A GENERAL FIRST MAIN THEOREM OF VALUE DISTRIBUTION. II

BY

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Dedicated to Marilyn Stoll

§ 3. The Levine form

Let V be a complex vector space of dimension $n+1$ with $n \in \mathbf{N}$. Suppose that a Hermitian product $(\,|\,)$ is given on V. On each $V[p]$, an associated Hermitian product $(\,|\,)$ is induced such that for every orthonormal base $\mathfrak{a} = (\mathfrak{a}_0, \ldots, \mathfrak{a}_p)$ the set

$$\{\mathfrak{a}_{\varphi(1)} \wedge \ldots \wedge \mathfrak{a}_{\varphi(p)} \mid \varphi \in \mathfrak{T}(p, n+1)\}$$

defines an orthonormal base of $V[p]$. If $0 \neq \mathfrak{x} \in V[p+1]$ and $0 \neq \mathfrak{y} \in V[q+1]$ with $p+q \leqslant n-1$, then the *projective distance* from $\mathfrak{x}$ to $\mathfrak{y}$ is defined by

$$\|\mathfrak{x}:\mathfrak{y}\| = \frac{|\mathfrak{x} \wedge \mathfrak{y}|}{|\mathfrak{x}|\,|\mathfrak{y}|}.$$

If $\xi \in \mathbf{P}(V[p+1])$ and $v \in \mathbf{P}(V[q+1])$, then the projective distance from ξ to v is well-defined by

$$\|\xi:v\| = \|\mathfrak{x}:\mathfrak{y}\| \quad \text{if} \quad \varrho(\mathfrak{x}) = \xi \text{ and } \varrho(\mathfrak{y}) = v,$$

where ϱ are the respective projections. Especially, this projective distance is defined as a real analytic function on $\mathfrak{G}^p(V) \times \mathfrak{G}^q(V)$ with

$$0 \leqslant \|\xi:v\| \leqslant 1 \text{ if } (\xi, v) \in \mathfrak{G}^p(V) \times \mathfrak{G}^q(V).$$

In the following, the vector space V, $V[p+1]$, $V[p+2]$ and $\mathbf{C}^r$ with $n-p=r$ will be considered. The natural projections are denoted by

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$$\varrho: V-\{0\}\to\mathbf{P}(V), \qquad \underline{\underline{\varrho}}: V[p+2]-\{0\}\to\mathbf{P}(V[p+2]),$$

$$\underline{\varrho}: V[p+1]-\{0\}\to\mathbf{P}(V[p+1], \quad \underset{\sim}{\varrho}:\mathbf{C}^r-\{0\}\to\mathbf{P}(\mathbf{C}^r).$$

The *euclidean forms* are denoted by

$$v \text{ and } v_s \text{ on } V, \qquad \underline{\underline{v}} \text{ and } \underline{\underline{v}}_s \text{ on } V[p+2],$$

$$\underset{\frown}{v} \text{ and } \underset{\frown}{v}_s \text{ on } V[p+1], \qquad \underset{\sim}{v} \text{ and } \underset{\sim}{v}_s \text{ on } \mathbf{C}^r.$$

The *projective forms* are denoted by

$$\omega \text{ and } \omega_s \text{ on } V-\{0\}, \qquad \ddot{\omega} \text{ and } \ddot{\omega}_s \text{ on } \mathbf{P}(V),$$

$$\underline{\omega} \text{ and } \underline{\omega}_s \text{ on } V[p+1]-\{0\}, \quad \underline{\ddot{\omega}} \text{ and } \underline{\ddot{\omega}}_s \text{ on } \mathbf{P}(V[p+1]),$$

$$\underline{\underline{\omega}} \text{ and } \underline{\underline{\omega}}_s \text{ on } V[p+2]-\{0\}, \quad \underline{\underline{\ddot{\omega}}} \text{ and } \underline{\underline{\ddot{\omega}}}_s \text{ on } \mathbf{P}(V[p+2]),$$

$$\underset{\sim}{\omega} \text{ and } \underset{\sim}{\omega}_s \text{ on } \mathbf{C}^r-\{0\}, \qquad \underset{\sim}{\ddot{\omega}} \text{ and } \underset{\sim}{\ddot{\omega}}_s \text{ on } \mathbf{P}(\mathbf{C}^r).$$

For $\alpha\in\mathfrak{G}^p(V)$, a holomorphic map

$$\pi_\alpha:\mathbf{P}(V)-\ddot{E}(\alpha)\to\mathbf{P}(V[p+2])$$

is defined by the following procedure: Take $\mathfrak{a}\in\varrho^{-1}(\alpha)$. Pick any $w\in\mathbf{P}(V)-\ddot{E}(\alpha)$. Take $\mathfrak{w}\in\varrho^{-1}(w)$. Then $\mathfrak{w}\wedge\mathfrak{a}\neq 0$. Define $\pi_\alpha(w)=\underline{\underline{\varrho}}(\mathfrak{w}\wedge\mathfrak{a})$. Then π_α is well defined and holomorphic in $\mathbf{P}(V)-\ddot{E}(\alpha)$ and meromorphic(1) on $\mathbf{P}(V)$ and maps $\mathbf{P}(V)-\ddot{E}(\alpha)$ into $\mathfrak{G}^{p+1}(V)$. Then π_α defines a lifting π_α^* of the forms on $\mathbf{P}(V[p+2])$ to forms on $\mathbf{P}(V)-\ddot{E}(\alpha)$. Define(2)

$$\Phi(\alpha)=\pi_\alpha^*(\underline{\underline{\ddot{\omega}}}) \text{ on } \mathbf{P}(V)-\ddot{E}(\alpha).$$

Then $\Phi(\alpha)$ is defined, real analytic, non-negative and of bidegree (1, 1) on $\mathbf{P}(V)-\ddot{E}(\alpha)$. Moreover,

$$d\Phi(\alpha)=0 \text{ on } \mathbf{P}(V)-\ddot{E}(\alpha).$$

It will be necessary, to use a simpler, but base dependent definition of $\Phi(\alpha)$:

LEMMA 3.1. *Let V be a complex vector space of dimension $n+1$ with a Hermitian product $(|)$. Let $0\leq p<n$ and $r=n-p$. Take $\alpha\in\mathfrak{G}^p(V)$. Let $\mathfrak{a}=(\mathfrak{a}_0,\dots,\mathfrak{a}_n)$ be an orthonormal base of V with $\alpha=\varrho(\mathfrak{a}_0\wedge\dots\wedge\mathfrak{a}_p)$. Define $\tilde{\sigma}_\alpha: V\to\mathbf{C}^r$ by*

$$\tilde{\sigma}_\alpha\Big(\sum_{\mu=0}^{n} z_\mu\mathfrak{a}_\mu\Big)=(z_{p+1},\dots,z_n).$$

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(1) Let G and H be complex spaces. Let A be open and dense in G. Let $f:A\to H$ be holomorphic. Let F be the closure of $\{(z,f(z))\,|\,z\in A\}$ in $G\times H$. Let $\pi:F\to G$ be the projection defined by $\pi(z,w)=z$. Then f is said to be *meromorphic* on G if and only if F is an analytic subset of $G\times H$ and if π is proper. See Remmert [14], Stoll [23] and Stein [17].

(2) See Levine [12] and Chern [3].

Then $E(\alpha)$ is the kernel of the linear map $\tilde{\sigma}_\alpha$. One and only one map

$$\sigma_\alpha : \mathbf{P}(V) - \ddot{E}(\alpha) \to \mathbf{P}(\mathbf{C}^r)$$

exists such that $\sigma_\alpha \circ \varrho = \underset{\sim}{\varrho} \circ \tilde{\sigma}_\alpha$ and this map σ_α is holomorphic on $\mathbf{P}(V) - \ddot{E}(\alpha)$ and meromorphic in $\mathbf{P}(V)$. Moreover,

$$\Phi(\alpha) = \sigma_\alpha^*(\underset{\sim}{\ddot{\omega}}).$$

Proof. Clearly, *Kern* $\tilde{\sigma}_\alpha = E(\alpha)$, because $E(\alpha)$ is the subspace spanned by $\mathfrak{a}_0, \ldots, \mathfrak{a}_p$. Clearly σ_α is well defined by $\sigma_\alpha \circ \varrho = \underset{\sim}{\varrho} \circ \tilde{\sigma}_\alpha$, holomorphic on $\mathbf{P}(V) - \ddot{E}(\alpha)$ and meromorphic in $\mathbf{P}(V)$. Now, define $\pi'_\alpha : V \to V[p+2]$ by setting

$$\pi'_\alpha(\mathfrak{w}) = \mathfrak{w} \wedge \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p \quad \text{if } \mathfrak{w} \in V.$$

Then π'_α is linear and $E(\alpha)$ its kernel. Moreover, $\pi_\alpha \circ \varrho = \underset{=}{\varrho} \circ \pi'_\alpha$. Because $\mathfrak{a}$ is an orthonormal base, it is

$$|\pi'_\alpha(\mathfrak{w})|^2 = |\mathfrak{w} \wedge \mathfrak{a}_0 \ldots \wedge \mathfrak{a}_p|^2 = \sum_{\mu=p+1}^{n} |w_\mu|^2 = |\tilde{\sigma}_\alpha(\mathfrak{w})|^2.$$

Now
$$\underset{=}{\omega}(\mathfrak{x}) = \tfrac{1}{2} d^\perp d \log |\mathfrak{x}| \quad \text{if } \mathfrak{x} \in V[p+2] - \{0\},$$

$$\underset{\sim}{\omega}(\mathfrak{z}) = \tfrac{1}{2} d^\perp d \log |\mathfrak{z}| \quad \text{if } \mathfrak{z} \in \mathbf{C}^r - \{0\}.$$

Hence
$$\tilde{\sigma}_\alpha^*(\underset{\sim}{\omega})(\mathfrak{w}) = \tfrac{1}{2} d^\perp d \log |\tilde{\sigma}_\alpha(\mathfrak{w})| = \tfrac{1}{2} d^\perp d \log |\pi'_\alpha(\mathfrak{w})| = (\pi'_\alpha)^*(\underset{=}{\omega})(\mathfrak{w}),$$

which implies
$$\varrho^*(\sigma^*(\underset{\sim}{\ddot{\omega}})) = \tilde{\sigma}_\alpha^*(\underset{\sim}{\varrho}^*(\underset{\sim}{\ddot{\omega}})) = \tilde{\sigma}_\alpha^*(\underset{\sim}{\omega}) = (\pi'_\alpha)^*(\underset{=}{\omega}) = (\pi'_\alpha)^*(\underset{=}{\varrho}^*(\underset{=}{\ddot{\omega}})) = \varrho^*(\pi_\alpha^*(\underset{=}{\ddot{\omega}})) = \mu^*(\Phi(\alpha));$$

because ϱ^* is injective, this implies $\sigma_\alpha^*(\underset{\sim}{\ddot{\omega}}) = \Phi(\alpha)$, q.e.d.

An easy, but important consequence is

LEMMA 3.2. *Let V be a complex vector space of dimension $n+1$ with a Hermitian product $(\,|\,)$. Let $0 \leqslant p < n$ and $r = n - p$. Take $\alpha \in \mathfrak{G}^p(V)$. Then*

$$\Phi(\alpha)^r = 0.$$

Proof. Apply Lemma 3.1 with the same notations. Then

$$\Phi(\alpha)^r = \sigma_\alpha^*(\underset{\sim}{\ddot{\omega}}^r) = \sigma_\alpha^*(0) = 0$$

because $\dim \mathbf{P}(\mathbf{C}^r) = r - 1$, q.e.d.

There is another way to define $\Phi(\alpha)$, namely:

LEMMA 3.3. *Let V be a complex vector space of dimension $n+1$ with a Hermitian product $(|)$. Let $0 \leqslant p < n$. Take $\alpha \in \mathfrak{G}^p(V)$. Then,*

$$\ddot{\omega}(w) = \Phi(\alpha)(w) = \tfrac{1}{2} dd^{\perp} \log \|w:\alpha\|,$$

if $w \in \mathbf{P}(V) - \ddot{E}(\alpha)$.

Proof. An $V - \{0\}$ is $\omega = \frac{1}{2} d^{\perp} \log |\mathfrak{w}|$. Pick $\mathfrak{a} \in \underline{\varrho}^{-1}(\alpha)$. Then $\pi'_\alpha(\mathfrak{w}) = \mathfrak{w} \wedge \mathfrak{a}$. Hence, if $w \in \mathbf{P}(V) - \ddot{E}(\alpha)$ and $\mathfrak{w} \in \varrho^{-1}(w)$, then

$$\begin{aligned}
\varrho^*(\ddot{\omega} - \Phi(\alpha)) &= \omega - \varrho^*(\pi_\alpha^*(\underline{\underline{\omega}})) = \omega - (\pi'_\alpha)^*(\varrho^*(\underline{\underline{\omega}})) \\
&= \omega - (\pi'_\alpha)^*(\underline{\underline{\omega}}) = \omega - \tfrac{1}{2} d^{\perp} d \log |\pi'_\alpha(\mathfrak{w})| \\
&= \tfrac{1}{2} dd^{\perp} \log \frac{1}{|\mathfrak{w}||\mathfrak{a}|} + \tfrac{1}{2} dd^{\perp} \log |\mathfrak{w} \wedge \mathfrak{a}| \\
&= \tfrac{1}{2} dd^{\perp} \log \frac{|\mathfrak{w} \wedge \mathfrak{a}|}{|\mathfrak{w}||\mathfrak{a}|} = \varrho^*(\tfrac{1}{2} dd^{\perp} \log \|w:\alpha\|).
\end{aligned}$$

Because ϱ^* is injective, this proves the assertion of the Lemma, q.e.d.

Again, let V be a complex vector space of dimension $n+1$ with a Hermitian product $(|)$. Let $0 \leqslant p < n$ and define $r = n - p$. For each $\alpha \in \mathfrak{G}^p(V)$, define the Levine(1) *form of order* r by

$$\Lambda(\alpha) = \frac{1}{(r-1)!} \sum_{\nu=0}^{r-1} \Phi(\alpha)^\nu \wedge \ddot{\omega}^{r-\nu-1}.$$

Then $\Lambda(\alpha)$ is a non-negative, real analytic form of bidegree $(r-1, r-1)$ on $\mathbf{P}(V) - \ddot{E}(\alpha)$ with $d\Lambda(\alpha) = 0$. For $r=1$ is $\Lambda(\alpha) = 1$. The following Lemma is due to Levine [12].

LEMMA 3.4.(2) *Let V be a complex vector space of dimension $n+1$ with an Hermitian product $(|)$. Let $0 \leqslant p < n$ and $r = n - p$. Take $\alpha \in \mathfrak{G}^p(V)$ then*

$$\tfrac{1}{2} dd^{\perp} \log \|w:\alpha\| \wedge \Lambda(\alpha)(w) = r \cdot \ddot{\omega}_r(w)$$

for $w \in \mathbf{P}(V) - \ddot{E}(\alpha)$.

Proof. According to Lemma 3.3 is

$$\begin{aligned}
\tfrac{1}{2} dd^{\perp} \log \|w:\alpha\| \wedge \Lambda(\alpha) &= \ddot{\omega} - \Phi(\alpha) \wedge \frac{1}{r!} \sum_{\nu=0}^{r-1} \Phi(\alpha)^\nu \wedge \ddot{\omega}^{r-1-\nu} \\
&= \frac{1}{(r-1)!} (\ddot{\omega}^r - \Phi(\alpha)^r) = r \cdot \ddot{\omega}_r
\end{aligned}$$

because $\Phi(\alpha)^r = 0$ according to Lemma 3.2, q.e.d.

Now, it shall be shown that certain integrals exist which involve the Levine form.

(1) See Levine [12] and Chern [3] (44).

(2) See Levine [12] (9).

LEMMA 3.5. *Let M be a pure m-dimensional complex manifold. Let V be a complex vector space of dimension $n+1$ with an Hermitian product $(\,|\,)$. Let $r\in\mathbf{N}$ with $p=n-r\geqslant 0$ and $q=m-r\geqslant 0$. Let $f: M\to\mathbf{P}(V)$ be a holomorphic map which is general of order r for $\alpha\in\mathfrak{G}^p(V)$. Let χ be a measurable differential form of bidegree $(q+1, q+1)$ on M with locally bounded, coefficients. Let K be a compact subset of M. Then*

$$\log\frac{1}{\|f:\alpha\|}\, f^*(\Lambda(\alpha))\wedge\chi$$

is integrable over K.

