

A general framework for the composition of quantum homomorphic encryption & quantum error correction

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Two essential primitives for universal, cloud-based quantum computation with security based on the laws of quantum mechanics, are quantum homomorphic encryption with information-theoretic security and quantum error correction. The former enables information-theoretic security of outsourced quantum computation, while the latter allows reliable and scalable quantum computations in the presence of errors. Previously these ingredients have been considered in isolation from one another. By establishing group-theoretic requirements that these two ingredients must satisfy, we provide a general framework for composing them. Namely, a quantum homomorphic encryption scheme enhanced with quantum error correction can directly inherit its properties from its constituent quantum homomorphic encryption and quantum error correction schemes. We apply our framework to both discrete- and continuous-variable models for quantum computation, such as Pauli-key and permutation-key encryptions in the qubit model, and displacement-key encryptions in a continuous-variable model based on Gottesman-Kitaev-Preskill codes.

I. INTRODUCTION

Future quantum computing infrastructure is likely to be accessible to most end users via cloud-based services, owing to the high infrastructure cost. Similarly, the high-value of the applications under demand makes security considerations paramount [1]. Quantum homomorphic encryption (QHE) is a class of cryptographic protocols allowing quantum computations to be performed on encrypted data without the server performing the computation having to first decrypt it, as is the case with conventional computational outsourcing. However, quantum computing differs from classical computing in that quantum error correction (QEC) is necessary to achieve fault-tolerance [2] under realistic noise processes. Here we reconcile these two formalisms, providing group-theoretic definitions for both QHE and QEC; if the chosen protocols satisfy these requirements, the security characteristics of the chosen QHE scheme carry across to the quantum error-corrected implementation.

A salient feature of quantum cryptographic protocols is that they can be imbued with security based solely on the fundamental laws of quantum mechanics. In contrast with traditional cryptographic schemes whose security depends on the assumed hardness of certain computational problems, quantum protocols with information-theoretic (IT) security remain provably robust against future developments in quantum computation or classical algorithms.

Quantum cryptographic protocols are typically studied in ideal scenarios where no environmental noise is

present. To combat environmental noise in these quantum protocols it is natural to consider their integration with quantum error correction (QEC) codes. However, for the QEC-enhanced quantum cryptographic scheme to remain secure, leakage of information about the encrypted data during the QEC process must be limited.

In the classical realm, a key attraction of homomorphic encryption (HE) [3] is that it serves as a cryptographic primitive from which a plethora of other cryptographic protocols can be derived [4]. Given the potential of quantum generalizations of HE to usher in a plethora of quantum cryptographic protocols, QHE schemes have been studied under both information-theoretic security [5–11] or with computational hardness assumptions [12, 13].

In this paper, we provide a systematic framework for integrating QEC and QHE. We begin by presenting an abstract mathematical framework for QHE and QEC from which we derive group-theoretic requirements for composability to be possible in Sec. II. We then present several concrete examples of satisfying QHE and QEC protocols in both discrete- (qubit) and continuous-variable (e.g optical) settings to illustrate the formalism in Sec. III. In the qubit model we demonstrate the utility of our framework by considering Pauli-key (Sec. III A 1) and permutation-key [7] encryptions (Sec. III A 3), and in the continuous-variable model we consider displacement-key [11] encryption (Sec. III B) applied to squeezed state encryption (Sec. III B 1) and Gottesman-Kitaev-Preskill (GKP) codes [14] (Sec. III B 2). In all cases we assume minimal quantum resources on the client side, limited to state preparation, teleportation and measurement, while the server is assumed to have arbitrary quantum resources. We conclude in Sec. IV. The key attraction of our result is that it does not require the reader to know the inner

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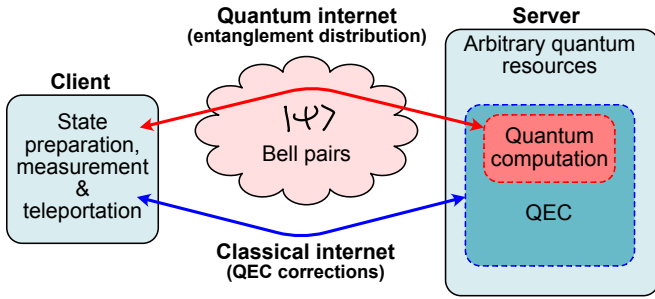


Figure 1: Communication model for outsourced quantum computation. The two parties Alice (the client) and Bob (the server) communicate both classical (blue) and quantum (red) information, where classical information is not secure. The classical communication that enables Bob to perform quantum error correction on the server side is unencrypted and should not disclose computational information. Alice is assumed to have minimal quantum resources, limited to state preparation, measurement and quantum state teleportation. Bob possesses arbitrary quantum computing infrastructure.

workings of either QHE or QEC. Simply speaking, any QEC-enhanced QHE scheme can inherit the properties of both QEC and QHE as long as certain conditions which we describe in Sec. II are satisfied.

QHE schemes enhanced with QEC can become applicable in a future era when the quantum internet becomes available [1]. With the advent of the quantum internet, entanglement distribution will be readily provided, and can be used to mediate quantum communication via quantum state teleportation [15]. Clients for cloud-based quantum computation need only use the minimal necessary hardware, such as single-qubit state preparation and the circuitry for state teleportation. In contrast, the server has arbitrary quantum capabilities, as shown in Fig. 1.

II. GENERAL GROUP THEORETIC CONSTRUCTION

Here we present generic group-theoretic definitions for both QHE and QEC such that when satisfied they may be directly composed with one another, enabling homomorphically encrypted, error-corrected quantum computation.

A. Quantum homomorphic encryption

The client, Alice, prepares the quantum state ρ which she would like to encrypt and subsequently outsource to the cloud for the execution of some computation. Before Alice sends her state to the server, she encrypts her state using the encryption process, $\text{Encr}_\kappa(\cdot)$, where $\kappa \in K$ denotes her chosen private key, chosen uniformly at random from a set of keys given by K . From Alice's perspective, who knows the key κ , her state is,

$$\text{Encr}_\kappa(\rho). \quad (1)$$

The server, Bob, or indeed any eavesdropper, who does not know Alice's private key, instead perceives a state mixed over all possible keys Alice may have chosen,

$$\text{Encr}(\rho) = \frac{1}{|K|} \sum_{\kappa \in K} \text{Encr}_\kappa(\rho). \quad (2)$$

In the case of CV encryptions, the discrete sum could take the form of a continuous integral, and the integrand weighted by an appropriate probability distribution.

We can now quantify the security of Alice's information from the server now in the following way. Take \mathcal{S} as the set of Alice's input density matrices. Then the maximal trace-distance between encrypted states is,

$$\Delta = \max_{\rho, \rho' \in \mathcal{S}} \frac{1}{2} \|\text{Encr}(\rho) - \text{Encr}(\rho')\|_1, \quad (3)$$

where $\|\cdot\|_1$ denotes the trace norm. The closer Δ is to zero, the less distinguishable Alice's distinct inputs are to the server, and the more secure her encryption.

