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A General Expression for Hall Coefficient Based on Fermi Liquid Theory

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A general expression for Hall conductivity including the effects of many-body interaction is derived on the basis of the Fermi liquid theory. It is exact as far as the most singular terms with respect to the quasiparticle damping are concerned. It is applicable for any types of interaction as far as the picture of Fermi liquid holds well.

§ 1. Introduction

The systems in which electron-electron interactions cannot be neglected, such as heavy fermion systems[1] and high-T_c oxide superconductors, are of current interest. The former system exhibits the large T^2-component of resistivity in its low temperature coherent regime. This behaviour is ascribed to electron-electron scattering with the Umklapp process. Therefore, Hall coefficient should also be affected by many-body effects. Until now, many-body effects have been treated only within crude approximations. In metals, it seems dangerous to treat them on the basis of such approximate calculations as alloy analogy and Hubbard decoupling which neglect the momentum dependence of the self-energy. Moreover, we should include the vertex corrections originating from electron-electron interactions in order not to violate the Ward identity. These motivated us to seek an exact formula for Hall coefficient including the many-body effects on the basis of the Fermi liquid theory.

Such a formula for conductivity was given by Éliashberg[2] in 1961. He collected all the terms which are, in the static limit, singular with respect to quasiparticle damping, namely, divergent terms as the quasiparticle damping goes to zero. The result is given by (for simplicity, we give a static conductivity per spin)

$$\sigma_{\mu\nu}=e^2\int\frac{d\boldsymbol{p}}{(2\pi)^3}\left\{\frac{1}{2\gamma_{\boldsymbol{p}}}\left(-\frac{df}{d\varepsilon}\right)_{\varepsilon=E(\boldsymbol{p})}\right\}v_\mu{}^*v_\nu{}^*$$
$$+\int\frac{d\boldsymbol{p}}{(2\pi)^3}\int\frac{d\boldsymbol{p}'}{(2\pi)^3}\left\{\frac{1}{2\gamma_{\boldsymbol{p}}}\left(-\frac{df}{d\varepsilon}\right)_{\varepsilon=E(\boldsymbol{p})}\right\}v_\mu{}^*\frac{a^2\mathcal{T}_{22}(\boldsymbol{p},\boldsymbol{p}')}{4i\gamma_{\boldsymbol{p}'}}v_\nu{}^{*\prime}\,. \qquad (1\cdot1)$$

Here $v_\mu{}^*$, $E(\boldsymbol{p})$ and $\gamma_{\boldsymbol{p}}$ represent the velocity, energy and damping constant of a quasiparticle of momentum $\boldsymbol{p}$, and f is the Fermi distribution function. The first term represents free quasiparticle propagation with damping, and the second term includes vertex correction arising from quasiparticle interaction $a^2\mathcal{T}_{22}$. Equation (1·1) can be written as

$$\sigma_{\mu\nu}=e^2\int\frac{d\boldsymbol{p}}{(2\pi)^3}\left\{\frac{1}{2\gamma_{\boldsymbol{p}}}\left(-\frac{df}{d\varepsilon}\right)_{\varepsilon=E(\boldsymbol{p})}\right\}v_\mu{}^*J_\nu\,, \qquad (1\cdot2)$$

where

$$J_\nu = v_\nu{}^* + \int \frac{d\boldsymbol{p}}{(2\pi)^3} \frac{a^2 \mathcal{T}_{22}(\boldsymbol{p}, \boldsymbol{p}')}{4i\gamma_{\boldsymbol{p}'}} v_\nu{}^{*\prime} . \tag{1·3}$$

This is compared with the expression obtained from the Boltzmann equation:

$$\sigma_{\mu\nu} = e^2 \tau \int \frac{d\boldsymbol{p}}{(2\pi)^3} \left(-\frac{df}{d\varepsilon} \right)_{\varepsilon = E(\boldsymbol{p})} v_\mu{}^* v_\nu{}^* . \tag{1·4}$$

If J_μ is proportional to $v_\mu{}^*$, i.e.,

$$J_\mu = v_\mu{}^* \cdot \chi(\boldsymbol{p}) , \tag{1·5}$$

we can define a transport relaxation time by

$$\tau_{\mathrm{tr}} = \frac{\chi(\boldsymbol{p})}{2\gamma_{\boldsymbol{p}}} , \tag{1·6}$$

and Eq. (1·2) reduces to Eq. (1·4).

In this paper, we devote ourselves to Hall conductivity. In this case, we must deal with a three-body interaction vertex. But this can be reduced to two-body interaction vertex by use of the Ward identity within our accuracy. The final result is given by

$$\sigma_{\mu\nu} = \frac{e^3}{c} H \int \frac{d\boldsymbol{p}}{(2\pi)^3} \left[J_\mu \frac{\partial J_\nu}{\partial p_\nu} - \frac{\partial J_\mu}{\partial p_\nu} J_\nu \right] v_\mu{}^* \frac{1}{(2\gamma_{\boldsymbol{p}})^2} \left(-\frac{df}{d\varepsilon} \right)_{\varepsilon = E(\boldsymbol{p})} , \quad (\mu = x,\ \nu = y) \tag{1·7}$$

while the Boltzmann equation gives[3]

$$\sigma_{\mu\nu} = \frac{e^3}{c} H \tau^2 \int \frac{d\boldsymbol{p}}{(2\pi)^3} \left[v_\mu{}^* \frac{\partial v_\nu{}^*}{\partial p_\nu} - \frac{\partial v_\mu{}^*}{\partial p_\nu} v_\nu{}^* \right] v_\mu{}^* \left(-\frac{df}{d\varepsilon} \right)_{\varepsilon = E(\boldsymbol{p})} . \quad (\mu = x,\ \nu = y) \tag{1·8}$$

If Eq. (1·5) holds, Eq. (1·7) reduces to Eq. (1·8) also in this case.

In the argument for Bloch electrons or tight binding models, we restrict ourselves to the case in which the result can be expressed only with single band quantities and start with the single band model. This is because we are mainly interested in the many-body effects and complications arising from interband effects are not of our concern.

In § 2, a formal expression for Hall conductivity is given on the basis of linear response theory. In § 3, the terms proportional to the magnetic field is calculated. These procedures are formulated by Fukuyama et al.[4] in their study of impurity effects on Hall coefficient. Then, following Éliashberg,[2] analytic continuation is performed and the most singular terms with respect to the quasiparticle damping are collected. Thus we get the final result (1·7). In § 4, some modifications needed to proceed to the case of Bloch electrons[5] are described. In § 5, some remarks on the results and the justification for the terms neglected in § 3 are given. The range of applicability is also discussed.

§ 2. Basic formula[4)]

We consider the situation in which a uniform static magnetic field $\boldsymbol{H}$ is applied along the z-axis, and the current is forced to flow in the x-direction. We assume reflection symmetries of the system with respect to xz-plane and yz-plane throughout this paper for simplicity.

The Hall coefficient R is generally given by

$$R=\frac{-\sigma_{yx}}{\sigma_{xx}\sigma_{yy}-\sigma_{xy}\sigma_{yx}}\frac{1}{H} \tag{2·1}$$

in terms of the conductivity tensor $\sigma_{\mu\nu}$ in the presence of the magnetic field $\boldsymbol{H}$. We consider a 'classical' or weak field limit (by which we mean $\omega_c\tau\ll1$ where ω_c is the cyclotron frequency and τ the electron mean free time) and retain only terms up to the first order in $\boldsymbol{H}$ in the prefactor of $1/H$, so that

$$R=\frac{\sigma_{xy}^{(1)}}{\sigma_{xx}^{(0)}\sigma_{yy}^{(0)}}\cdot\frac{1}{H} \tag{2·2}$$

is independent of $\boldsymbol{H}$. Here we denote the term of order $\boldsymbol{H}^m$ as $\sigma_{\mu\nu}{}^{(m)}$. The magnetic field free part $\sigma_{\mu\mu}{}^{(0)}$ ($\mu=x$ or y) has already been discussed[2)] and is given by Eq. (1·1). In this paper, we will discuss $\sigma_{xy}{}^{(1)}$, the part proportional to H in the transverse conductivity. The procedure of calculating $\sigma_{xy}{}^{(1)}$ is given by Fukuyama et al.[4)] and we shall follow them.

We introduce a magnetic field through the vector potential $\boldsymbol{A}(\boldsymbol{r})=\boldsymbol{A}_q e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$ and let $\boldsymbol{q}\to0$ later to obtain a uniform field. Also we calculate a static conductivity by introducing a uniform electric field of frequency ω and letting $\omega\to0$ at the end. According to Kubo formula, the conductivity tensor of our concern is given by

$$\sigma_{\mu\nu}(\boldsymbol{q},\omega)=\frac{1}{i\omega}[\Phi_{\mu\nu}(\boldsymbol{q},\omega+i0)-\Phi_{\mu\nu}(\boldsymbol{q},+i0)]. \tag{2·3}$$

We put $\mu=x$, $\nu=y$ throughout this paper. $\Phi_{\mu\nu}(\boldsymbol{q},\omega+i0)$ is obtained by the analytic continuation $\omega_\lambda\to\omega+i0$ from

$$\Phi_{\mu\nu}(\boldsymbol{q},\omega_\lambda)=\frac{1}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau' e^{\omega_\lambda(\tau-\tau')}\langle T_\tau \hat{J}_\mu{}^H(\boldsymbol{q},\tau)\hat{J}_\nu{}^H(\boldsymbol{0},\tau')\rangle_H\,, \tag{2·4}$$

where $\omega_\lambda=2\pi i\lambda T$, λ: positive integer, $T=\beta^{-1}$: temperature, and

$$\begin{aligned}
\hat{\boldsymbol{J}}^H(\boldsymbol{k})&=\hat{\boldsymbol{J}}(\boldsymbol{k})-\frac{e^2}{mc}\hat{\rho}(\boldsymbol{k}-\boldsymbol{q})\boldsymbol{A}_q\,,\\
\hat{\boldsymbol{J}}(\boldsymbol{k})&=\frac{1}{2mi}\int d\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}[\psi^\dagger(\boldsymbol{r})\boldsymbol{\nabla}\psi(\boldsymbol{r})-\boldsymbol{\nabla}\psi^\dagger(\boldsymbol{r})\cdot\psi(\boldsymbol{r})]\,,\\
\hat{\rho}(\boldsymbol{k})&=\int d\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,.
\end{aligned} \tag{2·5}$$

Here time evolution and thermal average are defined by the Hamiltonian:

$$\mathcal{H}_H = H_0 + H_{\rm int} - \frac{1}{c}\hat{\boldsymbol{J}}(-\boldsymbol{q})\cdot\boldsymbol{A}_{\boldsymbol{q}}\,. \tag{2·6}$$