Proof. It is sufficient to show, that for $\nu=0, 1, \ldots, r-1$, the form

$$\psi=\log\frac{1}{\|f:\alpha\|}\, f^*(\Phi(\alpha)^\nu)\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi$$

is integrable over a neighborhood of each point of K. Take $a\in K$. If $a\notin f^{-1}(\ddot{E}(\alpha))$, this form is real analytic in a neighborhood of a, hence integrable over any compact subneighborhood. Hence, suppose that $a\in f^{-1}(\ddot{E}(\alpha))$. Now, apply Lemma 2.5 with the same notations. Define $g=\varrho_0^{-1}\circ f\,|\,A$. Then $g=\mathfrak{a}_0+\sum_{\nu=1}^n f_\nu\mathfrak{a}_\nu$. Hence

$$\|f:\alpha\|=\frac{|g\wedge\mathfrak{a}_0\wedge\ldots\wedge\mathfrak{a}_p|}{|g|\cdot|\mathfrak{a}_0\wedge\ldots\wedge\mathfrak{a}_p|}=\frac{1}{|g|}\Big|\sum_{\mu=p+1}^{n} f_\mu\mathfrak{a}_\mu\wedge\mathfrak{a}_0\wedge\ldots\wedge\mathfrak{a}_p\Big|$$
$$=\frac{1}{|g|}\Big(\sum_{\mu=p+1}^{n}|f_\mu|^2\Big)^{\frac{1}{2}}\frac{|\varphi|}{|g|}.$$

Hence
$$\log\frac{1}{\|f:\alpha\|}=\log|g|-\log|\varphi|.$$

Define σ_α and $\tilde{\sigma}_\alpha$ as in Lemma 3.1. Then $\sigma_\alpha\circ g=\varphi$ and

$$f^*(\Phi(\alpha))=g^*\circ\varrho^*\circ\sigma_\alpha^*(\underset{\sim}{\ddot{\omega}})=g^*\circ\tilde{\sigma}_\alpha^*\circ\underset{\sim}{\varrho}^*(\underset{\sim}{\ddot{\omega}})$$
$$=(\sigma_\alpha\circ g)^*(\underset{\sim}{\omega})=\varphi^*(\underset{\sim}{\omega})=\tfrac{1}{2}d^\perp d\log|\varphi|=\frac{i}{2}\frac{(d\varphi|d\varphi)}{|\varphi|^2}-\frac{(d\varphi|\varphi\wedge(\varphi|d\varphi)}{|\varphi|^4}.$$

Define $\Omega(\mathfrak{z})=|\mathfrak{z}|^2\underset{\sim}{\omega}(\mathfrak{z})$ for $\mathfrak{z}\in\mathbf{C}^r-\{0\}$ and $\Omega(0)=0$. Then Ω is a non-negative form on $\mathbf{C}^r$ with locally bounded coefficients. Hence $f^*(\Phi(\alpha))=|\varphi|^{-2}\varphi^*(\Omega)$ almost everywhere on A, where $f^*(\Omega)$ is measurable and has locally bounded coefficients on A.

Let A_0 be compact neighborhood of a with $A_0\subseteq A$. Apply Proposition 1.7 twice using the tables

1.7	M	m	f	p	q	$\varkappa$	s	t	σ	τ	K	χ	φ
Here I	A	m	φ	r	q	1	ν	ν	$m-\nu$	$m-\nu$	A_0	$f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi$	$\Omega^{2\nu}$
Here II	A	m	φ	r	q	0	ν	ν	$m-\nu$	$m-\nu$	A_0	$\log\lvert g\rvert f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi$	$\Omega^{2\nu}$

Therefore, the forms

$$\log\frac{1}{|\varphi|}\frac{1}{|\varphi|^{2\nu}}\varphi^*(\Omega^{2\nu})\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi=-\log|\varphi|\,f^*(\Phi(\alpha))\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi$$

and
$$\frac{1}{|\varphi|^{2\nu}}\varphi^*(\Omega^{2\nu})\wedge\log|g|\,f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi=\log|g|\,f^*(\Phi(\alpha))\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi$$

are integrable over A_0. Hence, their sum ψ is also integrable over A_0, q.e.d.

LEMMA 3.6. *Let M be a pure m-dimensional complex manifold. Let V be a complex vector space of dimension $n+1$. Let $r\in\mathbf{N}$ with $p=n-r\geqslant 0$ and $q=m-r\geqslant 0$. Let $f:M\to\mathbf{P}(V)$ be an holomorphic map which is general of order r for $\alpha\in\mathfrak{G}^p(V)$. Let χ be a measurable differential form of degree $2q+1$ on M with locally bounded coefficients. Let K be a compact subset of M. Then*

$$d^\perp\log\|f:\alpha\|\wedge f^*(\Lambda(\alpha))\wedge\chi$$

is integrable over K.

Proof. It is sufficient to show that for $\nu=0,1,\ldots,r-1$ the form

$$\gamma=d^\perp\log\|f:\alpha\|\wedge f^*(\Phi(\alpha))^\nu\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi$$

is integrable over a neighborhood of each point of K. Take $a\in K$. If $a\notin f^{-1}(\ddot{E}(\alpha))$, the form γ is real analytic in a neighborhood of a, hence integrable over any compact subneighborhood. Hence, suppose that $a\in f^{-1}(\ddot{E}(\alpha))$. Now, apply Lemma 2.5 with the same notations. Define

$$g=\varrho_0^{-1}\circ f|A=\mathfrak{a}_0+\sum_{\nu=1}^{n}f_\nu\mathfrak{a}_\nu$$

and $\varphi=(f_{p+1},\ldots,f_n):A\to\mathbf{C}^r$. As in the proof of Lemma 3.5, it is

$$\log\|f:\alpha\|=\log|\varphi|-\log|g|$$

$$f^*(\Phi(\alpha))=\varphi^*(\underset{\sim}{\omega})\text{ on }A.$$

Moreover,
$$d^\perp\log\|f:\alpha\|=\frac{i}{2}\frac{(d\varphi|\varphi)-(\varphi|d\varphi)}{|\varphi|^2}-\frac{i}{2}\frac{(dg|g)-(g|dg)}{|g|^2}.$$

Define
$$\gamma_1=\frac{i}{2}\,|\varphi|^{-2}\,(d\varphi|\varphi)\wedge f^*(\Phi(\alpha)^\nu)\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi,$$

$$\gamma_2=\frac{-i}{2}\,|\varphi|^{-2}(\varphi|d\varphi)\wedge f^*(\Phi(\alpha)^\nu)\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi$$

$$\gamma_3=\frac{-i}{2}\,|g|^{-2}(dg|g)\wedge f^*(\Phi(\alpha)^\nu)\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi,$$

$$\gamma_4=\frac{i}{2}\,|g|^{-2}(g|dg)\wedge f^*(\Phi(\alpha)^\nu)\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi.$$

Now, measurable differential forms $\chi_{\mu\nu}$ of bidegee (μ,ν) with locally bounded coefficients exists such that
$$\chi=\sum_{\mu+\nu=2q+1}\chi_{\mu\nu}.$$

Then
$$\tilde{\chi}_1=f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi_{q,q+}\ ,$$
$$\tilde{\chi}_2=f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi_{q+1,q},$$
$$\tilde{\chi}_3=\frac{i}{2}\,\frac{(dg|g)}{|g|}\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi_{q,q+1},$$
$$\tilde{\chi}_4=\frac{i}{2}\,\frac{(g|dg)}{|g|}\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi_{q+1,q}$$

are measurable differential forms with locally bounded coefficients on A. Define $\Omega_1(0)=\Omega(0)=0$ and
$$\Omega_1(\mathfrak{z})=\frac{i}{2}\,\frac{(d\mathfrak{z}|\mathfrak{z})}{|\mathfrak{z}|}\wedge(|\mathfrak{z}|^2\underset{\sim}{\omega})^\nu,\quad\text{if }\mathfrak{z}\in\mathbb{C}^r-\{0\},$$
$$\Omega\ (\mathfrak{z})=(|\mathfrak{z}|^2\underset{\sim}{\omega})^\nu.$$

Then Ω_1 and Ω are measurable differential forms with locally bounded coefficient on A. A comparison of bidegrees shows that

$$\gamma_1=\frac{1}{|\varphi|^{2\nu+1}}\,\varphi^*(\Omega_1)\wedge\tilde{\chi}_1$$

$$\gamma_2=\frac{1}{|\varphi|^{2\nu+1}}\,\varphi^*(\overline{\Omega}_1)\wedge\tilde{\chi}_2$$

$$\gamma_3=\frac{-1}{|\varphi|^{2\nu}}\,\varphi^*(\Omega)\wedge\tilde{\chi}_3$$

$$\gamma_4=\frac{1}{|\varphi|^{2\nu}}\,\varphi^*(\Omega)\wedge\tilde{\chi}_4.$$

Let A_0 be a compact neighborhood of a with $A_0 \subseteq A$. Apply Proposition 1.7 four times using the table

1.7	M	m	f	p	q	$\varkappa$	s	t	σ	τ	K	χ	φ	h_Q
Here 1	A	m	φ	r	q	0	$\nu+1$	ν	$m-\nu-1$	$m-\nu$	A_0	$\tilde{\chi}_1$	Ω_1	1
Here 2	A	m	φ	r	q	0	ν	$\nu+1$	$m-\nu$	$m-\nu-1$	A_0	$\tilde{\chi}_2$	Ω_1	1
Here 3	A	m	φ	r	q	0	ν	ν	$m-\nu$	$m-\nu$	A_0	$\tilde{\chi}_1$	Ω	1
Here 4	A	m	φ	ɹ	q	0	ν	ν	$m-\nu$	$m-\nu$	A_0	$\tilde{\chi}_4$	Ω	1

Hence, $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are integrable over A_0. Therefore $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is integrable over A_0, q.e.d.

LEMMA 3.7. *Let M be a pure m-dimensional complex manifold. Let $0 \leqslant s \leqslant q < m$. Let χ be a non-negative form of bidegree (s, s) on a subset A of M. Let V be a complex vector space of dimension $n+1$ with an Hermitian product $(\,|\,)$. Let $r = m - q$ and $p = n - r$. Suppose that $0 \leqslant p < n$. Let $f: M \to \mathbf{P}(V)$ be a holomorphic map which is general of order r for $\alpha \in \mathfrak{G}^p(V)$. Let $a \in A - f^{-1}(\ddot{E}(\alpha))$. Then $f^*(\Lambda(\alpha)) \wedge \chi$ is a form of bidegree $(r-1+s,\, r-1+s)$ which is non-negative at a.*

Proof. Because

$$f^*(\Lambda(\alpha)) \wedge \chi = \frac{1}{(r-1)!} \sum_{\nu=0}^{r-1} f^*(\Phi(\alpha))^{\nu} \wedge f^*(\ddot{\omega})^{r-1-\nu} \wedge \chi$$

and because $f^*(\Lambda(\alpha))$ and $f^*(\ddot{\omega})$ are non-negative (II Lemma 2.5) and because both have bidegree $(1, 1)$, the form $f^*(\Lambda(\alpha)) \wedge \chi$ is non-negative at a, according to II Theorem 2.6, q.e.d.

§ 4. The First Main Theorem

Stokes' integral theorem will be used in the proof of the First Main Theorem. Let M be an oriented differentiable manifold of class C^∞ and pure real dimension m. Let S be a pure s-dimensional, oriented differential manifold of class C^k with $k \geqslant 1$. A map $f: S \to M$ of class C^k is said to be regular if and only if the rank of the Jacobian of f is the minimum of m and s for every point of S. The pure s-dimensional, oriented manifold S of class C^k is said to be a submanifold of class C^k of M if and only if S is a subspace of M and if the inclusion map $j: S \to M$ is of class C^k. The submanifold S is said to be smooth if and only if j is regular.

Let H be an open subset of M. The pure s-dimensional oriented manifold S of class C^k is said to be a *boundary manifold of M* if and only if the following properties are true:

1. *S is a relative open subset of $\bar{H}-H$.*
2. *$s=m-1$, and S is a smooth submanifold of class C^k of M.*
3. *If $a\in S$, then an open neighborhood U of a, an open subset U'' of $\mathbf{R}^{m-1}$, an interval $I=\{x\mid x\in\mathbf{R},\ |x|<\eta\}$ with $\eta>0$ and diffeomorphic maps*

$$\alpha: U\to I\times U'' \qquad \beta: U\cap S\to U''$$

of class C^k and with a positive Jacobian, and a surjective map $g:U\to I$ of class C^k exist such that

$$\alpha(x)=(g(x),\beta(x)) \quad \text{if } x\in U$$
$$U\cap H=\{x\mid x\in U,\ -\eta<g(x)<0\}$$
$$U\cap S=\{x\mid x\in U,\ g(x)=0\}$$
$$U\cap \bar{H}=\{x\mid x\in U,\ \eta>g(x)>0\}.$$

If S satisfies conditions 1 and 2, then one and only one orientation of S exists which makes S into a boundary manifold of class C^k. If M is a complex manifold of pure dimension m, these remarks apply also, because M can be considered as a real manifold of dimension $2m$.

Let M be a differential manifold of pure real dimension m. Then a differential form ξ of degree n on M satisfies locally a Lipschitz condition, if and only if for every $a\in M$, a diffeomorphic, orientation preserving map $\alpha: U\to U'$ of class C^∞ and functions ξ_φ on U and a constant L exist such that

1. *U is an open neighborhood of a in M and U' is open in $\mathbf{R}^m$.*
2. *If $\alpha=(x_1, \ldots, x_m)$, then*

$$\xi=\sum_{\varphi\in\mathfrak{T}(n,m)}\xi_\varphi\, dx_{\varphi(1)}\wedge\ldots\wedge dx_{\varphi(n)}$$

on U.

3. *If $x\in U$ and $x'\in U$, then*

$$|\xi_\varphi(x)-\xi_\varphi(x')|\leqslant L|\alpha(x)-\alpha(x')|.$$

If ξ satisfies locally a Lipschitz condition, then ξ is continuous, and almost everywhere $d\xi$ exists. Moreover, if $n=m-1$, then $d\xi$ is integrable over every compact subset of M.

THEOREM 4.1. (Stokes) *Let M be an oriented differentiable manifold of class C^∞ and with pure dimension m. Let $H\neq\emptyset$ be an open subset of M. Let S be empty or a boundary* *manifold of class C^k of H with $k\geqslant 1$. Let ξ be a form of degree $m-1$ on M with support T. Suppose that $\bar{H}\cap T$ is compact and $S\supseteq(\bar{H}-H)\cap T$. Then*

$$\int_S \xi = \int_H d\xi.$$

For a proof of this well-known Theorem see, for instance, [18] Satz 8.

For the rest of the paper, the following assumptions shall be made:

ASSUMPTION 4.2. *Let M be a pure m-dimensional complex manifold. Let V be a complex vector space of dimensional $n+1$ with an Hermitian product $(\,|\,)$. Let r be a positive integer with $p=n-r\geqslant 0$ and $q=m-r\geqslant 0$. Let $\alpha\in\mathfrak{G}^p(V)$. Suppose that $f:M\to \mathbf{P}(V)$ is a holomorphic map which is general of order r for α. On M, an exterior differential form χ of class C^1 and of bidegree (q, q) is given.*

If u and v are functions of class C^1 on M and if ξ is a differential form of bidegree $(m-1, m-1)$ on M, then $\partial u\wedge\partial v\wedge\xi$ has bidegree $(m+1, m-1)$ and $\bar\partial u\wedge\bar\partial v\wedge\xi$ has bidegree $(m-1, m+1)$. Hence both forms are zero. Therefore

$$\begin{aligned} du\wedge d^{\perp}v\wedge\xi &= i(\partial u+\bar\partial u)\wedge(\partial v-\bar\partial v)\wedge\xi \\ &= i(\bar\partial v\wedge\partial u+\bar\partial u\wedge\partial v)\wedge\xi. \end{aligned}$$

Hence

$$du\wedge d^{\perp}v\wedge\xi = dv\wedge d^{\perp}u\wedge\xi.$$

PROPOSITION 4.3.(1) *Let H be an open subset of M. Suppose that $\bar H$ is compact and that $S=\bar H-H$ is empty or a boundary manifold of H. Let $j:S\to M$ be the inclusion map. Let ψ be a function of class C^2 on M. Define*

$$\eta = \log\|f:\alpha\| f^*(\Lambda(\alpha))\wedge d^{\perp}\psi\wedge\chi.$$

Suppose that at least one of the following assumptions a) or b) or c) or d) is true:

a) The form η (that means $j^*(\eta)$) is integrable over S.
b) The form $j^*(\eta)$ is non-negative on S.
c) The form $j^*(\eta)$ is non-positive on S.
d) The form χ is non-negative. For every $a\in S$ an open neighborhood U of a exists such that $\psi(z)\geqslant\psi(a)$ if $z\in U\cap\bar H$.

Then η is integrable over S and

(1) For $r=1$, compare Stoll [21], Satz 6.2.

$$\int_S \log \|f:\alpha\| f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge \chi$$

$$= \int_H d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi + \int_H \log \|f:\alpha\| f^*(\Lambda(\alpha)) \wedge dd^\perp\psi \wedge \chi \tag{1}$$

$$- \int_H \log \|f:\alpha\| f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge d\chi.$$

Proof. At first, observe, that the existence of the integrals over H is implied by Lemma 3.5 and Lemma 3.6.

At first, it will be shown, that every point of $\bar{H}$ has an open neighborhood A such that the theorem holds if χ has compact support in A.

Take $a \in \bar{H}$. If $a \notin f^{-1}(\ddot{E}(\alpha))$, take an open neighborhood A of a such that $\bar{A} \cap f^{-1}(\ddot{E}(\alpha)) = \emptyset$. Suppose that χ has compact support in A, then all the integrands of (1) are continuous on M and η is of class C^1 on M. Therefore (1) is an immediate consequence of the Stokes integral theorem.