Alice would like to outsource computations $\mathcal{C} \in \mathcal{A}$, where $\mathcal{A} : \mathcal{S} \rightarrow \mathcal{S}$ is a group of unitary channels under composition. The server's task is to implement a quantum channel that corresponds to \mathcal{C} on the encrypted space, without knowledge of the key κ . Now let $\mathcal{S}_\kappa = \{\text{Encr}_\kappa(\rho) : \rho \in \mathcal{S}\}$ denote the set of quantum states encrypted by κ . Formally, the server implements the computation described by a quantum channel $\bar{\mathcal{C}}$ on density matrices in $\bar{\mathcal{S}}$, where

$$\bar{\mathcal{S}} = \bigcup_{\kappa \in K} \{\bar{\rho} \in \mathcal{S}_\kappa\} \quad (4)$$

denotes the set of all possible encrypted quantum states. From the client's perspective, the quantum state becomes,

$$\bar{\mathcal{C}} \circ \text{Encr}_\kappa(\rho). \quad (5)$$

Since the server has no knowledge of κ , the channel $\bar{\mathcal{C}}$ must be independent of κ . Let us assume that there exists a $\mathcal{C}_\kappa \in \mathcal{A}$ such that

$$\boxed{\mathcal{C}} \boxed{\mathcal{C}_\kappa} \boxed{\text{Encr}_\kappa} = \boxed{\text{Encr}_\kappa} \boxed{\bar{\mathcal{C}}}. \quad (6)$$

In the special case where \mathcal{C}_κ is the identity map, we obtain

$$\boxed{\mathcal{C}} \boxed{\text{Encr}_\kappa} = \boxed{\text{Encr}_\kappa} \boxed{\bar{\mathcal{C}}}. \quad (7)$$

The scheme is termed as quantum homomorphic encryption, because a group homomorphism φ relates $\bar{\mathcal{C}}$ to \mathcal{C} . Namely, there is a map φ such that for every $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{A}$, the computations $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_1 \circ \mathcal{C}_2$ satisfy the algebraic relationship,

$$\varphi(\mathcal{C}_1 \circ \mathcal{C}_2) = \varphi(\mathcal{C}_1) \circ \varphi(\mathcal{C}_2). \quad (8)$$

Here, $\varphi(\mathcal{C}_1), \varphi(\mathcal{C}_2)$ and $\varphi(\mathcal{C}_1 \circ \mathcal{C}_2)$ are what the server applies on $\bar{\mathcal{S}}$ to implement $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_1 \circ \mathcal{C}_2$ respectively. For every $\mathcal{C} \in \mathcal{A}$, we write $\bar{\mathcal{C}} = \varphi(\mathcal{C})$. The group homomorphism φ allows the server to compute any sequence $(\mathcal{C}_1, \dots, \mathcal{C}_s)$ using the sequence $(\varphi(\mathcal{C}_1), \dots, \varphi(\mathcal{C}_s))$. We now summarize the notation of homomorphism in QHE.

Definition 1 (Homomorphism in QHE). *Let \mathcal{A} and $\overline{\mathcal{A}}$ be a group of unitary channels that acts on \mathcal{S} and $\overline{\mathcal{S}}$ respectively. The map $\varphi : \mathcal{A} \rightarrow \overline{\mathcal{A}}$ is a group homomorphism if for every $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{A}$, we have $\varphi(\mathcal{C}_1 \circ \mathcal{C}_2) = \varphi(\mathcal{C}_1) \circ \varphi(\mathcal{C}_2)$.*

The server upon completing the computation on the cipherspace \mathcal{S} , returns the quantum state to the client. The client then decrypts this state with the decryption channel $\text{Decr}_{\kappa, \mathcal{C}}(\cdot)$. The group homomorphism φ that the server used before is only meaningful if after decryption, Alice recovers the correct state $\mathcal{C}(\rho)$. For the decryption to return the correct state, it suffices to require that

$$\boxed{\text{Encr}_{\kappa}} \boxed{\text{Decr}_{\kappa}} = \boxed{\mathcal{I}}, \quad (9)$$

where $\text{Decr}_{\kappa} = \text{Decr}_{\kappa, \mathcal{I}}$. Whenever Encr_{κ} is a unitary channel, the condition (9) is also equivalent to the requirement that $\text{Decr}_{\kappa} = \text{Encr}_{\kappa}^{\dagger}$.

The goal is that after decryption, Alice receives the correct state, no matter which secret key κ she uses, and which computation \mathcal{C} was evaluated. The following lemma tells us what decryption operation achieves this.

Lemma 1 (Correctness). *Suppose that*

$$\boxed{\text{Decr}_{\kappa, \mathcal{C}}} = \boxed{\text{Decr}_{\kappa}} \boxed{\mathcal{C}_{\kappa}^{\dagger}}. \quad (10)$$

Then

$$\boxed{\text{Encr}_{\kappa}} \boxed{\overline{\mathcal{C}}} \boxed{\text{Decr}_{\kappa, \mathcal{C}}} = \boxed{\mathcal{C}} \quad (11)$$

for every $\kappa \in K$ and $\mathcal{C} \in \mathcal{A}$.

Proof. Using (9) and (6), we find that

$$\boxed{\text{Encr}_{\kappa}} \boxed{\overline{\mathcal{C}}} \boxed{\text{Decr}_{\kappa}} = \boxed{\mathcal{C}} \boxed{\mathcal{C}_{\kappa}} \boxed{\text{Encr}_{\kappa}} \boxed{\text{Decr}_{\kappa}} = \boxed{\mathcal{C}} \boxed{\mathcal{C}_{\kappa}}, \quad (12)$$

from which the result can be deduced. \square

The proof of Lemma 1 shows that the structure of decryption channel can be derived using the structure of the encryption channel and the delegated computation.

Apart from the homomorphism of φ in a QHE scheme, the scheme also must be secure, correct and compact:

1. **Security:** We say that the scheme is ϵ -secure if the maximal trace-distance, Eq. (3), between encrypted states satisfies $\Delta \leq \epsilon$.
2. **Correctness:** The QHE scheme is correct if for every key $\kappa \in K$ and every computation $\mathcal{C} \in \mathcal{A}$, there is an encryption channel Encr_{κ} and decryption channel $\text{Decr}_{\kappa, \mathcal{C}}$ such that Eq. (11) holds.
3. **Compactness:** The QHE scheme is compact if the circuit complexity of the decryption channel is at most a polynomial in the key length.

Here, *compactness* is a condition that sometimes ensures that Alice's decryption $\text{Decr}_{\kappa, \mathcal{C}}(\cdot)$ is much easier to perform than the actual computation \mathcal{C} . In the most extreme case when $\text{Decr}_{\kappa, \mathcal{C}} = \text{Decr}_{\kappa}$, the decryption is independent of \mathcal{C} and compactness trivially holds.

B. Composability of QHE schemes

We may describe a QHE scheme using its input states \mathcal{S} , keys K , encryption maps $\text{Encr}_K = \{\text{Encr}_{\kappa} : \kappa \in K\}$, computations $\mathcal{A} : \mathcal{S} \rightarrow \mathcal{S}$, and group homomorphism φ . The decryption then depends only on the encryption operation and the delegated computation. From this perspective, the tuple,

$$(\mathcal{S}, K, \mathcal{A}, \varphi, \text{Encr}_K), \quad (13)$$

represents a QHE scheme. Now suppose that for $i = 1, \dots, m$, there are QHE schemes described by,

$$Q_i = (\mathcal{S}_i, K_i, \mathcal{A}_i, \varphi_i, \text{Encr}_{K_i}^{(i)}). \quad (14)$$

We say that the schemes Q_1, \dots, Q_m can be composed into a QHE scheme Q^* if:

1. There is a QHE scheme Q^* with set of input states $\mathcal{S}^* = \bigotimes_{i=1}^m \mathcal{S}_i$, set of encryption keys $K^* = \{(\kappa_1, \dots, \kappa_m) : \kappa_i \in K_i, i = 1, \dots, m\}$ and encryption maps Encr_{κ}^* of the form

$$\boxed{\text{Encr}_{\kappa}^*} = \begin{array}{c} \boxed{\text{Encr}_{\kappa_1}^{(1)}} \\ \vdots \\ \boxed{\text{Encr}_{\kappa_m}^{(m)}} \end{array}. \quad (15)$$

2. The set of computations \mathcal{A}^* that can be delegated in the QHE scheme Q^* contains every computation in $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_m$. Moreover, for every computation \mathcal{C}^* in \mathcal{A}^* , there exists computations $\mathcal{C}_i \in \mathcal{A}_i$ and a computation $\gamma \in \mathcal{A}^*$ such that

$$\boxed{\mathcal{C}^*} = \begin{array}{c} \boxed{\mathcal{C}_1} \\ \vdots \\ \boxed{\mathcal{C}_m} \end{array} \boxed{\gamma}. \quad (16)$$

Such a γ always exists if \mathcal{A}^* is a group under composition of the computation channels. The set of allowed computations \mathcal{A}^* can contain computations that are not necessarily in $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_m$.

