Faculty of Sciences and Mathematics
University of Niš

Available at: www.pmf.ni.ac.yu/org/filomat/welcome.htm

Filomat **19** (2005), 45 – 51

A GENERAL FIXED POINT THEOREM FOR FOUR WEAKLY COMPATIBLE MAPPINGS SATISFYING AN IMPLICIT RELATION

VALERIU POPA

(2000) AMS Mathematics Subject Classification: 54H25.
Key words and phrases: fixed point, compatible mappings, weakly compatible mappings, implicit relation.

Abstract

In this paper, using a combination of methods used in [5], [14], [15] and [17] the results of [1, Theorems 2.2 and 2.3] and [15, Theorem 3] are improved by removing the assumptions of continuity and reciprocally continuity, relaxing compatibility and compatibility of type (A) to weakly compatibility and replacing the completeness of the space with a set of four alternative conditions for four mappings satisfying an implicit relation.

1. Introduction

Let S and T be self mappings of a metric space (X,d). Jungck [7] defines S and T to be compatible if lim $d(STx_n,TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that lim Sx_n = lim $Tx_n = x$ for some $x \in X$. In 1993, Jungck, Murthy and Cho [9] defined S and T to be compatible of type (A) if lim $d(TSx_n,S^2x_n) = 0$ and lim $d(STx_n,T^2x_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that lim Sx_n = lim $Tx_n = x$ for some $x \in X$. By [9 Ex. 2.1. and Ex. 2.2.] it follows that the notions of compatible mappings and compatible mappings of type (A) are independent.

Recently, Pathak and Khan [12] introduced a new concept of compatible mappings of type (B) as a generalization of compatible mappings of type (A). We say that S and T are compatible of type (B) if

$$\lim d(STx_n,T^2x_n) \le \frac{1}{2}[\lim d(STx_n,St) + \lim d(St,S^2x_n)],$$

$$\lim d(TSx_n,S^2x_n) \le \frac{1}{2}[\lim d(TSx_n,Tt) + \lim d(Tt,T^2x_n)],$$

whenever $\{x_n\}$ is a sequence in X such that lim Sx_n = lim Tx_n = t for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B). By [12, Ex. 2.4] it follows that the implication is not reversible. In [13] the concept of compatible mappings of type (P) was introduced and compared with the concepts of compatible mappings of type (A). S and T are compatible of type (P) if lim $d(S^2x_n,T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that lim Sx_n = lim Tx_n = t for some $t \in X$.

Lemma 1. [7] (resp. [9], [12], [13]). Let S and T be compatible (resp. compatible of type (A), compatible of type (B), compatible of type (P)) self mappings of a metric space (X,d). If Sx = Tx for some $x \in X$, then STx = TSx.

In 1994, Pant [10] introduced the notion of pointwise R – weakly commuting mappings. It is proved in [11] that the notion of pointwise R – weak commutativity is eqivalent to commutativity at coincidence points.

Recently, Jungck [8] (resp. Dhage [2]) defines S and T to be weakly compatible (resp. coincidentally commuting) if Sx = Tx implies STx = TSx. Thus, S and T are weakly compatible or coincidentally commuting mappings if and only if S and T are pointwise R – weakly commuting mappings. It may, however, be noted that the notion of point-wise R-weakly commuting maps (1996) is older than the equivalent notions of weakly compatible maps (1996) and coincidently commuting maps (1999).

Remark 1. By Lemma 1 it follows that every compatible (compatible of type (A), compatible of type (B), compatible of type (P)) pair of mappings are weakly compatible.

The following example from [15] is an example of weakly compatible pair of mappings which is not compatible (compatible of type (A), compatible of type (P)).

Let X = [2,20] with the usual metric. Define

$$Tx = \begin{cases} x & \text{if } x = 2 \\ 12 + x & \text{if } 2 < x \le 5 \\ x - 3 & \text{if } 5 < x \le 20 \end{cases} ;\ Sx = \begin{cases} 2 & \text{if } x \in \{2\} \cup (5,20] \\ 8 & \text{if } 2 < x \le 5 \end{cases}.$$

S and T are weakly compatible since they commute at their coincidence point. S and T are not compatible of type (B).

Let us consider a decreasing sequence $\{x_n\}$ such that lim x_n = 5. Then lim Tx_n = 2, lim Sx_n = 2, lim STx_n = 8, lim T^2x_n = 14, lim S^2x_n = 2. Then

$$\lim d(STx_n,T^2x_n) = 6 > \frac{1}{2}[\lim d(STx_n,St)+\lim d(St,S^2x_n)] = \frac{1}{2}(6 + 0) = 3.$$

The following theorems are proved in [1].