Suppose that $a \in \bar{H} \cap f^{-1}(\ddot{E}(\alpha))$. Determine an open neighborhood A of a, holomorphic functions $f_1, \ldots, f_n$ and an orthonormal base $\mathfrak{a} = (\mathfrak{a}_0, \ldots, \mathfrak{a}_n)$ of V as in Lemma 2.5. Hence $\varrho(\mathfrak{a}_0) = f(a)$ and $\varrho(\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p) = \alpha$. Define

$$g = \mathfrak{a}_0 + \sum_{\nu=1}^{n} f_\nu \mathfrak{a}_\nu = \varrho_0^{-1} \circ f | A,$$

$$\varphi = (f_{p+1}, \ldots, f_n) : A \to \mathbf{C}^r.$$

Then $$\log \|f:\alpha\| = \log |\varphi| - \log |g|, \quad f^*(\Phi(\alpha)) = \varphi^*(\underset{\sim}{\omega}).$$

Take a function $\tilde{g}$ of class C^∞ on $\mathbf{R}$ such that $0 \leqslant \tilde{g}(x) \leqslant 1$ for all $x \in \mathbf{R}$ and $\tilde{g}(x) = 0$ if $x \leqslant 0$ and $\tilde{g}(x) = 1$ if $x \geqslant 1$. A constant $B > 0$ exists such that

$$2 |\tilde{g}'(x)| \leqslant B \quad \text{if } x \in \mathbf{R}.$$

For $\varrho > 0$, define a function g_ϱ of class C^∞ on $\mathbf{R}$ by setting

$$g_\varrho(x) = \tilde{g}\left(\frac{2x - \varrho}{\varrho}\right).$$

Then

a) For $x \in \mathbf{R}$ is $0 \leqslant g_\varrho(x) \leqslant 1$.
b) For $x \in \mathbf{R}$ is $\varrho |g_\varrho'(x)| \leqslant B$.
c) For $x \leqslant \varrho/2$ is $g_\varrho(x) = 0$.
d) For $x \geqslant \varrho$ is $g_\varrho(x) = 1$.

Define γ_ϱ on M by setting

$$\gamma_\varrho(z) = \begin{cases} g_\varrho(|\varphi(z)|) & \text{if } z \in A, \\ 1 & \text{if } z \in M - A. \end{cases}$$

Then $\gamma\varrho$ and $\gamma\varrho|S$ are measurable functions on M respectively S. Moreover, $\gamma_\varrho|A$ is of class C^∞. Moreover, $\gamma_\varrho|A$ is zero on the open neighborhood $A_\varrho = \{z \| \varphi(z)| < \varrho/2, z \in A\}$ of $A \cap \varphi^{-1}(0) = A \cap f^{-1}(\ddot{E}(\alpha))$.

Suppose that χ has compact support in A. Then $\gamma_\varrho \chi$ is a form of class C^1 on M which is identically zero in an open neighborhood of $f^{-1}(\ddot{E}(\alpha))$. Therefore Stoke's Theorem implies

$$\int_S \gamma_\varrho \log \|f:\alpha\| \, f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge \chi$$

$$= \int_H \gamma_\varrho d \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge \chi$$

$$+ \int_H \gamma_\varrho \log \|f:\alpha\| \, f^*(\Lambda(\alpha)) \wedge dd^\perp \psi \wedge \chi$$

$$- \int_H \gamma_\varrho \log \|f:a\| \, f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge d\chi$$

$$+ \int_H \log \|f:\alpha\| \, d\gamma_\varrho \wedge f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge d\chi.$$

According to Lemma 3.5 and Lemma 3.6, the forms

$$d \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge \chi = d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi,$$

$$\log \|f:\alpha\| f^*(\Lambda(\alpha)) \wedge dd^\perp \psi \wedge \chi \quad \text{and} \quad \log \|f:\alpha\| f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge d\chi$$

are integrable over $\bar{H}$, hence also over H.

If $z \in M - f^{-1}(\ddot{E}(\alpha))$, then $\gamma_\varrho(z) \to 1$ for $\varrho \to 0$, where $|\gamma_\varrho(z)| \leqslant 1$. Since $H \cap f^{-1}(\ddot{E}(\alpha))$ is a set of measure zero on H, this implies

$$\int_H \gamma_\varrho d \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge \chi \quad \to \quad \int_H d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi = I_1$$

for $\varrho \to 0$ and

$$\int_H \gamma_\varrho \log \|f:\alpha\| \, f^*(\Lambda(\alpha)) \wedge dd^\perp \psi \wedge \chi \quad \to \quad \int_H \log \|f:\alpha\| \, f^*(\Lambda(\alpha)) \wedge dd^\perp \psi \wedge \chi = I_2$$

for $\varrho \to 0$ and

$$\int_H \gamma_\varrho \log \|f:\alpha\| f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge d\chi \quad \to \quad \int_H \log \|f:\alpha\| f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge dx = I_3$$

for $\varrho \to 0$.

On A is $$d\gamma_\varrho = g_\varrho'(|\varphi|)\, d|\varphi| = g_\varrho'(|\varphi|) \frac{1}{2|\varphi|}((d\varphi|\varphi) + (\varphi|d\varphi)),$$

where $|g_\varrho'(|\varphi|)|\varphi|| \leqslant B$ on A for all $\varrho > 0$. Define

$$J_\nu(\varrho) = \int_H \log \|f:\alpha\|\, d\gamma_\varrho \wedge f^*(\Phi(\alpha)^\nu) \wedge f^*(\ddot{\omega}^{r-1-\nu}) \wedge d^\perp\psi \wedge \chi,$$

$$J\,(\varrho) = \int_H \log \|f:\alpha\|\, d\gamma_\varrho \wedge f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge \chi.$$

Then $$J(\varrho) = \frac{1}{(r-1)!} \sum_{\nu=0}^{r-1} J_\nu(\varrho).$$

Let T be the support of χ. Then T is a compact subset of A. For $0 < \varrho < 1$, define

$$L(\varrho) = \left\{ z \,|\, z \in T,\ \frac{\varrho}{2} \leqslant |(z)| \leqslant \varrho \right\}$$

On M define the continuous forms

$$\chi_1 = f^*(\ddot{\omega}^{r-1-\nu}) \wedge \bar{\partial}\psi \wedge \chi, \quad \chi_3 = \log |g| \chi_1,$$

$$\chi_2 = f^*(\ddot{\omega}^{r-1-\nu}) \wedge \partial\psi \wedge \chi, \quad \chi_4 = \log |g| \chi_2.$$

On $\mathbf{C}^r - \{0\}$ define the forms

$$\Omega_1(\mathfrak{z}) = \frac{i}{2} |\mathfrak{z}|^{2\nu-1} (d\mathfrak{z}|\mathfrak{z}) \wedge \underset{\sim}{\omega}^{2\nu} = \Omega_3(\mathfrak{z})$$

$$\Omega_2(\mathfrak{z}) = \frac{i}{2} |\mathfrak{z}|^{2\nu-1} (\mathfrak{z}|d\mathfrak{z}) \wedge \underset{\sim}{\omega}^{2\nu} = \Omega_4(\mathfrak{z}).$$

For $\mathfrak{z} = 0$ define $\Omega_\nu(0) = 0$ for $\nu = 1, 2, 3, 4$. Then Ω_ν for $\nu = 1, 2, 3, 4$ has locally bounded and measurable coefficients on $\mathbf{C}^r$. Define h_ϱ on A by

$$h_\varrho(z) = |\varphi(z)|\, g_\varrho'(|\varphi(z)|).$$

Then $|h_\varrho(z)| \leqslant B$ for all $\varrho > 0$ and all $z \in A$. Then

$$\log \|f:\alpha\|\, d\gamma_\varrho \wedge f^*(\Phi(\alpha)^\nu) \wedge f^*(\ddot{\omega}^{r-1-\nu}) \wedge d^\perp\psi \wedge \chi$$

$$= (\log|\varphi| - \log|g|)\, g_\varrho'(|\varphi|)\, \frac{1}{2|\varphi|}\, ((d\varphi|\varphi) + (\varphi|d\varphi)) \wedge \varphi^*(\underset{\sim}{\omega}^\nu) \wedge f^*(\ddot{\omega}^{r-1-\nu}) \wedge i(\partial\psi - \bar\partial\psi) \wedge \chi.$$

$$= -h_\varrho \log|\varphi|\, \frac{1}{|\varphi|^{2\nu+1}}\, \varphi^*(\Omega_1) \wedge \chi_1 + h_\varrho \log|\varphi|\, \frac{1}{|\varphi|^{2\nu+1}}\, \varphi^*(\Omega_2) \wedge \chi_2$$

$$+ h_\varrho \frac{1}{|\varphi|^{2\nu+1}}\, \varphi^*(\Omega_3) \wedge \chi_3 - h_\varrho\, \frac{1}{|\varphi|^{2\nu+1}}\, \varphi^*(\Omega_4) \wedge \chi_4$$

For $\lambda = 1, 2$, define $$J_\nu^\lambda(\varrho) = \int_{L(\varrho)} h_\varrho \log \frac{1}{|\varphi|}\, \frac{1}{|\varphi|^{2\nu+1}}\, \varphi^*(\Omega_\lambda) \wedge \chi_\lambda.$$

For $\lambda = 3, 4$, define $$J_\nu^\lambda(\varrho) = \int_{L(\varrho)} h_\varrho\, \frac{1}{|\varphi|^{2\nu+1}}\, \varphi^*(\Omega_\lambda) \wedge \chi_\lambda.$$

Because $d\gamma_\varrho$ has its support in $L(\varrho)$, this implies

$$J_\nu(\varrho) = +J_\nu^1(\varrho) - J_\nu^2(\varrho) + J_\nu^3(\varrho) - J_\nu^4(\varrho).$$

Now, apply Proposition 1.7 four times

1.7	M	m	f	p	q	$\varkappa$	s	t	σ	τ	K	χ	φ	h_ϱ
Here I	A	m	φ	r	q	1	$\nu+1$	ν	$m-\nu-1$	$m-\nu$	T	χ_1	Ω_1	h_ϱ
Here II	A	m	φ	r	q	1	ν	$\nu+1$	$m-\nu$	$m-\nu-1$	T	χ_2	Ω_2	h_ϱ
Here III	A	m	φ	r	q	0	$\nu+1$	ν	$m-\nu-1$	$m-\nu$	T	χ_3	Ω_3	h_ϱ
Here IV	A	m	φ	r	q	0	ν	$\nu+1$	$m-\nu$	$m-\nu-1$	T	χ_4	Ω_4	h_ϱ

Therefore $J_\nu^\lambda(\varrho) \to 0$ if $\varrho \to 0$ for $\lambda = 1, 2, 3, 4$ and $\nu = 0, 1, \ldots, r-1$. Hence $J_\nu(\varrho) \to 0$ if $\varrho \to 0$ for $\nu = 0, 1, \ldots, r-1$. Hence

$$J(\varrho) \to 0 \quad \text{for} \quad \varrho \to 0$$

Therefore, the following limit exist

$$\Gamma = \lim_{\varrho \to 0} \int_S \gamma_\varrho \log \|f:\alpha\|\, f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge \chi$$

with $$\Gamma = I_1 + I_2 - I_3$$

Now, the different cases a) b) c) and d) have to be considered:

a) Because $0 \leqslant \gamma_\varrho \leqslant 1$, because $\gamma_\varrho(z) \to 1$ for $\varrho \to 0$ if $z \in S - f^{-1}(\ddot{E}(\alpha))$, because $f^{-1}(\ddot{E}(\alpha)) \cap S$ is a set of measure zero on S and because η is integrable over S, it follows

$$\Gamma = \int_S \log \|f:\alpha\| \, f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge \chi,$$

which proves the theorem if the support of χ is compact and contained in A.

b) If $j^*(\eta)$ has a non-negative density on S, then

$$\int_S \gamma_\varrho j^*(\eta) \to \int_S j^*(\eta) \leqslant \infty$$

if $\varrho \to \infty$ by the same reasoning as in a), and

$$\infty > \Gamma = \lim_{\varrho \to 0} \int_S \gamma_\varrho j^*(\eta) = \int_S j^*(\eta) = \int_S \eta$$

which proves the theorem if the support of χ is compact and contained in A.

c) Replace ψ by $-\psi$ and the case reduces to the case b).

d) If χ is non-negative, so is $f^*(\Lambda(\alpha)) \wedge \chi$. According to Stoll [21] Satz 4.5, $j^*(\eta)$ is non-negative. Hence the case reduces to the case b).

Now, consider the general case. Finitely many points $a_1, ..., a_s$ in $\bar{H}$ and finitely many open sets $A_1, ..., A_s$ with $a_\sigma \in A_\sigma$ exist such that the Theorem is true if the support of χ is compact and contained in A_σ and such that $\bar{H} \subseteq A_1 \cup ... \cup A_s$. Take function $g_1, ..., g_s$ of class C^∞ on M such that $0 \leqslant g_\sigma \leqslant 1$ on M and such that the support of g_σ is compact and contained in A_σ and such that $\sum_{\sigma=1}^{s} g_\sigma(z) = 1$ if $z \in \bar{H}$. If a, b, c, or d is true for χ then a), b), c) or d) are true respectively for $g_\sigma \chi$. Hence

$$\int_S \log \|f:\alpha\| \, f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge g_\sigma \chi$$

$$= \int_H d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge g_\sigma \chi + \int_H \log \|f:\alpha\| \, f^*(\Lambda(\alpha)) \wedge dd^\perp \wedge g_\sigma \chi$$

$$- \int_H \log \|f:\alpha\| \, f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge d(g_\sigma \chi).$$

Because $\sum_{\sigma=1}^{s} g_\sigma(z) = 1$ for $z \in \bar{H}$, addition yields formula (1) with all integrals existing, q.e.d.

It is remarkable, that Stokes Theorem still holds for such a singular integrand as in formula (1). If ψ and $\log \|f:\alpha\|$ are exchanged, then the singularities become so strong as to invalidate the Stokes Theorem. However, a residue Theorem can be proved, which will be the base for the First Main Theorem.

Theorem 4.4. (*A residue formula*).(1) *The assumption 4.2 is made. Let H be an open subset of M. Suppose that $\bar{H}$ is compact and that $S=\bar{H}-H$ is empty or a boundary manifold of H. Let $j:S\to M$ be the inclusion map. Let ψ be a continuous function on $\bar{H}$, which satisfies locally a Lipschitz condition on $\bar{H}$. Let K be the support of $\psi\chi$ on S. Suppose that $S_\alpha = K\cap f^{-1}(\ddot{E}(\alpha))$ is a set of measure zero on $f^{-1}(\ddot{E}(\alpha))$. Suppose that*

$$\eta = \psi d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

is integrable over S. Then

$$\int_S \psi d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

$$= \int_H d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

$$- \int_H \psi d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge d\chi$$

$$+ 2r\int_H \psi f^*(\ddot{\omega}_r) \wedge \chi - \frac{2\pi^r}{(r-1)!}\int_{H\cap f^{-1}(\ddot{E}(\alpha))} \nu_f(z;\alpha)\,\psi\chi.$$

Remark: If $q=0$, then χ is a function and the last integral means summation over the finitely many points of $H\cap f^{-1}(\ddot{E}(\alpha))$.

Proof. At first, it will be shown, that every point of $\bar{H}$ has an open neighborhood A such that the theorem holds if χ has compact support in A. By Stokes' Theorem, this is trivial, if $a\in \bar{H} - f^{-1}(\ddot{E}(\alpha))$.

Take $a\in\bar{H}\cap f^{-1}(\ddot{E}(\alpha))$. Construct $A, f_1, \dots, f_n$, $\mathfrak{a}=(\mathfrak{a}_0, \dots, \mathfrak{a}_n)$, $g, \varphi, \tilde{g}, B, g_\varrho, \gamma_\varrho$ as in the beginning of the Proof of Proposition 4.3. Suppose that χ has compact support T in A. Then $\gamma_\varrho\chi$ is a form of class C^1 on M, which is identically zero in a neighborhood of $f^{-1}(\ddot{E}(\alpha))$. Therefore Stokes' Theorem implies:

$$\int_S \gamma_\varrho\psi\, d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

$$= \int_H \gamma_\varrho d\,\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

$$- \int_H \gamma_\varrho\psi\, d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge d\chi$$

(1) For $r=1$, see Stoll [21], Satz 6.3.

$$+\int_H \gamma_\varrho \psi\, dd^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

$$+\int_H \psi\, d\gamma_\varrho \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi.$$

Now, the limit of these integrals for $\varrho \to 0$ has to be studied. They are denoted in order by $I(\varrho)$, $I_1(\varrho)$, ..., $I_4(\varrho)$ such that

$$I(\varrho) = I_1(\varrho) + I_2(\varrho) + I_3(\varrho) + I_4(\varrho).$$

According to Lemma 3.4 and Lemma 3.6, the forms

$$d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi,$$
$$\psi \quad d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge d\chi,$$
$$\psi \quad dd^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi = 2r\psi f^*(\omega_r) \wedge \chi$$

are integrable over $\bar{H}$, hence also over H. Moreover,

$$j^*(\eta) = j^*(\psi d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha) \wedge \chi)$$

is integrable over S by assumption. On H, the set $H \cap f^{-1}(\ddot{E}(\alpha))$ and on S, the set $S \cap f^{-1}(\ddot{E}(\alpha))$ are sets of measure zero. Because $0 \leqslant \gamma_\varrho \leqslant 1$ and because $\gamma_\varrho(z) \to 1$ for $\varrho \to 0$ if $z \in \bar{H} - f^{-1}(\ddot{E}(\alpha))$, it follows that

$$I(\varrho) \to \int_S \psi d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha))\chi = I$$

$$I_1(\varrho) \to \int_H d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*\Lambda(\alpha)) \wedge \chi = I_1$$

$$I_2(\varrho) \to -\int_H \psi d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge d\chi = I_2$$

$$I_3(\varrho) \to 2r \int_H \psi f^*(\ddot{\omega}_r) \wedge \chi = I_3$$

for $\varrho \to 0$. Hence $I_4 = \lim_{\varrho \to 0} I_4(\varrho)$ exists and

$$I = I_1 + I_2 + I_3 + I_4.$$

It remains to compute the limit I_4. Define

$$L(\sigma) = \left\{ z \mid z \in T \text{ with } \frac{\varrho}{2} \leqslant |\varphi(z)| \leqslant \varrho \right\}.$$