H_0 describes the system of non-interacting electrons and $H_{\rm int}$ introduces the many-body interaction to it, whose explicit form need not be specified. For simplicity, we neglect the spin degeneracy, which will be recovered in the final expression (Eq. (4·7) or (4·8)) in a trivial way. To extract the part proportional to H, we write

$$\sigma_{\mu\nu}(\boldsymbol{q},\omega) = \frac{1}{i\omega}K^{\alpha}_{\mu\nu}(\boldsymbol{q},\omega)A_{\boldsymbol{q},\alpha}\,, \tag{2·7}$$

where

$$K^{\alpha}_{\mu\nu}(\boldsymbol{q},\omega) = \frac{e^2}{mc}\delta_{\nu\alpha}[\mathcal{L}_{\mu}(\boldsymbol{q},\omega+i0) - \mathcal{L}_{\mu}(\boldsymbol{q},+i0)]$$

$$+\frac{1}{c}[\mathcal{L}^{\alpha}_{\mu\nu}(\boldsymbol{q},\omega+i0) - \mathcal{L}^{\alpha}_{\mu\nu}(\boldsymbol{q},+i0)]\,, \tag{2·8}$$

$$\mathcal{L}_{\mu}(\boldsymbol{q},\omega_\lambda) = -\frac{1}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau' e^{\omega_\lambda(\tau-\tau')}\langle T_\tau \hat{J}_\mu(\boldsymbol{q},\tau)\hat{\rho}(-\boldsymbol{q},\tau')\rangle\,,$$

$$\mathcal{L}^{\alpha}_{\mu\nu}(\boldsymbol{q},\omega_\lambda) = \frac{1}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau'\int_0^\beta d\tau'' e^{\omega_\lambda(\tau-\tau'')}\langle T_\tau \hat{J}_\mu(\boldsymbol{q},\tau)\hat{J}_\alpha(-\boldsymbol{q},\tau')\hat{J}_\nu(\boldsymbol{0},\tau'')\rangle\,.$$

Here and hereafter time evolution and the thermal average are defined by the Hamiltonian

$$\mathcal{H} = H_0 + H_{\rm int}\,. \tag{2·9}$$

§ 3. Hall coefficient in nearly free electron system

In this section, we discuss the case in which the non-interacting electron dispersion is well described by $\varepsilon(\boldsymbol{p}) = \boldsymbol{p}^2/2m$ (m: the mass of an electron). The case in which $\varepsilon(\boldsymbol{p})$ has a general form will be argued in the next section.

The field operator is expanded in terms of a plane wave basis as

$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{r}}c_{\boldsymbol{p}}\,, \tag{3·1}$$

where V is the volume of the system and set to be unity hereafter. One-particle thermal Green function is defined by

$$\mathcal{G}(\boldsymbol{p},\varepsilon_n) = -\frac{1}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau' e^{\varepsilon_n(\tau-\tau')}\langle T_\tau c_{\boldsymbol{p}}(\tau)c_{\boldsymbol{p}}^\dagger(\tau')\rangle\,, \tag{3·2}$$

$$\varepsilon_n = (2n+1)\pi i T\,, \qquad c_{\boldsymbol{p}}(\tau) = e^{(\mathcal{H}-\mu N)\tau}c_{\boldsymbol{p}}e^{-(\mathcal{H}-\mu N)\tau}\,.$$

The renormalized vertices of two- and three-body interaction shown in Fig. 1 are written as

$$\Gamma\left(\begin{array}{cc|cc} \boldsymbol{p}^{+} & \boldsymbol{p}^{-} & \boldsymbol{p}^{-\prime} & \boldsymbol{p}^{+\prime} \\ \varepsilon_n{}^{+} & \varepsilon_n & \varepsilon_n{}' & \varepsilon_n{}^{+\prime} \end{array}\right), \tag{3·3}$$

$$\Gamma_3\left(\begin{array}{cc|cc|cc} \boldsymbol{p}^{+} & \boldsymbol{p}^{-} & \boldsymbol{p}^{-\prime} & \boldsymbol{p}^{+\prime} & \boldsymbol{p}'' & \boldsymbol{p}'' \\ \varepsilon_n{}^{+} & \varepsilon_n & \varepsilon_n{}' & \varepsilon_n{}^{+\prime} & \varepsilon_n{}'' & \varepsilon_n{}'' \end{array}\right), \tag{3·4}$$

where $\boldsymbol{p}^{\pm}=\boldsymbol{p}\pm\boldsymbol{q}/2$, $\boldsymbol{p}^{\pm\prime}=\boldsymbol{p}'\pm\boldsymbol{q}/2$, $\varepsilon_n{}^{+}=\varepsilon_n+\omega_\lambda$, etc. The renormalized vertices coupled to external fields are defined as

$$\Lambda_\mu\equiv\Lambda_\mu\left(\begin{array}{cc} \boldsymbol{p}^{+} & \boldsymbol{p}^{-} \\ \varepsilon_n{}^{+} & \varepsilon_n \end{array}\right)$$

$$=ev_\mu+T\sum_{\boldsymbol{p}',\varepsilon_n'}\Gamma\left(\begin{array}{cc|cc} \boldsymbol{p}^{+} & \boldsymbol{p}^{-} & \boldsymbol{p}^{-\prime} & \boldsymbol{p}^{+\prime} \\ \varepsilon_n{}^{+} & \varepsilon_n & \varepsilon_n{}' & \varepsilon_n{}^{+\prime} \end{array}\right)\mathcal{G}\left(\begin{array}{c} \boldsymbol{p}^{-\prime} \\ \varepsilon_n{}' \end{array}\right)\mathcal{G}\left(\begin{array}{c} \boldsymbol{p}^{+\prime} \\ \varepsilon_n{}^{+\prime} \end{array}\right)ev_\mu{}', \tag{3·5}$$

$$\Lambda_\nu\equiv\Lambda_\nu\left(\begin{array}{cc} \boldsymbol{p}^{\pm} & \boldsymbol{p}^{\pm} \\ \varepsilon_n & \varepsilon_n{}^{+} \end{array}\right)$$

$$=ev_\nu{}^{\pm}+T\sum_{\boldsymbol{p}',\varepsilon_n'}\Gamma\left(\begin{array}{cc|cc} \boldsymbol{p}^{\pm} & \boldsymbol{p}^{\pm} & \boldsymbol{p}' & \boldsymbol{p}' \\ \varepsilon_n & \varepsilon_n{}^{+} & \varepsilon_n{}^{+\prime} & \varepsilon_n{}' \end{array}\right)\mathcal{G}\left(\begin{array}{c} \boldsymbol{p}' \\ \varepsilon_n{}^{+\prime} \end{array}\right)\mathcal{G}\left(\begin{array}{c} \boldsymbol{p}' \\ \varepsilon_n{}' \end{array}\right)ev_\nu{}', \tag{3·6}$$

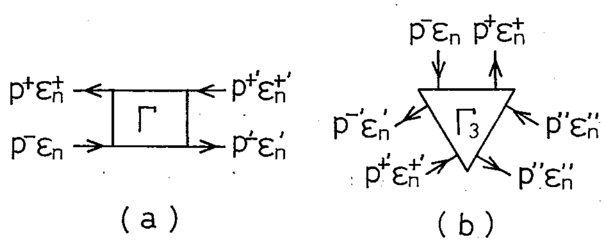


Fig. 1. The vertex functions of (a) two-body and (b) three-body interaction corresponding to (3·3) and (3·4), respectively.

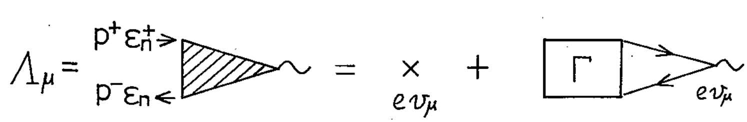


Fig. 2. The vertex function Λ coupled to an external field.

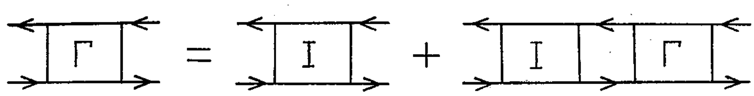


Fig. 3. Integral equation for Γ.

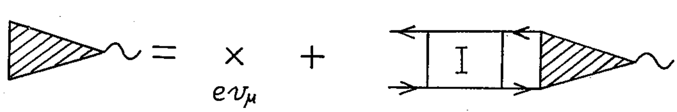


Fig. 4. Integral equation for Λ.

$$\Lambda_\alpha \equiv \Lambda_\alpha\begin{pmatrix} \boldsymbol{p}^- & \boldsymbol{p}^+ \\ \varepsilon_n & \varepsilon_n \end{pmatrix}$$

$$= ev_\alpha + T\sum_{\boldsymbol{p}',\varepsilon_n'} \Gamma\left(\begin{array}{cc|cc} \boldsymbol{p}^- & \boldsymbol{p}^+ & \boldsymbol{p}^{+\prime} & \boldsymbol{p}^{-\prime} \\ \varepsilon_n & \varepsilon_n & \varepsilon_n' & \varepsilon_n' \end{array}\right) \mathcal{G}\begin{pmatrix} \boldsymbol{p}^{+\prime} \\ \varepsilon_n' \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^{-\prime} \\ \varepsilon_n' \end{pmatrix} ev_\alpha' , \tag{3·7}$$

where $v_\mu = p_\mu/m$, $v_\mu' = p_\mu'/m$ and $v_\mu^\pm = p_\mu^\pm/m$. Note the symmetries

$$\Gamma\left(\begin{array}{cc|cc} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{array}\right) = \Gamma\left(\begin{array}{cc|cc} \boldsymbol{p}_3 & \boldsymbol{p}_4 & \boldsymbol{p}_1 & \boldsymbol{p}_2 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_1 & \varepsilon_2 \end{array}\right) = \Gamma\left(\begin{array}{cc|cc} \boldsymbol{p}_2 & \boldsymbol{p}_1 & \boldsymbol{p}_4 & \boldsymbol{p}_3 \\ \varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \end{array}\right). \tag{3·8}$$

