A General Framework of Continuation Methods for Complementarity Problems*

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September 1990

Abstract. A new class of continuation methods is presented which, in particular, solve linear complementarity problems with copositive-plus and L_*-matrices. Let $\boldsymbol{a}, \boldsymbol{b} \in R^n$ be nonnegative vectors. We embed the complementarity problem with a continuously differentiable mapping $\boldsymbol{f} : R^n \to R^n$ in an artificial system of equations

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = (\mu \boldsymbol{a}, \zeta \boldsymbol{b}) \text{ and } (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0} \ , \qquad (*)$$

where $\boldsymbol{F} : R^{2n} \to R^{2n}$ is defined by

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = (x_1 y_1, \ldots, x_n y_n, \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}))$$

and $\mu \geq 0$ and $\zeta \geq 0$ are parameters. A pair $(\boldsymbol{x}, \boldsymbol{y})$ is a solution of the complementarity problem if and only if it solves $(*)$ for $\mu = 0$ and $\zeta = 0$. A general idea of continuation methods founded on the system $(*)$ is as follows.

1. Choose n-dimensional vectors $\boldsymbol{a} \geq \boldsymbol{0}$ and $\boldsymbol{b} > \boldsymbol{0}$ such that the system $(*)$ has a trivial solution $(\boldsymbol{x}^1, \boldsymbol{y}^1)$ for some $\mu^1, \ \zeta^1 \geq 0$.

2. Trace solutions of $(*)$ from $(\boldsymbol{x}^1, \boldsymbol{y}^1)$ with $\mu = \mu^1$ and $\zeta = \zeta^1$ as the parameters μ and ζ are decreased to zero.

This idea provides a theoretical basis for various methods such as Lemke's method and a method of tracing the central trajectory of linear complementarity problems.

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*This research was done while M. Kojima and S. Mizuno were visiting at the IBM Almaden Research Center during the summer of 1990. Partial support from the Office of Naval Research under Contract N00014-87-C-0820 is acknowledged.

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1. Introduction

Let R^n denote the n-dimensional Euclidean space, and

$$\begin{aligned} R^n_+ &= \{\boldsymbol{x} \in R^n : \boldsymbol{x} \geq \boldsymbol{0}\}, \\ R^n_{++} &= \{\boldsymbol{x} \in R^n : \boldsymbol{x} > \boldsymbol{0}\}. \end{aligned}$$

Let $\boldsymbol{f} : R^n \to R^n$ be a C^1-mapping, i.e., $\boldsymbol{f}$ is continuously differentiable. We define the complementarity problem [4; 5; 6; 7; 13; 14; 24; 24; 30] with the mapping $\boldsymbol{f}$:

CP[$\boldsymbol{f}$] : Find a pair $(\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}$ such that

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}),\ (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0} \text{ and } x_i y_i = 0 \ (i = 1, \ldots, n).$$

We say that an $(\boldsymbol{x}, \boldsymbol{y})$ is a feasible solution (respectively, a strictly positive feasible solution) of CP[$\boldsymbol{f}$] if it satisfies $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $(\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}$ (respectively, $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $(\boldsymbol{x}, \boldsymbol{y}) > \boldsymbol{0}$). When $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}$ for some $\boldsymbol{M} \in R^{n \times n}$ and $\boldsymbol{q} \in R^n$, we call the problem *linear* and otherwise *nonlinear*. We define

LCP[$\boldsymbol{M}$,$\boldsymbol{q}$] : Find a pair $(\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}$ such that

$$\boldsymbol{y} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q},\ (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0} \text{ and } x_i y_i = 0 \ (i = 1, \ldots, n).$$

For every $\boldsymbol{x} \in R^n$, we denote by $\boldsymbol{X} = \operatorname{diag} \boldsymbol{x} \in R^{n \times n}$ the diagonal matrix with the coordinates of the vector $\boldsymbol{x}$. Define the mapping $\boldsymbol{F} : R^{2n} \to R^{2n}$ by

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} \\ \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \end{pmatrix}. \tag{1}$$

Let $\boldsymbol{a} \geq \boldsymbol{0}$ and $\boldsymbol{b} > \boldsymbol{0}$ be vectors in R^n. We embed the problem CP[$\boldsymbol{f}$] in an artificial system of equations:

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) \equiv \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} \\ \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \mu \boldsymbol{a} \\ \zeta \boldsymbol{b} \end{pmatrix} \quad \text{and } (\boldsymbol{x}, \boldsymbol{y}, \mu, \zeta) \geq \boldsymbol{0}. \tag{2}$$

Here $0 \leq \mu \in R$ and $0 \leq \zeta \in R$ are parameters or artificial variables. Obviously, a pair $(\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}$ solves CP[$\boldsymbol{f}$] if and only if it solves (2) for $\mu = 0$ and $\zeta = 0$.

The system (2) provides us with a general theoretical framework for various homotopy continuation methods [16; 17; 19; 20; 23; 24; 26] which are often called *path-following* methods. To design a continuation method, we need to specify

(i) how to choose an initial point $(\boldsymbol{x}^1, \boldsymbol{y}^1)$ together with initial values μ^1 and ζ^1 of the parameters μ and ζ satisfying (2), and

(ii) how to decrease the parameters μ and ζ from their initial values μ^1 and ζ^1 to zero.

As we will see below, (i) and (ii) are closely related. We discuss (ii) first.

In general, we prepare in advance two nonnegative continuous functions $\bar{\mu}(t)$ and $\bar{\zeta}(t)$ $(t \geq 0)$ such that $\bar{\mu}(0) = \bar{\zeta}(0) = 0$. The functions $\bar{\mu}$ and $\bar{\zeta}$ control the decrease of the parameters μ and ζ as t tends to 0:

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) \equiv \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} \\ \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \bar{\mu}(t)\boldsymbol{a} \\ \bar{\zeta}(t)\boldsymbol{b} \end{pmatrix} \quad \text{and} \quad (\boldsymbol{x}, \boldsymbol{y}, t) \geq \boldsymbol{0}.$$

Alternatively, we can change the parameters μ and ζ adaptively during the execution of the algorithm. In this paper, however, we restrict ourselves to simple cases where the change of the parameters μ and ζ is governed by linear functions:

$$\bar{\mu}(t) = \alpha t \quad \text{and} \quad \bar{\zeta}(t) = \beta t \quad \text{for every } t \in R_+.$$

Here α and β are nonnegative constants but at least one of them is positive. Redefining $\alpha\boldsymbol{a}$ to be $\boldsymbol{a}$ and $\beta\boldsymbol{b}$ to be $\boldsymbol{b}$, we may assume without loss of generality that $\alpha = 1$ if $\alpha > 0$ and $\beta = 1$ if $\beta > 0$, respectively. Thus we have three typical models:

(a) $\alpha = 0$ and $\beta = 1$. In this case (2) turns out to be

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) \equiv \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} \\ \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ t\boldsymbol{b} \end{pmatrix} \quad \text{and } (\boldsymbol{x}, \boldsymbol{y}, t) \geq \boldsymbol{0}. \tag{3}$$

This is the system of equations whose solution set is traced by Lemke's method [23; 24] for LCP[$\boldsymbol{M}, \boldsymbol{q}$]. Since $\boldsymbol{b} > \boldsymbol{0}$, the set

$$\{(\boldsymbol{x}, \boldsymbol{y}, t) = (\boldsymbol{0}, \boldsymbol{f}(\boldsymbol{0}) + t\boldsymbol{b}, t) : t \geq 0,\ \boldsymbol{f}(\boldsymbol{0}) + t\boldsymbol{b} \geq \boldsymbol{0}\}$$

forms a ray consisting of solutions of (3), from which Lemke's method starts. Several classes of linear complementarity problems are known to be solvable by Lemke's method. See, for example, [6; 30] for more details. The system (3) was also utilized in [7; 14] where the existence of solutions of CP[$\boldsymbol{f}$] was investigated.