Then
$$I_4(\varrho)=\int_{L(\varrho)}\psi\, d\gamma_\varrho \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha))\wedge\chi.$$

for $0<\varrho<1$. For $\nu=0, 1, \ldots, r-1$ define

$$J_\nu(\varrho)=\int_{L(\varrho)}\psi\, d\gamma \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Phi(\alpha)^\nu)\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\chi.$$

Then
$$I_4(\varrho)=\frac{1}{(r-1)!}\sum_{\nu=0}^{r-1} J^\nu(\varrho).$$

For $\nu=0, 1, \ldots, r-1$ define the continuous forms

$$\chi_1=f^*(\ddot{\omega}^{r-1-\nu})\wedge\psi\chi$$
$$\chi_2=\bar{\partial}\log|g|\wedge f^*(\ddot{\omega}^{r-1-\nu})\wedge\psi\chi$$
$$\chi_3=\bar{\partial}\log|g|\wedge f^*(\omega^{r-1-\nu})\wedge\psi\chi$$

on A. On $\mathbf{C}^r-\{0\}$, define

$$\Omega_1(\mathfrak{z})=(d\mathfrak{z}\,|\,\mathfrak{z})\wedge(\mathfrak{z}\,|\,d\mathfrak{z})\wedge|\mathfrak{z}|^{2\nu-2}\,\underset{\sim}{\omega}^{2\nu},$$

$$\Omega_2(\mathfrak{z})=(d\mathfrak{z}\,|\,\mathfrak{z})\wedge|\mathfrak{z}|^{2\nu-1}\,\underset{\sim}{\omega}^{2\nu},$$

$$\Omega_3(\mathfrak{z})=(\mathfrak{z}\,|\,d\mathfrak{z})\wedge|\mathfrak{z}|^{2\nu-1}\,\underset{\sim}{\omega}^{2\nu}.$$

For $\mathfrak{z}=0$ define $\Omega_\lambda(\mathfrak{z})=0$ for $\lambda=1, 2, 3$. Then Ω_λ for $\lambda=1, 2, 3$ has locally bounded and measurable coefficients on $\mathbf{C}^r$. On A define h_ϱ by setting

$$h_\varrho(z)=|\varphi(z)|\,g_\varrho'(|\varphi(z)|)$$

Then $|h_\varrho(z)|\leqslant B$ for all $\varrho>0$ and all $z\in A$. On A is

$$d\gamma_\varrho=\tfrac{1}{2}\,g_\varrho'(|\varphi|)\,\frac{(d\varphi\,|\,\varphi)+(\varphi\,|\,d\varphi)}{|\varphi|},$$

$$d^\perp\log\|f:\alpha\|=d^\perp\log|\varphi|-d^\perp\log|g|,$$

where
$$((d\varphi\,|\,\varphi)+(\varphi\,|\,d\varphi))\wedge d^\perp\log|\varphi|$$
$$=\frac{i}{2\,|\varphi|^2}\,((d\varphi\,|\,\varphi)+(\varphi\,|\,d\varphi))\wedge((d\varphi\,|\,\varphi)-(\varphi\,|\,d\varphi))$$
$$=-\frac{i}{|\varphi|^2}\,(d\varphi\,|\,\varphi)\wedge(\varphi\,|\,d\varphi).$$

Therefore

$$\psi d\gamma_\varrho \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Phi(\alpha)^\nu) \wedge f^*(\ddot{\omega}^{r-1-\nu}) \wedge \chi$$

$$= -\frac{i}{2} g_\varrho'(|\varphi|) \frac{1}{|\varphi|^3} (d\varphi|\varphi) \wedge (\varphi|d\varphi) \wedge \varphi^*(\underset{\sim}{\omega}^\nu) \wedge \chi_1 + \frac{i}{2} g_\varrho'(|\varphi|) \frac{1}{|\varphi|} (d\varphi|\varphi) \wedge \varphi^*(\underset{\sim}{\omega}^\nu) \wedge \chi_2$$

$$- \frac{i}{2} g_\varrho'(|\varphi|) \frac{1}{|\varphi|} (\varphi|d\varphi) \wedge \varphi^*(\underset{\sim}{\omega}^\nu) \wedge \chi_3$$

$$= -\frac{i}{2} h_\varrho \frac{1}{|\varphi|^{2\nu+2}} \varphi^*(\Omega_1) \wedge \chi_1 + \frac{i}{2} h_\varrho \frac{1}{|\varphi|^{2\nu+1}} \varphi^*(\Omega_2) \wedge \chi_2 - \frac{i}{2} h_\varrho \frac{1}{|\varphi|^{2\nu+1}} \varphi^*(\Omega_2) \wedge \chi_3.$$

For $0 < \varrho < 1$ and $\nu = 0, 1, \dots, r = 1$ define

$$J_\nu^1(\varrho) = \int_{L(\varrho)} h_\varrho \frac{1}{|\varphi|^{2\nu+2}} \varphi^*(\Omega_1) \wedge \chi_1$$

$$J_\nu^2(\varrho) = \int_{L(\varrho)} h_\varrho \frac{1}{|\varphi|^{2\nu+1}} \varphi^*(\Omega_2) \wedge \chi_2$$

$$J_\nu^3(\varrho) = \int_{L(\varrho)} h_\varrho \frac{1}{|\varphi|^{2\nu+1}} \varphi^*(\Omega_2) \wedge \chi_3.$$

Then
$$J_\nu(\varrho) = \frac{-i}{2} J_\nu^1(\varrho) + \frac{i}{2} J_\nu^2(\varrho) - \frac{i}{2} J_\nu^3(\varrho).$$

Now, apply Proposition 1.7 three times:

1.7	M	m	f	p	q	$\varkappa$	s	t	σ	τ	K	χ	φ	h_ϱ
Here I	A	m	φ	r	q	0	$\nu+1$	$\nu+1$	$m-\nu-1$	$m-\nu-1$	T	χ_1	Ω_1	h_ϱ
Here II	A	m	φ	r	q	0	$\nu+1$	ν	$m-\nu-1$	$m-\nu$	T	χ_2	Ω_2	h_ϱ
Here III	A	m	φ	r	q	0	ν	$\nu+1$	$m-\nu$	$m-\nu-1$	T	χ_3	Ω_3	h_ϱ

where in I the case $\nu = r-1$ has to be excluded. Therefore

$$J_\nu^\lambda(\varrho) \to 0 \quad \text{for} \quad \varrho \to 0,$$

if $\lambda = 1$ and $\nu = 0, 1, \dots, r-2$, and if $\lambda = 2, 3$ and $\nu = 0, 1, \dots, r-1$.
Hence

$$J_\nu(\varrho) \to 0 \quad \text{for} \quad \varrho \to 0 \quad \nu = 0, 1, \dots, r-2.$$

Hence
$$I_4 = -\frac{i}{2} \frac{1}{(r-1)!} \lim_{\varrho \to 0} J_{r-1}^1(\varrho).$$

Let A_0 be an open neighborhood of T with $\bar{A}_0$ compact and with $\bar{A}_0 \subseteq A$. Define $\lambda_\varrho(\mathfrak{z}) = g_\varrho(|\mathfrak{z}|)$. Then

$$\frac{i}{2(r-1)!} J^1_{r-1}(\varrho) = \int_{A_0 \cap H} d^\perp \log |\varphi| \wedge d\lambda_\varrho \circ \varphi \wedge \varphi^*(\underset{\sim}{\omega}_{r-1}) \wedge \psi \chi.$$

Here, $A_0 \cap H$ is an open subset of A with $\overline{A_0 \cap H} \subseteq \bar{A}_0 \subseteq A$ where $\overline{A_0 \cap H}$ is compact. Let K_0 be the support of $\psi\chi$ in $\overline{A_0 \cap H} - A_0 \cap H$. Then

$$\overline{A_0 \cap H} - A_0 \cap H \subseteq \bar{A}_0 \cap \bar{H} - A_0 \cap H \subseteq ((\bar{A}_0 - A_0) \cap \bar{H}) \cup (A_0 \cap S).$$

Because $K_0 \cap (\bar{A}_0 - A_0) = \emptyset$ and because $K_0 \cap S \subseteq K \cap A$, it is

$$K_0 \cap \varphi^{-1}(0) \subseteq K_0 \cap (A_0 \cap S) \cap \varphi^{-1}(0) \subseteq K \cap A \cap f^{-1}(\ddot{E}(\alpha)).$$

Hence $K_0 \cap \varphi^{-1}(0)$ is a set of measure zero on $\varphi^{-1}(0)$, respectively empty if $q=0$. Lemma 1.8 implies

$$I_4 = -\frac{i}{2(r-1)!} \lim_{\varrho \to 0} J^1_{r-1}(\varrho) = -\frac{2\pi^r}{(r-1)!} \int_{A_0 \cap H \cap \varphi^{-1}(0)} \nu_\varphi \psi \chi \text{ for } \varrho \to 0.$$

According to Lemma 2.5 is $\nu_\varphi(z) = \nu_f(z; \alpha)$ for every simple point of $\varphi^{-1}(0) \cap A = A \cap f^{-1}(\ddot{E}(\alpha))$. Because χ has compact support in A_0, this implies

$$I_4 = -\frac{2\pi^r}{(r-1)!} \int_{A_0 \cap H \cap \varphi^{-1}(0)} \nu_\varphi \psi \chi = -\frac{2\pi^r}{(r-1)!} \int_{H \cap f^{-1}(\ddot{E}(\alpha))} \nu_\varrho(z:\alpha)\, \psi \chi$$

which proves the Theorem if χ has compact support in A.

Now, consider the general case. Finitely many points $a_1, \ldots, a_s$ in $\bar{H}$ and finitely many open sets $A_1, \ldots, A_s$ with $a_\sigma \in A_\sigma$ exist such that the Theorem is true if the support of χ is compact and contained in A_σ and such that $\bar{H} \subseteq A_1 \cup \ldots \cup A_s$. Take functions $g_1, \ldots, g_s$ of class C^∞ on M such that $0 \leqslant g_\sigma \leqslant 1$ on M and such that the support of g_σ is compact and contained in A_σ, and such that $\sum_{\sigma=1}^s g_\sigma(z) = 1$ if $z \in \bar{H}$. Then

$$\begin{aligned} &\int_S \psi\, d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge g_\sigma \chi \\ &\quad = \int_H d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge g_\sigma \chi \\ &\qquad - \int_H \psi\, d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge d(g_\sigma \chi) \\ &\qquad + 2r \int_H \psi f^*(\ddot{\omega}_r) \wedge g_\sigma \chi - \frac{2\pi^r}{(r-1)!} \int_{H \cap f^{-1}(\ddot{E}(\alpha))} \nu_f(z:\alpha)\, \psi g_\sigma \chi. \end{aligned}$$

Because $\sum_{\sigma=1}^s g_\sigma(z) = 1$ for $z \in \bar{H}$, addition proves the theorem, q.e.d.

For any number $n \in \mathbf{N}$, define

$$W(n) = \frac{\pi^n}{n!}$$

as the volume of the ball of radius 1 in $\mathbf{C}^n$. By setting $\psi \equiv 1$ in Theorem 4.4, the so called unintegrated First Main Theorem is obtained:

THEOREM 4.5. (*The unintegrated First Main Theorem.*) *The assumptions 4.2 are made. Let H be an open subset of M such that $\bar{H}$ is compact and such that $\bar{H} - H = S$ is empty or a boundary manifold of H. Suppose that $S \cap f^{-1}(\ddot{E}(\alpha))$ is a set of measure zero on $f^{-1}(\ddot{E}(\alpha))$. Suppose that*

$$d^{\perp} \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

is integrable over S. Suppose that $d\chi = 0$ on H. Then

$$\frac{1}{2\pi} \frac{1}{W(r-1)} \int_H d^{\perp} \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi + \int_{H \cap f^{-1}(\ddot{E}(\alpha))} \nu_f(z;\alpha)\,\chi$$
$$= \frac{1}{W(r)} \int_H f^*(\ddot{\omega}_r) \wedge \chi.$$

Remark 1. If $q = 0$, then χ is a function and the integral over $H \cap f^{-1}(\ddot{E}(\alpha))$ means a sum over this finite set.

Remark 2. If χ is non-negative, both integrals

$$n_f(H;\alpha) = \int_{H \cap f^{-1}(\ddot{E}(\alpha))} \nu_f(z;\alpha)\,\chi$$

$$A_f(H) = \frac{1}{W(r)} \int_H f^*(\ddot{\omega}_r) \wedge \chi$$

are non-negative, where upon

$$\mu_f(S;\alpha) = \frac{1}{2\pi W(r-1)} \int_S d^{\perp} \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

does not have to have a fixed sign. It is

$$\mu_f(S;\alpha) + n_f(H;\alpha) = A_f(H).$$

Remark 3. If $q = 0$, and $\chi = 1$ the Theorem is due to Levine [12]. *Observe, that then*

$$n_f(H;\alpha) = \sum_{z \in H} \nu_f(z;\alpha).$$

If $q=0$, and χ satisfies the assumptions, then χ is a function of class C^1 with $d\chi=0$. Hence χ is constant on any component of H. Hence Theorem 4.5 is not more general then Levine [12], if $q=0$.

Proof. Set $\psi=1$ in Theorem 4.4, q.e.d. If $M=H$ is compact, then $S=\phi$. Hence the following consequence is obtained:

THEOREM 4.6. *The assumptions 4.2 are made. Suppose that M is compact. Suppose that $d\chi=0$ on M. Then*

$$\int_{f^{-1}(\ddot{E}(\alpha))} \nu_f(z;\alpha)\,\chi=\frac{1}{W(r)}\int_M f^*(\ddot{\omega}_r)\wedge\chi$$

which means that $n_f(M,\alpha)=\int_{f^{-1}(\ddot{E}(\alpha))}\nu_f(z,\alpha)\,\chi$ is constant for every $\alpha\in\mathfrak{G}^p(V)$ for which f is general of order r. Especially, if $r=n$ and if f is q-fibering then

$$\int_{f^{-1}(\alpha)} \nu_f(z;\alpha)\chi=\frac{1}{W(r)}\int_M f^*(\ddot{\omega}_r)\wedge\chi$$

for all $\alpha\in\mathbf{P}(V)$.

Therefore Theorem 4.6 is a generalization of II Theorem 3.8 if M is compact and $f:M\to\mathbf{P}(V)$ is q-fibering. Theorem 4.6 asserts in this case that the fiber integral is not only continuous but constant and even gives the value of the constant. Therefore the question arises, if the fiber integral is constant provided M is compact, $f:M\to N$ holomorphic and q-fibering with $\dim M-\dim N=q$ and $d\chi=0$.(1)

ASSUMPTIONS 4.7.(2)

The assumptions 4.2 are made. In addition it is assumed:

1. *In M, open subsets G and g with compact closures $\bar{G}$ and $\bar{g}$ are given, where $\bar{g}\subseteq G$.*
2. *$\Gamma=\bar{G}-G$ is a boundary manifold of G.*
3. *$\gamma=\bar{g}-g$ is a boundary manifold of g.*
4. *The form χ is non-negative on M. Moreover, $d\chi=0$ on M.*
5. *A continuous function ψ on M is given such that*
   a) *$\psi\,|\,(\bar{G}-g)$ is of class C^2 on $(\bar{G}-g)$.*
   b) *For $z\in M-G$ is $\psi(z)=0$. For $z\in\bar{g}$ is $\psi(z)=R=$ constant.*
   c) *For $z\in\bar{G}-g$ is $0\leqslant\psi(z)\leqslant R$.*
6. *For $n\in\mathbf{N}$ is $W(n)=\pi^n/n!$*
7. *Define the compensation functions by*

(1) Added in proof: As it is easily seen, this is true and is a consequence of the continuity of the fiber integral.

(2) Compare Stoll [21], IV page 77 and V page 80.

$$m_f(\gamma; \alpha) = \frac{1}{2\pi} \frac{1}{W(r-1)} \int_\gamma \log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge \chi$$

$$m_f(\Gamma; \alpha) = \frac{1}{2\pi} \frac{1}{W(r-1)} \int_\Gamma \log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge \chi,$$

where $d^\perp \psi$ on γ and Γ is formed as the continuous continuation of $d^\perp \psi$ on $G - \bar{g}$

8. *Define the valance function by*

$$N_f(G; \alpha) = \int_{f^{-1}(\ddot{E}(\alpha))} \nu_f(z; \alpha)\, \psi \chi$$

9. *Define the characteristic function by*

$$T_f(G) = \frac{1}{W(r)} \int_G \psi f^*(\ddot{\omega}_r) \wedge \chi.$$

10. *Define the deficit by*

$$\Delta_f(G; \alpha) = \frac{1}{2\pi} \frac{1}{W(r-1)} \int_G \log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge dd^\perp \psi \wedge \chi.$$

If these assumptions are made, then

$$\nu_f(z; \alpha) \psi \chi \geqslant 0 \quad \text{and} \quad \psi f^*(\omega_r) \wedge \chi \geqslant 0.$$

Hence

$$N_f(G; \alpha) \geqslant 0 \quad \text{and} \quad T_f(G) \geqslant 0.$$

According to Stoll [21] Satz 4.5 is

$$\log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge d^\perp \psi \wedge \chi \geqslant 0$$

along Γ and γ. Hence

$$m_f(\gamma, \alpha) \geqslant 0 \quad \text{and} \quad m_f(\Gamma, \alpha) \geqslant 0.$$

Hence the valence function, compensation functions, and characteristic functions are non-negative if the assumption 4.7 is made.

