here we consider infinitely many impulses on infinite time interval  $\mathbb{R}_+$ .

Let  $X = L_2(\Omega)$ ,  $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ . Define  $Ax = (\Delta_y - 3I)x$ ,  $x \in D(A)$ . By Theorem 2.5 of [28],  $A$  is just the infinitesimal generator of a contraction  $C_0$ -semigroup in  $L_2(\Omega)$ , that is,  $\|T(t)\| \leq e^{-3t}$  for  $t \geq 0$ , so  $m = 1$  and  $\omega = -3 < 0$ .

Denote  $x(\cdot)(y) = x(\cdot, y)$ ,  $f(\cdot, x)(y) = 0$  and  $g_i(\cdot, x)(y) = (\sin i) \cdot x(y)$ , then the problem (4.11) can be abstracted into

$$(4.12) \quad \begin{cases} x'(t) = Ax(t), t \in (2i + 1, 2(i + 1)), & i \in \{0\} \cup \mathbb{N}, \\ x(2i^+) = x(2i^-), & i \in \mathbb{N}, \\ x(t) = (\sin i)x(2i^-), & t \in (2i, 2i + 1], i \in \mathbb{N}. \end{cases}$$



Clearly,  $(H'_0)$ – $(H'_3)$  are satisfied, and  $L_g = 1$ ,  $\alpha = \gamma = 1$ ,  $L_f = 0$ ,  $\omega + ML_f = -3$ ,  $\omega = e^{-3} < 1$  and by Remark 4.3, we can take  $\varphi(t) = e^{\omega' t}$ ,  $\omega' > -3$  and  $\psi = 1$ . Then  $(H'_4)$  holds for  $c_\varphi = e^3/(\omega' + 3)$ . Thus, applying Theorem 4.2, the equation (4.11) is Ulam–Hyers–Rassias stable with respect to  $(e^{\omega' t}, 1)$  on  $\mathbb{R}_+$  with  $\omega' > -3$  and

$$c_{f, \mathbb{N}, g, \varphi} = \frac{e^3}{\omega' + 3} \left( 2 + \frac{1}{1 - e^{-3}} \right).$$

## REFERENCES

- [1] S. AFONSO, E.M. BONOTTO, M. FEDERSON AND L. GIMENES, *Boundedness of solutions of retarded functional differential equations with variable impulses via generalized ordinary differential equations*, Math. Nachr. **285** (2012), 545–561.
- [2] ———, *Stability of functional differential equations with variable impulsive perturbations via generalized ordinary differential equations*, Bull. Sci. Math. **137** (2013), 189–214.
- [3] S.M. AFONSO, E.M. BONOTTO, M. FEDERSON AND Š. SCHWABIK, *Discontinuous local semiflows for Kurzweil equations leading to LaSalle's invariance principle for differential systems with impulses at variable times*, J. Differential Equations **250** (2011), 2969–3001.
- [4] N.U. AHMED, *Existence of optimal controls for a general class of impulsive systems on Banach space*, SIAM J. Control Optimal **42** (2003), 669–685.
- [5] N.U. AHMED, K.L. TEO AND S.H. HOU, *Nonlinear impulsive systems on infinite dimensional spaces*, Nonlinear Anal. **54** (2003), 907–925.
- [6] SZ. ANDRÁS AND J.J. KOLUMBÁN, *On the Ulam–Hyers stability of first order differential systems with nonlocal initial conditions*, Nonlinear Anal. **82** (2013), 1–11.
- [7] SZ. ANDRÁS AND A.R. MÉSZÁROS, *Ulam–Hyers stability of dynamic equations on time scales via Picard operators*, Appl. Math. Comput. **219** (2013), 4853–4864.
- [8] D.D. BAINOV, V. LAKSHMIKANTHAM AND P.S. SIMEONOV, *Theory of impulsive differential equations*, Series in Modern Applied Mathematics, vol. 6, World Scientific, Singapore, 1989.
- [9] D.D. BAINOV AND P.S. SIMEONOV, *Integral inequalities and applications*, Kluwer Academic Publishers, Dordrecht, 1992.
- [10] M. BENCHOHRA, J. HENDERSON AND S. NTOUYAS, *Impulsive differential equations and inclusions*, Contemporary Mathematics and Its Applications, vol. 2, Hindawi, New York, USA, 2006.
- [11] L. CĂDARIU, *Stabilitatea Ulam–Hyers–Bourgin pentru ecuatii functionale*, Ed. Univ. Vest Timișoara, Timișara, 2007.
- [12] Y.K. CHANG, AND W.T. LI, *Existence results for second order impulsive functional differential inclusions*, J. Math. Anal. Appl. **301** (2005), 477–490.
- [13] D.S. CIMPEAN AND D. POPA, *Hyers–Ulam stability of Euler's equation*, Appl. Math. Lett. **24** (2011), 1539–1543.
- [14] Z. FAN, *Impulsive problems for semilinear differential equations with nonlocal conditions*, Nonlinear Anal. **72** (2010), 1104–1109.
- [15] Z. FAN AND G. LI, *Existence results for semilinear differential equations with nonlocal and impulsive conditions*, J. Funct. Anal. **258** (2010), 1709–1727.