(b) $\alpha = 1$ and $\beta = 0$. In this case (2) turns out to be

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) \equiv \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} \\ \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} t\boldsymbol{a} \\ \boldsymbol{0} \end{pmatrix} \quad \text{and } (\boldsymbol{x}, \boldsymbol{y}, t) \geq \boldsymbol{0}. \tag{4}$$

Let $\boldsymbol{a} > \boldsymbol{0}$. Suppose $\boldsymbol{f} : R^n \to R^n$ has the form $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}$ for some positive semi-definite $\boldsymbol{M} \in R^{n \times n}$ and $\boldsymbol{q} \in R^n$. That is, we consider LCP[$\boldsymbol{M}, \boldsymbol{q}$] with a positive semi-definite matrix $\boldsymbol{M}$. We assume that LCP[$\boldsymbol{M}, \boldsymbol{q}$] has a strictly positive feasible solution. Then (4) has a unique solution $(\boldsymbol{\xi}(t), \boldsymbol{\eta}(t))$ for every

$t > 0$, which is smooth with respect to $t > 0$. Furthermore, the solution curve $\{(\boldsymbol{\xi}(t), \boldsymbol{\eta}(t)) : t > 0\}$ converges to a solution of LCP[$\boldsymbol{M}, \boldsymbol{q}$]. When we take $\boldsymbol{a} = (1, \ldots, 1)^T \in R^n$, the trajectory is called the *path of centers* or the *central trajectory*, which was originally studied in the context of linear and convex programs [34; 35] and later extended to LCP[$\boldsymbol{M}, \boldsymbol{q}$]. The existence of the path of centers leading to a solution of LCP[$\boldsymbol{M}, \boldsymbol{q}$] was shown independently in [25; 26; 33]. See also [16; 20]. The path of centers has played an essential role in the design of many interior point path-following methods for linear programs [12; 22; 28; 32], convex quadratic programs [11; 29] and LCP[$\boldsymbol{M}, \boldsymbol{q}$] [18; 21].

(c) $\alpha = 1$ and $\beta = 1$. In this case (2) turns out to be

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) \equiv \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} \\ \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} t\boldsymbol{a} \\ t\boldsymbol{b} \end{pmatrix} \quad \text{and } (\boldsymbol{x}, \boldsymbol{y}, t) \geq \boldsymbol{0}. \tag{5}$$

The homotopy continuation method given in [17] for the nonlinear CP[$\boldsymbol{f}$] utilizes this system. Let $\boldsymbol{x}^1 > \boldsymbol{0}$ and take a sufficiently large $\boldsymbol{y}^1$ such that $\boldsymbol{y}^1 - \boldsymbol{f}(\boldsymbol{x}^1) > \boldsymbol{0}$. Define $\boldsymbol{a} = \boldsymbol{X}^1\boldsymbol{y}^1$, $\boldsymbol{b} = \boldsymbol{y}^1 - \boldsymbol{f}(\boldsymbol{x}^1)$ and $t^1 = 1$. Then the point $(\boldsymbol{x}^1, \boldsymbol{y}^1, t^1)$ satisfies (5). The existence of the trajectory starting from $(\boldsymbol{x}^1, \boldsymbol{y}^1, t^1)$ and leading to a solution of CP[$\boldsymbol{f}$] was shown in [19] when $\boldsymbol{f}$ is a monotone mapping, and in [20] when $\boldsymbol{f}$ is a uniform P-function. The existence of the trajectory as well as a numerical method for tracing it was studied in [17] for more general P_0-function cases.

It is interesting to compare (3) of **(a)** with (5) of **(c)**. Both contain the subsystem $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) + t\boldsymbol{b}$. The only difference lies in the choice of $\boldsymbol{a}$; if we take $\boldsymbol{a}$ to be $\boldsymbol{0}$ in $\boldsymbol{X}\boldsymbol{y} = t\boldsymbol{a}$ of (5), we obtain (3). This implies that the model **(a)** is an extreme variant of **(c)**. On the other hand, Kojima, Megiddo and Noma [17] took a strictly positive $\boldsymbol{a}$ in their homotopy continuation method for CP[$\boldsymbol{f}$], which may be regarded as another extreme variant of **(c)**. One purpose of the present paper is to investigate general cases where some components of $\boldsymbol{a}$ are zero and the others are positive.

So far, the studies of both the interior point path-following method in the model **(b)** and the homotopy continuation method in the model **(c)** were limited to the class of complementarity problems with P_0-functions (P_0-matrices in linear cases). See [17; 18; 19; 20]. On the other hand, Lemke's method [24] in **(a)** solves linear complementarity problems with larger classes of matrices, some of which are not contained in the class P_0. The classes of L_*-matrices [6] and copositive-plus matrices [23] fall in this category. Another purpose of this paper is to fill this gap. We will apply the model **(c)** to LCP[$\boldsymbol{M}, \boldsymbol{q}$] with an L_*-matrix $\boldsymbol{M}$ and a copositive-plus matrix $\boldsymbol{M}$.

2. Compactifying the domain of the parameter t

Define $\boldsymbol{G} : R^{2n} \to R^{2n}$ by

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} - \boldsymbol{a} \\ \boldsymbol{y} - \boldsymbol{b} \end{pmatrix} .$$

Let $\boldsymbol{H} : R^{2n} \times [0,1] \to R^{2n}$ be a convex homotopy between the mappings $\boldsymbol{F} : R^{2n} \to R^{2n}$ and $\boldsymbol{G} : R^{2n} \to R^{2n}$ given by

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, \theta) &\equiv (1-\theta)\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}) + \theta \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) \\ &= \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} - \theta \boldsymbol{a} \\ \boldsymbol{y} - (1-\theta)\boldsymbol{f}(\boldsymbol{x}) - \theta \boldsymbol{b} \end{pmatrix} . \end{aligned} \tag{6}$$

Consider the system

$$\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, \theta) = \boldsymbol{0}, \quad (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0} \quad \text{and} \quad \theta \in [0,1]. \tag{7}$$

This system serves as a continuous deformation from the artificial system of equations

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0} \text{ and } (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0},$$

which has a known solution $(\boldsymbol{B}^{-1}\boldsymbol{a}, \boldsymbol{b})$ (where $\boldsymbol{B} = \text{diag}\, \boldsymbol{b}$) into the system

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0} \text{ and } (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0},$$

which is equivalent to CP[$\boldsymbol{f}$].