THEOREM 4.8. *The First Main Theorem. The assumption* 4.7 *are made. Then*

$$N_f(G; \alpha) + m_f(\Gamma; \alpha) - m_f(\gamma; \alpha) = T_f(G) + \Delta_f(G; \alpha).$$

Proof. Let $\tilde{\gamma}$ be γ with the opposite orientation. Define $H = \bar{G} - g$. Then $S = \Gamma \cup \tilde{\gamma}$ is a boundary manifold of H and $\bar{H} - H = S$. Let λ be a function of class C^∞ on M with compact support in G and with $\lambda | \bar{g} = 1$ and such that $0 \leqslant \lambda \leqslant 1$ on M. Then

$$\eta_1 = \log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge \lambda\chi$$

has density 0 along Γ and a non-positive density along γ. According to Proposition 4.3, η_1 is integrable along $\tilde{\gamma}$, hence along γ. Observe, that along γ

$$\eta_1 = \log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge \chi$$

and that η_1 has a non-negative density on γ. Hence $m_f(\gamma; \alpha)$ exists. Define

$$\eta_2 = \log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge (1-\lambda)\chi.$$

Then η_2 has a non-negative density on Γ and density 0 on γ. According to Proposition 4.3, η_2 is integrable along Γ. Observe that along Γ is

$$\eta_2 = \log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge \chi$$

and has a non-negative density on Γ. Hence $m_f(\Gamma; \alpha)$ exists. Therefore, $\log 1/\|f:\alpha\| f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge \chi$ is integrable along $\Gamma \cup \tilde{\gamma}$, but does not have a fixed sign along $\Gamma \cup \tilde{\gamma}$. According to Proposition 4.3 is

$$\frac{1}{2\pi}\frac{1}{W(r-1)}\int_S \log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge d^\perp\psi \wedge \chi$$

$$= -\frac{1}{2\pi}\frac{1}{W(r-1)}\int_H d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

$$+\frac{1}{2\pi}\frac{1}{W(r-1)}\int_H \log \frac{1}{\|f:\alpha\|} f^*(\Lambda(\alpha)) \wedge dd^\perp\psi \wedge \chi.$$

Because ψ is constant on $\bar{g}$ this implies

$$m_f(\Gamma; \alpha) - m_f(\gamma; \alpha) = -\frac{1}{2\pi W(r-1)}\int_G d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi + \Delta_f(G, \alpha),$$

where especially all the integrals involved exist. According to Stoll [21] p. 62 Hilfssatz 1, the function ψ satisfies locally a Lipschitz condition on M. The support of $\psi\chi$ on Γ is empty and

$$\psi\, d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

is 0 along Γ, hence integrable over Γ. Hence Theorem 4.4 can be applied with H replaced by G and S replaced by Γ which implies[1]

$$0 = \frac{1}{2\pi W(r-1)} \int_G d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi + \frac{1}{W(r)} \int_G \psi f^*(\ddot{\omega}_r) \wedge \chi - \int_{G\cap f^{-1}(E(\alpha))} \nu_f(z;\alpha)\,\psi\chi$$

or

$$N_f(G;\alpha) = \frac{1}{2\pi}\frac{1}{W(r-1)} \int_G d\psi \wedge d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi + T_f(G).$$

Addition implies

$$N_f(G;\alpha) + m_f(\Gamma;\alpha) - m_f(\gamma;\alpha) = T_f(G) + \Delta_f(G;\alpha) \qquad \text{q.e.d.}$$

Now, it shall be shown, that special known versions of the First Main Theorem can be obtained from Theorem 4.8.

1. Stoll [21]: Let $r=1$. Then $q=m-1$. Suppose that χ is positive definite. Then $\psi=\psi(G)$ can be chosen uniquely to G such that $dd^\perp\psi \wedge \chi = 0$ on $G-\bar{g}$ and such that

$$\frac{1}{2\pi}\int_\gamma d^\perp\psi \wedge \chi = \frac{1}{2\pi}\int_\Gamma d^\perp\psi\wedge\chi = 1.$$

Then $\Delta_f(G,\alpha)=0$. Then Satz 8.2 of Stoll [21] follows for holomorphic maps. Satz 8.2 holds also for meromorphic maps $f: M \to \mathbf{P}(V)$.

2. H. Weyl and J. Weyl [29]. (Special case of 1): Take $m=r=1=\chi$. Then $q=0$. Take $dd^\perp\psi=0$ in $H=G-\bar{g}$. Then ψ is a harmonic function on H. Adjust R such that

$$\frac{1}{2\pi}\int_\Gamma d^\perp\psi = \frac{1}{2\pi}\int_\gamma d^\perp\psi = 1.$$

Then (4.2) p. 182 of H. Weyl and J. Weyl [29] follows with $\Delta_f=0$.

3. H. Kneser [8], (Special case of 1): Take $r=1=n$. Then $q=m-1$. Take $M=\mathbf{C}^m$. Take $\chi=v_{m-1}$. Take

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[1] If Theorem 4.4 would be applied separately to $G-\bar{g}$ and g, then the ugly question of the exstence of the integral

$$\int_\gamma \psi d^\perp \log \|f:\alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

would arise. This difficulty is avoided by the application of Theorem 4.4 to G with ψ satisfying locally a Lipschitz condition. The same trick was already used in Stoll [21] Satz 6.5.

$$G = \{\mathfrak{z} \mid |\mathfrak{z}| < r\} \text{ and } g = \{\mathfrak{z} \mid |\mathfrak{z}| < r_0\}$$

with $0 < r_0 < r$. On $G - \bar{g} = H$ define

$$\psi(\mathfrak{z}) = \frac{1}{2m-2}\left(\frac{1}{|\mathfrak{z}|^{2m-2}} - \frac{1}{r^{2m-2}}\right)$$

Then the First Main Theorem of H. Kneser [8], p. 32 follows in a somewhat modified formulation (see Stoll [19], p. 212 (1.7)).

4. S. S. Chern [3]. Take $r = m = n$. Then $q = p = 0$. Take $M = \mathbf{C}^m$. Take $\chi = 1$. Define G, g and ψ as in 3. Then the First Main Theorem of Chern [3], p. 15 follows where

Here	$N_f(G, \alpha)$	$m_f(G, \alpha)$	$m_f(\gamma, \alpha)$	$T_f(G)$	$\Delta_f(G, \alpha)$
There	$N(r, A)$	$I(r, A)$	$I(r_0, A) = \text{const}$	$T(r)$	$S(r, A) - 2m \int_{r_0}^{r} I(r, A) \frac{dr}{r}$

Surprisingly, the same "choice" of ψ occurs in 3 and 4. Of course, the original approach in 3 and 4 did not include a choice. Also Theorem 4.8 gives a clearer picture of the integrands of N_f, m_f, T_f and A_f then that was possible in [3].

If $M = \mathbf{C}^m$ and $G = \{\mathfrak{z} \mid |\mathfrak{z}| < r\}$ and $g = \{\mathfrak{z} \mid |\mathfrak{z}| < r_0\}$ with $0 < r_0 < r$, then Theorem 4.8 suggests other choices for ψ and χ such that Δ_f has a non-negative integrand; for instance $\psi = \log r/|\mathfrak{z}|$ on $\bar{G} - g$ and v_q on $\mathbf{C}^m$ gives $dd^\perp\psi \wedge \chi = 2\omega \wedge v_q$ on $\bar{G} - g$; or $\psi = \frac{1}{4}(r^2 - |\mathfrak{z}|^2)$ on $\bar{G} - g$ and v_q on $\mathbf{C}^m$ gives $dd^\perp\psi \wedge \chi = (q+1)v_{q+1}$ on $\bar{G} - g$.

§ 5. Open maps into the projective space

Let V be a complex vector space of dimension $n+1$. Suppose that an Hermitian scalar product $(\,|\,)$ on V is given. Then $\Phi(\alpha)$ and $\Lambda(\alpha)$ are formed for $r = n$ that is $p = 0$. For $\mathfrak{a} \in V - \{0\}$ define

$$\xi_{\mathfrak{a}}(\mathfrak{w}) = \frac{1}{|\mathfrak{w} \wedge \mathfrak{a}|}\left((d\mathfrak{w} \mid \mathfrak{a}) - \frac{(\mathfrak{w} \mid \mathfrak{a})}{|\mathfrak{w}|^2}(d\mathfrak{w} \mid \mathfrak{w})\right)$$

for $\mathfrak{z} \in V - E\,(\mathfrak{a})$. If $\lambda \in \mathbf{C} - \{0\}$, then

$$\xi_{\lambda\mathfrak{a}}(\mathfrak{w}) = \frac{\bar{\lambda}}{|\lambda|}\,\xi_{\mathfrak{a}}(\mathfrak{w}).$$

If g is a complex valued function of class C^1 on an open subset U of V and if $g(\mathfrak{w}) \neq 0$ for $\mathfrak{w} \in U$, then

$$\xi_{\mathfrak{a}}(g(\mathfrak{w}) \cdot \mathfrak{w}) = \frac{g(\mathfrak{w})}{|g(\mathfrak{w})|}\,\xi_{\mathfrak{a}}(\mathfrak{w}).$$

For $\mathfrak{a} \in V - \{0\}$ define

$$\tau_{\mathfrak{a}}(\mathfrak{w}) = \frac{i}{2}\, \xi_{\mathfrak{a}}(\mathfrak{w}) \wedge \overline{\xi_{\mathfrak{a}}(\mathfrak{w})}$$

for $\mathfrak{w} \in V - E(\mathfrak{a})$. Because $\tau_{\lambda\mathfrak{a}}(g(\mathfrak{w})\mathfrak{w}) = \tau_{\mathfrak{a}}(\mathfrak{w})$, one and only one form $\ddot{\tau}_\alpha$ of bidegree (1, 1) exists on $\mathbf{P}(V) - \ddot{E}(\alpha)$ such that $\varrho^*(\ddot{\tau}_\alpha) = \tau$ where $\varrho: V \to \{0\} \to \mathbf{P}(V)$ is the natural projection and where $\varrho(\mathfrak{a}) = \alpha$. The form $\tau_{\mathfrak{a}}$ is non-negative and idempotent, i.e. $\tau_{\mathfrak{a}} \wedge \tau_{\mathfrak{a}} = 0$. Hence $\ddot{\tau}_\alpha$ is non-negative and

$$\ddot{\tau}_\alpha \wedge \ddot{\tau}_\alpha = 0.$$

LEMMA 5.1.[1] *On* $\mathbf{P}(V) - \ddot{E}(\alpha)$ *is*

$$\ddot{\omega} - \ddot{\tau}_\alpha = \|\alpha : w\|^2 \Phi(\alpha).$$

Proof. Define $\pi'_\alpha: V \to V[2]$ by $\pi'_\alpha(\mathfrak{w}) = \mathfrak{w} \wedge \mathfrak{a}$ where $\mathfrak{a} \in \varrho^{-1}(\alpha)$. Let $\underline{\underline{\varrho}}: V[2] - \{0\} \to \mathbf{P}(V[2])$ be the natural projection. Then $\pi_\alpha: \mathbf{P}(V) - \ddot{E}(\alpha) \to \mathbf{P}(V[2])$ is well defined by $\pi_\alpha \circ \varrho = \underline{\underline{\varrho}} \circ \pi'_\alpha$. According to the definition of $\Phi(\alpha)$ is

$$\begin{aligned}
\varrho^*(\Phi(\alpha))\,(\mathfrak{w}) &= \varrho^*(\pi^*_\alpha(\underline{\underline{\ddot{\omega}}}))\,(\mathfrak{w}) = (\pi'_\alpha)^*(\underline{\underline{\varrho}}^*(\underline{\underline{\ddot{\omega}}}))\,(\mathfrak{w}) \\
&= (\pi'_\alpha)^*(\omega)\,(\mathfrak{w}) = \tfrac{1}{2}\, d^\perp d \log \underline{|}\,\mathfrak{w} \wedge \mathfrak{a}\,| \\
&= \frac{i}{2}\, \partial\bar\partial \log\, [(\mathfrak{w}\,|\,\mathfrak{w})\,|\mathfrak{a}|^2 - (\mathfrak{w}\,|\,\mathfrak{a})\,(\mathfrak{a}\,|\,\mathfrak{w})] \\
&= \frac{i}{2}\, \partial \frac{1}{|\mathfrak{w} \wedge \mathfrak{a}|^2}\, [(\mathfrak{w}\,|\,d\mathfrak{w})\,|\mathfrak{a}|^2 - (\mathfrak{w}\,|\,\mathfrak{a})\,(\mathfrak{a}\,|\,d\mathfrak{w})] \\
&= \frac{i}{2} \cdot \frac{1}{|\mathfrak{w} \wedge \mathfrak{a}|^2}\, [(d\mathfrak{w}\,|\,d\mathfrak{w})\,|\mathfrak{a}|^2 - (d\mathfrak{w}\,|\,\mathfrak{a}) \wedge (\mathfrak{a}\,|\,d\mathfrak{w})] \\
&\quad - \frac{i}{2} \cdot \frac{1}{|\mathfrak{w} \wedge \mathfrak{a}|^4}\, [(d\mathfrak{w}\,|\,\mathfrak{w})\,|\mathfrak{a}|^2 - (d\mathfrak{w}\,|\,\mathfrak{a})\,(\mathfrak{a}\,|\,\mathfrak{w})]\, [(\mathfrak{w}\,|\,d\mathfrak{w})\,|\mathfrak{a}|^2 - (\mathfrak{w}\,|\,\mathfrak{a})\,(\mathfrak{a}\,|\,d\mathfrak{w})].
\end{aligned}$$

Hence

$$\begin{aligned}
&\varrho^*(\ddot{\omega})\,(\mathfrak{w}) - \|\,\mathfrak{w} : \mathfrak{a}\,\|^2\, \varrho^*(\Phi(\alpha))\,(\mathfrak{w}) \\
&= \frac{i}{2}\, \frac{(d\mathfrak{w}\,|\,d\mathfrak{w})}{|\mathfrak{w}|^2} - \frac{i}{2}\, \frac{(d\mathfrak{w}\,|\,\mathfrak{w}) \wedge (\mathfrak{w}\,|\,d\mathfrak{w})}{|\mathfrak{w}|^4} \\
&\quad - \frac{i}{2}\, \frac{(d\mathfrak{w}\,|\,d\mathfrak{w})}{|\mathfrak{w}|^2} + \frac{i}{2}\, \frac{(d\mathfrak{w}\,|\,\mathfrak{a}) \wedge (\mathfrak{a}\,|\,d\mathfrak{w})}{|\mathfrak{w}|^2\,|\mathfrak{a}|^2}
\end{aligned}$$

[1] See Chern [3] (42).

$$+\frac{i}{2}\frac{1}{|\mathfrak{w}\wedge\mathfrak{a}|^2|\mathfrak{w}|^2|\mathfrak{a}|^2}[(d\mathfrak{w}\,|\,\mathfrak{w})\,|\mathfrak{a}|^2-(d\mathfrak{w}\,|\,\mathfrak{a})\,(\mathfrak{a}\,|\,\mathfrak{w})]\wedge[(\mathfrak{w}\,|\,d\mathfrak{w})\,|\mathfrak{a}|^2-(\mathfrak{w}\,|\,\mathfrak{a})\,(\mathfrak{a}\,|\,d\mathfrak{w})]$$

$$=\frac{i}{2}\frac{(d\mathfrak{w}\,|\,\mathfrak{a})\wedge(\mathfrak{a}\,|\,d\mathfrak{w})}{|\mathfrak{w}\wedge\mathfrak{a}|^2}+\frac{i}{2}\frac{|(\mathfrak{w}\,|\,\mathfrak{a})|^2}{|\mathfrak{w}\wedge\mathfrak{a}|^2}\frac{(d\mathfrak{w}\,|\,\mathfrak{w})\wedge(\mathfrak{w}\,|\,d\mathfrak{w})}{|\mathfrak{w}|^4}$$

$$-\frac{i}{2}\frac{(\mathfrak{w}\,|\,\mathfrak{a})}{|\mathfrak{w}\wedge\mathfrak{a}|^2}\frac{(d\mathfrak{w}\,|\,\mathfrak{w})\wedge(\mathfrak{a}\,|\,d\mathfrak{w})}{|\mathfrak{w}|^2}-\frac{i}{2}\frac{(\mathfrak{a}\,|\,\mathfrak{w})}{|\mathfrak{w}\wedge\mathfrak{a}|^2}\frac{(d\mathfrak{w}\,|\,\mathfrak{a})\wedge(\mathfrak{w}\,|\,d\mathfrak{w})}{|\mathfrak{w}|^2}$$

$$=\frac{i}{2}\frac{1}{|\mathfrak{w}\wedge\mathfrak{a}|^2}\left[(d\mathfrak{w}\,|\,\mathfrak{a})-\frac{(\mathfrak{w}\,|\,\mathfrak{a})}{|\mathfrak{w}|^2}(d\mathfrak{w}\,|\,\mathfrak{w})\right]\wedge\left[(\mathfrak{a}\,|\,d\mathfrak{w})-\frac{(\mathfrak{a}\,|\,\mathfrak{w})}{|\mathfrak{w}|^2}(\mathfrak{w}\,|\,d\mathfrak{w})\right]$$

$$=\frac{i}{2}\xi_{\mathfrak{a}}(\mathfrak{w})\wedge\overline{\xi_{\mathfrak{a}}(\mathfrak{w})}=\varrho^*(\ddot{\tau}_\alpha)(\mathfrak{w}).$$

Because ϱ^* is injective, this proves the Lemma; q.e.d.