3. There is a homomorphism $\varphi^* : \mathcal{A}^* \rightarrow \overline{\mathcal{A}^*}$, such that for every sequence of computations $(\mathcal{C}_1^*, \dots, \mathcal{C}_k^*)$ with $\mathcal{C}_i^* \in \mathcal{A}^*$, we have

$$\varphi^*(\mathcal{C}_1^* \circ \dots \circ \mathcal{C}_k^*) = \varphi^*(\mathcal{C}_1^*) \circ \dots \circ \varphi^*(\mathcal{C}_k^*). \quad (17)$$

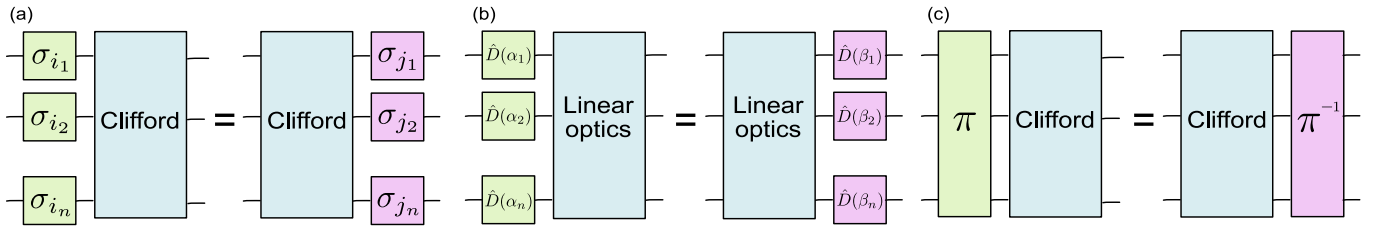


Figure 2: Commutation properties of the encryption (green) and decryption (purple) operations through quantum computations (cyan). (a) Random single-qubit Pauli operators (σ_i) commute through Clifford circuits to form an equivalent set of single-qubit Pauli operators at the output (σ_j). (b) Random single-mode displacement operators $\hat{D}(\alpha_i)$ similarly commute through linear optics circuits to form an equivalent set of displacements, $\hat{D}(\beta_i)$, at the output. (c) A random permutation π , which applies to the subsets of rails forming the blocks shown in Fig. 4, commutes through a Clifford circuit. Here the permutation. (a,b,c) In all cases the output operators can be efficiently classically computed from the input operators.

4. If Q_i is ϵ_i -secure for every $i = 1, \dots, m$, then there is a function f such that Q^* is $f(\epsilon_1, \dots, \epsilon_m)$ -secure.
5. Q^* is correct if Q_1, \dots, Q_m are correct.
6. Q^* is compact if Q_1, \dots, Q_m are compact.

and for any key $\kappa = (\kappa_1, \dots, \kappa_m) \in K^*$, the decryption operation of Q^* is

We denote the composed QHE scheme as a tuple

$$Q^* = (\mathcal{S}^*, K^*, \mathcal{A}^*, \varphi^*, \text{Encr}_{K^*}^*). \quad (18)$$

In Theorem 2, we make explicit what the decryption operation can be, provided that (1) \mathcal{A}^* is a group, and (2) all encryption operations in $\text{Encr}_{K^*}^*$ belong to \mathcal{A}^* .

Theorem 2. Let $Q^* = (\mathcal{S}^*, K^*, \mathcal{A}^*, \varphi^*, \text{Encr}_{K^*}^*)$ be composed from the QHE schemes Q_1, \dots, Q_m . Suppose that \mathcal{A}^* is a group under composition, and all encryption operations in $\text{Encr}_{K^*}^*$ belong to \mathcal{A}^* . For every $j = 1, \dots, m$, let \mathcal{C}_{κ_j} be a feasible computation for Q_j . Then for any delegated computation $\mathcal{C}^* \in \mathcal{A}^*$ that can be written as

$$\begin{array}{c} \boxed{\mathcal{C}_{\kappa_1}} \\ \vdots \\ \boxed{\mathcal{C}_{\kappa_m}} \end{array} \gamma = \gamma \gamma_{\mathcal{C}_{\kappa_1}, \dots, \mathcal{C}_{\kappa_m}}, \quad (19)$$

$$\boxed{\text{Decr}_{\kappa}^*} \mathcal{C}^* = \boxed{\text{Encr}_{\kappa}^{*\dagger}} \gamma_{\kappa}^* \gamma_{\mathcal{C}_{\kappa_1}, \dots, \mathcal{C}_{\kappa_m}}, \quad (20)$$

where γ_{κ}^* is a quantum channel that depends on only γ and κ .

Proof. From the composability of the QHE protocols Q_1, \dots, Q_m , we know (15) holds. Let us denote $\text{Decr}_{\kappa}^* = (\text{Encr}_{\kappa}^*)^\dagger$. Then,

$$\begin{aligned} \boxed{\text{Encr}_{\kappa}^*} \boxed{\mathcal{C}^*} \boxed{\text{Decr}_{\kappa}^*} &= \begin{array}{c} \boxed{\text{Encr}_{\kappa_1}^{(1)}} \boxed{\bar{\mathcal{C}}_1} \\ \vdots \\ \boxed{\text{Encr}_{\kappa_m}^{(m)}} \boxed{\bar{\mathcal{C}}_m} \end{array} \bar{\gamma} \boxed{\text{Decr}_{\kappa}^*} = \begin{array}{c} \boxed{\mathcal{C}_1} \boxed{\mathcal{C}_{\kappa_1}} \\ \vdots \\ \boxed{\mathcal{C}_m} \boxed{\mathcal{C}_{\kappa_m}} \end{array} \boxed{\text{Encr}_{\kappa}^*} \bar{\gamma} \boxed{\text{Decr}_{\kappa}^*} \\ &= \begin{array}{c} \boxed{\mathcal{C}_1} \\ \vdots \\ \boxed{\mathcal{C}_m} \end{array} \boxed{\gamma} \boxed{\gamma_{\kappa}} \boxed{\text{Encr}_{\kappa}^*} \boxed{\text{Decr}_{\kappa}^*} = \begin{array}{c} \boxed{\mathcal{C}_1} \\ \vdots \\ \boxed{\mathcal{C}_m} \end{array} \boxed{\gamma} \boxed{\gamma_{\mathcal{C}_{\kappa_1}, \dots, \mathcal{C}_{\kappa_m}}} \boxed{\gamma_{\kappa}} = \boxed{\mathcal{C}^*} \boxed{\gamma_{\mathcal{C}_{\kappa_1}, \dots, \mathcal{C}_{\kappa_m}}} \boxed{\gamma_{\kappa}} \end{aligned} \quad (21)$$

Now we outline how we derive (21).

1. The first equality: We use the composability of the QHE schemes Q_1, \dots, Q_m to expand the encryption operation of Q^* in terms of the encryption operations of Q_1, \dots, Q_m . Next we use the decomposition of \mathcal{C}^* in terms of $\mathcal{C}_1, \dots, \mathcal{C}_m$ and γ as given in (16). Note that here, we use the notation $\bar{\mathcal{C}} = \varphi^*(\mathcal{C})$, $\bar{\mathcal{C}}_i = \varphi^*(\mathcal{C}_i)$ and $\bar{\gamma} = \varphi^*(\gamma)$.
2. The second equality: From (6), since Q_i is a QHE scheme for every $i = 1, \dots, m$ there exists a \mathcal{C}_{κ_i} such that

$$\bar{\mathcal{C}}_i \circ \text{Encr}_{\kappa_i} = \text{Encr}_{\kappa_i} \circ \mathcal{C}_{\kappa_i} \circ \mathcal{C}_i. \quad (22)$$

3. The third equality: Since \mathcal{A}^* is a group, we know that γ must be in \mathcal{A}^* . From (6), since Q^* is a QHE scheme, for every $\gamma \in \mathcal{A}^*$, there must be a γ_{κ} such that

$$\boxed{\gamma \quad \gamma_{\kappa} \quad \text{Encr}_{\kappa}^*} = \boxed{\text{Encr}_{\kappa}^* \quad \bar{\gamma}}. \quad (23)$$

4. The fourth equality: Since Decr_{κ}^* is the inverse of Encr_{κ}^* , application of Encr_{κ}^* followed by Decr_{κ}^* results in the identity operation. Next we use the identity in (19).
5. The fifth equality: We use the decomposition of \mathcal{C}^* as given by (16).