We will show below that (7) is equivalent to (5). Define $\boldsymbol{\psi} : R^{2n} \times R_+ \to R^{2n} \times [0,1)$ by

$$\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}, t) = \left(\boldsymbol{x},\ \tfrac{1}{1+t}\boldsymbol{y},\ \tfrac{t}{1+t}\right) \quad \text{for every} \quad (\boldsymbol{x}, \boldsymbol{y}, t) \in R^{2n} \times R_+.$$

Apparently, $\boldsymbol{\psi}$ is a diffeomorphism from $R^{2n} \times R_+$ onto $R^{2n} \times [0,1)$. We have

(i) $(\boldsymbol{x}, \boldsymbol{y}, t)$ is a solution of (5) if and only if $\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}, t)$ is a solution of (7),

and

(ii) every solution $(\boldsymbol{x}, \boldsymbol{y}, \theta)$ of (7) such that $\theta < 1$ is mapped diffeomorphically to a solution $\boldsymbol{\psi}^{-1}(\boldsymbol{x}, \boldsymbol{y}, \theta) = \left(\boldsymbol{x}, \tfrac{1}{1-\theta}\boldsymbol{y}, \tfrac{\theta}{1-\theta}\right)$ of (5).

To show the equivalence between (5) and (7), we also need to consider solutions of (7) on the hyperplane $\{(\boldsymbol{x}, \boldsymbol{y}, \theta) : \theta = 1\}$. Recall that we have assumed $\boldsymbol{b} > \boldsymbol{0}$. Hence, if we fix θ to be 1, then (7) has a unique solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 1) = (\boldsymbol{B}^{-1}\boldsymbol{a}, \boldsymbol{b}, 1)$. This solution of

(7) corresponds to a "limit" of solutions of (5) rather than a particular solution thereof, as we show below.

We observe that

$$D\boldsymbol{H}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 1) = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{B}^{-1}\boldsymbol{A} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}$$

(i.e., the Jacobian matrix of the mapping $\boldsymbol{H}$ with respect to the vector $(\boldsymbol{x}, \boldsymbol{y})$ at $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 1) = (\boldsymbol{B}^{-1}\boldsymbol{a}, \boldsymbol{b}, 1)$) is nonsingular. Here $\boldsymbol{A} = \operatorname{diag} \boldsymbol{a}$, $\boldsymbol{B} = \operatorname{diag} \boldsymbol{b}$, and $\boldsymbol{I} \in R^{n \times n}$ is the identity. Hence, by the implicit function theorem, for every θ sufficiently close to 1, (7) has a unique solution $(\boldsymbol{x}(\theta), \boldsymbol{y}(\theta), \theta)$, which is smooth in the parameter θ, in a neighborhood of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 1)$ such that $(\boldsymbol{x}(1), \boldsymbol{y}(1)) = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$. Thus, there always exists a trajectory of the form

$$T_\delta = \{(\boldsymbol{x}(\theta), \boldsymbol{y}(\theta), \theta) : 1 - \delta < \theta \leq 1\}$$

in a neighborhood of the known solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 1)$ for some $\delta > 0$. Therefore,

(iii) the set

$$\{\boldsymbol{\psi}^{-1}(\boldsymbol{x}(\theta), \boldsymbol{y}(\theta), \theta) : 1 - \delta < \theta < 1\} = \left\{\left(\boldsymbol{x}(\theta), \tfrac{1}{1-\theta}\boldsymbol{y}(\theta), \tfrac{\theta}{1-\theta}\right) : 1 - \delta < \theta < 1\right\}$$

forms a trajectory consisting of solutions of (5) such that $\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}, t)$ converges to a unique solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 1)$ of (7) on the hyperplane $\{(\boldsymbol{x}, \boldsymbol{y}, \theta) : \theta = 1\}$ along the trajectory as t tends to infinity.

We can also see that

(iv) if $\{(\boldsymbol{x}^p, \boldsymbol{y}^p, t^p)\}$ is a sequence of solutions of (5) such that t^p tends to infinity and $\boldsymbol{x}^p$ converges to some $\hat{\boldsymbol{x}} \in R^n$ as p tends to infinity, then $\boldsymbol{\psi}(\boldsymbol{x}^p, \boldsymbol{y}^p, t^p) \in T_\delta$ for every sufficiently large p and $\boldsymbol{\psi}(\boldsymbol{x}^p, \boldsymbol{y}^p, t^p)$ converges to the unique solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 1) = (\boldsymbol{B}^{-1}\boldsymbol{a}, \boldsymbol{b}, 1)$ of (7) on the hyperplane $\{(\boldsymbol{x}, \boldsymbol{y}, \theta) : \theta = 1\}$.

Roughly speaking, the domain $[0, \infty]$ of the parameter t in (5) has been compactified into the domain $[0, 1]$ of the parameter θ in (7). In the remainder of the paper, we will deal with (7) instead of (5) since the former is mathematically easier to handle.

3. Existence of a trajectory

Let S denote the set of solutions $(\boldsymbol{x}, \boldsymbol{y}, \theta)$ of (7) such that $\theta > 0$;

$$S = \{(\boldsymbol{x}, \boldsymbol{y}, \theta) : \boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, \theta) = \boldsymbol{0},\ (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0},\ 0 < \theta \leq 1\}.$$

The unique solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 1) = (\boldsymbol{B}^{-1}\boldsymbol{a}, \boldsymbol{b}, 1)$ of (7) on the hyperplane $\{(\boldsymbol{x}, \boldsymbol{y}, \theta) : \theta = 1\}$, as well as the trajectory T_δ emanating from the point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 1)$, are contained in the set S. Let T denote the connected component of S which contains T_δ.

The following theorem ensures that the set T generically forms a trajectory.

Theorem 3.1. *Let $\boldsymbol{a} \in R^n_+$ be fixed. Then, for almost every $\boldsymbol{b} \in R^n_{++}$, the set T forms a trajectory, a 1-dimensional manifold which is homeomorphic to $(0,1]$, such that*

$$T = \{(\boldsymbol{\xi}(s), \boldsymbol{\eta}(s), \tau(s)) : 0 < s \leq 1\}$$

and $\lim_{s\to 0} \tau(s) = 0$ whenever T is bounded. Here $\boldsymbol{\xi} : (0,1] \to R^n$, $\boldsymbol{\eta} : (0,1] \to R^n$ and $\tau : (0,1] \to (0,1]$ are piecewise C^1-mappings, or C^1-mappings when $\boldsymbol{a} > \mathbf{0}$.

Proof: The proof of the theorem is divided into two parts. First, we reformulate the set S in terms of the solution set of a system consisting of n piecewise C^1 equations and $n+1$ variables. Later, we will utilize the notion of a regular value of a piecewise C^1-mapping to show that generically the set of the solutions of the system of piecewise C^1 equations is a disjoint union of 1-dimensional piecewise smooth manifolds. The first part is interesting in its own right. But the second part, which requires some other notions such as a polyhedral subdivision of R^n and a piecewise C^1-mapping on it, would be lengthy but rather standard in the theory of continuation methods [2; 3; 10], so we omit the details of the second part. See, for example, [1; 15].