If we define the irreducible vertex $\Gamma^{(\mathrm{I})}$ by

$$\Gamma\left(\begin{array}{cc|cc} \boldsymbol{p}^+ & \boldsymbol{p}^- & \boldsymbol{p}^{-\prime} & \boldsymbol{p}^{+\prime} \\ \varepsilon_n^+ & \varepsilon_n & \varepsilon_n' & \varepsilon_n^{+\prime} \end{array}\right) = \Gamma^{(\mathrm{I})}\left(\begin{array}{cc|cc} \boldsymbol{p}^+ & \boldsymbol{p}^- & \boldsymbol{p}^{-\prime} & \boldsymbol{p}^{+\prime} \\ \varepsilon_n^+ & \varepsilon_n & \varepsilon_n' & \varepsilon_n^{+\prime} \end{array}\right)$$

$$+ T\sum_{\boldsymbol{p}'',\varepsilon_n''} \Gamma^{(\mathrm{I})}\left(\begin{array}{cc|cc} \boldsymbol{p}^+ & \boldsymbol{p}^- & \boldsymbol{p}^{-\prime\prime} & \boldsymbol{p}^{+\prime\prime} \\ \varepsilon_n^+ & \varepsilon_n & \varepsilon_n'' & \varepsilon_n^{+\prime\prime} \end{array}\right) \mathcal{G}\begin{pmatrix} \boldsymbol{p}^{-\prime\prime} \\ \varepsilon_n'' \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^{+\prime\prime} \\ \varepsilon_n^{+\prime\prime} \end{pmatrix} \Gamma\left(\begin{array}{cc|cc} \boldsymbol{p}^{+\prime\prime} & \boldsymbol{p}^{-\prime\prime} & \boldsymbol{p}^{-\prime} & \boldsymbol{p}^{+\prime} \\ \varepsilon_n^{+\prime\prime} & \varepsilon_n'' & \varepsilon_n' & \varepsilon_n^{+\prime} \end{array}\right), \tag{3·9}$$

the integral equation for Λ_μ is given by

$$\Lambda_\mu\begin{pmatrix} \boldsymbol{p}^+ & \boldsymbol{p}^- \\ \varepsilon_n^+ & \varepsilon_n \end{pmatrix} = ev_\mu + T\sum_{\boldsymbol{p}',\varepsilon_n'} \Gamma^{(\mathrm{I})}\left(\begin{array}{cc|cc} \boldsymbol{p}^+ & \boldsymbol{p}^- & \boldsymbol{p}^{-\prime} & \boldsymbol{p}^{+\prime} \\ \varepsilon_n^+ & \varepsilon_n & \varepsilon_n' & \varepsilon_n^{+\prime} \end{array}\right) \mathcal{G}\begin{pmatrix} \boldsymbol{p}^{-\prime} \\ \varepsilon_n' \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^{+\prime} \\ \varepsilon_n^{+\prime} \end{pmatrix} \Lambda_\mu\begin{pmatrix} \boldsymbol{p}^{+\prime} & \boldsymbol{p}^{-\prime} \\ \varepsilon_n^{+\prime} & \varepsilon_n' \end{pmatrix}. \tag{3·10}$$

These relations are shown in Figs. 2~4, where Λ is denoted by a shaded triangle and Γ (or $\Gamma^{(\mathrm{I})}$) by a rectangle with the letter Γ (or I) inside it.

Using these functions, Eq. (2·8) is written as

$$K^\alpha_{\mu\nu} = (\mathrm{i}) + (\mathrm{ii}) + (\mathrm{iii}) + (\mathrm{iv}), \tag{3·11}$$

$$(\mathrm{i}) = \frac{e^2}{mc}\delta_{\nu\alpha} T\sum_{\boldsymbol{p},\varepsilon_n} \Lambda_\mu\begin{pmatrix} \boldsymbol{p}^+ & \boldsymbol{p}^- \\ \varepsilon_n^+ & \varepsilon_n \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^+ \\ \varepsilon_n^+ \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^- \\ \varepsilon_n \end{pmatrix},$$

$$(\mathrm{ii}) = \frac{1}{c} T\sum_{\boldsymbol{p},\varepsilon_n} \Lambda_\mu\begin{pmatrix} \boldsymbol{p}^+ & \boldsymbol{p}^- \\ \varepsilon_n^+ & \varepsilon_n \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^- \\ \varepsilon_n \end{pmatrix} \Lambda_\alpha\begin{pmatrix} \boldsymbol{p}^- & \boldsymbol{p}^+ \\ \varepsilon_n & \varepsilon_n \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^+ \\ \varepsilon_n \end{pmatrix} \Lambda_\nu\begin{pmatrix} \boldsymbol{p}^+ & \boldsymbol{p}^+ \\ \varepsilon_n & \varepsilon_n^+ \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^+ \\ \varepsilon_n^+ \end{pmatrix},$$

$$(\mathrm{iii}) = \frac{1}{c} T\sum_{\boldsymbol{p},\varepsilon_n} \Lambda_\mu\begin{pmatrix} \boldsymbol{p}^+ & \boldsymbol{p}^- \\ \varepsilon_n^+ & \varepsilon_n \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^- \\ \varepsilon_n \end{pmatrix} \Lambda_\nu\begin{pmatrix} \boldsymbol{p}^- & \boldsymbol{p}^- \\ \varepsilon_n & \varepsilon_n^+ \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^- \\ \varepsilon_n^+ \end{pmatrix} \Lambda_\alpha\begin{pmatrix} \boldsymbol{p}^- & \boldsymbol{p}^+ \\ \varepsilon_n^+ & \varepsilon_n^+ \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^+ \\ \varepsilon_n^+ \end{pmatrix},$$

$$(\mathrm{iv}) = \frac{1}{c} T\sum_{\boldsymbol{p},\varepsilon_n} T\sum_{\boldsymbol{p}',\varepsilon_n'} T\sum_{\boldsymbol{p}'',\varepsilon_n''} \Gamma_3^{(\mathrm{I})}\left(\begin{array}{cc|cc|cc} \boldsymbol{p}^+ & \boldsymbol{p}^- & \boldsymbol{p}' & \boldsymbol{p}' & \boldsymbol{p}^{-\prime\prime} & \boldsymbol{p}^{+\prime\prime} \\ \varepsilon_n^+ & \varepsilon_n & \varepsilon_n' & \varepsilon_n^{+\prime} & \varepsilon_n'' & \varepsilon_n'' \end{array}\right) \mathcal{G}\begin{pmatrix} \boldsymbol{p}^+ \\ \varepsilon_n^+ \end{pmatrix} \Lambda_\mu\begin{pmatrix} \boldsymbol{p}^+ & \boldsymbol{p}^- \\ \varepsilon_n^+ & \varepsilon_n \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^- \\ \varepsilon_n \end{pmatrix}$$

$$\times \mathcal{G}\begin{pmatrix} \boldsymbol{p}' \\ \varepsilon_n' \end{pmatrix} \Lambda_\nu\begin{pmatrix} \boldsymbol{p}' & \boldsymbol{p}' \\ \varepsilon_n' & \varepsilon_n^{+\prime} \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}' \\ \varepsilon_n^{+\prime} \end{pmatrix} \cdot \mathcal{G}\begin{pmatrix} \boldsymbol{p}^{-\prime\prime} \\ \varepsilon_n'' \end{pmatrix} \Lambda_\alpha\begin{pmatrix} \boldsymbol{p}^{-\prime\prime} & \boldsymbol{p}^{+\prime\prime} \\ \varepsilon_n'' & \varepsilon_n'' \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^{+\prime\prime} \\ \varepsilon_n'' \end{pmatrix}.$$

Diagrammatic expressions for these terms are shown in Fig. 5. The 'irreducible' vertex function $\Gamma_3^{(\mathrm{I})}$ [Fig. 6(a)] of three-body interaction is defined such that it cannot be decomposed as (b) nor (c) nor (d) in Fig. 6, i.e., it contains no 'two-particle-reducible' parts in any of its particle-hole channels.

3.1. *Extraction of the $\boldsymbol{q}$-linear terms*[4)]

In order to obtain the terms proportional to $\boldsymbol{H}=i\boldsymbol{q}\times\boldsymbol{A}$, we need to extract the part linear in $\boldsymbol{q}$ from $K_{\mu\nu}^{\alpha}$. Since the final expression should depend on $\boldsymbol{A}$ only through $\boldsymbol{H}$, i.e., be gauge invariant, we expect to get the $\boldsymbol{q}$-dependence of $K_{\mu\nu}^{\alpha}$ in the form $q_\mu\delta_{\nu\alpha}-q_\nu\delta_{\mu\alpha}$.

The vertex Λ_ν coupled to the uniform electric field which does not change the momentum is expanded as [Fig. 7]

$$\Lambda_\nu\begin{pmatrix}\boldsymbol{p}^{\pm} & \boldsymbol{p}^{\pm}\\ \varepsilon_n & \varepsilon_n\end{pmatrix}\cong\Lambda_\nu{}^{\circ}\pm\frac{1}{2}q_\rho\cdot\partial_\rho\Lambda_\nu{}^{\circ}\,, \tag{3·12}$$

where

$$\Lambda_\nu{}^{\circ}=\Lambda_\nu|_{q=0}=\Lambda_\nu\begin{pmatrix}\boldsymbol{p} & \boldsymbol{p}\\ \varepsilon_n & \varepsilon_n{}^{+}\end{pmatrix}, \tag{3·13}$$

and ∂_ρ denotes the differentiation with respect to p_ρ. The symbol $\cong$ expresses the equality up to the first order in $\boldsymbol{q}$. We write the vertex Λ_μ related to the observed current, which changes both momentum and energy, as [Fig. 8]

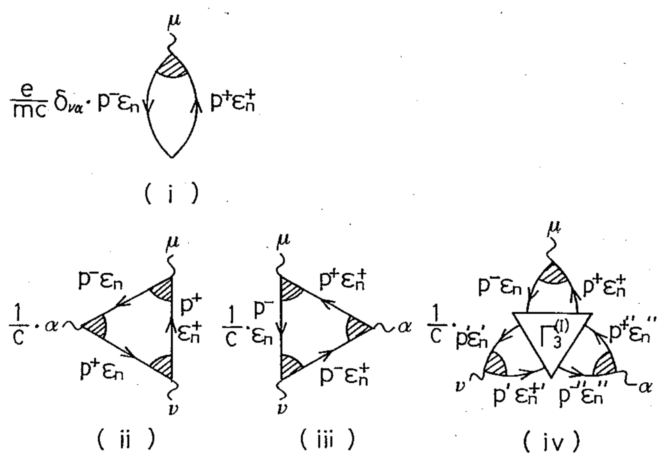


Fig. 5. Contributions to $K_{\mu\nu}^{\alpha}(\boldsymbol{q},\omega_\lambda)$.

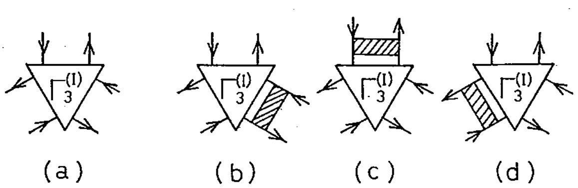


Fig. 6. Definition of $\Gamma_3^{(\mathrm{I})}$ (a), which cannot be decomposed as (b) nor (c) nor (d).

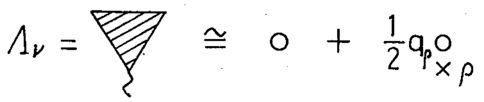


Fig. 7. The $\boldsymbol{q}$-linear extraction from ν-vertex. The circle means the vertex with $\boldsymbol{q}=0$. The cross with the letter ρ means differentiation with respect to p_ρ.

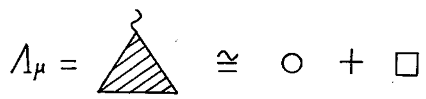


Fig. 8. The $\boldsymbol{q}$-linear extraction from μ-vertex. The square represents the $\boldsymbol{q}$-linear part of Λ_μ.