Eq. (21) implies that the decryption operation for the composed QHE scheme must have the form

$$\text{Decr}_{\kappa, \mathcal{C}^*} = \gamma_{\mathcal{C}_{\kappa_1}, \dots, \mathcal{C}_{\kappa_m}} \circ \gamma_{\kappa}^{\dagger} \circ (\text{Encr}_{\kappa}^*)^{\dagger}, \quad (24)$$

where $\gamma_{\mathcal{C}_{\kappa_1}, \dots, \mathcal{C}_{\kappa_m}}$ is defined in (II.g). \square

C. Quantum error correction

1. Algebraic conditions

There are two approaches for making QEC compatible with QHE:

1. We encode quantum data into a QEC code before encrypting it within a QHE scheme.
2. We encrypt the data into a QHE scheme before encoding it into a QEC code.

Here we provide the algebraic conditions such that these two approaches are equivalent and therefore directly composable.

Consider composable QHE schemes Q_1, \dots, Q_q and $\tilde{Q}_1, \dots, \tilde{Q}_r$ where,

$$\begin{aligned} Q_i &= (\mathcal{S}_i, K_i, \mathcal{A}_i, \varphi_i, \text{Encr}_{K_i}^{(i)}), \\ \tilde{Q}_j &= (\tilde{\mathcal{S}}_j, \tilde{K}_j, \tilde{\mathcal{A}}_j, \tilde{\varphi}_j, \widetilde{\text{Encr}}_{\tilde{K}_j}^{(j)}), \end{aligned} \quad (25)$$

and let us denote the composition of these schemes as

$$Q^{\#} = (\mathcal{S}^{\#}, K^{\#}, \mathcal{A}^{\#}, \varphi^{\#}, \text{Encr}_{K^{\#}}^{\#}). \quad (26)$$

Now, a QEC code protects quantum data by applying a strategically chosen encoding channel on the quantum data, mapping it into a protected quantum state [16]. If error rates are sufficiently low, QEC protocols can recover the uncorrupted quantum data. QEC protocols involve performing measurements and corrections based on the measurement outcomes.

Since we consider two distinct versions of QEC here, we use distinct notations for the different encoding channels. We denote the encoding channel that acts before and after QHE as Enc and $\widetilde{\text{Enc}}$ respectively. Associated with the encodings Enc and $\widetilde{\text{Enc}}$ are homomorphisms L and \bar{L} such that for every $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{A}^{\#}$ and $\bar{\mathcal{C}}_1, \bar{\mathcal{C}}_2 \in \mathcal{A}^{\#}$, we have,

$$\begin{aligned} L(\mathcal{C}_2 \circ \mathcal{C}_1) &= L(\mathcal{C}_2) \circ L(\mathcal{C}_1), \\ \bar{L}(\bar{\mathcal{C}}_2 \circ \bar{\mathcal{C}}_1) &= \bar{L}(\bar{\mathcal{C}}_2) \circ \bar{L}(\bar{\mathcal{C}}_1), \end{aligned} \quad (27)$$

and,

$$\begin{aligned} \text{Enc} \circ \mathcal{C}_2 \circ \mathcal{C}_1 &= L(\mathcal{C}_2 \circ \mathcal{C}_1) \circ \text{Enc}, \\ \widetilde{\text{Enc}} \circ \bar{\mathcal{C}}_2 \circ \bar{\mathcal{C}}_1 &= \bar{L}(\bar{\mathcal{C}}_2 \circ \bar{\mathcal{C}}_1) \circ \widetilde{\text{Enc}}. \end{aligned} \quad (28)$$

The outputs of the homomorphisms L and \bar{L} are logical operators corresponding to the QEC codes with encoding channels Enc and $\widetilde{\text{Enc}}$ respectively.

Here, the QHE schemes Q_i and \tilde{Q}_j play different roles before we apply the QEC encoding channel $\widetilde{\text{Enc}}$. Only Q_1, \dots, Q_j contains the actual encrypted quantum data; the QHE schemes $\tilde{Q}_1, \dots, \tilde{Q}_r$ encrypt only ancilla states. Since the encoded QHE scheme $Q^{\#}$ inherits its security from the unencoded QHE scheme, we can quantify its security in terms of the security of the constituent QHE schemes Q_1, \dots, Q_j and $\tilde{Q}_1, \dots, \tilde{Q}_r$.

We now provide sufficient conditions for which encoding before and after encryption can be made equivalent.

Lemma 3. *Let $Q^{\#}$ be a composable QHE scheme as denoted in (26). Suppose that for every $\mathcal{C} \in \mathcal{A}^{\#}$ and every $\kappa \in K^{\#}$, there is a $f(\kappa, \mathcal{C}) \in K^{\#}$ such that,*

$$\text{Encr}_{\kappa}^{\#} \circ \mathcal{C} = \varphi^{\#}(\mathcal{C}) \circ \text{Encr}_{f(\kappa, \mathcal{C})}^{\#}. \quad (29)$$

Suppose that $\text{Enc} \in \mathcal{A}^{\#}$ and suppose that for every $\mathcal{C} \in \mathcal{A}^{\#}$, we have $L(\mathcal{C}) \in \mathcal{A}^{\#}$. Suppose that for every $\kappa \in K^{\#}$, there exists a $\lambda \in K^{\#}$ such that for every $\mathcal{C} \in \mathcal{A}^{\#}$, we have,

$$\widetilde{\text{Enc}} \circ \text{Encr}_{\kappa} \circ \mathcal{C} = \text{Encr}_{\lambda} \circ \widetilde{\text{Enc}} \circ \mathcal{C}. \quad (30)$$

Then the following holds.

1. For every $\kappa \in K^{\#}$, there exists a $\lambda \in K^{\#}$ such that for every $\mathcal{C} \in \mathcal{A}^{\#}$, we have,

$$\begin{aligned} \varphi^{\#}(L(\mathcal{C})) \circ \text{Encr}_{f(\kappa, L(\mathcal{C}))}^{\#} \circ \text{Enc} \\ = \bar{L}(\varphi(\mathcal{C})) \circ \widetilde{\text{Enc}} \circ \text{Encr}_{f(\lambda, \mathcal{C})}^{\#}. \end{aligned} \quad (31)$$

$$\overline{L}(\varphi(\mathcal{C})) \circ \text{Encr}_\kappa = \varphi^\#(L(\mathcal{C})) \circ \text{Encr}_{f(\kappa, \mathcal{C})}. \quad (32)$$

2. Furthermore, if $\overline{L}(\varphi(\mathcal{C})) = \varphi^\#(L(\mathcal{C}))$, then for every $\kappa \in K^\#$, there exists a $\lambda \in K^\#$ such that for every $\mathcal{C} \in \mathcal{A}^\#$, we have,

$$f(f(\kappa, L(\mathcal{C})), \text{Enc}) = f(\lambda, \mathcal{C}). \quad (33)$$

$$\begin{aligned} \text{Encr}_\kappa \circ \text{Enc} &= \text{Enc} \circ \mathcal{C}' \circ \text{Encr}_\lambda^\# \\ &= \text{Enc} \circ \text{Encr}_\lambda^\# \circ \mathcal{C}. \end{aligned} \quad (34)$$

Proof. For the first part, we note the following algebraic relations. Using (29), the definitions of the homomorphisms L and \overline{L} , and the fact that $\mathcal{C}, \text{Enc} \in \mathcal{A}^\#$, we have

$$\begin{aligned} \text{Encr}_\kappa^\# \circ \text{Enc} \circ \mathcal{C} &= \text{Encr}_\kappa^\# \circ L(\mathcal{C}) \circ \text{Enc} \\ &= \varphi^\#(L(\mathcal{C})) \circ \text{Encr}_{f(\kappa, L(\mathcal{C}))}^\# \circ \text{Enc}. \end{aligned} \quad (35)$$

Similarly we have,

$$\begin{aligned} \overline{\text{Enc}} \circ \text{Encr}_\lambda^\# \circ \mathcal{C} &= \overline{\text{Enc}} \circ \varphi(\mathcal{C}) \circ \text{Encr}_{f(\lambda, \mathcal{C})}^\# \\ &= \overline{L}(\varphi(\mathcal{C})) \circ \overline{\text{Enc}} \circ \text{Encr}_{f(\lambda, \mathcal{C})}^\#. \end{aligned} \quad (36)$$

From this we obtain the first result.