For every $\alpha \in R$ and $\boldsymbol{u} = (u_1, \ldots, u_n)^T \in R^n$, we use the notation

$$\alpha^+ = \max\{0, \alpha\}, \quad \alpha^- = \min\{0, \alpha\} \text{ and } \boldsymbol{u}^\pm = (u_1^\pm, \ldots, u_n^\pm).$$

The correspondences $\boldsymbol{u} \to \boldsymbol{u}^+$ and $\boldsymbol{u} \to \boldsymbol{u}^-$ should be regarded as piecewise linear mappings from R^n into itself. For every $\boldsymbol{u} \in R^n$, obviously,

$$\boldsymbol{u}^+ \geq \mathbf{0}, \ -(\boldsymbol{u}^-) \geq \mathbf{0} \text{ and } \ u_i^+ u_i^- = 0 \ (i = 1, \ldots, n).$$

With $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ we can rewrite CP[$\boldsymbol{f}$] as the system consisting of n piecewise C^1 equations and n variables $u_1, \ldots, u_n$:

$$\boldsymbol{u}^- + \boldsymbol{f}(\boldsymbol{u}^+) = \mathbf{0}.$$

This formulation of CP[$\boldsymbol{f}$] was given in [8]. See also [27]. When we consider LCP[$\boldsymbol{M}, \boldsymbol{q}$], the system above turns out to be piecewise linear:

$$\boldsymbol{u}^- + \boldsymbol{M}\boldsymbol{u}^+ + \boldsymbol{q} = \mathbf{0}.$$

Smale [33] proposed a "regularization" of the piecewise linear system for applying Newton's Method to LCP[$\boldsymbol{M}, \boldsymbol{q}$]. According to the analysis given in [16] on Smale's regularization technique, we will apply the regularization technique to CP[$\boldsymbol{f}$], and derive another representation of the set S of solutions $(\boldsymbol{x}, \boldsymbol{y}, \theta)$ of (7) such that $\theta > 0$. For every $\alpha \geq 0$, $\boldsymbol{a} = (a_1, \ldots, a_n)^T \in R^n_+$, $\nu \in R$ and $\boldsymbol{u} = (u_1, \ldots, u_n)^T \in R^n$, define

$$\begin{aligned} \varphi^\pm(\nu; \alpha) &= \frac{\nu \pm \sqrt{\nu^2 + 4\alpha}}{2} \quad \text{and} \\ \boldsymbol{\Phi}^\pm(\boldsymbol{u}; \boldsymbol{a}) &= (\varphi^\pm(u_1; a_1), \ldots, \varphi^\pm(u_n; a_n)). \end{aligned}$$

Then $\boldsymbol{\Phi}^+(\boldsymbol{u};\boldsymbol{a})$ and $\boldsymbol{\Phi}^-(\boldsymbol{u};\boldsymbol{a})$ are piecewise C^1 mappings (or C^1 mappings when $\boldsymbol{a} > \boldsymbol{0}$) from R^n into itself such that

$$\varphi^+(u_i;a_i) \geq 0\ ,\ -\varphi^-(u_i;a_i) \geq 0\ ,\ \text{and } \varphi^+(u_i;a_i)(-\varphi^-(u_i;a_i)) = a_i$$

$(i = 1, \dots, n)$. Specifically,

$$\boldsymbol{\Phi}^\pm(\boldsymbol{u};\boldsymbol{0}) = \boldsymbol{u}^\pm \ \text{ for every } \boldsymbol{u} \in R^n.$$

Now we consider the system

$$\boldsymbol{\Phi}^-(\boldsymbol{u};\theta\boldsymbol{a}) + (1-\theta)\boldsymbol{f}(\boldsymbol{\Phi}^+(\boldsymbol{u};\theta\boldsymbol{a})) + \theta\boldsymbol{b} = \boldsymbol{0} \ \text{ and } (\boldsymbol{u},\theta) \in R^n \times [0,1]. \tag{8}$$

The system (8) is equivalent to (7) in the sense that $(\boldsymbol{u},\theta)$ is a solution of (8) if and only if $(\boldsymbol{x},\boldsymbol{y},\theta) = (\boldsymbol{\Phi}^+(\boldsymbol{u};\theta\boldsymbol{a}), -\boldsymbol{\Phi}^-(\boldsymbol{u};\theta\boldsymbol{a}), \theta)$ is a solution of (7). To prove the theorem, we are only concerned with the set of solutions $(\boldsymbol{u},\theta)$ of (8) with $\theta > 0$. Hence, defining the piecewise C^1-mapping $\boldsymbol{P} : R^n \times (0,1] \to R^n$ by

$$\boldsymbol{P}(\boldsymbol{u},\theta;\boldsymbol{a}) = \frac{\boldsymbol{\Phi}^-(\boldsymbol{u};\theta\boldsymbol{a}) + (1-\theta)\boldsymbol{f}(\boldsymbol{\Phi}^+(\boldsymbol{u};\theta\boldsymbol{a}))}{\theta} \ \text{ for every } (\boldsymbol{u},\theta) \in R^n \times (0,1],$$

we will rewrite (8) as

$$\boldsymbol{P}(\boldsymbol{u},\theta;\boldsymbol{a}) = -\boldsymbol{b} \ \text{ and } (\boldsymbol{u},\theta) \in R^n \times (0,1].$$

When the vector $\boldsymbol{a}$ is strictly positive, the mapping $\boldsymbol{P} : R^n \times (0,1] \to R^n$ is C^1 over R^n. When some of the components of $\boldsymbol{a} \geq \boldsymbol{0}$ are zero, however, the mapping $\boldsymbol{P}$ is generally a piecewise C^1-mapping such that it is C^1 on each set of the form $Q \times (0,1]$, where Q denotes an orthant of R^n. Let $\hat{S}$ denote the set of solutions of the system above:

$$\hat{S} = \{(\boldsymbol{u},\theta) \in R^n \times (0,1] \ : \ \boldsymbol{P}(\boldsymbol{u},\theta;\boldsymbol{a}) = -\boldsymbol{b}\}.$$

Then $(\boldsymbol{u},\theta) \in \hat{S}$ if and only if $(\boldsymbol{\Phi}^+(\boldsymbol{u};\theta\boldsymbol{a}), -\boldsymbol{\Phi}^-(\boldsymbol{u};\theta\boldsymbol{a}), \theta) \in S$. Note that the correspondence

$$(\boldsymbol{u},\theta) \in \hat{S} \longrightarrow (\boldsymbol{\Phi}^+(\boldsymbol{u};\theta\boldsymbol{a}), -\boldsymbol{\Phi}^-(\boldsymbol{u};\theta\boldsymbol{a}), \theta) \in S$$

is one-to-one and piecewise C^1. Specifically, the set T corresponds to the set

$$\hat{T} = \{(\boldsymbol{u},\theta) : \boldsymbol{u} = \boldsymbol{x} - \boldsymbol{y},\ (\boldsymbol{x},\boldsymbol{y},\theta) \in T\}.$$

Conversely, the set T can be represented as

$$T = \{(\boldsymbol{\Phi}^+(\boldsymbol{u};\theta\boldsymbol{a}), -\boldsymbol{\Phi}^-(\boldsymbol{u};\theta\boldsymbol{a}), \theta) : (\boldsymbol{u},\theta) \in \hat{T}\}.$$

We also see that T is bounded if and only if $\hat{T}$ is.