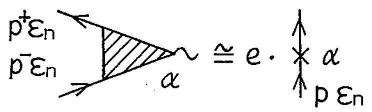


Fig. 9. The reduction of α-vertex combined with two Green functions both sides of it by means of the Ward identity. The cross with the letter α denotes differentiation with respect to p_α.

$$\Lambda_\mu\begin{pmatrix} \boldsymbol{p}^+ & \boldsymbol{p}^- \\ \varepsilon_n{}^+ & \varepsilon_n \end{pmatrix} \cong \Lambda_\mu{}^\circ + \Lambda_\mu{}^\triangle \,. \tag{3·14}$$

Here $\Lambda_\mu{}^\circ$ and $\Lambda_\mu{}^\triangle$ are the zeroth and the first order part of Λ_μ in $\boldsymbol{q}$, respectively. The vertex Λ_α, which couples to the external magnetic field and preserves the frequency argument, has no $\boldsymbol{q}$-linear terms. This can be seen from Eqs. (3·7) and (3·8). So, combined with the Green functions of both sides of it, it can be written as [Fig. 9]

$$\mathcal{G}\begin{pmatrix} \boldsymbol{p}^- \\ \varepsilon_n \end{pmatrix} \Lambda_\alpha \begin{pmatrix} \boldsymbol{p}^- & \boldsymbol{p}^+ \\ \varepsilon_n & \varepsilon_n \end{pmatrix} \mathcal{G}\begin{pmatrix} \boldsymbol{p}^+ \\ \varepsilon_n \end{pmatrix} \cong \Lambda_\alpha \begin{pmatrix} \boldsymbol{p} & \boldsymbol{p} \\ \varepsilon_n & \varepsilon_n \end{pmatrix} \mathcal{G}^2\begin{pmatrix} \boldsymbol{p} \\ \varepsilon_n \end{pmatrix}$$
$$= e \cdot \partial_\alpha \mathcal{G}(\boldsymbol{p}, \varepsilon_n) \,. \tag{3·15}$$

In the last equality, we used the Ward identity:[4),6)~8)]

$$\partial_\mu \Sigma(\boldsymbol{p}, \varepsilon_n) = T \sum_{\boldsymbol{p}', \varepsilon_n'} \Gamma^{(1)} \left(\begin{array}{cc|cc} \boldsymbol{p} & \boldsymbol{p} & \boldsymbol{p}' & \boldsymbol{p}' \\ \varepsilon_n & \varepsilon_n & \varepsilon_n' & \varepsilon_n' \end{array} \right) \cdot \partial_\mu{}' \mathcal{G}(\boldsymbol{p}', \varepsilon_n') \,. \tag{3·16}$$

Finally, the $\boldsymbol{q}$-linear part of the integral equation (3·10) [Fig. 4] is given by [Fig. 10]

$$\Lambda_\mu{}^\triangle \cong T \sum_{\boldsymbol{p}', \varepsilon_n'} \Gamma^{(1)\circ} \mathcal{G}' \mathcal{G}'(+) \Lambda_\mu{}^{\triangle\prime} + \frac{1}{2} q_\rho T \sum_{\boldsymbol{p}', \varepsilon_n'} \Gamma^{(1)} [\mathcal{G}' \overleftrightarrow{\partial}_\rho{}' \mathcal{G}'(+)] \Lambda_\mu{}^{\circ\prime}$$
$$+ T \sum_{\boldsymbol{p}', \varepsilon_n'} \Gamma^{(1)\triangle} \mathcal{G}' \mathcal{G}'(+) \Lambda_\mu{}^{\circ\prime} \,. \tag{3·17}$$

Here we put

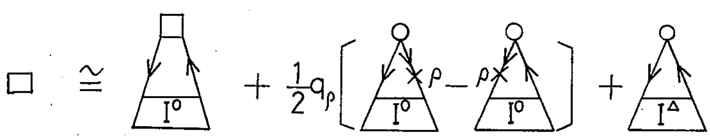


Fig. 10. $\boldsymbol{q}$-linear part of Fig. 4. I° represents the irreducible vertex with $\boldsymbol{q}=0$, and I the $\boldsymbol{q}$-linear part of the vertex.

$$\mathcal{G}=\mathcal{G}(\boldsymbol{p},\varepsilon_n)\,,\qquad \mathcal{G}(+)=\mathcal{G}(\boldsymbol{p},\varepsilon_n{}^+)\,,$$

$$\mathcal{G}'=\mathcal{G}(\boldsymbol{p}',\varepsilon_n')\,,\qquad \Lambda_\nu{}^{\circ\prime}=\Lambda_\nu\begin{pmatrix}\boldsymbol{p}' & \boldsymbol{p}'\\ \varepsilon_n' & \varepsilon_n{}^{+\prime}\end{pmatrix},\ \text{etc.}$$

and defined the "alternate" differentiation as

$$[A\overleftrightarrow{\partial}_\rho B]=A\frac{\partial B}{\partial p_\rho}-\frac{\partial A}{\partial p_\rho}B\,. \tag{3·18}$$

With these elementary procedures, the $\boldsymbol{q}$-linear terms of (i)~(iv) are derived as follows:

$$\text{(i)}\quad \cong\frac{e^2}{mc}\delta_{\nu\alpha}T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu{}^\triangle\mathcal{G}\,\mathcal{G}(+)+\frac{e^2}{2mc}\delta_{\nu\alpha}q_\rho T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu{}^\circ[\mathcal{G}\overleftrightarrow{\partial}_\rho\mathcal{G}(+)]\,,$$

$$\text{(ii)}\quad \cong\frac{e}{c}T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu\begin{pmatrix}\boldsymbol{p}^+ & \boldsymbol{p}^-\\ \varepsilon_n{}^+ & \varepsilon_n\end{pmatrix}\cdot\partial_\alpha\mathcal{G}\cdot\Lambda_\nu\begin{pmatrix}\boldsymbol{p}^+ & \boldsymbol{p}^+\\ \varepsilon_n & \varepsilon_n{}^+\end{pmatrix}\mathcal{G}(\boldsymbol{p}^+\cdot\varepsilon_n{}^+)$$

$$\cong\frac{e}{c}T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu{}^\triangle\Lambda_\nu{}^\circ\cdot\partial_\alpha\mathcal{G}\cdot\mathcal{G}(+)+\frac{e}{2c}q_\rho T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu{}^\circ(\partial_\rho\Lambda_\nu{}^\circ)\cdot\partial_\alpha\mathcal{G}\cdot\mathcal{G}(+)$$

$$+\frac{e}{2c}q_\rho T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu{}^\circ\Lambda_\nu{}^\circ\cdot\partial_\alpha\mathcal{G}\cdot\partial_\rho\mathcal{G}(+)\,,$$

$$\text{(iii)}\quad \cong\frac{e}{c}T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu\begin{pmatrix}\boldsymbol{p}^+ & \boldsymbol{p}^-\\ \varepsilon_n{}^+ & \varepsilon_n\end{pmatrix}\mathcal{G}(\boldsymbol{p}^-,\varepsilon_n)\Lambda_\nu\begin{pmatrix}\boldsymbol{p}^- & \boldsymbol{p}^-\\ \varepsilon_n & \varepsilon_n{}^+\end{pmatrix}\cdot\partial_\alpha\mathcal{G}(+)$$

$$\cong\frac{e}{c}T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu{}^\triangle\Lambda_\nu{}^\circ\mathcal{G}\cdot\partial_\alpha\mathcal{G}(+)-\frac{e}{2c}q_\rho T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu{}^\circ(\partial_\rho\Lambda_\nu{}^\circ)\mathcal{G}\cdot\partial_\alpha\mathcal{G}(+)$$

$$-\frac{e}{2c}q_\rho T\sum_{\boldsymbol{p},\varepsilon_n}\Lambda_\mu{}^\circ\Lambda_\nu{}^\circ\cdot\partial_\rho\mathcal{G}\cdot\partial_\alpha\mathcal{G}(+)\,,$$

$$\text{(iv)}\quad \cong\frac{e}{c}T\sum_{\boldsymbol{p},\varepsilon_n}T\sum_{\boldsymbol{p}',\varepsilon_n'}T\sum_{\boldsymbol{p}'',\varepsilon_n''}\Gamma_3{}^{(\mathrm{I})\circ}\mathcal{G}(\boldsymbol{p}^+,\varepsilon_n{}^+)\Lambda_\mu\begin{pmatrix}\boldsymbol{p}^+ & \boldsymbol{p}^-\\ \varepsilon_n{}^+ & \varepsilon_n\end{pmatrix}\mathcal{G}(\boldsymbol{p}^-,\varepsilon_n)$$

$$\times\mathcal{G}(\boldsymbol{p}',\varepsilon_n')\Lambda_\nu\begin{pmatrix}\boldsymbol{p}' & \boldsymbol{p}'\\ \varepsilon_n' & \varepsilon_n{}^{+\prime}\end{pmatrix}\mathcal{G}(\boldsymbol{p}',\varepsilon_n{}^{+\prime})\cdot\partial_\alpha''\mathcal{G}''$$

$$+\frac{e}{c}T\sum_{\boldsymbol{p},\varepsilon_n}T\sum_{\boldsymbol{p}',\varepsilon_n'}T\sum_{\boldsymbol{p}'',\varepsilon_n''}\Gamma_3{}^{(\mathrm{I})\triangle}\cdot\Lambda_\mu{}^\circ\mathcal{G}(+)\mathcal{G}\cdot\Lambda_\nu{}^{\circ\prime}\mathcal{G}'\mathcal{G}'(+)\cdot\partial_\alpha''\mathcal{G}''\,.$$