Now, if $\overline{L}(\varphi(\mathcal{C})) = \varphi^\#(L(\mathcal{C}))$, we have,

$$\text{Enc} \circ \text{Encr}_{f(f(\kappa, L(\mathcal{C})), \text{Enc})}^\# = \overline{\text{Enc}} \circ \text{Encr}_{f(\lambda, \mathcal{C})}^\#, \quad (37)$$

which implies that for all $\mathcal{C} \in \mathcal{A}^\#$, we have (33), which shows the second result. \square

Note that in the extreme case where $f(\kappa, \mathcal{C}) = \kappa$ for every $\mathcal{C} \in \mathcal{A}^\#$, (33) simplifies to $\lambda = \kappa$.

To derive the decryption channel for the encoded $Q^\#$, note that in the assumption (29) used in Lemma 3 implies the condition (6) that is used in Theorem 2 whenever $\text{Enc} \in \mathcal{A}^\#$ and $\mathcal{A}^\#$ is a group under composition. Hence we can use Theorem 2 to derive the corresponding decryption channel.

We now illustrate how one can encode the encrypted state of $Q^\#$ into a QEC code. Suppose that each encrypted state in Q_i and \tilde{Q}_j is an m -qubit state. Then, the overall encrypted state of $Q^\#$ is a $(q+r)m$ -qubit state. Treating quantum information in Q_j and \tilde{Q}_j as data and ancilla registers respectively, we can use $\overline{\text{Enc}}$, the encoding circuit of a $[[[(q+r)m, qr, d]]$ stabilizer code, to encode the encrypted quantum data. Here, $(q+r)m$ denotes the length of the code, qr counts the number of encoded logical qubits, and d denotes the distance of the QEC code. We like to emphasize that as long as the set of allowed delegated computations and the encoding circuit for a QEC code is carefully chosen, we need not restrict our attention to stabilizer codes [17]. Namely, Lemma 3 can also be used to establish the equivalence of encoding before and after encryption for non-stabilizer codes [18, 19], such as permutation-invariant codes [20–25].

For QEC to be fully compatible with QHE, the server must implement the QEC procedures without knowledge of the secret key. In Section II C 2, we sketch how this can be done with some examples.

2. Examples

To help conceptualise the above theoretical framework, we now consider the composition of QEC and QHE in the context of two concrete examples:

- Pauli-key encoding with stabilizer error correcting codes, described in Sec. III A 1.
- Permutation-key encoding, described in Sec. III A 3.

a. Pauli-key encoding with stabilizer codes: QEC in general relies on syndrome measurements to detect errors and subsequently project them back into the logical qubit space. In an outsourced setting this necessarily requires classical interaction between the client and server such that the client can selectively reveal the decrypted syndrome measurement outcome. However this naturally raises the question as to whether revealing this information compromises the security of logical data.

In general this is not the case since syndrome qubits reveal only information about the codespace, which commutes with the logical qubit space.

Taking stabiliser codes as an example, where syndrome measurements are performed as per Fig. 3, we see that the syndrome qubits reveal correlations between qubits in the codespace which are only dependent upon errors, not logical qubits, which by definition commute with the stabilisers. Therefore, if in Fig. 3 Alice were to reveal the decryption key associated with the operator P_c , this does not directly reveal any information about the individual P_i operators, only their product, which relates to a parity checksum that is invariant under the encoded logical qubit space.

Take the simplest example of the 3-qubit repetition code, where the generating stabilisers are ZZI and IZZ, and the logical basis states are given by $|000\rangle$ and $|111\rangle$. While measurement of an individual phase-flip operator, Z, directly measures the logical qubit, measurement of an IZZ operator and its permutations only reveals pairwise parity, which is invariant under logical operations since the logical basis states always exhibit pairwise parity in the physical qubits. Measuring these operators can therefore diagnose errors (when odd parities are encountered) while revealing no information about the logical qubit.

In general, so long as the codespace's stabilizers commute with logical operators, which they necessarily do by definition, publicly revealing the decryption of these checksums does not compromise the security of logical data.

b. Permutation-key encryption: In the case of permutation key encryption we take a known code which is randomised under the encryption process via permutation operators. In the absence of Alice revealing any

information the server, the server’s knowledge of the state is limited to some ϵ which is obtained upon measuring the entire state.

For the server to perform syndrome measurements, the client must reveal what the decrypted syndrome is. However in this instance this necessarily reveals some additional information beyond ϵ about the overall secret key that also encodes the logical qubit space. The distinction with the previous example is that this process reveals not only information about the syndrome, but also some logical information.

To minimise leakage of logical information, the client can encrypt classical bits in advance and provide them to the server, selectively revealing which of these encrypted bits the server should employ. This prevents information about the secret key from being leaked if the client were to reveal additional unencrypted information. The introduction of these additional resources compensate for information leaked during syndrome measurement to preserve the desired level of ϵ .

Hence, there is a resource trade-off whereby additional ancillary qubits are introduced to preserve the desired secrecy level.

III. APPLICATION

In this section we demonstrate how our scheme applies to several examples, considering both discrete- and continuous-variable scenarios.

A. Clifford circuits

1. Pauli-key encryption

An example of a circuit-based model for quantum computation that naturally fits our group-theoretic construction is that of Clifford circuits encrypted using Pauli-key encryption. Clifford circuits by definition commute with the Pauli group, satisfying the commutation relation,

$$U_C \cdot \bigotimes_{i=1}^n \sigma_{f(i)} = \bigotimes_{i=1}^n \sigma_{g(i)} \cdot U_C, \quad (38)$$

where U_C is an arbitrary n -qubit Clifford circuit, and $f(n), g(n) \in \{0, 1, 2, 3\}$ map qubit labels to Pauli gates at the input and output respectively. Importantly, $f(n)$ and $g(n)$ can be efficiently classically calculated from one another for any known U_C .

The encryption process proceeds by applying randomly chosen Pauli operators independently to each qubit prior to the circuit, represented by $f(n)$. This acts as Alice’s private key, requiring $2n$ classical bits. This can be inverted after the computation via the associated function $g(n)$, thereby perfectly recovering the computational state.

From Bob’s perspective, who does not know the private key $f(n)$, on each qubit this simulates a perfect

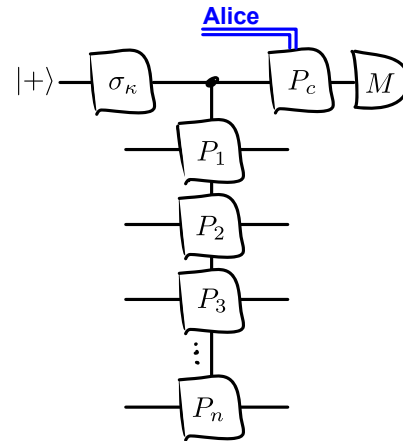


Figure 3: Circuit for performing a stabilizer measurement on encrypted data. The circuit performs an n -qubit stabilizer measurement given by the Pauli sequence $K = \bigotimes_{i=1}^n P_i$ with measurement outcome M . The $|+\rangle$ state has been encrypted with Pauli key σ_κ . In an error corrected circuit such stabilizer measurements are performed repeatedly over the course of the execution of the computation. Whenever the server performs such a measurement it feed-forwards to a subsequent classically-controlled correction operation. To perform this, Alice must communicate the Pauli correction, P_c , associated with the ancillary qubit prior to Bob performing his measurement, thereby requiring classical interaction between client and server. Note that the controlled- K operation, when decomposed into more elementary gates, requires only Clifford operations and is compatible with outsourced Clifford schemes. If the measurement M is a Pauli, the correction P_c needn’t be physically implemented and can instead be absorbed into a classical correction of the measurement outcome.

depolarising channel that contains no information about Alice’s state. Without knowledge of the private key, the encrypted n -qubits is the n -qubit maximally mixed state

$$\text{Encr}(\rho) = \frac{\sigma_0^{\otimes n}}{2^n}. \quad (39)$$

Clifford circuits acting on Clifford states are known to be efficiently simulatable and not universal for quantum computation. However with the addition of T -gates Clifford circuits become universal. Although introducing T -gates directly would undermine the commutation relation given in Eq. (38), utilising ancillary magic state (eigenstates of the T -gate) inputs we are able to teleport the action of T -gates into Clifford circuits without directly utilising any non-Clifford operations. Using this magic state injection trick we can maintain consistency with our model whilst enabling quantum computation beyond Clifford circuits. However, the requirements of compactness places an upper bound on the number of T -gates that can be performed in this way.