Consequently, the theorem follows from the result on regular values of piecewise C^1-mappings.

(a)' Almost every $-\boldsymbol{b} < \mathbf{0}$ is a regular value of the piecewise C^1-mapping $\boldsymbol{P}$.

(b)' If $-\boldsymbol{b}$ is a regular value of the piecewise C^1-mapping $\boldsymbol{P}$ then $\hat{S}$ is disjoint union of smooth 1-dimensional manifolds; specifically its connected component $\hat{T}$ forms a piecewise smooth trajectory (or a smooth trajectory when $\boldsymbol{a} > \mathbf{0}$) such that either $\|\boldsymbol{u}\|$ tends to infinity or θ tends to 0 along the trajectory $\hat{T}$.

∎

In view of Theorem 3.1, we know that the set T generically forms a smooth or piecewise smooth trajectory. Furthermore, if the trajectory T is bounded, we guarantee that it will lead us to a solution of CP[$\boldsymbol{f}$]. The boundedness of S, which ensures the boundedness of T, will be discussed in the next section. In general, the trajectory T may not converge to any $(\boldsymbol{x}, \boldsymbol{y}, 0)$. It should be noted, however, that since T is bounded, there exists at least one limit point as θ tends to 0 along the trajectory, and every limit point is a solution of CP[$\boldsymbol{f}$].

4. Sufficient conditions for boundedness of the trajectory T

The following theorem can be derived easily from the Theorem of [20] and the relations (i) – (iv) of (2) and (7) which we established in Section 2.

Theorem 4.1. *Let $\boldsymbol{a} \geq \mathbf{0}$ and $\boldsymbol{b} > \mathbf{0}$. Suppose that $\boldsymbol{f} : R^n \rightarrow R^n$ is a uniform P-function, i.e., there exists a positive number γ satisfying*

$$\max_i \, (x_i^1 - x_i^2)(f_i(\boldsymbol{x}^1) - f_i(\boldsymbol{x}^2)) \geq \gamma \|\boldsymbol{x}^1 - \boldsymbol{x}^2\|^2 \quad \text{for every } \boldsymbol{x}^1,\ \boldsymbol{x}^2 \in R^n.$$

Then the set S is bounded. Furthermore, for each fixed $\theta \in [0,1]$, (7) has a unique solution $(\boldsymbol{\xi}(\theta), \boldsymbol{\eta}(\theta))$, which is continuous with respect to the parameter $\theta \in [0,1]$; hence the set T, as well as the set S can be written as

$$T = S = \{(\boldsymbol{\xi}(\theta), \boldsymbol{\eta}(\theta), \theta) : 0 < \theta \leq 1\}. \tag{9}$$

We call a continuous mapping $\boldsymbol{f} : R^n \rightarrow R^n$ *monotone* if

$$(\boldsymbol{x}^1 - \boldsymbol{x}^2)^T (\boldsymbol{f}(\boldsymbol{x}^1) - \boldsymbol{f}(\boldsymbol{x}^2)) \geq 0 \quad \text{for every } \boldsymbol{x}^1,\ \boldsymbol{x}^2 \in R^n.$$

The problem CP[$\boldsymbol{f}$] with a monotone function $\boldsymbol{f}$ has an important application to convex programs. See, for example, [13; 14].

Theorem 4.2. *Let $\boldsymbol{a} \geq \mathbf{0}$ and $\boldsymbol{b} > \mathbf{0}$. Suppose that the mapping $\boldsymbol{f} : R^n \rightarrow R^n$ is monotone and that* CP[$\boldsymbol{f}$] *has a strictly positive feasible solution. Then S is bounded. If $\boldsymbol{a} > \mathbf{0}$ then, for each fixed $\theta \in (0,1]$, (7) has a unique solution $(\boldsymbol{\xi}(\theta), \boldsymbol{\eta}(\theta))$, which is continuous with respect to the parameter $\theta \in (0,1]$; hence the set T as well as the set S can be written as in* (9).

Proof: Let $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ be a strictly positive feasible solution of CP[$\boldsymbol{f}$]. Define the positive numbers ϵ and ω by

$$\begin{aligned} \epsilon &= \min\{b_i,\ \tilde{x}_i,\ \tilde{y}_i\ :\ i = 1, \dots, n\}, \\ \omega &= \max\{b_i,\ \tilde{x}_i,\ \tilde{y}_i\ :\ i = 1, \dots, n\}. \end{aligned}$$

Suppose that $(\boldsymbol{x}, \boldsymbol{y}, \theta) \in S$. Then, by the monotonicity of $\boldsymbol{f}$, we have

$$\begin{aligned} 0 &\leq (1-\theta)(\boldsymbol{x} - \tilde{\boldsymbol{x}})^T(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\tilde{\boldsymbol{x}})) \\ &= (\boldsymbol{x} - \tilde{\boldsymbol{x}})^T(\boldsymbol{y} - \theta\boldsymbol{b} - (1-\theta)\tilde{\boldsymbol{y}}). \end{aligned} \tag{10}$$

Let $\boldsymbol{e} = (1, \dots, 1)^T \in R^n$. Then

$$\begin{aligned} \epsilon(\boldsymbol{e}^T\boldsymbol{x} + \boldsymbol{e}^T\boldsymbol{y}) &\leq (\theta\boldsymbol{b} + (1-\theta)\tilde{\boldsymbol{y}})^T\boldsymbol{x} + \tilde{\boldsymbol{x}}^T\boldsymbol{y} \quad \text{(by the definition of } \epsilon) \\ &\leq \boldsymbol{x}^T\boldsymbol{y} + \tilde{\boldsymbol{x}}^T(\theta\boldsymbol{b} + (1-\theta)\tilde{\boldsymbol{y}}) \quad \text{(by (10))} \\ &\leq \boldsymbol{e}^T\boldsymbol{a} + n\omega^2 \quad \text{(by } \boldsymbol{X}\boldsymbol{y} = \theta\boldsymbol{a} \text{ and the definition of } \omega). \end{aligned}$$

Thus we have shown that S is contained in the bounded set

$$\{(\boldsymbol{x}, \boldsymbol{y}, \theta) \in R^{2n+1}_{+} : \boldsymbol{e}^T\boldsymbol{x} + \boldsymbol{e}^T\boldsymbol{y} \leq (\boldsymbol{e}^T\boldsymbol{a} + n\omega^2)/\epsilon,\ \theta \leq 1\}.$$

The second assertion of the theorem follows from Corollary 1.2 of [19] and the relations (i) – (iv) of the (2) and (7) which we established in Section 2. ∎