Theorem 1 [1]. Let {S,I} and {T,J} be compatible pairs of a complete metric space (X,d) into itself such that

(a) $T(X) \subset I(X)$, $S(X) \subset J(X)$,

(b) For all x, y in X, with a, b $\ge$ 0, a + b < 1, either

(1) $$d(Sx,Ty) \le \frac{a[d(Ix,Sx)d(Ix,Ty) + d(Jy,Ty)d(Jy,Sx)]}{d(Ix,Sx) + d(Jy,Ty)} + bd(Ix,Jy)$$ whenever $d(Ix,Sx) + d(Jy,Ty) \neq 0$, or

(1') d (Sx,Ty) = 0 whenever d (Ix,Sx) + d (Jy,Ty) = 0.

If one of S, T, I and J is continuous then S, T, I and J have a common fixed point z in X.

Further, z is the unique common fixed point of S and I, and T and J.

Theorem 2 [1]. Let {S,I} and {T,J} be compatible of type (A) pairs of mappings of a complete metric space (X,d) into itself such that condition (a) and (b) of Theorem 1 are satisfied. If one of S, T, I and J is continuous then S, T, I and J have a common fixed point z in X. Further, z is the unique common fixed point of S and I, and T and J.

2. Implicit relations

Let K_6 be the set of all real continuous functions $F(x_1,\ldots,x_6)$: $R_+^6 \to R$ with $t_3 + t_4 \neq 0$ satisfying the following conditions:

(K_1) F is decreasing in variables t_5 and t_6,

(K_2) there exists $h \in [0,1)$ such that for every $u,v \geq 0$ with

(K_a) $F(u,v,v,u,u+v,0) \leq 0$ or

(K_b) $F(u,v,u,v,0,u+v) \leq 0$

we have $u \leq hv$.

Ex. 1. $F(t_1,\ldots,t_6) = t_1 - \dfrac{a[t_3t_5 + t_4t_6]}{t_3 + t_4} - bt_2$, where $0 \leq a + b < 1$.

(K_1) Obviously.

(K_2) Let $F(u,v,v,u,u+v,0) = u - \dfrac{av(u+v)}{u+v} - bv \leq 0$.

Then $u \leq hv$, where $h = a + b < 1$.

Similarly, $F(u,v,u,v,0,u+v) \leq 0$ implies $u \leq hv$.

Ex. 2. $F(t_1,\ldots,t_6) = t_1 - \dfrac{a[t_3^2 + t_4^2]}{t_3 + t_4} - bt_2 - ct_5t_6 \leq 0$, where $0 < c + b < 1$, $a > 0$ and $c \geq 0$.

(K_1) Obviously.

(K_2) Let $F(u,v,v,u,u+v,0) = u - \dfrac{a(u^2 + v^2)}{u+v} - bv \leq 0$, which implies

$$u^2(1-a) + uv(1-b) - (a+b)v^2 \leq 0.$$

If $v = 0$, then $u = 0$, a contradiction. Then $f(t) = t^2(1-a) + t(1-b) - (a+b) \leq 0$, where $t = \dfrac{u}{v}$, $f(0) < 0$ and $f(1) = 2[1-(a+b)] > 0$. Let $h \in (0,1)$ be the root of the equation $f(t) = 0$, then $f(t) < 0$ for $t \leq h$ and thus $u \leq hv$.

Similarly, $F(u,v,u,v,0,u+v) \leq 0$ implies $u \leq hv$.

Ex. 3. $F(t_1,\ldots,t_6) = t_1 - \dfrac{ct_3t_4 + bt_5t_6}{t_3 + t_4}$, where $1 \leq c < 2$.

(K_1) Obviously.

(K_2) Let $u > 0$ and $F(u,v,v,u,u+v,0) = u - \dfrac{cuv}{u+v} \leq 0$. Then $u^2 + uv - cuv \leq 0$ which implies $u \leq hv$, where $0 \leq h = c - 1 < 1$.

Similarly, $F(u,v,u,v,0,u+v) \leq 0$ implies $u \leq hv$.

If $u = 0$ and $v > 0$ then $u \leq hv$.

Ex. 4. $F(t_1,\dots,t_6) = t_1^3 + t_1^2 + t_1 - \frac{(bt_5 + ct_6)^2}{t_3 + t_4}$ where $1 \leq b < \frac{\sqrt{2}}{2}$, $0 \leq c < \frac{\sqrt{2}}{2}$.

(K_1) Obviously.

(K_2) $F(u,v,v,u,u+v,0) = u^3 + u^2 + u - \frac{b^2(u+v)^2}{u+v} \leq 0$ which implies $u - b^2(u + v) \leq 0$, hence $u \leq h_1v$, where $h_1 = \frac{b^2}{1-b^2} < 1$.

Similarly, $F(u,v,u,v,0,u + v) \leq 0$ implies $u \leq h_2v$, where $h_2 = \frac{c^2}{1-c^2} < 1$. Then $u \leq hv$, where $h = \max\{h_1,h_2\}$.