- [16] M. FEČKAN, Y. ZHOU AND J. WANG, *On the concept and existence of solutions for impulsive fractional differential equations*, Commun. Nonlinear Sci. Numer. Simulat. **17** (2012), 3050–3060.
- [17] M. FRIGON AND D. O'REGAN, *Existence results for first-order impulsive differential equations*, J. Math. Anal. Appl. **193** (1995), 96–113.
- [18] ———, *Impulsive differential equations with variable times*, Nonlinear Anal. **26** (1996), 1913–1922.
- [19] ———, *First order impulsive initial and periodic problems with variable moments*, J. Math. Anal. Appl. **233** (1999), 730–739.
- [20] B. HEGYI AND S. M. JUNG, *On the stability of Laplace's equation*, Appl. Math. Lett. **26** (2013), 549–552.
- [21] E. HERNÁNDEZ AND D. O'REGAN, *On a new class of abstract impulsive differential equations*, Proc. Amer. Math. Soc. **141** (2013), 1641–1649.
- [22] D.H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. **27** (1941), 222–224.
- [23] D.H. HYERS, G. ISAC AND TH.M. RASSIAS, *Stability of functional equations in several variables*, Birkhäuser, 1998.
- [24] S.M. JUNG, *Hyers–Ulam–Rassias stability of functional equations in mathematical analysis*, Hadronic Press, Palm Harbor, 2001.
- [25] ———, *Hyers–Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **17** (2004), 1135–1140.
- [26] J. LIU, *Nonlinear impulsive evolution equations*, Dyn. Contin. Discrete Impuls. Syst. **6** (1999), 77–85.
- [27] N. LUNGU AND D. POPA, *Hyers–Ulam stability of a first order partial differential equation*, J. Math. Anal. Appl. **385** (2012), 86–91.
- [28] A. PAZY, *Semigroup of linear operators and applications to partial differential equations*, Springer–Verlag, New York, 1983.
- [29] M. PIERRI, D. O'REGAN AND V. ROLNIK, *Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses*, Appl. Math. Comput. **219** (2013), 6743–6749.
- [30] TH.M. RASSIAS, *On the stability of linear mappings in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
- [31] I.A. RUS, *Ulam stability of ordinary differential equations*, Studia Univ. “Babeş Bolyai” Mathematica **54** (2009), 125–133.
- [32] ———, *Ulam stabilities of ordinary differential equations in a Banach space*, Carpathian J. Math. **26** (2010), 103–107.
- [33] A.M. SAMOILENKO AND N.A. PERESTYUK, *Impulsive differential equations*, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, vol. 14, World Scientific, Singapore, 1995.
- [34] P. SATTAYATHAM, *Strongly nonlinear impulsive evolution equations and optimal control*, Nonlinear Anal. **57** (2004), 1005–1020.
- [35] S.M. ULAM, *A collection of mathematical problems*, Interscience Publishers, New York, 1968.
- [36] J. WANG, M. FEČKAN AND Y. ZHOU, *On the new concept of solutions and existence results for impulsive fractional evolution equations*, Dyn. Partial Differ. Equ. **8** (2011), 345–361.

- [37] ———, *Ulam's type stability of impulsive ordinary differential equations*, J. Math. Anal. Appl. **395** (2012), 258–264.

*Manuscript received May 12, 2014*

*accepted September 16, 2015*

JINRONG WANG  
Department of Mathematics  
Guizhou University  
Guiyang, Guizhou 550025, P.R. CHINA  
and  
School of Mathematics and Computer Science  
Guizhou Normal College  
Guiyang, Guizhou 550018, P.R. CHINA  
*E-mail address:* sci.jrwang@gzu.edu.cn, wangjinrong@gznc.edu.cn

MICHAL FEČKAN  
Department of Mathematical Analysis and Numerical Mathematics  
Faculty of Mathematics, Physics and Informatics  
Comenius University in Bratislava  
Mlynská Dolina  
842 48 Bratislava, SLOVAKIA  
and  
Mathematical Institute  
Slovak Academy of Sciences  
Štefánikova 49  
814 73 Bratislava, SLOVAKIA  
*E-mail address:* Michal.Feckan@fmph.uniba.sk