In the remainder of this section, we consider LCP[$\boldsymbol{M}, \boldsymbol{q}$] with $\boldsymbol{M} \in R^{n\times n}$ and $\boldsymbol{q} \in R^n$. Then the mapping $\boldsymbol{H} : R^{2n} \times [0, 1] \to R^{2n}$ defined by (6) turns out to be

$$\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, \theta) = \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} - \theta\boldsymbol{a} \\ \boldsymbol{y} - (1-\theta)(\boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}) - \theta\boldsymbol{b} \end{pmatrix}.$$

The matrix $\boldsymbol{M}$ is called a *P-matrix* if all its principal minors are positive, and a *positive semi-definite matrix* if $\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} \geq 0$ for every $\boldsymbol{x} \in R^n$. Suppose $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}$ (where $\boldsymbol{q} \in R^n$). It is well-known that $\boldsymbol{M}$ is a P-matrix (respectively, positive semi-definite) if and only if $\boldsymbol{f}$ is a uniform P-function (respectively, a monotone mapping). Therefore, as a corollary of the theorems above, we obtain:

Corollary 4.3. *Let $\boldsymbol{a} \geq \boldsymbol{0}$ and $\boldsymbol{b} > \boldsymbol{0}$. Suppose*

(i) *$\boldsymbol{M}$ is a P-matrix, or*
(ii) *$\boldsymbol{M}$ is a positive semi-definite matrix and* LCP[$\boldsymbol{M}, \boldsymbol{q}$] *has a strictly positive feasible solution.*

Then the set $S = \{(\boldsymbol{x}, \boldsymbol{y}, \theta) : \boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, \theta) = \boldsymbol{0},\ (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0},\ 0 < \theta \leq 1\}$ is bounded.

The results above will be generalized in Theorems 4.5 and 4.7.

Lemma 4.4. *Let* $\boldsymbol{a} \geq \mathbf{0}$ *and* $\boldsymbol{b} > \mathbf{0}$*. Suppose that the set* $S = \{(\boldsymbol{x}, \boldsymbol{y}, \theta) : \boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, \theta) = \mathbf{0},\ (\boldsymbol{x}, \boldsymbol{y}) \geq \mathbf{0},\ 0 < \theta \leq 1\}$ *is unbounded. Then there exist* $\delta \geq 0$ *and* $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in R^{2n}$ *such that*

$$\boldsymbol{e}^T \boldsymbol{\xi} = 1,\ \xi_i \eta_i = 0\ (i = 1, \ldots, n), \quad \boldsymbol{\eta} = \boldsymbol{M}\boldsymbol{\xi} + \delta \boldsymbol{b} \text{ and } (\boldsymbol{\xi}, \boldsymbol{\eta}) \geq \mathbf{0}. \tag{11}$$

Proof: By the assumption, there exists a sequence $\{(\boldsymbol{x}^p, \boldsymbol{y}^p, \theta^p)\} \subset S$ such that $\lim_{p \to \infty} \boldsymbol{e}^T \boldsymbol{x}^p = \infty$. Hence, for $p = 1, 2, \ldots$, we have

$$x_i^p y_i^p = \theta^p a_i \quad (i = 1, \ldots, n), \tag{12}$$

$$\boldsymbol{y}^p = (1 - \theta^p)(\boldsymbol{M}\boldsymbol{x}^p + \boldsymbol{q}) + \theta^p \boldsymbol{b}, \tag{13}$$

$$(\boldsymbol{x}^p, \boldsymbol{y}^p) \geq \mathbf{0}. \tag{14}$$

Since θ^p lies in the interval $(0, 1]$ $(p = 1, 2, \ldots)$, we can take a subsequence of $\{(\boldsymbol{x}^p, \boldsymbol{y}^p, \theta^p)\}$ such that θ^p converges to some $\theta^* \in [0, 1]$ along the subsequence. For simplicity of notation, we assume that the sequence itself converges to some $\theta^* \in [0, 1]$. We first deal with the case that $0 \leq \theta^* < 1$. From the relations (12), (13) and (14) above, we have

$$\frac{x_i^p}{\boldsymbol{e}^T \boldsymbol{x}^p} \frac{y_i^p}{\boldsymbol{e}^T \boldsymbol{x}^p} = \frac{\theta^p a_i}{(\boldsymbol{e}^T \boldsymbol{x}^p)^2} \quad (i = 1, \ldots, n),$$

$$\frac{\boldsymbol{y}^p}{\boldsymbol{e}^T \boldsymbol{x}^p} = \frac{(1 - \theta^p)(\boldsymbol{M}\boldsymbol{x}^p + \boldsymbol{q}) + \theta^p \boldsymbol{b}}{\boldsymbol{e}^T \boldsymbol{x}^p},$$

$$\frac{(\boldsymbol{x}^p, \boldsymbol{y}^p)}{\boldsymbol{e}^T \boldsymbol{x}^p} \geq \mathbf{0}.$$

Choosing an appropriate subsequence if necessary, we may assume without loss of generality that $\dfrac{\boldsymbol{x}^p}{\boldsymbol{e}^T \boldsymbol{x}^p}$ converges to some $\boldsymbol{\xi} \in R^n$ such that $\boldsymbol{e}^T \boldsymbol{\xi} = 1$. Hence, taking the limit in the above relations as p tends to infinity, we have

$$\xi_i \eta_i' = 0\ (i = 1, \ldots, n), \quad \boldsymbol{\eta}' = (1 - \theta^*)\boldsymbol{M}\boldsymbol{\xi} \ \text{ and } \ (\boldsymbol{\xi}, \boldsymbol{\eta}') \geq \mathbf{0}$$

for some $\boldsymbol{\eta}'$. Thus, letting $\boldsymbol{\eta} = \dfrac{\boldsymbol{\eta}'}{1 - \theta^*}$ and $\delta = 0$, we obtain (11).

Now we deal with the case that $\theta^* = 1$. Assume that $\|(1 - \theta^p)\boldsymbol{x}^p\|$ converges to zero. Then we see from (13) that $\boldsymbol{y}^p$ converges to $\boldsymbol{b}$. Hence, it follows from (12) that $\boldsymbol{x}^p$ converges to $\boldsymbol{B}^{-1}\boldsymbol{a}$. This contradicts the assumption that the sequence $\{(\boldsymbol{x}^p, \boldsymbol{y}^p, \theta^p)\}$ is unbounded. Therefore we only have to deal with the case where either for some $\kappa > 0$,

$$\lim_{p \to \infty} (1 - \theta^p) \boldsymbol{e}^T \boldsymbol{x}^p = \kappa \tag{15}$$

or

$$\lim_{p\to\infty}(1-\theta^p)\boldsymbol{e}^T\boldsymbol{x}^p = \infty \tag{16}$$