Here $\Gamma_3{}^{(\mathrm{I})\circ}$ and $\Gamma_3{}^{(\mathrm{I})\triangle}$ mean $\Gamma_3{}^{(\mathrm{I})}$ with $\boldsymbol{q}=0$ and the $\boldsymbol{q}$-linear part of $\Gamma_3{}^{(\mathrm{I})}$, respectively. Making use of the Ward identity:

$$(\partial_\alpha+\partial_\alpha')\Gamma^{(\mathrm{I})}\left(\begin{array}{cc|cc}\boldsymbol{p} & \boldsymbol{p} & \boldsymbol{p}' & \boldsymbol{p}'\\ \varepsilon_n{}^+ & \varepsilon_n & \varepsilon_n' & \varepsilon_n{}^{+\prime}\end{array}\right)$$

$$=T\sum_{p'',\varepsilon_n''}\Gamma_3^{(\mathrm{I})}\left(\begin{matrix}\boldsymbol{p} & \boldsymbol{p}\\ \varepsilon_n{}^+ & \varepsilon_n\end{matrix}\middle|\begin{matrix}\boldsymbol{p}' & \boldsymbol{p}'\\ \varepsilon_n{}' & \varepsilon_n{}^{+\prime}\end{matrix}\middle|\begin{matrix}\boldsymbol{p}'' & \boldsymbol{p}''\\ \varepsilon_n{}'' & \varepsilon_n{}''\end{matrix}\right)\cdot\partial_\alpha{}''\mathcal{G}(\boldsymbol{p}'',\varepsilon_n{}''),\tag{3·19}$$

which can be proved in a similar way to Eq. (3·16), we can rewrite (iv) as

$$(\mathrm{iv})\cong\frac{e}{c}T\sum_{p,\varepsilon_n}T\sum_{p',\varepsilon_n'}(\partial_\alpha+\partial_\alpha{}')\Gamma^{(\mathrm{I})\circ}\Lambda_\mu{}^\triangle\mathcal{G}\,\mathcal{G}(+)\cdot\Lambda_\nu{}^{\circ\prime}\mathcal{G}'\mathcal{G}'(+)\tag{iv-a}$$

$$+\frac{e}{2c}q_\rho T\sum_{p,\varepsilon_n}T\sum_{p',\varepsilon_n'}(\partial_\alpha+\partial_\alpha{}')\Gamma^{(\mathrm{I})\circ}\Lambda_\mu{}^\circ[\mathcal{G}\overleftrightarrow{\partial}_\rho\mathcal{G}(+)]\cdot\Lambda_\nu{}^{\circ\prime}\mathcal{G}'\mathcal{G}'(+)\tag{iv-b}$$

$$+\frac{e}{c}T\sum_{p,\varepsilon_n}T\sum_{p',\varepsilon_n'}T\sum_{p'',\varepsilon_n''}\Gamma_3^{(\mathrm{I})\triangle}\Lambda_\mu{}^\circ\mathcal{G}\,\mathcal{G}(+)\cdot\Lambda_\nu{}^\circ\mathcal{G}'\mathcal{G}'(+)\cdot\partial_\alpha{}''\mathcal{G}''.\tag{iv-c}$$

Using Eq. (3·10) for the ∂_α-terms and integrating partially for the $\partial_\alpha{}'$-terms, we get

$$(\mathrm{iv\text{-}a})=\frac{e}{c}T\sum_{p,\varepsilon_n}\Lambda_\mu{}^\triangle\mathcal{G}\,\mathcal{G}(+)\cdot\partial_\alpha\Lambda_\nu{}^\circ-\frac{e^2}{mc}\delta_{\nu\alpha}T\sum_{p,\varepsilon_n}\Lambda_\mu{}^\triangle\mathcal{G}\,\mathcal{G}(+)$$

$$-\frac{e}{c}T\sum_{p,\varepsilon_n}T\sum_{p',\varepsilon_n'}\Gamma^{(\mathrm{I})\circ}\Lambda_\mu{}^\triangle\mathcal{G}\,\mathcal{G}(+)\cdot\partial_\alpha{}'[\Lambda_\nu{}^{\circ\prime}\mathcal{G}'\mathcal{G}'(+)],$$

$$(\mathrm{iv\text{-}b})=\frac{e}{2c}q_\rho T\sum_{p,\varepsilon_n}\Lambda_\mu{}^\circ[\mathcal{G}\overleftrightarrow{\partial}_\mu\mathcal{G}(+)]\cdot\partial_\alpha\Lambda_\nu{}^\circ-\frac{e^2}{2mc}\delta_{\nu\alpha}q_\rho T\sum_{p,\varepsilon_n}\Lambda_\mu{}^\circ[\mathcal{G}\overleftrightarrow{\partial}_\rho\mathcal{G}(+)]$$

$$-\frac{e}{2c}q_\rho T\sum_{p,\varepsilon_n}T\sum_{p',\varepsilon_n'}\Gamma^{(\mathrm{I})\circ}\Lambda_\mu{}^\circ[\mathcal{G}\overleftrightarrow{\partial}_\rho\mathcal{G}(+)]\partial_\alpha{}'[\Lambda_\nu{}^{\circ\prime}\mathcal{G}'\mathcal{G}'(+)].$$

The second terms of (iv-a) and (iv-b) cancel with (i). From Eq. (3·17), the last terms of (iv-a) and (iv-b) yield

$$-\frac{e}{c}T\sum_{p,\varepsilon_n}\Lambda_\mu{}^\triangle\partial_\alpha[\Lambda_\nu{}^\circ\mathcal{G}\,\mathcal{G}(+)]$$

$$+\frac{e}{c}T\sum_{p,\varepsilon_n}T\sum_{p',\varepsilon_n'}\Gamma^{(\mathrm{I})\triangle}\Lambda_\mu{}^\circ\mathcal{G}\cdot\mathcal{G}(+)\cdot\partial_\alpha{}'[\Lambda_\nu{}^{\circ\prime}\mathcal{G}'\mathcal{G}'(+)],\tag{3·20}$$

whose first term cancels with the first terms of (ii), (iii) and (iv-a).

The resulting terms are (A) the first term of (iv-b) and the second terms of (ii) and (iii); (B) the last terms of (ii) and (iii); (C) (iv-c); and (D) the second term of Eq. (3·20):

$$A=\frac{e}{2c}q_\rho T\sum_{p,\varepsilon_n}\Lambda_\mu{}^\circ[\mathcal{G}\overleftrightarrow{\partial}_\rho\mathcal{G}(+)]\cdot\partial_\alpha\Lambda_\nu{}^\circ-\frac{e}{2c}q_\rho T\sum_{p,\varepsilon_n}\Lambda_\mu{}^\circ[\mathcal{G}\overleftrightarrow{\partial}_\alpha\mathcal{G}(+)]\partial_\rho\Lambda_\nu{}^\circ,$$

$$B=\frac{e}{2c}q_\rho T\sum_{p,\varepsilon_n}\Lambda_\mu{}^\circ[\partial_\alpha\mathcal{G}\cdot\partial_\rho\mathcal{G}(+)-\partial_\rho\mathcal{G}\cdot\partial_\alpha\mathcal{G}(+)]\Lambda_\nu{}^\circ,$$

$$C=\frac{e}{c}T\sum_{p,\varepsilon_n}T\sum_{p',\varepsilon_n'}T\sum_{p'',\varepsilon_n''}\Gamma_3^{(\mathrm{I})\triangle}\Lambda_\mu{}^\circ\mathcal{G}\,\mathcal{G}(+)\cdot\Lambda_\nu{}^{\circ\prime}\mathcal{G}'\mathcal{G}'(+)\cdot\partial_\alpha{}''\mathcal{G}'',$$

$$D=\frac{e}{c}T\sum_{p,\varepsilon_n}T\sum_{p',\varepsilon_n'}\Gamma^{(\mathrm{I})\triangle}\Lambda_\mu{}^\circ\mathcal{G}\,\mathcal{G}(+)\cdot\partial_\alpha{}'[\Lambda_\nu{}^{\circ\prime}\mathcal{G}'\mathcal{G}'(+)].$$

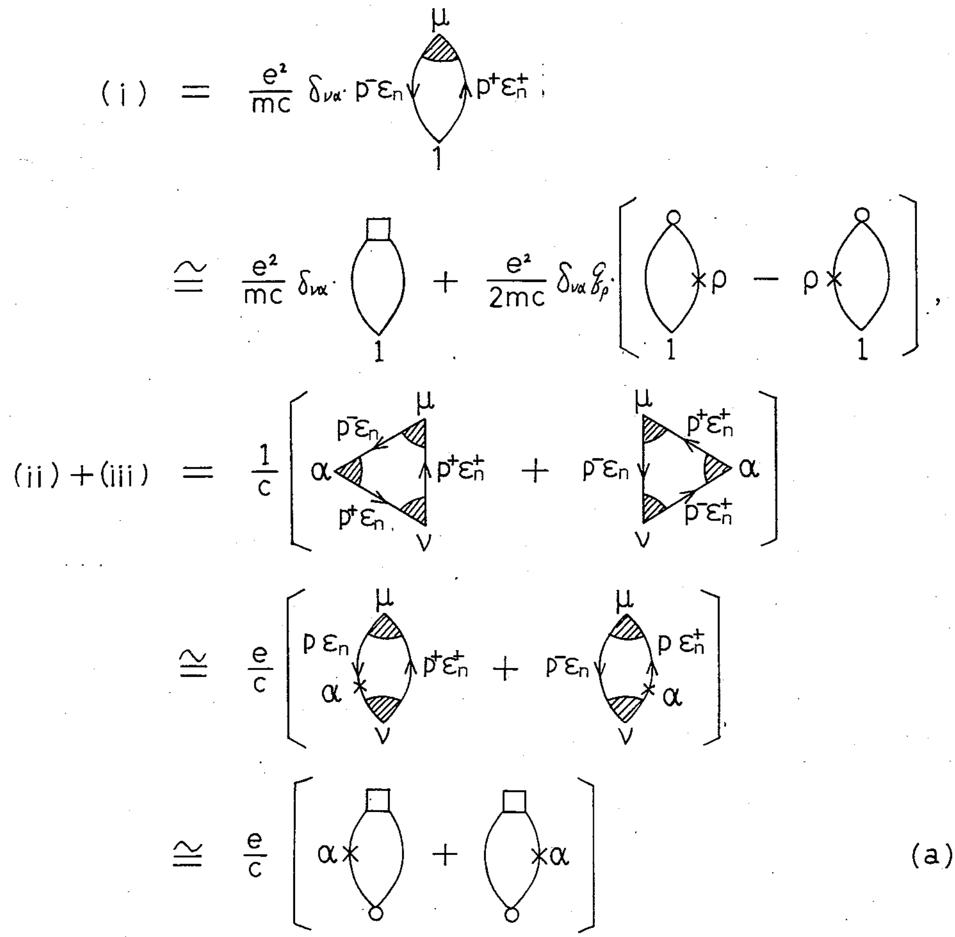
We neglect C and D whose $\boldsymbol{q}$-linear terms arise from $\Gamma_3^{(\mathrm{I})}$ and $\Gamma^{(\mathrm{I})}$. The validity of this approximation will be discussed in § 5.1. With the replacement: $q_\rho\partial_\rho \to \delta_{\nu\alpha}q_\mu\partial_\mu + \delta_{\mu\alpha}q_\nu\partial_\nu$ which is valid under the assumed reflection symmetries of the system, we finally get as $A+B$

$$K^{\alpha}_{\mu\nu}(\omega_\lambda)=\frac{e}{4c}(q_\mu\delta_{\nu\alpha}-q_\nu\delta_{\mu\alpha})T\sum_{p,\varepsilon_n}\{[\Lambda_\mu{}^\circ\overleftrightarrow{\partial}_\nu\Lambda_\nu{}^\circ][\mathcal{G}\overleftrightarrow{\partial}_\mu\mathcal{G}(+)] + [\Lambda_\nu{}^\circ\overleftrightarrow{\partial}_\mu\Lambda_\mu{}^\circ][\mathcal{G}\overleftrightarrow{\partial}_\nu\mathcal{G}(+)]\}\,. \tag{3·21}$$

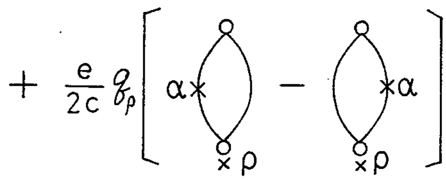
This has the desired gauge-invariant form. When x- and y-directions are equivalent, this reduces to

$$K^{\alpha}_{\mu\nu}(\omega_\lambda)=\frac{e}{2c}(q_\mu\delta_{\nu\alpha}-q_\nu\delta_{\mu\alpha})T\sum_{p,\varepsilon_n}[\Lambda_\mu{}^\circ\overleftrightarrow{\partial}_\nu\Lambda_\nu{}^\circ][\mathcal{G}\overleftrightarrow{\partial}_\mu\mathcal{G}(+)]\,. \tag{3·22}$$

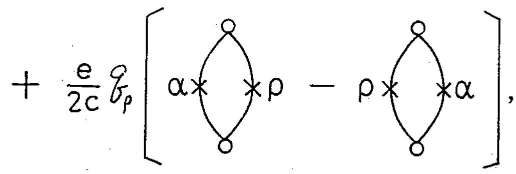
The above procedure is seen more intuitively in the diagrammatic calculation and we give it in the following. In the diagram, the upper (lower) vertex corresponds to $\Lambda_\mu(\Lambda_\nu)$ except (i) whose lower vertex is simply 1, and the right (left) line corresponds to the Green function with frequency $\varepsilon_n{}^+(\varepsilon_n)$.