Other examples are presented in [15].

S and T are said to be reciprocally continuous [11] if $\lim TSx_n = Tt$ and $\lim STx_n = St$ whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ for some $t \in X$. If S and T are both continuous then they are obviously reciprocally continuous, but the converse is not true. There exists reciprocally continuous mappings S and T such that S and T are non-continuous [11]. The following theorem is proved in [15].

Theorem 3 [15]. Let (S,I) and (T,J) a weakly compatible pair of self-mappings on a complete metric space (X,d) such that

(a) $S(X) \subset J(X)$, $T(X) \subset I(X)$;

(b) $F(d(Sx,Ty),d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),d(Jy,Sx)) \leq 0$

for all $x, y \in X$ with $d(Ix,Sx) + d(Jy,Ty) \neq 0$, where $F \in K_6$, or

(b') $d(Sx,Ty) = 0$ if $d(Ix,Sx) + d(Jy,Ty) = 0$;

(c) (S,I) and (T,J) is a compatible pair of reciprocally continuous mappings. Then S, T, I and J have a unique common fixed point.

In this paper, using a combination of methods used in [5], [14], [15] and [17] the results of Theorems 1 – 3 are improved by removing the assumptions of continuity and reciprocally continuity, relaxing compatibility and compatibility of type (A) to weakly compatibility and replacing the completeness of the space with a set of four alternative conditions for four mappings satisfying an implicit relation.

3. Main results

Theorem 4. Let S, T, I and J be self mappings of a metric space (X,d) such that

(a) $S(X) \subset J(X)$ and $T(X) \subset I(X)$;

(b) The pairs (S,I) and (T,J) are weakly compatible;

(c) $F(d(Sx,Ty),d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),d(Jy,Sx)) \leq 0$

for all $x, y \in X$ with $d(Ix,Sx) + d(Jy,Ty) \neq 0$, where $F \in K_6$, or

(c') $d(Sx,Ty) = 0$ if $d(Ix,Sx) + d(Jy,Ty) = 0$.

If one of S(X), T(X), I(X) and J(X) is a complete subspace of X, then S, T, I and J have a common fixed point u. Further, u is the unique common fixed point of S and I, and T and J.

Proof. Let x_0 be an arbitrary point in X. Then since (a) holds, we can inductively define the sequences $\{x_n\}$ and $\{y_n\}$ by

$$y_{2n} = Sx_{2n} = Jx_{2n+1}; \quad y_{2n+1} = Tx_{2n+1} = Ix_{2n+2} \text{ for } n = 0, 1, 2, ...$$

(i). If

$$d(Ix_{2n},Sx_{2n}) + d(Jx_{2n+1},Tx_{2n+1}) \neq 0 \text{ and } d(Ix_{2n},Sx_{2n}) + d(Jx_{2n-1},Tx_{2n-1}) \neq 0$$

for $n = 0, 1, 2, ...$ as in the proof of [15, Theorem 3] it follows that $\{y_n\}$ is a Cauchy sequence.

Now suppose that J(X) is a complete subspace of X, then the sequence $y_{2n} = Jx_{2n+1}= Sx_{2n}$ is a Cauchy sequence in J(X) and hence has a limit u. Let $v\in J^{-1}u$, then Jv = u. Since $\{y_{2n}\}$ is convergent and $y_{2n+1} = Tx_{2n+1} = Ix_{2n+2}$ also converges to u. To prove that u=Tv, assume that d(u,Tv)>0. Setting $x =x_{2n}$ and y=v in (c) we have

$$F(d(Sx_{2n},Tv),d(Ix_{2n},Jv),d(Ix_{2n},Sx_{2n}),d(Jv,Tv),d(Ix_{2n},Tv),d(Jv,Sx_{2n})) \leq 0$$

and letting n tend to infinity we get

$$F(d(u,Tv),0,0,d(u,Tv),d(u,Tv),0) \leq 0$$

which implies by (K_a) that d (u,Tv) = 0, a contradiction. Then u = Tv.

Since $T(X) \subset I(X)$, u = Tv implies $u \in I(X)$. Let $w \in I^{-1}u$, then Iw = u. Now using the earlier arguments one can show that Sw = u. Therefore, u = Jv = Tv = Iw = Sw. If one assumes that I(X) is complete then analogous arguments establish the earlier conclusion. The remaining two cases are essentially the same as the previous cases. Indeed, if S(X) is complete then by (a) $u \in S(X)\subset J(X)$. Similarly, if T(X) is complete, $u \in T(X)\subset I(X)$.