In the Pauli-key QHE scheme, the compactness requirement constrains the number of T -gates that can be implemented. Since the complexity of the decryption procedure increases exponentially with the number of T -gates, if the complexity of the decryption algorithm is to

remain polynomial in the number of qubits, the number of T -gates must be at most logarithmic in the number of qubits.

This Pauli-key encryption scheme can also support QEC in an n -qubit stabilizer code that encodes k qubits, where the encoding into a QEC code occurs after encryption. For this, note that any sequence of Pauli-key encryption schemes are composable. Because of this, we can compose a k -qubit Pauli-key encryption scheme (with randomly chosen keys) with a $(n - k)$ -qubit trivial Pauli encryption scheme (with all Pauli keys equal to the identity operation σ_0). Since any stabilizer code has a Clifford encoding circuit, we can then use an encoding circuit for any n qubit stabilizer code that encodes k logical qubits to protect the encrypted quantum data into an n -qubit stabilizer code. Since the QEC procedures of a stabilizer code are also Clifford operations which belong to the allowed set of computations of the composed scheme, the server can perform all of the QEC operations without any assistance from Alice.

2. IQP circuits

As shown by [26], a special case of Pauli-key encryption applies to IQP circuits [27], a class of quantum sampling problems whose gates are diagonal (i.e commuting) in the computational basis. IQP circuits begin with an n -qubit input state comprising $|\pm\rangle$ states, to which a diagonal n -qubit gate is applied, comprising controlled-phase and T -gates. Finally the qubits are measured in the diagonal basis yielding n -bit measurement samples.

This sampling problem is characterised by the probability distribution,

$$P(\vec{x}, \vec{y}) = |\langle x_1, \dots, x_n | H^{\otimes n} C H^{\otimes n} | y_1, \dots, y_n \rangle|^2, \quad (40)$$

for input and output bit-strings \vec{x} and \vec{y} respectively, where $C = \text{diag}(p_1, \dots, p_{2^n})$ is a diagonal n -qubit circuit and $|p_i| = 1 \forall i$.

Since the input states are restricted to $|\pm\rangle$, encryption using all four Pauli operators is not required, and randomly applying I or Z gates (i.e a dephasing channel) suffices to yield a maximally mixed state.

Hence, for IQP circuits comprising gates diagonal in the logical basis, phase encryption yields,

$$\text{Encr}(\rho) = \frac{1}{2^n} \bigotimes_{i=1}^n \sum_{\kappa_i \in \{0,1\}} Z^{\kappa_i} \rho Z^{\kappa_i}, \quad (41)$$

which is the maximally mixed state when the input qubits are restricted to $|\pm\rangle$. Since the encryption and computation operators commute, the input and output keys are identical.

3. Permutation-key encryption

Now, we revisit a quantum homomorphic encryption scheme [7] that supports delegated Clifford gates and

encrypts using a permutation-key. Since this quantum homomorphic encryption scheme can be understood using the formalism of quantum error correcting codes, it is simple to amend to also support quantum error correction.

A key advantage of the permutation-key encryption scheme over the Pauli-key encryption scheme is that the decryption algorithm of the quantum data in the permutation-key encryption scheme is independent of the delegated Clifford computations. This prevents the server from revealing too much information about the computation to the client.

In the permutation-key encryption scheme of [7], each qubit of quantum data [Fig. 4(a)],

$$\frac{1}{2} \sum_{j=0}^3 a_j \sigma_j, \quad (42)$$

is first mapped to [Fig. 4(b)],

$$\frac{1}{2^m} \sum_{j=0}^3 a_j \sigma_j^{\otimes m}, \quad (43)$$

Applying $2m - 2$ CNOT gates [7] on the data qubit and $m - 1$ maximally mixed states achieves this mapping.

In addition, this map takes a data qubit into a random stabilizer code where the randomization is over all stabilizer codes that encode one logical qubit and supports transversal logical Clifford gates. Whenever $m - 1$ is divisible by four, each m qubit block of quantum data now supports logical Clifford computations that are equal to transversal application of the underlying Clifford operations. Second, the client prepares an additional m maximally mixed qubits, and applies a secret permutation π_κ amongst the $2m$ qubits [Fig. 4(c)].

The trace distance between the encrypted quantum data from the server's point of view and the maximally mixed state is at most,

$$\Delta(r, m) = \sqrt{\frac{2^r}{|K|}} = \sqrt{\frac{2^r}{\binom{2m}{m}}}, \quad (44)$$

and approaches 0 exponentially fast in m [7].

Without knowledge of the permutation-key, the server does not know which m of the $2m$ qubits hold the actual quantum data. Nonetheless, the server is still able perform a homomorphic evaluation of Clifford gates on the encrypted quantum data. The server can achieve this via applying identical Clifford gates on each of the $2m$ qubits, because the action of the m identical Clifford gates on sets of qubits that do not hold quantum data is trivial.

The server can also evaluate T -gates on the encrypted quantum data via gate teleportation. By preparing an encrypted magic state,

$$|T\rangle = T|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle), \quad (45)$$

performing transversal CNOTs, and measuring qubits transversally in the computational basis as described in

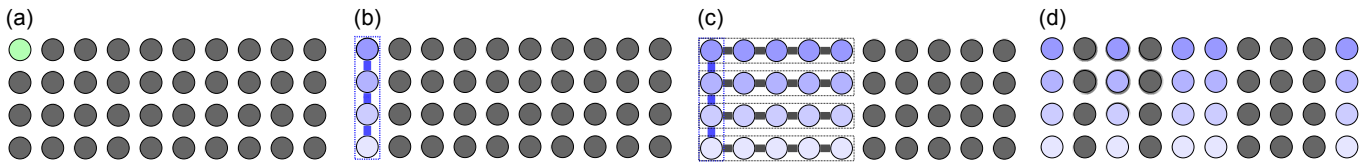


Figure 4: Obtaining a permutation-key encryption scheme that supports quantum error correction. This corresponds to the encryption scheme shown in Fig. 2(c). Here, we illustrate a single data qubit encoded into a 4-qubit code across the rows, mapped into a random quantum code, and finally encrypted with a permutation-key. (a) Data qubit is on the first row and column. (b) A QEC encoding applies on the first column. (c) The first half of each row is encoded identically into quantum codes that support transversal Cliffords. (d) The columns are permuted according to the permutation key κ . Here (a,b,c) represent the encoding, and (d) represents the encryption. In this example, composition is independent of the order of the QHE and QEC operations.

[7], a T -gate on the encrypted quantum data can be effected with probability $1/2$. Using multiple copies of encrypted secrets, Ref. [7] showed that we have a non-deterministic scheme for QHE with some T -gates.

In this paper, we explain that by augmenting the QHE protocol with classical communication, the permutation-key QHE scheme for implementing T -gates in [7] can be made deterministic. For every encrypted magic state, Alice also prepares an encrypted $|0\rangle$ state and an encrypted $|1\rangle$ state, and has these encrypted states assigned in random rows. Using these three encrypted ancilla states, Alice can help the server perform deterministic T -gates. For this, the server first transmits classical bits containing the measurement outcomes on the encrypted magic state to Alice. Second, Alice uses her knowledge of the secret permutation key to evaluate the parity of a subset of these classical bits that correspond to the secret permutation. Third, based on this parity, Alice tells the server the which row, where an encrypted $|0\rangle$ or encrypted $|1\rangle$ state resides, should be used to perform a controlled correction for the gate-teleportation process. Without the permutation key, the classical label is completely independent of the quantum data [28] and of the private key. Hence in spite of the server's classical communication with Alice, the server learns no additional information about the quantum data. With this method, the server can perform any number of T -gates, provided that Alice prepares the encrypted magic states on demand. The only caveat is that the security of the scheme worsens exponentially with the number of encrypted T -gates consumed (see Eq. (44)).