(a)

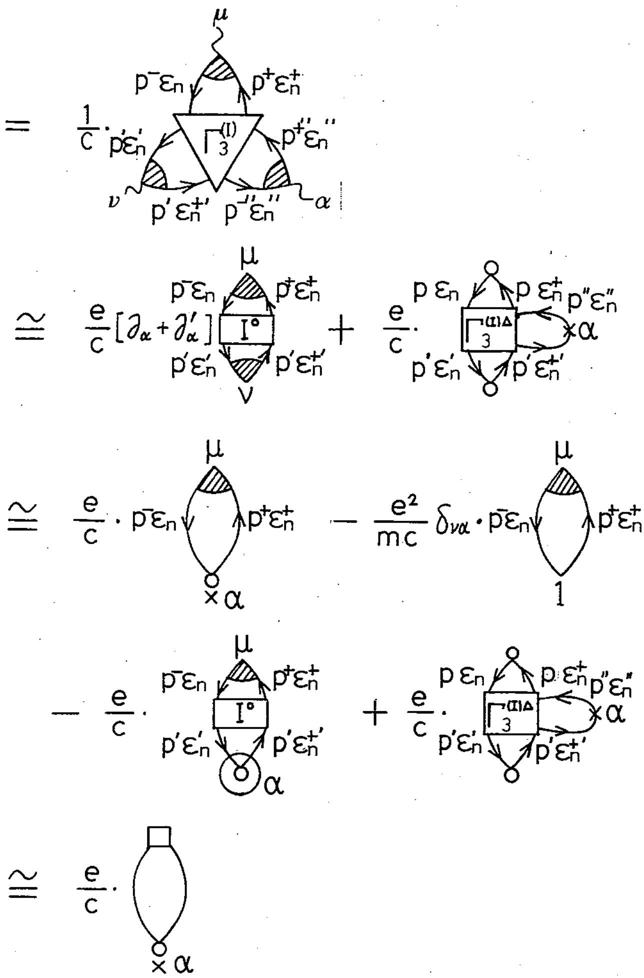


(b)

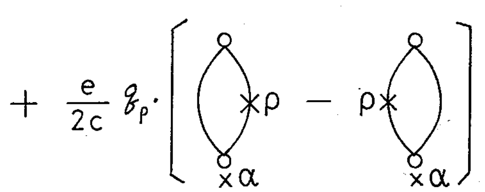


(c)

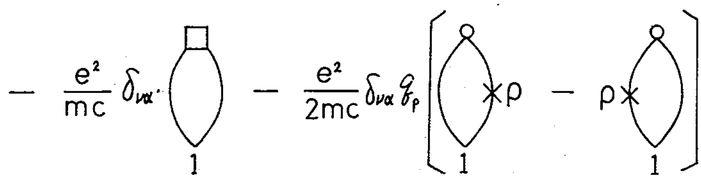
(iv)



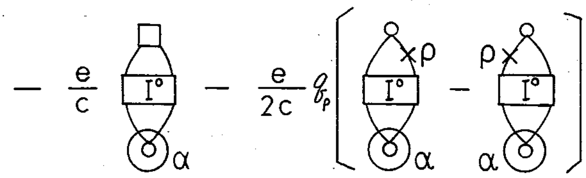
(d)



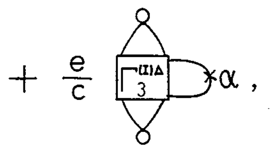
(e)



(f)

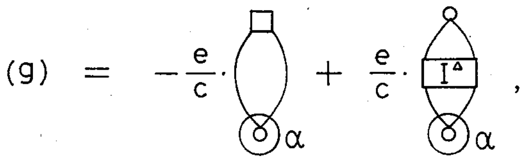


(g)



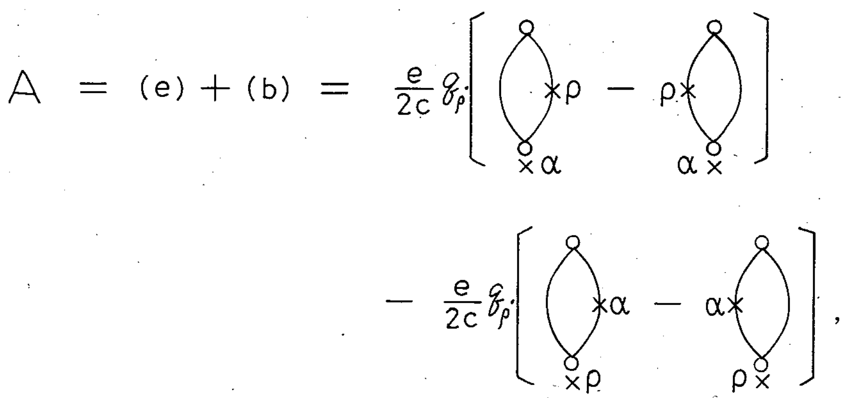
(h)

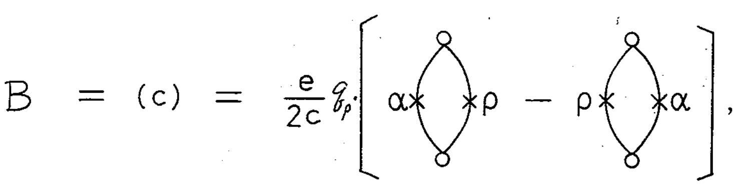
where  represents $\partial_\alpha'[\Lambda_\nu^{\circ\prime}\mathscr{G}'\mathscr{G}'(+)]$. (i) cancels with (f). From Fig. 10, (
becomes

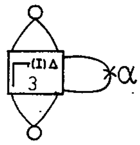


whose first term cancels with (a)+(d).

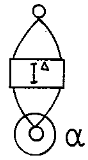
Thus, we finally get





$C \;=\; (h) \;=\; \frac{e}{c}\cdot$ 

,

$D \;=\;$ 2nd term of (g) $\;=\; \frac{e}{c}\cdot$ 

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3.2. *Main terms in the Fermi liquid*[2)]

Now, we shall perform the analytic continuation of Eq. (3·21) or (3·22). Since we are concerned only with energy variables in this subsection, some momentum variables are omitted for simplicity. Also, momentum derivatives are irrelevant and the problem of the analytic continuation of Eq. (3·22) is equivalent to that of

$$L_{\mu\nu}(\omega_\lambda)\equiv T\sum_{\varepsilon_n}\Lambda_\mu{}^\circ\Lambda_\nu{}^\circ\mathcal{G}(+)\mathcal{G}\ . \tag{3·23}$$

Thus we consider Eq. (3·23) for the moment. Note that, from Eq. (3·8), $\Lambda_\mu{}^\circ$ and $\Lambda_\nu{}^\circ$ are different components of an identical vector. $\Lambda_\mu{}^\circ$ is defined by Eq. (3·5) with $\boldsymbol{q}=0$. The analytic property of Γ was examined by Éliashberg in Ref. 2). According to that, one can see that $\Lambda_\mu{}^\circ$ has branch cuts $\mathrm{Im}\varepsilon=0$ and $\mathrm{Im}(\varepsilon+\omega)=0$ by which the whole ε-plane is divided into three regions [Fig. 11]. From each region, $\Lambda_\mu{}^\circ$ is analytically continued to the function $J_\mu{}^{(1)}$, $J_\mu{}^{(2)}$ or $J_\mu{}^{(3)}$ defined on the real axis, where

$$J_\mu{}^{(l)}(\varepsilon;\,\omega)=ev_\mu+\sum_{p'}ev_\mu{}'\int_{-\infty}^{\infty}\frac{d\varepsilon'}{4\pi i}\sum_{m=1}^{3}\mathcal{T}_{lm}(\varepsilon,\,\varepsilon';\,\omega)g_m(\varepsilon';\,\omega)\,,\ (l=1,2,3) \tag{3·24}$$

$$\left.\begin{aligned} g_1(\varepsilon;\,\omega)&=G^R(\varepsilon+\omega)G^R(\varepsilon)\\ g_2(\varepsilon,\,\omega)&=G^R(\varepsilon+\omega)G^A(\varepsilon)\\ g_3(\varepsilon;\,\omega)&=G^A(\varepsilon+\omega)G^A(\varepsilon)\end{aligned}\right\}\ , \tag{3·25}$$

and the definition of effective vertices $\mathcal{T}_{lm}(\varepsilon,\,\varepsilon;\,\omega)$ [see Fig. 12] at finite temperature is given in Ref. 2) as Eq. (12). Thus Eq. (3·23) is analytically continued to the real frequency through the 'retarded' function as

$$L_{\mu\nu}(\omega)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{4\pi i}\Big[\mathrm{th}\frac{\varepsilon}{2T}K_{\mu\nu}^{(1)}(\varepsilon;\,\omega) + \left(\mathrm{th}\frac{\varepsilon+\omega}{2T}-\mathrm{th}\frac{\varepsilon}{2T}\right)K_{\mu\nu}^{(2)}(\varepsilon;\,\omega)-\mathrm{th}\frac{\varepsilon+\omega}{2T}K_{\mu\nu}^{(3)}(\varepsilon;\,\omega)\Big]\,, \tag{3·26}$$

$$K_{\mu\nu}^{(l)}(\varepsilon;\,\omega)=J_\mu{}^{(l)}(\varepsilon;\,\omega)J_\nu{}^{(l)}(\varepsilon;\,\omega)g_l(\varepsilon;\,\omega)\,. \tag{3·27}$$

Next, in order to clarify the physical meaning, we shall make an approximation which is based on Fermi liquid picture. One-particle Green functions can be written as

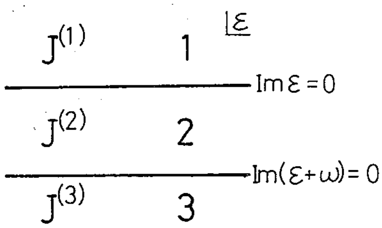


Fig. 11. The analytic region of $\Lambda_\mu^\circ(\varepsilon)$ as a function of a complex variable ε. From each region, Λ_μ° is analytically continued to the whole plane through the functions $J_\mu^{(l)}$.