By u = Jv = Tv and weak compatibility of (J,T) we have

$$Tu = TJv = JTv = Ju.$$

By u = Iw = Sw and weak compatibility of (I,S) we have

$$Su =SIw = ISw = Iu.$$

Then d (Iw,Sw) + d (Ju,Tu) = 0 and by (c') we have d (Sw,Tu)=0, i. e. d(u,Tu)=0, which implies u = Tu. Similarly one can show that u = Su. Then

$$u = Tu = Ju = Su = Iu.$$

Let z be other fixed point of T and J. Then

$$d (Su,Iu) + d (Tz,Jz) = d (u,u) + d (z,z) = 0.$$

Therefore, by (c') we have d (Su,Tz) = d (u,z) = 0. Hence, u = z. Similarly, u is the unique fixed point of S and I.

(ii). If

$$d(Ix_{2n},Sx_{2n}) + d(Jx_{2n+1},Tx_{2n+1}) = 0$$

and

$$d(Ix_{2n},Sx_{2n}) + d(Jx_{2n-1},Tx_{2n-1}) = 0$$

for some n = 1, 2, ... the proof is identical with the proof of case (ii) by [15, Theorem 3]

Corollary 1. Theorems 1 and 2.

Proof. It is follows from Theorem 4 and Ex. 1 .

Corollary 2. Theorem 3.

Proof. It is follows from Theorem 4.

If I = J = id there we obtain the following theorem.

Theorem 5. Let S and T self mappings of a complete metric space such that for all x, y in X either

$$F(d(Sx,Ty),d(x,y),d(x,Sx),d(y,Ty),d(x,Ty),d(y,Sx)) \leq 0$$

if d (x,Sx) + d(y,Ty) $\neq$ 0 or d(Sx,Ty) = 0 otherwise. Then S and T have a unique common fixed point.

Remark 2. By Theorem 5 and Ex. 2 for b = c = 0 we obtain Theorem 2 of [3]. By Theorem 5 and Ex. 3 we obtain Theorem 3 of [4] because the condition (K_2) is not necessary in the proof.

References

[1] Z. Ahmad and A. J. Asad, Common fixed point of compatible and (A) type compatible mappings, Math. Notae, 4 (1999 - 2002), 67 – 74.

[2] B. C. Dhage, On common fixed points and coincidentally commuting mappings in D – metric spaces, Indian J. Pure Appl. Math. 30 (4) (1999), 395 – 406.

[3] B. Fisher, Common fixed point and constant mappings satisfying a rational inequality, Math. Seminar Notes 6 (1978), 29 – 35.

[4] B. Fisher, Theorems of mappings satisfying a rational inequality, Comment. Math. Univ. Carolinae 19 (1978), 37 – 44.

[5] M. Imdad, S. Kumar and M. S. Khan, Remarks on some fixed point theorems satisfying implicit relations, Radovi Mat. 11 (2002), 135 – 143.

[6] G. S. Jeong and B. E. Rhoodes, Some remarks for improving fixed point theorems for more that two mappings, Indian J. Pure Appl. Math. 28 (9) (1997), 1177 – 1196.

[7] G. Jungck, Compatible mappings and common fixed points, Internat J. Math. Math. Sci. 9 (1986), 771 – 779.

[8] G. Jungck, Compatible fixed point for non-continuous non-self mappings on non-numeric spaces, Far East J.Math. Sci. 4(2)(1996),192–212.

[9] G. Jungck, P. P. Murthy and Y. J, Cho, Compatible mappings of type (A) and common fixed points, Math. Japonica, 36 (1993), 381 – 390.

[10] R. P. Pant, Common fixed points for non-commuting mappings, J. Math. Anal. Appl. 188 (1994), 436-440.

[11] R. P. Pant, Common fixed points for four mappings, Bull. Calcutta Math. Soc. 9 (1998), 281 – 286.

[12] H. K. Pathak and M. S. Khan, Compatible mappings of type (B) and common fixed point theorems of Gregus type, Czechoslovak Math. J., 45 (120) (1995), 685 – 698.

[13] H. K. Pathak, Y. J. Cho, S. M. Kang and B. S. Lee, Fixed point theorems for compatible mappings of type (P) and application to dynamic programming, Le Matematiche (Fasc. I), 50 (1995), 15 – 33.

[14] V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, Demonstratio Mathematica, 32 (1999), 157 – 163.

[15] V. Popa, Some fixed point theorems for weakly compatible mappings, Radovi Mat. 10 (2001), 245 – 252.

[16] V. Popa, Coincidence and fixed point theorems for noncontinuous hybrid contractions, Nonlinear Analysis Forum, 7 (2) (2002), 153 – 158.

[17] S. L. Singh, V. Chadha and S. N. Mishra, Remarks on recent fixed point theorems for compatible mappings, Internat J. Math. Math. Sci. 19, 4 (1996), 801 – 804.

Department of Mathematics
University of Bacău
600114 Bacău, RUMANIA
E-mail: vpopa@ub.ro