If $\alpha\in\mathbf{P}(V)$ and $r=n$, then,

$$\Lambda(\alpha)=\frac{1}{(n-1)!}\sum_{\nu=0}^{n-1}\Phi(\alpha)^\nu\wedge\ddot{\omega}^{n-1-\nu}$$

$$=\frac{1}{(n-1)!}\sum_{\nu=0}^{n-1}(\ddot{\omega}-\ddot{\tau}_\alpha)^\nu\frac{1}{\|w:\alpha\|^{2\nu}}\wedge\ddot{\omega}^{n-1-\nu}$$

$$=\frac{1}{(n-1)!}\sum_{\nu=0}^{n-1}(\ddot{\omega}^\nu-\nu\ddot{\tau}_\alpha\wedge\ddot{\omega}^{\nu-1})\frac{\ddot{\omega}^{n-1-\nu}}{\|w:\alpha\|^{2\nu}}$$

$$=\sum_{\nu=0}^{n-1}\frac{1}{\|w:\alpha\|^{2\nu}}\ddot{\omega}_{n-1}-\frac{1}{n-1}\sum_{\nu=0}^{n-1}\frac{\nu}{\|w:\alpha\|^{2\nu}}\ddot{\tau}_\alpha\wedge\ddot{\omega}_{n-2}.$$

Hence
$$\Lambda(\alpha)=\sum_{\nu=0}^{n-1}\frac{1}{\|w:\alpha\|^{2\nu}}\ddot{\omega}_{n-1}-\frac{1}{n-1}\sum_{\nu=0}^{n-1}\frac{\nu}{\|w:\alpha\|^{2\nu}}\ddot{\tau}_\alpha\wedge\ddot{\omega}_{n-2}. \tag{1}$$

Let M and N be oriented pure-dimensional manifolds of class C^∞. Let m be the real dimension of M and let n be the real dimension of N. Let $C\subseteq M$ and $A\subseteq N$, where A is measurable. Let η be a form of degree n on A: Suppose that for every $u\in A$ a form $\gamma(u)$ of degree p is given on C. Take $z\in C$. Let $\mathfrak{S}(M)$ be the oriented C^∞-structure of M. If $\alpha\in\mathfrak{S}(M)$ with $z\in U_\alpha$, then

$$\gamma(u)(z)=\sum_{\varphi\in\mathfrak{T}(p,m)}\gamma_\varphi(z,u;\alpha)\,dx^\alpha_{\varphi(1)}\wedge\ldots\wedge dx^\alpha_{\varphi(p)}.$$

Then $\gamma(u)\eta$ is said to be integrable over A at z if all the integrals $\int_A\gamma_\varphi(z,u;\alpha)\eta$ exist. Clearly, this definition does not depend on the choice of α, and

$$\int_A \gamma(u)(z)\,\eta = \sum_{\varphi\in\mathfrak{T}(p,m)} \left(\int_A \gamma_\varphi(z,u;\alpha)\,\eta\right) dx^\alpha_{\varphi(1)} \wedge \ldots \wedge dx^\alpha_{\varphi(p)}$$

is independent of the choice of α. If $\gamma(u)\eta$ is integrable over A for every $z\in C$, then $\gamma(u)\eta$ is said to be integrable over A on C; moreover, a form $\int_A\gamma(u)\eta$ of degree p is defined on C by setting

$$\left(\int_A \gamma(u)\,\eta\right)(z) = \int_A \gamma(u)(z)\,\eta \quad \text{for } z\in C.$$

Obviously, the following rules hold:

a) If $\gamma_1(u)\eta$ and $\gamma_2(u)\eta$ are integrable over A on C, so is $(\gamma_1(u)+\gamma_2(u))\eta$ and

$$\int_A (\gamma_1(u)+\gamma_2(u))\,\eta = \int_A \gamma_1(u)\,\eta + \int_A \gamma_2(u)\,\eta.$$

b) If $\gamma(u)\eta$ is integrable over A on C and if ϱ is a form of bidegree q on C, then $(\gamma(u)\wedge\varrho)\eta$ is integrable over A on C and

$$\int_A (\gamma(u)\wedge\varrho)\,\eta = \left(\int_A \gamma(u)\,\eta\right)\wedge\varrho.$$

c) If $\gamma(u)\eta$ is integrable over A on C if M' is a pure m'-dimensional oriented manifold of class C^∞, if $f: M'\to M$ is a map of class C^1 and if $C' = f^{-1}(C)$, then $f^*(\gamma(u))\eta$ is integrable over A on C' and

$$\int_A f^*(\gamma(u))\,\eta = f^*\left(\int_A \gamma(u)\,\eta\right).$$

d) If $\gamma(u)\eta$ is integrable over A on C, if η is non-negative on A, if C is measurable and if ϱ is a measurable form of degree $m-p$ on C such that $\gamma(u)\wedge\varrho$ is integrable over C and such that $(\int_C|\gamma(u)\wedge\varrho|)\eta$ is integrable over A, then

$$\int_A\left(\int_C \gamma(u)\wedge\varrho\right)\eta = \int_C\left(\int_A \gamma(u)\,\eta\right)\wedge\varrho.$$

Now, the integral meanvalue

$$\frac{1}{W(n)}\int_{\alpha\in\mathbf{P}(V)} \log\frac{1}{\|w:\alpha\|}\,\Lambda(\alpha)\wedge\ddot{\omega}_n(\alpha)$$

shall be computed. Some preparations are necessary:

LEMMA 5.2. *Define*

$$g(x) = (\log x) \sum_{\nu=0}^{n-1} \nu x^\nu \quad \textit{for } x > 0$$

and

$$I_\lambda = \frac{1}{\pi^{n+1}} \int_{\mathbf{C}^{n+1}} e^{-|a_0|^2 - \dots - |a_n|^2} g\left(\frac{|a_0|^2 + \dots + |a_n|^2}{|a_1|^2 + \dots + |a_n|^2}\right) \frac{|a_\lambda|^2}{|a_1|^2 + \dots + |a_n|^2} v_{n+1}(\mathfrak{a}),$$

then

$$I_\lambda = \sum_{\nu=0}^{n-1} \frac{(n-1-\nu)}{(\nu+1)^2}.$$

Proof. Without loss of generality $\lambda = 1$. Substitute

$$a_\nu = \sqrt{t_\nu}\, e^{i\varphi_\nu}, \quad \frac{i}{2} da_\nu \wedge d\bar{a}_\nu = \tfrac{1}{2} dt_\nu \wedge d\varphi_\nu$$

and
$$I_1 = \int_0^\infty \dots \int_0^\infty e^{-t_0 - \dots - t_n} g\left(\frac{t_0 + \dots + t_n}{t_1 + \dots + t_n}\right) \frac{t_1}{t_1 + \dots + t_n} dt_1 \dots dt_n.$$

Now substitute [1]

$$t_0 = \tau(s_1 + \dots + s_n) \quad 0 > \tau < 1 \quad 0 < s_\nu < \infty$$

$$t_\nu = (1-\tau)\, s_\nu \qquad \text{if } \nu = 1, \dots, n$$

$$\tau = \frac{t_0}{t_0 + \dots + t_n} \qquad s_\nu = t_\nu \frac{t_0 + \dots + t_n}{t_1 + \dots + t_n},$$

then
$$t_0 + \dots + t_n = s_1 + \dots + s_n \quad (1-\tau) = \frac{t_1 + \dots + t_n}{t_0 + \dots + t_n}$$

$$\frac{\partial(t_0, \dots, t_n)}{\partial(\tau, s_1, \dots, s_n)} = (s_1 + \dots + s_n)(1-\tau)^{n-1}.$$

Hence
$$I_1 = \int_0^1 g\left(\frac{1}{1-\tau}\right)(1-\tau)^{n-1} d\tau \int_0^\infty \int_0^\infty e^{-s_1 - \dots - s_n} s_1\, ds_1 \dots ds_n$$

$$= \int_0^1 \log \frac{1}{1-\tau} \sum_{\nu=0}^{n-1} \nu(1-\tau)^{n-1-\nu} d\tau$$

$$= \int_0^1 \log \frac{1}{1-\tau} \sum_{\nu=0}^{n-1} (n-1-\nu)(1-\tau)^\nu d\tau = \sum_{\nu=0}^{n-1} \frac{(n-1-\nu)}{(\nu+1)^2}, \text{ q.e.d.}$$

LEMMA 5.3. *Define $g(x)$ as in Lemma 5.2. For $\lambda \neq \varrho$ the integral*

$$I_{\lambda\varrho} = \frac{1}{\pi^{n+1}} \int_{\mathbf{C}^{n+1}} e^{-|a_0|^2 - \dots - |a_n|^2} g\left(\frac{|a_0|^2 + \dots + |a_n|^2}{|a_1|^2 + \dots + |a_n|^2}\right) \frac{a_\lambda \bar{a}_\varrho}{|a_1|^2 + \dots + |a_n|^2} v_{n+1}(\mathfrak{a})$$

exists and is zero.

[1] See Weyl [29] p. 227.

Proof. Obviously $|a_\lambda \bar{a}_\varrho| \leqslant \frac{1}{2}|a_\lambda|^2 + |a_\varrho|^2$ implies that $I_{\lambda\varrho}$ exists and $|I_{\lambda\varrho}| \leqslant \frac{1}{2}(I_\lambda + I_\varrho)$. The substitution

$$a'_\mu = \begin{cases} a_\mu & \text{if } \mu \neq \lambda \\ -a_\lambda & \text{if } \mu = \lambda \end{cases}$$

shows that $I_{\lambda\varrho} = -I_{\lambda\varrho}$. Hence $I_{\lambda\varrho} = 0$, q.e.d.

LEMMA 5.4. *Let V be a complex vector space of dimension $(n+1)$ with an Hermitian product $(\,|\,)$. Let $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ be in V. For $\mathfrak{a} \in V$, define*

$$\begin{aligned} L(\mathfrak{a}, \mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = (\mathfrak{u}|\mathfrak{a})(\mathfrak{a}|\mathfrak{v}) + |\mathfrak{w}|^{-4}|(\mathfrak{w}|\mathfrak{a})|^2(\mathfrak{u}|\mathfrak{w})(\mathfrak{w}|\mathfrak{v}) \\ - |\mathfrak{w}|^{-2}(\mathfrak{a}|\mathfrak{w})(\mathfrak{u}|\mathfrak{a})(\mathfrak{w}|\mathfrak{v}) \\ - |\mathfrak{w}|^{-2}(\mathfrak{w}|\mathfrak{a})(\mathfrak{a}|\mathfrak{v})(\mathfrak{u}|\mathfrak{w}) \end{aligned}$$

Define

$$g(x) = \log x \sum_{\nu=0}^{n-1} \nu x^\nu \quad \text{for } x > 0.$$

Then the integral

$$A(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \frac{1}{\pi^{n+1}} \int_V e^{-|\mathfrak{a}|^2} g\left(\frac{1}{\|\mathfrak{w} : \mathfrak{a}\|^2}\right) \frac{L(\mathfrak{a}, \mathfrak{u}, \mathfrak{v}, \mathfrak{w})}{|\mathfrak{w} \wedge \mathfrak{a}|^2} v_{n+1}(\mathfrak{a})$$

exists and is

$$A(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \left[\frac{(\mathfrak{u}|\mathfrak{v})}{|\mathfrak{w}|^2} - \frac{1}{|\mathfrak{w}|^2}(\mathfrak{u}|\mathfrak{w})(\mathfrak{w}|\mathfrak{v})\right] \sum_{\nu=0}^{n-1} \frac{(n-1-\nu)}{(\nu+1)^2}.$$

Proof. An orthonormal base $\mathfrak{e} = (\mathfrak{e}_0, \ldots, \mathfrak{e}_n)$ of V exists such that

$$\mathfrak{w} = w_0 \mathfrak{e}_0, \quad \mathfrak{u} = u_0 \mathfrak{e}_0 + u_1 \mathfrak{e}_1, \quad \mathfrak{v} = v_0 \mathfrak{e}_0 + v_1 \mathfrak{e}_1 + v_2 \mathfrak{e}_2.$$

Then $\mathfrak{a} = \sum_{\nu=0}^n a_\nu \mathfrak{e}_\nu$ and

$$L(\mathfrak{a}, \mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = |a_1|^2 u_1 \bar{v}_1 + \bar{a}_1 a_2 u_1 \bar{v}_2,$$

$$|\mathfrak{w} \wedge \mathfrak{a}|^2 = |w_0|^2 (|a_1|^2 + \ldots + |a_n|^2),$$

$$\|\mathfrak{w} : \mathfrak{a}\|^2 = \frac{|a_0|^2 + \ldots + |a_n|^2}{|a_1|^2 + \ldots + |a_n|^2}.$$

Hence $A(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$ exists and

$$A(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = I_1 \cdot \frac{u_1 \bar{v}_1}{|w_0|^2} = \sum_{\nu=0}^{n-1} \frac{n-1-\nu}{(\nu+1)^2} \frac{u_1 \bar{v}_1}{|w_0|^2}$$

Now

$$\frac{(\mathfrak{u}|\mathfrak{v})}{|\mathfrak{w}|^2} - \frac{(\mathfrak{u}|\mathfrak{w})(\mathfrak{w}|\mathfrak{v})}{|\mathfrak{w}|^4} = \frac{1}{|w_0|^2} u_1 \bar{v}_1,$$

which proves the lemma, q.e.d.

LEMMA 5.5. *For* $w \in \mathbf{P}(V)$ *define*

$$A(w) = \frac{1}{W(n)} \int_{\alpha \in \mathbf{P}(V)} \log \frac{1}{\|w:\alpha\|} \sum_{\nu=1}^{n-1} \frac{\nu}{\|w:\alpha\|^{2\nu}} \ddot{\tau}_\alpha(w)\, \ddot{\omega}_n(\alpha),$$

then

$$A(w) = \tfrac{1}{2} \sum_{\nu=0}^{n-1} \frac{(n-1-\nu)}{(\nu+1)^2} \ddot{\omega}(w).$$

Proof. Define $A_0(\mathfrak{w}) = \varrho^*(A)(\mathfrak{w})$. Then [1]

$$A_0(\mathfrak{w}) = \frac{1}{2\pi^{n+1}} \int_V e^{-|\mathfrak{a}|^2} \log \frac{1}{\|\mathfrak{w}:\mathfrak{a}\|^2} \sum_{\nu=0}^{n-1} \frac{\nu}{\|\mathfrak{w}:\mathfrak{a}\|^{2\nu}} \tau_\mathfrak{a}(\mathfrak{w})\, v_{n+1}(\mathfrak{a}).$$

Now

$$\begin{aligned}\tau_\mathfrak{a}(\mathfrak{w}) &= \frac{i}{2} \xi_\mathfrak{a}(\mathfrak{w}) \wedge \overline{\xi_\mathfrak{a}(\mathfrak{w})} \\ &= \frac{i}{2} \frac{(d\mathfrak{w}\,|\,\mathfrak{a}) \wedge (\mathfrak{a}\,|\,d\mathfrak{w})}{|\mathfrak{w} \wedge \mathfrak{a}|^2} + \frac{i}{2} \frac{|(\mathfrak{w}\,|\,\mathfrak{a})|^2}{|\mathfrak{w} \wedge \mathfrak{a}|^2} \frac{(d\mathfrak{w}\,|\,\mathfrak{w}) \wedge (\mathfrak{w}\,|\,\mathfrak{w})}{|\mathfrak{w}|^4} \\ &\quad - \frac{i}{2} \frac{(\mathfrak{w}\,|\,\mathfrak{a})}{|\mathfrak{w} \wedge \mathfrak{a}|^2} \frac{(d\mathfrak{w}\,|\,\mathfrak{w}) \wedge (\mathfrak{a}\,|\,d\mathfrak{w})}{|\mathfrak{w}|^2} - \frac{i}{2} \frac{(\mathfrak{a}\,|\,\mathfrak{w})}{|\mathfrak{w} \wedge \mathfrak{a}|^2} \frac{(d\mathfrak{w}\,|\,\mathfrak{a}) \wedge (\mathfrak{w}\,|\,d\mathfrak{w})}{|\mathfrak{w}|^2}.\end{aligned}$$

Hence, if $\alpha \in \mathfrak{S}(V)$ is a coordinate system, then

$$\tau_\mathfrak{a}(\mathfrak{w}) = \frac{i}{2} \sum_{\mu,\nu=0}^{n} L(\mathfrak{a}, \mathfrak{w}_{z_\mu}, \mathfrak{w}_{z_\nu}, \mathfrak{w}) \frac{1}{|\mathfrak{w} \wedge \mathfrak{a}|^2} dz_\mu \wedge d\bar{z}_\nu.$$

Hence

$$\begin{aligned}A_0(\mathfrak{w}) &= \frac{i}{4} \sum_{\mu,\nu=0}^{n} A(\mathfrak{w}_{z_\mu}, \mathfrak{w}_{z_\nu}, \mathfrak{w})\, dz_\mu \wedge d\bar{z}_\nu \\ &= \sum_{\nu=0}^{n-1} \frac{(n-1-\nu)}{(\nu+1)^2} \frac{i}{4|\mathfrak{w}|^4} \sum_{\mu,\nu=0}^{n} (|\mathfrak{w}|^2 (\mathfrak{w}_{z_\mu}\,|\,\mathfrak{w}_{z_\nu}) - (\mathfrak{w}_{z_\mu}\,|\,\mathfrak{w})(\mathfrak{w}\,|\,\mathfrak{w}_{z_\nu})\, dz_\mu \wedge d\bar{z}_\nu \\ &= \sum_{\nu=0}^{n-1} \frac{(n-1-\nu)}{(\nu+1)^2} \frac{i}{4|\mathfrak{w}|^4} (|\mathfrak{w}|^2 (d\mathfrak{w}\,|\,d\mathfrak{w}) - (d\mathfrak{w}\,|\,\mathfrak{w}) \wedge (\mathfrak{w}\,|\,d\mathfrak{w})) \\ &= \frac{1}{2} \sum_{\nu=0}^{n-1} \frac{n-1-\nu}{(\nu+1)^2} \omega(\mathfrak{w})\end{aligned}$$

or

$$A(w) = \frac{1}{2} \sum_{\nu=0}^{n-1} \frac{n-1-\nu}{(\nu+1)^2} \ddot{\omega}(w), \text{ q.e.d.}$$

LEMMA 5.6. *If* $w \in \mathbf{P}(V)$, *define*

$$B(w) = \frac{1}{W(n)} \int_{\alpha \in \mathbf{P}(V)} \log \frac{1}{\|w:\alpha\|} \sum_{\nu=0}^{n-1} \frac{1}{\|w:\alpha\|^{2\nu}} \ddot{\omega}_n(\alpha).$$

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[1] See Weyl [29] p. 128 and Stoll [18] Hilfssatz 1 p. 142.