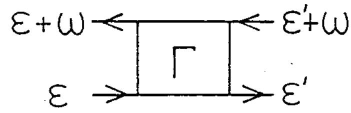


Fig. 12. $\mathcal{T}_{lm}(\varepsilon, \varepsilon'; \omega)$.

$$G^R(\varepsilon)=[G^A(\varepsilon)]^*=\frac{a}{\varepsilon-E(\boldsymbol{p})+i\gamma_p}, \tag{3·28}$$

where $E(\boldsymbol{p})$, γ_p and a are the energy, the damping constant and the wave-function renormalization factor of a quasiparticle of momentum $\boldsymbol{p}$. It is assumed that the temperature is sufficiently low and if $\varepsilon\sim T$ and $E(\boldsymbol{p})\sim T$, then $\gamma_p \ll T$. We collect all the terms in Eq. (3·26) that are proportional to or higher order in $1/\gamma_p$, i.e., divergent as the quasiparticle damping approaches zero. For $\omega \ll T$, g_l's behave like

$$g_1(\varepsilon;\omega)=[g_3(\varepsilon;\omega)]^*\sim\left(\frac{a}{\varepsilon-E(\boldsymbol{p})+i0}\right)^2, \tag{3·29}$$

$$g_2(\varepsilon;\omega)\sim\frac{2\pi i a^2\delta(\varepsilon-E(\boldsymbol{p}))}{\omega+2i\gamma_p}. \tag{3·30}$$

After ε-integration (or $\boldsymbol{p}$-integration for impurity scattering), g_1 and g_3 leave no singularities arising from the smallness of γ_p, while g_2-section contributes to $1/\gamma_p$-singularity. Since the irreducible vertex part $\Gamma^{(1)}$ [or $\mathcal{T}^{(1)}$] has no singular factor $1/\gamma_p$, we should collect all the terms that have at least one g_2-section.

On the other hand, in order to obtain a static conductivity $\sigma(\omega\to 0)$, we must extract the ω-linear terms from Eq. (3·26). The second term of Eq. (3·26) is already linear in ω, so we can put $\omega=0$ in $K_{\mu\nu}^{(2)}$. The first and the last terms of Eq. (3·26) give no contributions to Eq. (3·22). This is understood as follows.

In these cases, a g_2-section should be picked up in either $J_\mu^{(l)}$ or $J_\nu^{(l)}$ ($l=1, 3$), since we collect diagrams with at least one g_2-section. Then, the factor ω is always present in the Fermi distribution function of the form $\mathrm{th}((\varepsilon+\omega)/2T)-\mathrm{th}(\varepsilon/2T)=(\omega/2T)\cdot\mathrm{ch}^{-2}(\varepsilon/2T)$ as seen from Eq. (12) in Ref. 2), and consequently we are allowed to put $\omega=0$ in g_1 or g_3 in Eq. (3·27). Returning to Eq. (3·22), these g_1- and g_3-sections are subject to momentum derivatives and vanish:

$$[G^{R(A)}(\boldsymbol{p},\varepsilon)\overleftrightarrow{\partial}_\mu G^{R(A)}(\boldsymbol{p},\varepsilon)]=0.$$

Now, we introduce the 'irreducible' vertex $\mathcal{T}_{lm}^{(0)}$ with respect to g_2-section by

$$\mathcal{T}_{lm}^{(0)}(\varepsilon,\varepsilon';\omega)=\mathcal{T}_{lm}^{(1)}(\varepsilon,\varepsilon';\omega)+\sum_{p''}\int_{-\infty}^{\infty}\frac{d\varepsilon''}{4\pi i}\sum_{k=1,3}\mathcal{T}_{lk}^{(1)}(\varepsilon,\varepsilon'',\omega)$$
$$\times g_k(\varepsilon'';\omega)\mathcal{T}_{km}^{(0)}(\varepsilon'',\varepsilon';\omega). \tag{3·31}$$

Here $\mathcal{T}^{(1)}$ corresponds to $\Gamma^{(1)}$ in the same way that $\mathcal{T}$ corresponds to Γ. Denoting $\mathcal{T}_{lm}^{(0)}$'s by shaded rectangles and $\mathcal{T}_{22}$ (neither $\mathcal{T}_{22}^{(0)}$ nor $\mathcal{T}_{22}^{(1)}$) by a shaded circle, $\mathcal{T}_{2j}$'s ($j \neq 2$) are expressed as shown in Fig. 13, and $J_\mu^{(2)}$ as shown in Fig. 14. From the last line of Fig. 14, we obtain the relation

$$J_\mu^{(2)}(\boldsymbol{p}, \varepsilon) = Q_\mu(\boldsymbol{p}, \varepsilon) + \sum_{\boldsymbol{p}'} \int_{-\infty}^{\infty} \frac{d\varepsilon'}{4\pi i} \mathcal{T}_{22}(\boldsymbol{p}\varepsilon|\boldsymbol{p}'\varepsilon') g_2(\boldsymbol{p}', \varepsilon'; \omega=0) Q_\mu(\boldsymbol{p}', \varepsilon'), \quad (3\cdot32)$$

where

$$Q_\mu(\boldsymbol{p}, \varepsilon) = ev_\mu + \sum_{\boldsymbol{p}'} ev_\mu' \int_{-\infty}^{\infty} \frac{d\varepsilon'}{4\pi i} \sum_{j=1,3} \mathcal{T}_{2j}^{(0)}(\boldsymbol{p}\varepsilon|\boldsymbol{p}'\varepsilon') g_j(\boldsymbol{p}', \varepsilon'; \omega=0). \quad (3\cdot33)$$

This $Q_\mu(\boldsymbol{p}, \varepsilon)$ is connected with the quasiparticle velocity $v_\mu^* = \partial_\mu E(\boldsymbol{p})$ by

$$Q_\mu = e \frac{v_\mu^*}{a}. \quad (3\cdot34)$$

This is seen from the analytically-continued Ward identity:[2),8)]

$$\frac{\partial}{\partial p_\mu} [G^R(\boldsymbol{p}, \varepsilon)]^{-1} = -v_\mu - \sum_{\boldsymbol{p}'} v_\mu' \int_{-\infty}^{\infty} \frac{d\varepsilon'}{4\pi i} \sum_{j=1,3} \mathcal{T}_{1j}(\boldsymbol{p}\varepsilon|\boldsymbol{p}'\varepsilon') g_j(\boldsymbol{p}', \varepsilon'; \omega=0), \quad (3\cdot35)$$

and the replacement: $\mathcal{T}_{2j}^{(0)} \to (\mathcal{T}_{1j} + \mathcal{T}_{3j})/2$ which can be justified for $\varepsilon \sim T$.[2)]

Thus Eq. (3·17) is analytically continued as

$$K_{\mu\nu}^{a}(\omega) \cong -\omega \frac{e}{2c}(q_\mu \delta_{\nu a} - q_\nu \delta_{\mu a}) \sum_{\boldsymbol{p}} \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi i} \left(-\frac{df}{d\varepsilon}\right) [J_\mu^{(2)} \overleftrightarrow{\partial}_\nu J_\nu^{(2)}][G^R \overleftrightarrow{\partial}_\mu G^A], \quad (3\cdot36)$$

where f is the Fermi distribution function. With Eq. (3·28), the ε-integration is performed as

$$\int_{-\infty}^{\infty} d\varepsilon A(\varepsilon) \left[\frac{1}{(\varepsilon - E(\boldsymbol{p}))^2 + \gamma_{\boldsymbol{p}}^2}\right]^2 \approx A(E(\boldsymbol{p})) \frac{\pi}{2\gamma_{\boldsymbol{p}}^3}, \quad (3\cdot37)$$

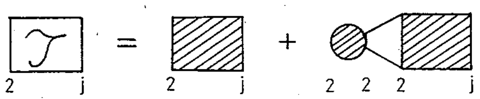


Fig. 13. The expression for $\mathcal{T}_{2j}(j \neq 2)$ in terms of $\mathcal{T}_{2j}^{(0)}(j \neq 2)$ (shaded rectangles) and $\mathcal{T}_{22}$(shaded circle).

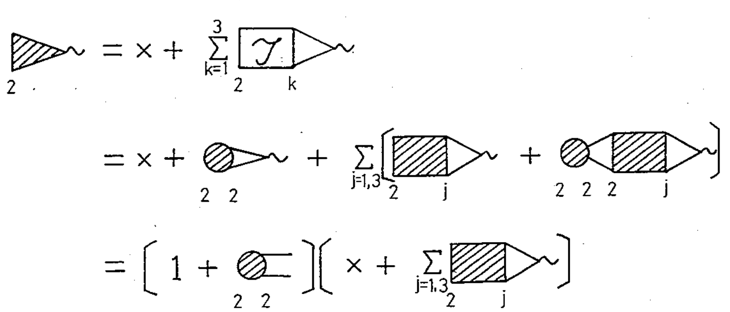


Fig. 14. The expression for $J_\mu^{(2)}$. The cross means the bare current vertex ev_μ.

where $A(\varepsilon)$ is a smooth function over the scale $\varepsilon \sim \gamma_p$. This holds even for $A(\varepsilon) = -df/d\varepsilon$, because this function does not change appreciably over the scale $\varepsilon \sim \gamma_p \ll T$. Finally we get the result as

$$\sigma_{\mu\nu} = \frac{e^3}{c} H \int \frac{d\boldsymbol{p}}{(2\pi)^3} [J_\mu \overleftrightarrow{\partial}_\nu J_\nu] v_\mu^* \frac{1}{(2\gamma_p)^2} \left(-\frac{df}{d\varepsilon}\right)_{\varepsilon = E(\boldsymbol{p})}, \qquad (\mu = x,\ \nu = y) \tag{3·38}$$

where

$$J_\mu \equiv \frac{a}{e} J_\mu^{(2)}(\boldsymbol{p}, E\boldsymbol{p})$$

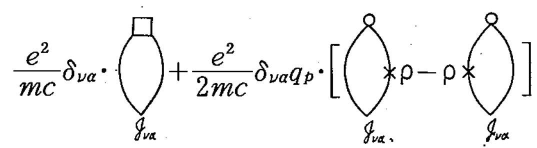
$$= v_\mu^* + \int \frac{d\boldsymbol{p}}{(2\pi)^3} a^2 \mathcal{T}_{22}(\boldsymbol{p}E(\boldsymbol{p}) | \boldsymbol{p}'E(\boldsymbol{p}')) \frac{v_\mu^{*\prime}}{4i\gamma_{p'}}. \tag{3·39}$$

The second term of Eq. (3·39) represents the vertex correction arising from quasi-particle interactions. Note that two velocities in Eq. (3·38) are subject to this vertex correction. Note also that Eqs. (2·38) and (2·39) are derived for the static ($\omega=0$) limit or in the 'first sound regime' rather than in the 'zero sound regime': $\gamma_p \ll \omega$. In the latter case,[2] the second term of Eq. (2·39) is replaced by the backflow term which is well-known in the usual Fermi liquid theory at zero temperature.[6),7)]

§ 4. Extension to 'Bloch electron' system

Now we turn to the case of 'Bloch electron' which has an arbitrary dispersion relation $\varepsilon(\boldsymbol{p})$.