On the other hand, it follows from (12), (13) and (14) that

$$\begin{aligned}
&\frac{(1-\theta^p)x_i^p}{(1-\theta^p)\boldsymbol{e}^T\boldsymbol{x}^p}\frac{y_i^p}{(1-\theta^p)\boldsymbol{e}^T\boldsymbol{x}^p} = \frac{(1-\theta^p)\theta^p a_i}{((1-\theta^p)\boldsymbol{e}^T\boldsymbol{x}^p)^2}\ (i=1,\dots,n),\\
&\frac{\boldsymbol{y}^p}{(1-\theta^p)\boldsymbol{e}^T\boldsymbol{x}^p} = \frac{\boldsymbol{M}(1-\theta^p)\boldsymbol{x}^p + (1-\theta^p)\boldsymbol{q} + \theta^p\boldsymbol{b}}{(1-\theta^p)\boldsymbol{e}^T\boldsymbol{x}^p},\\
&\frac{((1-\theta^p)\boldsymbol{x}^p, \boldsymbol{y}^p)}{(1-\theta^p)\boldsymbol{e}^T\boldsymbol{x}^p} \geq \boldsymbol{0}.
\end{aligned}$$

We may further assume without loss of generality that $\dfrac{(1-\theta^p)\boldsymbol{x}^p}{(1-\theta^p)\boldsymbol{e}^T\boldsymbol{x}^p}$ converges to some $\boldsymbol{\xi}$. Thus, taking the limit as p tends to infinity above, we obtain (11) with $\delta = \dfrac{1}{\kappa}$ if (15) occurs and $\delta = 0$ if (16) occurs. This completes the proof. ∎

A matrix $\boldsymbol{M} \in R^{n\times n}$ is called an L_**-matrix* if for every nonzero $\boldsymbol{\xi} \geq \boldsymbol{0}$, there is an index i such that $\xi_i > 0$ and $[\boldsymbol{M}\boldsymbol{\xi}]_i > 0$, where $[\boldsymbol{M}\boldsymbol{\xi}]_i$ denotes the ith component of the vector $\boldsymbol{M}\boldsymbol{\xi}$. The corresponding class L_* contains the class of P-matrices since the latter are characterized by the condition that for every nonzero $\boldsymbol{\xi} \in R^n$, there is an index i such that $\xi_i[\boldsymbol{M}\boldsymbol{\xi}]_i > 0$ (see [9]). If $\boldsymbol{M}$ is an L_*-matrix, LCP$[\boldsymbol{M}, \boldsymbol{q}]$ always has a solution for any $\boldsymbol{q}$ (see [6]).

A matrix $\boldsymbol{M} \in R^{n\times n}$ is called *copositive* if $\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} \geq 0$ for every $\boldsymbol{x} \geq \boldsymbol{0}$. The matrix $\boldsymbol{M}$ is called *copositive-plus* if it is copositive and

$$\boldsymbol{x} \geq \boldsymbol{0} \text{ and } \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} = 0 \text{ always imply } \boldsymbol{x}^T(\boldsymbol{M}+\boldsymbol{M}^T)\boldsymbol{x} = \boldsymbol{0}\ .$$

The class of copositive-plus matrices contains the class of positive semi-definite matrices. It is well-known that LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has a solution if and only if it is feasible, i.e., there is an $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ such that $\hat{\boldsymbol{y}} = \boldsymbol{M}\hat{\boldsymbol{x}} + \boldsymbol{q}$ and $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \geq \boldsymbol{0}$. It should be noted that the existence of a solution depends on the constant vector $\boldsymbol{q}$. But Lemma 4.4 does not involve the constant vector $\boldsymbol{q}$. This suggests that we cannot apply Lemma 4.4 directly to LCP$[\boldsymbol{M}, \boldsymbol{q}]$ to show the boundedness of S. We need to transform LCP$[\boldsymbol{M}, \boldsymbol{q}]$ into an equivalent linear complementarity problem, to which we will apply Lemma 4.4.

We assume below that the matrix $\boldsymbol{M}$ is either an L_*-matrix or a copositive-plus one.

Theorem 4.5. *Let $\boldsymbol{a} \geq \boldsymbol{0}$ and $\boldsymbol{b} > \boldsymbol{0}$. Suppose that $\boldsymbol{M}$ is an L_*-matrix. Then the set $S = \{(\boldsymbol{x}, \boldsymbol{y}, \theta) : \boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, \theta) = \boldsymbol{0},\ (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0},\ 0 < \theta \leq 1\}$ is bounded.*

Proof: Assume, on the contrary, that S is unbounded. By Lemma 4.4, there exist a nonnegative number δ and an $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in R^{2n}$ satisfying (11). It follows that

$$\boldsymbol{e}^T \boldsymbol{\xi} = 1, \ \boldsymbol{\xi} \geq \boldsymbol{0} \ \text{ and } \xi_i [\boldsymbol{M}\boldsymbol{\xi}]_i = -\xi_i \delta b_i \leq 0 \ (i = 1, \ldots, n).$$

This contradicts the assumption that $\boldsymbol{M}$ is an L_*-matrix. ∎

Consider now the problem LCP$[\boldsymbol{M}, \boldsymbol{q}]$ with a copositive-plus matrix. Let

$$\boldsymbol{M}' = \boldsymbol{M} + \boldsymbol{q}\boldsymbol{q}^T.$$

The following lemma shows that LCP$[\boldsymbol{M}, \boldsymbol{q}]$ is equivalent to LCP$[\boldsymbol{M}', \boldsymbol{q}]$ whenever $\boldsymbol{M}$ is copositive-plus.

Lemma 4.6. *Suppose $\boldsymbol{M}$ is copositive-plus.*

(i) *If there is a nonzero $\boldsymbol{\xi} \in R^n$ such that*

$$\boldsymbol{\xi} \geq \boldsymbol{0}, \ \boldsymbol{M}\boldsymbol{\xi} \geq \boldsymbol{0}, \ \boldsymbol{\xi}^T \boldsymbol{M} \boldsymbol{\xi} = \boldsymbol{0} \ \textit{and} \ \boldsymbol{q}^T \boldsymbol{\xi} < 0,$$

LCP$[\boldsymbol{M}, \boldsymbol{q}]$ *has no feasible solution.*

(ii) *If there is a nonzero $\boldsymbol{\xi} \in R^n$ such that*

$$\boldsymbol{\xi} \geq \boldsymbol{0}, \ \boldsymbol{M}\boldsymbol{\xi} \geq \boldsymbol{0}, \ \boldsymbol{\xi}^T \boldsymbol{M} \boldsymbol{\xi} = \boldsymbol{0} \ \textit{and} \ \boldsymbol{q}^T \boldsymbol{\xi} \leq 0,$$

LCP$[\boldsymbol{M}, \boldsymbol{q}]$ *has no strictly positive feasible solution.*

(iii) *If $(\boldsymbol{x}, \boldsymbol{y})$ is a solution of* LCP$[\boldsymbol{M}, \boldsymbol{q}]$ *then $1 - \boldsymbol{q}^T \boldsymbol{x} \geq 1$ and $(\boldsymbol{x}', \boldsymbol{y}') = \dfrac{(\boldsymbol{x}, \boldsymbol{y})}{1 - \boldsymbol{q}^T \boldsymbol{x}}$ is a solution of* LCP$[\boldsymbol{M}', \boldsymbol{q}]$.