Then
$$B(w)=\frac{n}{2}\sum_{\nu=0}^{n-1}\frac{1}{(\nu+1)^2}.$$

Proof. Take $\mathfrak{w}\in V$ with $\varrho(\mathfrak{w})=w$. Then

$$B(w)=\frac{1}{2\pi^{n+1}}\int_V e^{-|\mathfrak{a}|^2}\log\frac{1}{\|\mathfrak{w}:\mathfrak{a}\|^2}\sum_{\nu=0}^{n-1}\frac{1}{\|\mathfrak{w}:\mathfrak{a}\|^{2\nu}}\,v_{n+1}(\mathfrak{a}).$$

Using the same substitutions as in the Proof of Lemma 5.2, it follows

$$\begin{aligned}B(w)&=\frac{1}{2}\int_0^\infty\!\!\int_0^\infty e^{-t_0-\dots-t_n}\log\frac{t_0+\dots+t_n}{t_1+\dots+t_n}\sum_{\nu=0}^{n-1}\left(\frac{t_0+\dots+t_n}{t_1+\dots+t_n}\right)^\nu dt_0\dots dt_n\\&=\frac{1}{2}\int_0^1\log\frac{1}{1-\tau}\sum_{\nu=0}^{n-1}\left(\frac{1}{1-\tau}\right)^\nu(1-\tau)^{n-1}\int_0^\infty\!\!\int_0^\infty e^{-s_1-\dots-s_n}(s_1+\dots+s_n)\,ds_1\dots ds_n\\&=\frac{n}{2}\int_0^1\log\frac{1}{1-\tau}\sum_{\nu=0}^{n-1}(1-\tau)^\nu\,dt\\&=\frac{n}{2}\sum_{\nu=0}^{n-1}\frac{1}{(\nu+1)^2},\quad\text{q.e.d.}\end{aligned}$$

PROPOSITION 5.7. *For $w\in\mathbf{P}(V)$ is*

$$\frac{1}{W(n)}\int_{\mathbf{P}(V)}\log\frac{1}{\|w:\alpha\|}\,\Lambda(\alpha)\wedge\ddot{\omega}_n(\alpha)=\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\,\ddot{\omega}_{n-1}.$$

Proof. According to (1) is

$$\begin{aligned}\frac{1}{W(n)}\int_{\mathbf{P}(V)}\log\frac{1}{\|w:\alpha\|}\,\Lambda(\alpha)\wedge\ddot{\omega}_n(\alpha)&=B(w)\ddot{\omega}_{n-1}-\frac{1}{n-1}A(w)\wedge\ddot{\omega}_{n-2}\\&=\frac{n}{2}\sum_{\nu=0}^{n-1}\frac{1}{(\nu+1)^2}\ddot{\omega}_{n-1}-\frac{1}{2(n-1)}\sum_{\nu=0}^{n-1}\frac{n-1-\nu}{(\nu+1)^2}\ddot{\omega}\wedge\ddot{\omega}_{n-2}\\&=\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\ddot{\omega}_{n-1}(w),\quad\text{q.e.d.}\end{aligned}$$

Now, the assumptions 4.2 and 4.7 with $r=n$ and $p=0$ are made. Then

$$\log\frac{1}{\|f:\alpha\|}\,f^*(\Lambda(\alpha))\wedge d^\perp\psi\wedge\chi$$

has a non-negative densitiy along γ and Γ. Hence

$$\frac{1}{W(n)}\int_{\alpha\in\mathbf{P}(V)}\left(\frac{1}{2\pi}\frac{1}{W(n-1)}\int_{\Gamma}\log\frac{1}{\|f:\alpha\|}f^*(\Lambda(\alpha))\wedge d^{\perp}\psi\wedge\chi\right)\ddot{\omega}_n(\alpha)$$

$$=\frac{1}{2\pi}\frac{1}{W(n-1)}\int_{\Gamma}f^*\left(\frac{1}{W(n)}\int_{\alpha\in\mathbf{P}(V)}\log\frac{1}{\|w:\alpha\|}\Lambda(\alpha)\,\ddot{\omega}_n(\alpha)\right)\wedge d^{\perp}\psi\wedge\chi$$

$$=\frac{1}{2\pi}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\frac{1}{2\pi}\frac{1}{W(n-1)}\int_{\Gamma}d^{\perp}\psi\wedge f^*(\ddot{\omega}_{n-1})\wedge\chi.$$

Define the *mean compensation* function $\mu_f(\Gamma)$ and $\mu_f(\gamma)$ by

$$\mu_f(\Gamma)=\frac{1}{2\pi}\frac{1}{W(n-1)}\int_{\Gamma}d^{\perp}\psi\wedge f^*(\ddot{\omega}_{n-1})\wedge\chi,$$

$$\mu_f(\gamma)=\frac{1}{2\pi}\frac{1}{W(n-1)}\int_{\gamma}d^{\perp}\psi\wedge f^*(\ddot{\omega}_{n-1})\wedge\chi.$$

Then these integrals have a non-negative integrand and

$$\frac{1}{W(n)}\int_{\mathbf{P}(V)}m_f(\gamma,\alpha)\,\ddot{\omega}_n(\alpha)=\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\mu_f(\gamma)$$

$$\frac{1}{W(n)}\int_{\mathbf{P}(V)}m_f(\Gamma,\alpha)\,\ddot{\omega}_n(\alpha)=\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\mu_f(\Gamma).$$

THEOREM 5.8. *Suppose that the assumptions 4.2 and 4.7 hold with $r=n$ and $p=0$. Then*

$$T_f(G)=\frac{1}{W(n)}\int_{\mathbf{P}(V)}N_f(G,\alpha)\,\ddot{\omega}_n(\alpha).$$

Moreover, define the mean deficit by

$$\underline{\Delta}_f(G)=\frac{1}{2\pi}\frac{1}{W(n-1)}\int_{G}dd^{\perp}\psi\wedge f^*(\ddot{\omega}_{n-1})\wedge\chi,$$

then
$$\frac{1}{W(n)}\int_{\mathbf{P}(V)}\Delta_f(G,\alpha)\,\ddot{\omega}_n(\alpha)=\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\underline{\Delta}_f(G)$$

and
$$\underline{\Delta}_f(G)=\mu_f(\Gamma)-\mu_f(\gamma).$$

Proof. The map f is open and q-fibering. According to II Proposition 2.2 and its Remark is

$$\frac{1}{W(n)}\int_{\mathbf{P}(V)} N_f(G,\alpha)\,\ddot{\omega}_n(\alpha)$$

$$=\frac{1}{W(n)}\int_{\mathbf{P}(V)}\left(\int_{f^{-1}(\alpha)\cap G}\nu_f\psi\chi\right)\ddot{\omega}_n(\alpha)$$

$$=\frac{1}{W(n)}\int_G \psi f^*(\ddot{\omega}_n)\wedge\chi = T_f(G).$$

Taking the mean value over the First Main Theorem implies

$$T_f(G)+\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}(\mu_f(\Gamma)-\mu_f(\gamma))$$

$$=T_f(G)+\frac{1}{W(n)}\int_{\mathbf{P}(V)}\Delta_f(G,\alpha,)\,\ddot{\omega}_n(\alpha).$$

Stokes Theorem implies

$$\mu_f(\Gamma)-\mu_f(\gamma)=\underline{\Delta}_f(G).$$

Hence
$$\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\underline{\Delta}_f(G)=\frac{1}{W(n)}\int_{\mathbf{P}(V)}\Delta_f(G,\alpha)\,\ddot{\omega}_n(\alpha),\quad\text{q.e.d.}$$

The last formula could also have been proved by exchanging the order of integration, without the use of the First Main Theorem.

THEOREM 5.9. *Suppose that the assumptions 4.2 and 4.7 hold with* $r=n$ *and* $p=0$. *Suppose that* u *is a non-negative continuous form of bidegree* $(1,1)$ *on* $\bar{G}$ *such that* $dd^\perp\psi\leqslant u$ *on* $G-\bar{g}$. *Define*

$$\underline{\Delta}_f(G,u)=\frac{1}{2\pi}\frac{1}{W(n-1)}\int_{G-\bar{g}}u\wedge f^*(\ddot{\omega}_{n-1})\wedge\chi.$$

Define
$$b(G)=\frac{1}{W(n)}\int_{f(G)}\ddot{\omega}_n(\alpha)$$

as the normed volume of the image of G. *Then*

$$0\leqslant(1-b(G))\,T_f(G)\leqslant\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}(\underline{\Delta}_f(G,u)+\mu_f(\gamma)).$$

Proof. Of course $b(G)$ exists, because $f(G)$ is an open non-empty subset of $\mathbf{P}(V)$ and

$$0<b(G)\leqslant\frac{1}{W(n)}\int_{\mathbf{P}(V)}\ddot{\omega}_n=1.$$

Define δ on $\mathbf{P}(V)$ by $\delta(\alpha)=1$ if $\alpha\in f(G)$ and $\delta(\alpha)=0$ if $\alpha\notin f(G)$. If $\alpha\notin f(G)$, then $N_f(G,\alpha)=0$. Hence

$$\frac{1}{W(n)}\int_{\mathbf{P}(V)}\delta(\alpha)\,N_f(G,\alpha)\,\ddot{\omega}(\alpha)=\frac{1}{W(n)}\int_{\mathbf{P}(V)}N_f(G,\alpha)\,\ddot{\omega}_n(\alpha)=T_f(G).$$

Because $dd^{\perp}\psi\leqslant u$, it is $0\leqslant u-dd^{\perp}\psi$ and

$$0\leqslant\log\frac{1}{\|f:\alpha\|}\,(u-dd^{\perp}\psi)\wedge f^*(\Lambda(\alpha))\wedge\chi.$$

Hence
$$\Delta_f(G,\alpha)\leqslant\frac{1}{2\pi}\,\frac{1}{W(n-1)}\int_{G-\bar{g}}\log\frac{1}{\|f:\alpha\|}\,u\wedge f^*(\Lambda(\alpha))\wedge\chi.$$

Hence
$$N_f(G,\alpha)\leqslant T_f(G)+m_f(\gamma,\alpha)+\Delta_f(G,\alpha)$$

implies
$$\begin{aligned}T_f(G)&\leqslant b(G)\,T_f(G)+\frac{1}{W(n)}\int_{\mathbf{P}(V)}\delta(\alpha)\,m_f(\gamma,\alpha)\,\ddot{\omega}_n(\alpha)\\&\quad+\frac{1}{W(n)}\int_{\mathbf{P}(V)}\delta(\alpha)\,\frac{1}{2\pi}\,\frac{1}{W(n-1)}\int_{G-\bar{g}}\log\frac{1}{\|f:\alpha\|}\,f^*(\Lambda(\alpha))\wedge u\wedge\chi\\&\leqslant b(G)\,T_f(G)+\frac{1}{W(n)}\int_{\mathbf{P}(V)}m_f(\gamma,\alpha)\,\ddot{\omega}_n(\alpha)\\&\quad+\frac{1}{2\pi}\,\frac{1}{W_{n-1}}\int_{G-\bar{g}}\left(\frac{1}{W(n)}\int_{\mathbf{P}(V)}\log\frac{1}{\|f:\alpha\|}\,f^*(\Lambda(\alpha))\,\ddot{\omega}_n(\alpha)\right)\wedge u\wedge\chi\end{aligned}$$

or
$$T_f(G)\leqslant b(G)\,T_f(G)+\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\,(\mu_f(\gamma)+\underline{\Delta}_f(G,u)),\ \text{q.e.d.}$$

Let g be fixed, and let $\mathfrak{G}$ be the set of all open subsets G of M such that $G\supset\bar{g}$ and $\bar{G}$ or compact, and such that $\Gamma=\bar{G}-G$ is a boundary manifold of G. Any function $s(G)$ on $\mathfrak{G}$ can be considered as a Moore–Smith sequence, in respect to the set $\mathfrak{G}$ which is directed by $\subseteq$. Define the total deficit by

$$D_f=\overline{\lim_{G\in\mathfrak{G}}}\,\frac{1}{T_f(G)}\,(\mu_f(\gamma)+\Delta_f(G,u)).$$

Because
$$b(G)\to b_f=\frac{1}{W(n)}\int_{f(M)}\ddot{\omega}_n$$

Theorem 5.9 seems to imply

$$(1-b_f)\leqslant\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\cdot D_f. \tag{2}$$

Especially, if $D_f=0$, then f maps onto almost all of $\mathbf{P}(V)$ and if $D_f<\infty$, then the measure of the complement of $f(M)$ is estimated.

However, there are certain shortcomings to this reasoning. First of all, $T_f(G)=0$ may happen. Second, $T_f(G)$, $\mu_f(\gamma)$ and $\underline{\Delta}_f(G, u)$ all depend on ψ respectively u. For each fixed G, there are infinitely many possible choices of ψ and for each choice of ψ infinitely many possible choices of u. Hence estimate (2) becomes meaningful only, if some reasonable, a priori choice of ψ and u is made. Three examples of such choices shall be given here.

1. Example (Stoll [21]). Here, M is a connected complex manifold of dimension m. The form χ is of class C^∞ and positive on M with bidegree $(m-1, m-1)$ such that $d\chi=0$. The set g is open in M with compact closure $\bar{g}$. The boundary $\gamma=\bar{g}-g$ is a boundary manifold of g. Let $\mathfrak{G}$ be the set of all open subsets G of M, such that $\bar{g}\subset G$ and $\bar{G}$ is compact and such that $\Gamma=\bar{G}-G$ is a boundary manifold of G. For $G\in\mathfrak{G}$, the function ψ is uniquely defined on M as a continuous function on M which is of class C^∞ on $\bar{G}-g$ such that

a) $\psi(z)=0$ if $z\in M-G$,

b) $\psi(z)=R(G)=\text{constant}>0$ if $z\in\bar{g}$,

c) $dd^\perp\psi\wedge\chi=0$ on $G-\bar{g}$,

(d)
$$\frac{1}{2\pi}\int_\Gamma d^\perp\psi\wedge\chi=\frac{1}{2\pi}\int_\gamma d^\perp\psi\wedge\chi=1.$$

Here $R(G)>0$ is uniquely defined and $0\leqslant\psi\leqslant R(G)$ on M. The vector space V is $\mathbf{C}^2$; and $f: M-\mathbf{P}(V)=\mathbf{P}$ is a meromorphic, non-constant function on M without points of indetermination. Then $n=1=r$, $q=m-1$, $p=0$ and

$$T_f(G)=\frac{1}{\pi}\int_G\psi f^*(\ddot{\omega})\wedge\chi$$

$$N_f(G,\alpha)=\int_{f^{-1}(\alpha)\cap G}\nu_f(z;\alpha)\,\psi\,\chi$$

$$m_f(\Gamma,\alpha)=\frac{1}{2\pi}\int_\Gamma\log\frac{1}{\|f:\alpha\|}\,d^\perp\psi\wedge\chi,$$

$$m_f(\gamma,a)=\frac{1}{2\pi}\int_\gamma\log\frac{1}{\|f:\alpha\|}\,d^\perp\psi\wedge\chi,$$

$$\Delta_f(G,\alpha)=0,$$

$$\mu_f(G)=\mu_f(\gamma)=1.$$

Define
$$T_f(M)=\sup\{T_f(G)\,|\,G\in\mathfrak{G}\}\leqslant\infty.$$

Because $T_f(G)$ is monotonically increasing in G, it converges

$$T_f(G) \to T_f(M) \quad \text{for } G \to M.$$

Then Theorem 5.9 implies:

THEOREM 5.10. *Under the assumptions of the Examples* 1, *the following statement is true*

$$0 \leqslant (1 - b_f) \leqslant \frac{1}{2\,T_f(M)} \quad \textit{if } T_f(M) < \infty$$

$$b_f = 1 \qquad \textit{if } T_f(M) = \infty$$

Of course, if f has points of indetermination, then $f(M) = \mathbf{P}$ and $b_f = 1$. If $R(M) = \sup\{R(G) \mid G \in \mathfrak{G}\}$ is infinite, then $T_f(M) = \infty$. (Stoll [21], Satz 11.4.) Such manifolds where called of global capacity zero.