Simple replacement for $\varepsilon(\boldsymbol{p})$ and v_μ, which have been $\boldsymbol{p}^2/2m$ and p_μ/m in the previous section, with those of a Bloch electron leads to the expression obtained from Eq. (3·21), with extra terms:



$$= \frac{e^2}{mc} T \sum_{p,\varepsilon_n} \Lambda_\mu{}^\triangle \mathcal{G}\, \mathcal{G}(+) \mathcal{I}_{\nu\alpha} + \frac{e^2}{2mc} q_\rho T \sum_{p,\varepsilon_n} \Lambda_\mu{}^\circ [\mathcal{G} \overleftrightarrow{\partial}_\rho \mathcal{G}(+)] \mathcal{I}_{\nu\alpha}, \tag{4·1}$$

where

$$\frac{1}{m} \mathcal{I}_{\nu\alpha} = \frac{1}{m} \delta_{\nu\alpha} - \frac{\partial^2 \varepsilon(\boldsymbol{p})}{\partial p_\nu \partial p_\alpha}. \tag{4·2}$$

This term is not gauge invariant in general. What is wrong with this prescription?

The above inconsistency arises from the fact that the current operator appearing in the conductivity formula has finite interband matrix elements in general.[9)] This may be interpreted as interband transitions violate the current conservation within a single band and consequently the gauge invariance of the single-band formula. Nevertheless we expect that when the band in which the whole Fermi surface lies is energetically far apart from any other bands, the conductivity can be expressed only with single band quantities in a gauge-invariant manner. To treat this situation self-consistently, we start with the single band model explained below. Though this picture may not be suitable in the real situation, this method makes possible a consistent calculation for some model systems which are interesting in the context of many body problem.

We assume that all the physics, even the intermediate states, are described by the Hamiltonian defined in a single-band subspace

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{\rm int}\,, \tag{4·3}$$

$$\mathcal{H}_0=\sum_{p}\varepsilon(\boldsymbol{p})c_{p}{}^{\dagger}c_{p}-\frac{1}{c}\hat{\boldsymbol{j}}_{-q}\cdot\boldsymbol{A}_q\,,$$

$$\hat{\boldsymbol{j}}_{-q}=\sum_{p}\boldsymbol{v}_{p}c_{p+}^{\dagger}c_{p-}+\mathcal{O}(q^2)\,.$$

The summation over $\boldsymbol{p}$ is performed within the first Brillouin zone. As we saw in the previous section, we are interested in the linear terms in $\boldsymbol{q}$ and neglect higher order terms. In expectation of the gauge-invariant results, we determine the conserved current operator by means of the equation of continuity:

$$\frac{\partial}{\partial t}\hat{\rho}_q+i\boldsymbol{q}\cdot\hat{\boldsymbol{j}}_q{}^H=0\,. \tag{4·4}$$

Here

$$\hat{\rho}_q\cong e\sum_{p}c_{p-}^{\dagger}c_{p+} \tag{4·5}$$

is the $\boldsymbol{q}$-component of the charge density operator and $\hat{\boldsymbol{j}}_q{}^H$ is that of the current operator to be determined. The result is

$$(\hat{j}_q{}^H)^\mu=e\sum_{p}v_p{}^\mu c_{p-}^{\dagger}c_{p+}-\frac{e^2}{c}A_{q,\nu}\sum_{p}\frac{\partial^2\varepsilon(\boldsymbol{p})}{\partial p_\mu\partial p_\nu}c_p{}^{\dagger}c_p\,. \tag{4·6}$$

Working with Eqs. (4·3) and (4·6), we can see that the gauge non-invariant terms (4·1) indeed vanish and get the same expression as Eq. (3·38):

$$\sigma_{\mu\nu}=\frac{e^3}{c}H\sum_{\sigma}\int\frac{d\boldsymbol{p}}{(2\pi)^3}[J_\mu{}^\sigma\overleftrightarrow{\partial}_\nu J_\nu{}^\sigma]v_\mu{}^{*\sigma}\frac{1}{(2\gamma_{\boldsymbol{p}})^2}\left(-\frac{df}{d\varepsilon}\right)_{\varepsilon=E(\boldsymbol{p})}\,,\quad(\mu=x,\ \nu=y) \tag{4·7}$$

$$J_\mu{}^\sigma=v_\mu{}^{*\sigma}+\sum_{\sigma'}\int\frac{d\boldsymbol{p}}{(2\pi)^3}\frac{a^2[\mathcal{T}_{22}(\boldsymbol{p},\boldsymbol{p}')]^{\sigma\sigma'}}{4i\gamma_{\boldsymbol{p}'}}v_\mu{}^{*\prime\sigma'}\,,$$

also in the case of general dispersion. Here we take into account the spin degrees of freedom and the summation over $\boldsymbol{p}$ is performed in the first Brillouin zone. In this formula, the equivalence of the x- and y-directions of the system is assumed. In case that x- and y-directions are not equivalent, we must use the 'symmetrized' formula

$$\sigma_{\mu\nu}=\frac{e^3}{2c}H\sum_{\sigma}\int\frac{d\boldsymbol{p}}{(2\pi)^3}\{[J_\mu{}^\sigma\overleftrightarrow{\partial}_\sigma J_\nu{}^\sigma]v_\mu{}^{*\sigma}+(\mu\leftrightarrow\nu)\}\frac{1}{(2\gamma_{\boldsymbol{p}})^2}\left(-\frac{df}{d\varepsilon}\right)_{\varepsilon=E(\boldsymbol{p})}$$
$$(\mu=x,\ \nu=y) \tag{4·8}$$

corresponding to Eq. (3·21).

§ 5. Discussion

5.1. *The validity of neglecting C and D*

As we saw in the preceding sections, the main term in $\sigma_{xy}^{(1)}$ behaves like $(1/\gamma_p)^2$, while that of $\sigma_{xx}^{(0)}$ like $1/\gamma_p$. So the Hall coefficient (2·2) remains finite in the limit $\gamma_p \to 0$, and as far as this value is concerned, we have only to collect these most singular terms. For σ_{xy}, the $(1/\gamma_p)^2$-singularity arises from the "alternately" differentiated g_2-section:

$$\int_{-\infty}^{\infty}[G^R \overleftrightarrow{\partial}_\mu G^A]d\varepsilon \propto \gamma_p \int_{-\infty}^{\infty}\left[\frac{1}{(\varepsilon - E(\boldsymbol{p}))^2 + \gamma_p^2}\right]^2 d\varepsilon$$

$$\propto \frac{1}{\gamma_p^2}. \tag{5·1}$$

In the following, we shall show that this order of singularity is not contained in the diagrams whose $\boldsymbol{q}$-linear terms are extracted from the vertex part $\Gamma_3^{(1)}$ or $\Gamma^{(1)}$, namely, C and D in § 3. Any single electron-hole propagator appearing explicitly in these diagrams (we are not concerned with those included in Λ's), when analytically continued, contributes at most $1/\gamma_p$ (the case when continued to g_2). This is in contrast to the case of Eq. (5·1). When two*) pair propagators are continued to g_2 in such a diagram, the vertex between them is $\mathcal{T}_{22}$ which is of order γ_p.**) Thus the diagrams C and D are at most of order $1/\gamma_p$ and can be neglected compared with the main term of order $(1/\gamma_p)^2$.

5.2. *The range of applicability*

Though we have been mainly concerned with electron-electron interaction in the text, there are other kinds of interaction which are popular in actual systems such as electron-phonon interaction and scattering from impurities. The former case can be treated in the same way by reinterpreting the interaction line as the phonon propagator. For the case of impurity scattering, the interaction line carries no frequency and the analytic property of the vertex function becomes simpler than that discussed in this paper (i.e., the whole $\mathrm{Im}\varepsilon - \mathrm{Im}\varepsilon'$ plane shrinks to the line $\mathrm{Im}\varepsilon = \mathrm{Im}\varepsilon'$). Anyway the same expression is obtained.

Thus we conclude that our result can be applicable to any interaction discussed above or their combinations, as far as the system remains to be Fermi liquid and the $1/\gamma_p$-term can be neglected in comparison with the $(1/\gamma_p)^2$-term. It is also assumed that interband effects can be neglected.

5.3. *Some remarks*

First, we note that γ_p and $\mathcal{T}_{22}$ must be determined according to the Ward identity which means local conservation of particle current. This is crucial for transport coefficients.[10]

*) Note that the pair-propagator connected to the α-vertex in C cannot be continued to g_2.

**) This is seen from the Ward identity, e.g., Eq. (2·20) in Ref. 8).

Second, we point out that under the consistency between γ_p and $\mathcal{T}_{22}$ mentioned above, the quantity σ_{xy} is divergent if we consider only the electron-electron interaction and do not take the Umklapp processes into account.[11] This is because under these conditions, the electron system couples to no momentum reservoir to which the momentum is released, and consequently the total momentum is conserved. Thus in this case, Umklapp processes must be taken into account.[10]

§ 6. Summary and conclusions

In this paper we derived the general expression for Hall conductivity based on the theory of Fermi liquid, in which many-body effects are included.

Starting with Kubo formula for conductivity in the presence of a magnetic field, the terms proportional to the strength of the magnetic field are extracted according to Fukuyama et al.[4] In this procedure, it is assumed that contributions from the diagrams whose $\boldsymbol{q}$-linear parts are extracted from vertex functions can be neglected. This assumption proves to be valid as far as the $(1/\gamma_p)^2$-terms are concerned, which is the main contribution in the Fermi liquid. It should be noted that owing to this approximation, we can get the expression containing only two-body interaction vertices for the Hall conductivity, which originally contained the three-body interaction vertex. After analytic continuation[2] and collection of $(1/\gamma_p)^2$-terms, we get Eq. (4·8) as a result. This is applicable to any kind of interaction such as electron-electron interaction, electron-phonon interaction and impurity scattering, so long as the picture of Fermi liquid holds well.

In a future study, we will apply the general expression obtained here to strongly correlated systems, such as heavy fermion and high T_c superconducting systems, and clarify the effects of electron interactions on the Hall effect.

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