The security of the permutation-key scheme worsens exponentially with the number r of T -gates and ancilla qubits used. However, the security of the permutation-key scheme improves exponentially with m . Hence if we increase m together with r , we can guarantee a constant level of security. Now let the size of the quantum input be s , which corresponds to s rows of quantum data. The decryption algorithm involves permuting $2m$ columns of qubits, where r is the number of rows. Hence, the number of swaps needed to perform the permutation for the decryption is $O((r+s)m)$. Since m is linear in r , the complexity of decryption is $O(r^2 + rs)$. Thus, as long as r is polynomial in s , the complexity of the decryption is

polynomial in s , and is compact. The permutation-key scheme is therefore compact for a polynomial number of T -gates (far more favourable than the Pauli-key scheme).

In Fig. 4, we illustrate how we can obtain a permutation-key encryption scheme that is compatible with QEC. To use an n -qubit QEC code in a QHE scheme that permutes $2m$ qubits, Alice encodes every data qubit into a QEC code on mn qubits, located on n rows and m columns. This QEC code has a concatenated structure with an inner and an outer code, and has a distance that is at least the distance of its inner code. The inner and outer codes have encoding circuits that apply row-wise and column-wise respectively. The role of the inner code and the outer codes differ. While the inner code gives the scheme its QEC capabilities, the outer code gives the scheme a column-wise transversal structure. Namely, if the inner code has logical bit-flip and phase-flip operators

$$\begin{aligned} X_L &= P_1 \otimes \cdots \otimes P_n, \\ Z_L &= Q_1 \otimes \cdots \otimes Q_n, \end{aligned} \quad (46)$$

the concatenated code will have logical

$$\begin{aligned} X_L &= P_1^{\otimes m} \otimes \cdots \otimes P_n^{\otimes m}, \\ Z_L &= Q_1^{\otimes m} \otimes \cdots \otimes Q_n^{\otimes m}, \end{aligned} \quad (47)$$

where the tensor product with respect to m applies column-wise. Here, the outer code is highly random because of its many maximally mixed states as inputs, has a distance of 1, and is useless for QEC. In contrast, the inner code can be any stabilizer code with a non-trivial distance. After encoding into this concatenated code on m columns, Alice encrypts this data by randomly permuting these m columns amongst the $2m$ column available to Alice.

Now QEC involving stabilizer codes only requires Clifford operations and measurements. Note that the server can perform the Clifford part of the QEC procedure easily because such operations are automatically allowed by the permutation-key QHE scheme. However, performing the measurement part of the QEC protocol is more challenging. While the server can measure the encrypted states, the server will not obtain much meaningful information from the measurement statistics. In this case, for every measured encrypted qubit, the server can communicate

the corresponding $2m$ classical bits to Alice over a classical communication channel. Based on these $2m$ classical bits, Alice can compute the correct parity on m of the bits. Alice then tells the server which one of the many encrypted $|0\rangle$ and encrypted $|1\rangle$ ancillas to use. Since the rows labels for these encrypted ancillas are randomly assigned, the server does not receive any additional information on this parity bit. The server can next use the encrypted ancilla to perform conditional Pauli operations on the overall concatenated code to complete the QEC procedure.

In the paradigm of fault-tolerant quantum computation, we accept the inevitability that each elementary gate fails with some probability. In spite of noise, carefully designed fault-tolerant quantum circuitry can perform reliable quantum computation. Quantum information will be encoded into a fault-tolerant family of quantum error correction codes $\Omega = \{\mathcal{Q}_t : t \geq 1\}$ that corrects an increasing number of errors t . Suppose that each physical gate fails independently with probability p_0 , and we use a quantum code \mathcal{Q}_t that corrects t errors. We say that Ω has a fault-tolerant threshold p_Ω if for every positive integer t , the logical failure probability $p_{t,\Omega}$ of each fault-tolerant gate in \mathcal{Q}_t satisfies the inequality,

$$p_{t,\Omega} \leq a_\Omega (p_0/p_\Omega)^t, \quad (48)$$

for some positive a_Ω . Our framework is compatible with mainstream fault-tolerant quantum computation schemes that are based on Clifford computations and gate-teleportation.

Fault-tolerant syndrome extraction allow us to reliably extract information about errors as they occur. Compared to conventional non-fault-tolerant syndrome extraction, fault-tolerant syndrome extraction has additional overhead in terms of the number of ancillary qubits required. Recent advances in flag-fault-tolerant quantum computation [29, 30] allows the number of these ancilla to be bounded for each round of fault-tolerant syndrome extraction. In the Aliferis, Gottesman and Preskill framework of fault-tolerant quantum computation [31], each fault-tolerant gadget is both preceded and followed by rounds of fault-tolerant syndrome extraction. Hence, for the fault-tolerant evaluation of a quantum circuit of depth `depth` on r qubits, the total number of fault-tolerant syndrome extractions required for fault-tolerant evaluation is at most $r(\text{depth} + 1)$. Since each fault-tolerant syndrome extraction requires at most,

$$A_{n,t} = 2tn[t(t+2) + n] \binom{t}{2} \leq t^3 n(t^2 + 2t + n), \quad (49)$$

ancillary qubits [30]¹ for a length n quantum error correction code that corrects t errors, the total number of

¹ This equation does not directly appear in this reference, but through private communication with Chao was shown to follow from these results.

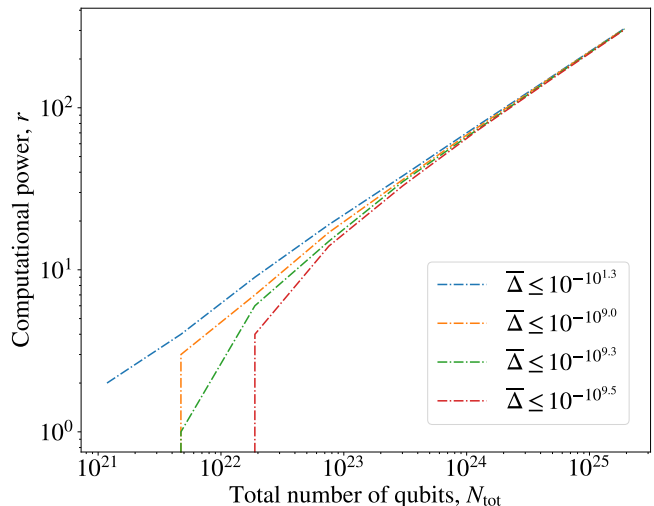


Figure 5: Tradeoffs between total number of physical qubits (N_{tot}) and computational power (number of logical qubits r) in a fault-tolerant setting. The base physical error probability is $p_0 = 10^{-6}$, and the target failure probability per logical gadget is $\bar{p} \leq 10^{-30}$. We consider a family of quantum codes correcting t errors (e.g family of surface codes) with threshold $p_\Omega = 10^{-3}$, coefficient $a_\Omega = 10$, and length $n = (2t + 1)^2$. Here, $\bar{\Delta}$ quantifies the security of the delegated quantum computation scheme, with $\bar{\Delta} = 0$ corresponding to perfect security.

logical ancillary qubits that we require for the entire computation will be at most $3r(\text{depth} + 1)A_{n,t}$, where the factor of three arises from the need to use encrypted $|0\rangle$ and $|1\rangle$ states. Hence for the QEC enhanced scheme, the trace distance between the encrypted state as perceived by the server and the maximally mixed state is at most,

$$\bar{\Delta} = \Delta(r + 3r(\text{depth} + 1)A_{n,t}, m). \quad (50)$$

The total number of qubits used in the scheme is thus,

$$N_{\text{tot}} = 2mr(1 + 3(\text{depth} + 1)A_{n,t}). \quad (51)$$

With techniques provided in App. A, these relationships can be visualised as shown in Fig. 5.