THEOREM 5.11. *If the global capacity of M in respect to χ and g is infinite, then every non-constant meromorphic function f on M assumes almost all values on M.*

2. EXAMPLE (Chern [3] for $\varkappa = m$ and $m = n$). Here, V is a complex vector space of dimension $n+1$. Define $M = \mathbf{C}^m$ with $m \geqslant n$. Define $q = m - n$. Choose $\chi = \hat{v}_q$ on $\mathbf{C}^m$, where the euclidean form v_q on $\mathbf{C}^m$ is denoted by $\hat{v}_q$ to distinguish it from the corresponding form on V. Take $r_0 > 0$ and define

$$g = \{\mathfrak{z} \mid |\mathfrak{z}| < r_0\} \qquad \text{in } \mathbf{C}^m,$$

$$\gamma = \{\mathfrak{z} \mid |\mathfrak{z}| = r_0\}.$$

For $r \geqslant r_0$ define

$$G(r) = \{\mathfrak{z} \mid |\mathfrak{z}| < r\},$$

$$\Gamma(r) = \{\mathfrak{z} \mid |\mathfrak{z}| = r\}.$$

Take any non-negative integer $\varkappa$. For $\mathfrak{z} \in \bar{G}(r) - g$ define

$$\psi_\varkappa(\mathfrak{z}) = \frac{1}{2\varkappa - 2}\left(\frac{1}{|\mathfrak{z}|^{2\varkappa-2}} - \frac{1}{r^{2\varkappa-2}}\right) \quad \text{if } \varkappa \neq 1$$

$$\psi_1(\mathfrak{z}) = \log \frac{r}{|\mathfrak{z}|} \quad \text{if } \varkappa = 1.$$

For $\mathfrak{z} \in \mathbf{C}^m - \bar{G}(r)$ define $\psi_\varkappa(\mathfrak{z}) = 0$ for $\mathfrak{z} \in \bar{g}$ define $\psi_\varkappa(\mathfrak{z})$ by the constant value of ψ on $\bar{g} - g$.

Let $f : \mathbf{C}^m \to \mathbf{P}(V)$ be an open holomorphic map. Then f is general of order s for every s in $1 \leqslant s \leqslant n$. The characteristic for this order is

$$T_{s,f}(r) = T_{s,f}(G(r)) = \frac{1}{W(s)} \int_{G(r)} \psi_\varkappa f^*(\ddot{\omega}_s) \wedge \hat{v}_{m-s}.$$

Define

$$A_{s,f}(r) = A_{s,f}(G(r)) = \frac{1}{W(s)} \int_{G(r)} f^*(\ddot{\omega}_s) \wedge \hat{v}_{m-s}.$$

Obviously, $T_{s,f}$ and $A_{s,f}$ are monotonically increasing continuous non-negative functions of r if $r > r_0$. Now

$$f^*(\ddot{\omega}_s) \wedge \hat{v}_{m-s} = g\hat{v}_m,$$

where g is a non-negative C^∞-function on $\mathbf{C}^m$. Let $\Gamma(1)$ be the unit sphere in $\mathbf{C}^m$ considered as a boundary manifold of the unit ball. Let σ be the euclidean volume element of $\Gamma(1)$. Then

$$h_s(t) = \frac{1}{W(s)} \int_{\mathfrak{v} \in \Gamma(1)} g(t\mathfrak{v})\, \sigma(\mathfrak{v})$$

is a continuous function of t in $0 \leqslant t < \infty$. Then

$$A_{s,f}(r) = \int_0^r t^{2m-1} h_s(t)\, dt.$$

Hence $A_{s,f}(r)$ is a differential function of r for $r \geqslant 0$ with the continuous derivative

$$A'_{s,f}(r) = r^{2m-1} h_s(r).$$

If $\varkappa \neq 1$ and $r \geqslant r_0$, then

$$T_{s,f}(r) = \int_0^r \frac{1}{2\varkappa - 2}\left(\frac{1}{t^{2\varkappa-2}} - \frac{1}{r^{2\varkappa-2}}\right) t^{2m-1} h_s(t)\, dt$$

$$+ \frac{1}{2\varkappa-2}\left(\frac{1}{r_0^{2\varkappa-2}} - \frac{1}{r^{2\varkappa-2}}\right) A_{s,f}(r_0).$$

Hence, $T_{s,f}(r)$ is a differential function of r for $r \geqslant r_0$ with the continuous derivative

$$T'_{s,f}(r) = \frac{1}{r^{2\varkappa-1}} \int_{r_0}^r t^{2m-1} h(t)\, dt + \frac{1}{r^{2\varkappa-1}} A_{s,f}(r_0)$$

or

$$T'_{s,f}(r) = \frac{1}{r^{2\varkappa-1}} A_{s,f}(r).$$

Therefore

$$T_{s,f}(r) = \int_{r_0}^r A_{s,f}(t) \frac{dt}{t^{2\varkappa-1}} \quad \text{if } \varkappa \neq 1.$$

If $\varkappa = 1$ and $r \geqslant r_0$, then

$$T_{s,f}(r) = \int_{r_0}^r \log\frac{r}{t}\, t^{2m-1} h(t)\, dt + \log\frac{r}{r_0} A_{s,f}(r_0).$$

Again $T_{s,f}(r)$ is a differentiable function of r if $r \geqslant r_0$ with the continuous derivative

$$T'_{s,f}(r) = \frac{1}{r} \int_{r_0}^r t^{2m-1} h(t)\, dt + \frac{1}{r} A_{s,f}(r_0)$$

or
$$T'_{s,f}(r)=\frac{1}{r}A_{s,f}(r).$$
Therefore, if $r\geqslant r_0$ and $\varkappa\geqslant 0$, then
$$T_{s,f}(r)=\int_{r_0}^{r}A_{s,f}(t)\frac{dt}{t^{2\varkappa-1}}.$$
If $r_0<|\mathfrak{z}|<r$, then
$$dd^{\perp}\psi_{\varkappa}=i\frac{(d\mathfrak{z}\,|\,d\mathfrak{z})}{|\mathfrak{z}|^{2\varkappa}}-i\varkappa\frac{(d\mathfrak{z}\,|\,\mathfrak{z})\wedge(\mathfrak{z}\,|\,d\mathfrak{z})}{|\mathfrak{z}|^{2\varkappa+2}}.$$
Define $u=2|\mathfrak{z}|^{-2\varkappa}\hat{v}$ on $\bar{G}(r)-g$, then $dd^{\perp}\psi_{\varkappa}\leqslant u$ on $\bar{G}(r)-g$. Then $\hat{v}\wedge\hat{v}_q=(q+1)\hat{v}_{q+1}$ implies
$$\underline{\Delta}_f(r)=\underline{\Delta}_f(G(r),u)=\frac{q+1}{\pi W(n-1)}\int_{G(r)-\bar{g}}\frac{1}{|\mathfrak{z}|^{2\varkappa}}f^*(\ddot{\omega}_{n-1})\wedge\hat{v}_{q+1}.$$
Hence
$$\Delta_f(r)=\frac{(q+1)}{\pi}\int_{r}^{r}t^{2m-2\varkappa-1}h_{n-1}(t)\,dt$$
$$=\frac{q+1}{\pi}\int_{r_0}^{r}A'_{n-1,f}(t)\frac{dt}{t^{2\varkappa-1}}.$$
Define
$$b(r)=\frac{1}{W(1)}\int_{f(G(r))}\ddot{\omega}_n$$
and
$$b(\infty)=\frac{1}{W(n)}\int_{f(\mathbf{C}^m)}\ddot{\omega}_n.$$
According to Theorem 5.9 is
$$0\leqslant(1-b(r))\,T_{n,f}(r)\leqslant\frac{1}{2}\sum_{\nu=0}^{n-1}\frac{1}{\nu+1}\left(\frac{q+1}{\pi}\int_{r_0}^{r}A'_{n-1,f}(t)\frac{dt}{t^{2\varkappa-1}}+\mu_f(\gamma)\right),$$
where $\mu_f(\gamma)$ does not depend on r.

This implies immediately the following generalization of a theorem of Chern [3].

THEOREM 5.12. *Suppose that the assumptions of Example 2 are made. Suppose that*
$$T_{n,f}(r)\to\infty\quad\text{for } r\to\infty$$
$$\frac{1}{T_{n,f}(r)}\int_{r_0}^{r}A'_{n-1,f}\frac{dt}{t^{2\varkappa-1}}\to 0\ \text{ for } r\to\infty$$
then $\mathbf{P}(V)-f(\mathbf{C}^m)$ *is a set of measure zero, i.e. f assumes almost every value in* $\mathbf{P}(V)$.

This Theorem is due to S. S. Chern [3] if $q=0$ and $\varkappa=m$. It connects the characteristic of f as a map general of order n with the characteristic of f as a map general of order $n-1$.

3. EXAMPLE. A pair (M, h) is said to be a Levi-manifold if and only if

a) *M is a connected, non-compact, m-dimensional complex manifold.*

b) *h is a non-negative function of class C^∞ on M.*

c) *h is plurisubharmonic, that means $d^\perp dh \geqslant 0$.*

d) *h is proper, that means for every $r > 0$ the set*

$$G(r) = \{z \mid h(z) < r\}$$

has a compact closure.

The Levi-manifold is called *strict* if h is a strict plurisubharmonic function, that is, $d^\perp dh > 0$. The connected, non-compact complex manifold M is said to be *pseudo-convex* if and only if a real function h exists such that (M, h) is a Levi-manifold. The connected complex manifold M is a Stein manifold, if and only if a real function h exists such that (M, h) is a strict Levi-manifold.(1)

Now, suppose that a Levi-manifold (M, h) is given with $\dim M = m$. As before define $G(r) = \{z \mid h(z) < r\}$ if $0 < r \in \mathbf{R}$. Define $\Gamma(r) = \{z \mid h(z) = r\}$ if $0 < r \in \mathbf{R}$. Take $a \in M$. Then $h(M)$ contains every $r \geqslant h(a)$. A set E of measure 0 exists in $\mathbf{R}$ such that $dh \neq 0$ on $\Gamma(r)$ if $r \in \mathbf{R} - E$ and $r \geqslant h(a)$. Take $r_0 > h(a)$ with $r_0 \in \mathbf{R} - E$. Then $g = G(r_0)$ is open, non-empty and relative compact in M with $\gamma = \Gamma(r_0) = \bar{g} - g$ as boundary manifold. Define

$$\mathfrak{R}_0 = \{r \mid r_0 < r \in \mathbf{R} - E\}.$$

For every $r \in \mathfrak{R}_0$ is $G(r)$ an open, relative compact neighborhood of $\bar{g}$ whose boundary $\Gamma(r) = \bar{G}(r) - G(r)$ can be considered as a boundary manifold of $G(r)$.

For s in $0 \leqslant s \leqslant m$ define

$$\chi_s = \frac{1}{s!}\, d^\perp dh \wedge \ldots \wedge d^\perp dh \quad (s\text{-times})$$

(for $s = 0$, this means $\chi_s = 1$). Then χ_s is a non-negative form of bidegree (s, s) and of class C^∞ on M. Obviously, $d\chi_s = 0$.

For $r \geqslant r_0$ define $\psi = \psi_r$ by

$$\psi(z) = \begin{cases} r - r_0 & \text{if } z \in \bar{g} \\ r - h(z) & \text{if } z \in \bar{G}(r) - \bar{g} \\ 0 & \text{if } z \in M - \bar{G}(r). \end{cases}$$

Then ψ is continuous on M and of class C^∞ on $\overline{G(r)} - g$. Moreover, $0 \leqslant \psi \leqslant r - r_0$ on M. Then, on $\overline{G(r)} - g$ is

(1) See Grauert [5], [6] and Narasimhan [13].

$$dd^{\perp}\psi = d^{\perp} dh$$

$$dd^{\perp}\psi \wedge \chi_s = (s+1)\chi_{s+1}.$$

Let V be a complex vector space of dimension $n+1$ with $0 < n \leqslant m$. Define $q = m-n$ Let $f: M \to \mathbf{P}(V)$ be an open and holomorphic map. Then f is general of order s for each s in $1 \leqslant s \leqslant n$. The characteristic of order s for f on $G(r)$ is

$$T_{s,f}(r) = T_{s,f}(G(r)) = \frac{1}{W(s)} \int_{G(r)} \psi_r f^*(\ddot{\omega}_s) \wedge \chi_{m-s}$$

for $r \geqslant r_0$. Define

$$A_{s,f}(r) = A_{s,f}(G(r)) = \frac{1}{W(s)} \int_{G(r)} f^*(\ddot{\omega}_s) \wedge \chi_{m-s}.$$

Obviously, $T_{s,f}$ and $A_{s,f}$ are non-negative, monotonically increasing and continuous function of r if $r \geqslant r_0$. Define $I = \{t \mid r_0 \leqslant t \leqslant r\}$. On $(\bar{G}(r) - g) \times I$ define

$$\lambda(z,t) = \begin{cases} 1 & \text{if } h(z) \leqslant t \\ 0 & \text{if } h(z) > t. \end{cases}$$

Then

$$\{t \mid \lambda(z,t) \neq 0\} = \{t \mid h(z) \leqslant t \leqslant r\}$$

$$\{z \mid \lambda(z,t) \neq 0\} = \overline{G(t)} \quad \text{if } t \in \mathfrak{R}_0.$$

Hence

$$\int_{r_0}^{r} A_{s,f}(t)\,dt = \int_{r_0}^{r} \frac{1}{W(s)} \int_{G(r)-g} \lambda(z,t) f^*(\ddot{\omega}_s) \wedge \chi_{m-s}\,dt + (r-r_0) A_{s,f}(r_0)$$

$$= \frac{1}{W(s)} \int_{G(r)-g} \left(\int_{r_0}^{r} \lambda(z,t)\,dt \right) f^*(\ddot{\omega}_s) \wedge \chi_{m-s} + (r-r_0) A_{s,f}(r_0)$$

$$= \frac{1}{W(s)} \int_{G(r)-g} (r - h(z)) f^*(\ddot{\omega}_0) \wedge \chi_{m-s} + (r-r_0) A_{s,f}(r_0) = T_{s,f}(r).$$

Hence

$$T'_{s,f}(r) = \int_{r_0}^{r} A_{s,f}(t)\,dt$$

is a differentiable function of r for $r \geqslant r_0$ with derivative

$$T_{s,f}(r) = A_{s,f}(r).$$

For $r > r_0$ and $s = n$ and $u = dd^{\perp}\psi_r$ is

$$\underline{\Delta}_f(r) = \underline{\Delta}_f(G(r), u)) = \frac{(q+1)}{2\pi} \frac{1}{W(n-1)} \int_{G(r)-\bar{g}} f^*(\ddot{\omega}_{n-1}) \wedge \chi_{q+1}$$

or
$$\underline{\Delta}_f(r) = \frac{q+1}{2\pi}\,(A_{n-1,f}(r) - A_{n-1,f}(r_0)).$$

Define
$$b(r) = \frac{1}{W(n)} \int_{f(G(r))} \ddot{\omega}_n,$$
$$b(M) = \frac{1}{W(n)} \int_{f(M)} \ddot{\omega}_n.$$

Then $0 \leqslant b(r) \leqslant b(M) \leqslant 1$ and $b(r) \to b(M)$ for $r \to \infty$. According to Theorem 5.9 is

$$0 \leqslant (1 - b(r))\,T_{n,f}(r)$$
$$\leqslant \frac{1}{2} \sum_{\nu=1}^{n-1} \frac{1}{\nu+1} \left(\frac{q+1}{2\pi}\,(A_{n-1,f}(r) - A_{n-1,f}(r_0)) + \mu_f(\gamma) \right),$$

where $\mu_f(\gamma)$ does not depend on r.

Suppose that
$$T_{n,f}(r) \to \infty \quad \text{for } r \to \infty.$$

Define
$$\delta_f = \overline{\lim_{r\to\infty}}\, \frac{T'_{n-1,f}(r)}{T_{n,f}(r)} = \overline{\lim_{r\to\infty}}\, \frac{A_{n-1,f}(r)}{T_{n,f}(r)}.$$

Then the following defect theorem is proved.

THEOREM 5.13. *Suppose that the assumptions of example 3 are made, especially that $T_{n,f}(r) \to \infty$ for $r \to \infty$. Then*

$$0 \leqslant (1 - b(M)) \leqslant \frac{m-n+1}{4\pi} \sum_{\nu=0}^{n-1} \frac{1}{\nu+1}\,\delta_f$$

Especially, f assumes almost every value of $\mathbf{P}(V)$ (i.e., $b(M) = 1$) if

$$\frac{A_{n-1,f}(r)}{T_{n,f}(r)} \to 0 \quad \text{for } r \to \infty. \tag{*}$$

This Theorem generalizes the theorem of Chern [3] to Levi-manifolds for open holomorphic maps into the complex projective space.

In the case $n = 1$,
$$A_{0,f}(r) = M(r) = \int_{G(r)} \chi_m$$

does not depend on f, and is the measure of $G(r)$ in respect to the[1] "semi-Kaehler metric" χ_1 on M and (*) merely requires that $T_{1,f}(r)$ increases stronger than this measure $M(r)$.

[1] Of course, χ_1 is a Kaehler metric only if $\chi_1 > 0$, where upon here only $\chi_1 \geqslant 0$ is required.

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