B. Continuous variable linear optics

Quantum computation can also be implemented using continuous variable (CV) architectures [32] rather than discrete variables (i.e qubits). Considering CV linear optics as an example, optical displacement operators are easily seen to satisfy our desired commutation relations,

$$\hat{U}_{\text{LO}} \cdot \bigotimes_{i=1}^n \hat{D}_i(\alpha_i) = \bigotimes_{i=1}^n \hat{D}_i(\beta_i) \cdot \hat{U}_{\text{LO}}, \quad (52)$$

where U_{LO} is an n -mode passive linear optics unitary, and $\hat{D}_i(\alpha_i)$ is the optical displacement operator acting on the

i th mode with coherent amplitude α_i . The displacement operator can equivalently be represented in terms of a complex coherent displacement amplitude α , or in the position (\hat{p}) and momentum (\hat{q}) degrees of freedom,

$$\begin{aligned}\hat{D}(\alpha) &= \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}), \\ \hat{D}(x, p) &= \exp(ip\hat{x} - ix\hat{p}).\end{aligned}\quad (53)$$

The input and output displacement amplitudes are related by simple matrix inversion, which is classically efficient for Alice to solve,

$$\hat{U}_{\text{LO}} \cdot \vec{\alpha} = \vec{\beta}. \quad (54)$$

This commutation property can be further generalised [33]. Namely, arbitrary Gaussian operations (i.e those described by Hamiltonians at most quadratic in the creation and annihilation operators) may be decomposed into layers of linear optics networks and single-mode squeezers (SMSs) of the form,

$$\hat{S}(z) = \exp\left[\frac{1}{2}(z^*\hat{a}^2 - z(\hat{a}^\dagger)^2)\right], \quad z = re^{i\theta}. \quad (55)$$

However since displacement operators commute through SMSs yielding distinct but easily calculated displacement parameters,

$$\hat{D}(\alpha)\hat{S}(z) = \hat{S}(z)\hat{D}(\gamma), \quad (56)$$

where $\gamma = \alpha \cosh(r) + \alpha^* e^{i\theta} \sinh(r)$, it immediately follows that displacement key encryption satisfies the desired commutation relation for general Gaussian networks.

In summary, a displacement-key encryption scheme uses displacement operators with amplitudes α_i on the i mode to encrypt. When $\kappa(\alpha)$ is a continuous key-space, the eavesdropper perceives the state

$$\text{Encr}(\rho) = \int \kappa(\alpha) \hat{D}_i(\alpha_i) \rho \hat{D}_i^\dagger(\alpha_i) d^2\alpha, \quad (57)$$

when ρ is the input state. To have quantum error correction capabilities, we can first encode ρ into a GKP code, and second, for the encryption, we can choose the keys κ to perform logical Pauli operations, thereby reducing us to Pauli-key encryption.

1. Squeezed state encoding

Highly squeezed states can be directly employed to approximate qubits, given that in the limit of infinite squeezing the \hat{x} and \hat{p} eigenstates are orthogonal and therefore a legitimate qubit basis. However infinite squeezing requires infinite energy and with finite squeezing a logical basis can only be approximated. Nonetheless, in certain regimes, this approximate encoding is sufficient for universal fault-tolerance quantum computation [34–36].

With such an encoding, universality may be achieved using Gaussian operations (assuming some additional non-Gaussian input states), which commute with displacement operators, thereby lending itself to our requirements. In this formalism stabilizer codes are replaced with nullifier codes, given by linear combinations of the position and momentum operators across different modes of the form,

$$\hat{n}^x = \sum_{k=1}^N \alpha_k \hat{x}_k, \quad \hat{n}^p = \sum_{k=1}^N \beta_k \hat{p}_k, \quad (58)$$

where k denotes optical mode number and $\alpha_k, \beta_k \in \{-1, 0, 1\}$. These nullifiers similarly commute with displacement key operators following from Eq. (53).

2. GKP encoding

A special case of CV encoding is using Gottesman-Knill-Preskill (GKP) states [14], which encode logical (discrete) qubits into CV space via discretisation of the continuous degrees of freedom.

Ref. [14] describes the full details of how this form of encoding achieves universal quantum computation. For our purposes, we focus on the GKP encoding. Here it suffices to note that under GKP encoding, logical Pauli \hat{X} and \hat{Z} gates are implemented using position and momentum shifts,

$$\begin{aligned}\hat{Z} &= e^{2\pi i \hat{q}/n\alpha}, \\ \hat{X} &= e^{-i\hat{p}\alpha},\end{aligned}\quad (59)$$

which upon inspection correspond to special cases of the displacement operator given in Eq. (53).

The remaining gates necessary to perform universal Clifford operations are readily implemented deterministically using non-linear optical operations. And similar to before, the trick of using magic state injection to implement T -gates may be employed to achieve universality.

It immediately follows via this logical mapping that the Pauli-key encryption naturally maps across to displacement-key encryption as per Eq. (52). Similarly, since GKP codes effectively map continuous-variable state to discrete-variable qubit encoding one could similarly employ GKP-encoded qubits to the permutation-key encryption presented in Sec. III A 3.

IV. CONCLUSION

We have presented a general group-theoretic formalism for unifying quantum error correction with homomorphic encryption. So long as the underlying QEC and homomorphic encryption primitives satisfy the required commutation relations it immediately follows that concatenation is possible, under which security can be preserved.

We demonstrated the applicability of our approach by application to stabilizer codes with Pauli-key encryption, permutation-key encryption, and optical networks with displacement-key encryption, thereby demonstrating its viability in the context of well-known discrete- and continuous-variable encryption techniques.

Given that large-scale quantum computation necessarily requires quantum error correction and that for economic reasons cloud-based deployment of quantum computation will inevitably dominate the future quantum landscape, the ability to reconcile these two techniques is essential. Simultaneously, the high value of future cloud-based quantum computations mandates the assurance of information security of computational results.

Our technique applies broadly to many quantum computing architectures and associated quantum error correction techniques where client-side quantum resources are assumed to be minimal, typically restricted to only single-qubit state preparation and communication, some-

thing expected to be enabled by a future global quantum internet.

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Appendix A: Derivations for permutation-key encryption

We consider a code family with $n = (2t + 1)^2$ qubits that corrects t errors. Setting \bar{p} as the logical failure rate, we note that for the inequality $a_\Omega(p_0/p_\Omega)^t \leq \bar{p}$ to hold,

it suffices to have,

$$t \geq \left\lceil \frac{\log(\bar{p}/a_\Omega)}{\log(p_0/p_\Omega)} \right\rceil. \quad (\text{A1})$$

Since $\bar{\Delta}$ decreases with increasing m , we pick the maximum m such that,

$$2mr(1 + 3(\text{depth} + 1)A_{n,t}) \leq N_{\text{tot}} \quad (\text{A2})$$

holds, which gives,

$$m = \left\lfloor \frac{N_{\text{tot}}}{2r(1 + 3(\text{depth} + 1)A_{n,t})} \right\rfloor. \quad (\text{A3})$$

The tradeoff that we numerically investigate is the following: for a target logical gate failure probability of \bar{p} and a maximum number of qubits N_{tot} , what is the maximum computation power we can have at a fixed base noise rate p_0 ? To impose $\Delta \leq 2^{-k}$, it suffices to set,

$$2m \geq r(1 + 3(\text{depth} + 1)A_{n,t}) + 2k, \quad (\text{A4})$$

which gives us an upper bound on r . We numerically find the largest r that satisfies this inequality along with the inequality,

$$2mr(1 + 3(1 + \text{depth})A_{n,t}) \leq N_{\text{tot}}. \quad (\text{A5})$$

Using the two inequalities above, we have

$$r \leq \frac{\min\{2m - 2k, N_{\text{tot}}/(2m)\}}{1 + 3(1 + \text{depth})A_{n,t}}, \quad (\text{A6})$$

for which the maximum r occurs when $2m - 2k = \lfloor N_{\text{tot}}/(2m) \rfloor$. Given this, we choose,

$$m = \left\lfloor \frac{\sqrt{k^2 + N_{\text{tot}}} + k}{2} \right\rfloor, \quad (\text{A7})$$

from which we find that the maximum r is approximately,

$$\sqrt{k^2 + N_{\text{tot}}} - k = O(\sqrt{N_{\text{tot}}}). \quad (\text{A8})$$

Fig. 5 shows how the computational power of our scheme increases with the total number available qubits.