(iv) *Suppose that $(\boldsymbol{x}', \boldsymbol{y}')$ is a solution of the* LCP$[\boldsymbol{M}', \boldsymbol{q}]$. *If $1 + \boldsymbol{q}^T \boldsymbol{x}' > 0$ then*

$$(\boldsymbol{x}, \boldsymbol{y}) = \frac{(\boldsymbol{x}', \boldsymbol{y}')}{1 + \boldsymbol{q}^T \boldsymbol{x}'}$$

is a solution of the LCP$[\boldsymbol{M}, \boldsymbol{q}]$. *If $1 + \boldsymbol{q}^T \boldsymbol{x}' \leq 0$ then* LCP$[\boldsymbol{M}, \boldsymbol{q}]$ *has no feasible solution.*

Proof: (i) and (ii): Since $\boldsymbol{M}$ is copositive-plus, we see from the assumption that $(\boldsymbol{M} + \boldsymbol{M}^T)\boldsymbol{\xi} = \boldsymbol{0}$. Hence, by the second relation of (i) (or (ii)), we have $\boldsymbol{\xi}^T \boldsymbol{M} \leq \boldsymbol{0}$. If, on the contrary, LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has a feasible solution or, respectively, a strictly positive feasible solution $(\boldsymbol{x}, \boldsymbol{y})$, then

$$0 \leq \boldsymbol{\xi}^T \boldsymbol{y} = \boldsymbol{\xi}^T \boldsymbol{M} \boldsymbol{x} + \boldsymbol{q}^T \boldsymbol{\xi} < 0$$

or, respectively,

$$0 < \boldsymbol{\xi}^T \boldsymbol{y} = \boldsymbol{\xi}^T \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}^T \boldsymbol{\xi} \leq 0 \ .$$

This is a contradiction. Thus we have shown (i) and (ii).

(iii): Since $\boldsymbol{M}$ is copositive-plus, we have $\boldsymbol{x}^T \boldsymbol{M}\boldsymbol{x} \geq 0$. On the other hand, we see $0 = \boldsymbol{x}^T \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}^T \boldsymbol{x}$. Hence $\boldsymbol{q}^T \boldsymbol{x} \leq 0$, which implies $1 - \boldsymbol{q}^T \boldsymbol{x} \geq 1$. Obviously, $(\boldsymbol{x}', \boldsymbol{y}') \geq \boldsymbol{0}$ and $x_i' y_i' = 0$ $(i = 1, \ldots, n)$. We also see that

$$\begin{aligned} \boldsymbol{M}'\boldsymbol{x}' + \boldsymbol{q} &= \boldsymbol{M}\frac{\boldsymbol{x}}{1 - \boldsymbol{q}^T \boldsymbol{x}} + \frac{\boldsymbol{q}^T \boldsymbol{x}}{1 - \boldsymbol{q}^T \boldsymbol{x}}\boldsymbol{q} + \boldsymbol{q} \\ &= \boldsymbol{M}\frac{\boldsymbol{x}}{1 - \boldsymbol{q}^T \boldsymbol{x}} + \frac{1}{1 - \boldsymbol{q}^T \boldsymbol{x}}\boldsymbol{q} = \boldsymbol{y}' \ . \end{aligned}$$

Thus we have shown that $(\boldsymbol{x}', \boldsymbol{y}')$ is a solution of the LCP$[\boldsymbol{M}', \boldsymbol{q}]$.

(iv): The first assertion of (iv) is easily verified. To see the second assertion of (iv), assume that $1 + \boldsymbol{q}^T \boldsymbol{x}' \leq 0$. Obviously $\boldsymbol{q}^T \boldsymbol{x}' \leq -1$. By the definition of $\boldsymbol{M}'$,

$$\boldsymbol{y}' = \boldsymbol{M}\boldsymbol{x}' + (1 + \boldsymbol{q}^T \boldsymbol{x}')\boldsymbol{q} \ .$$

Hence

$$0 = (\boldsymbol{x}')^T \boldsymbol{y}' = (\boldsymbol{x}')^T \boldsymbol{M}\boldsymbol{x}' + (1 + \boldsymbol{q}^T \boldsymbol{x}')\boldsymbol{q}^T \boldsymbol{x}'.$$

Since $\boldsymbol{M}$ is copositive-plus, we also have $(\boldsymbol{x}')^T \boldsymbol{M}\boldsymbol{x}' \geq 0$. Hence

$$1 + \boldsymbol{q}^T \boldsymbol{x}' = -\frac{(\boldsymbol{x}')^T \boldsymbol{M}\boldsymbol{x}'}{\boldsymbol{q}^T \boldsymbol{x}'} \geq 0,$$

which together with $1 + \boldsymbol{q}^T \boldsymbol{x}' \leq 0$ implies $1 + \boldsymbol{q}^T \boldsymbol{x}' = 0$. Therefore,

$$\boldsymbol{x}' \geq \boldsymbol{0}, \ \boldsymbol{y}' = \boldsymbol{M}\boldsymbol{x}' \geq \boldsymbol{0}, \ (\boldsymbol{x}')^T \boldsymbol{M}\boldsymbol{x}' = 0 \text{ and } \boldsymbol{q}^T \boldsymbol{x}' < 0.$$

By (i), we conclude that LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has no feasible solutions. ∎

Let

$$S' = \{(\boldsymbol{x}, \boldsymbol{y}, \theta) \in R_+^{2n} \times (0, 1] : \boldsymbol{H}'(\boldsymbol{x}, \boldsymbol{y}, \theta) = \boldsymbol{0}\},$$

where

$$\boldsymbol{H}'(\boldsymbol{x}, \boldsymbol{y}, \theta) = \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} - \theta \boldsymbol{a} \\ \boldsymbol{y} - (1 - \theta)(\boldsymbol{M}'\boldsymbol{x} + \boldsymbol{q}) - \theta \boldsymbol{b} \end{pmatrix} \ .$$

Now we are ready to apply Lemma 4.4 to LCP$[\boldsymbol{M}', \boldsymbol{q}]$.

Theorem 4.7. *Let* $\boldsymbol{a} \geq \boldsymbol{0}$ *and* $\boldsymbol{b} > \boldsymbol{0}$. *Suppose that*

(i) $\boldsymbol{M}$ *is copositive-plus, and*
(ii) LCP$[\boldsymbol{M}, \boldsymbol{q}]$ *has a strictly positive feasible solution.*

Then S' is bounded.

Proof: Assume, on the contrary, that S' is unbounded. Then, by Lemma 4.4, we can find a nonnegative δ and $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in R^{2n}$ such that

$$\boldsymbol{e}^T\boldsymbol{\xi} = 1,\ \xi_i\eta_i = 0\ (i = 1, \dots, n),\quad \boldsymbol{\eta} = \boldsymbol{M}'\boldsymbol{\xi} + \delta\boldsymbol{b} \text{ and } (\boldsymbol{\xi}, \boldsymbol{\eta}) \geq \boldsymbol{0}.$$

Hence, by the definition of $\boldsymbol{M}'$,

$$0 = \boldsymbol{\xi}^T\boldsymbol{\eta} = \boldsymbol{\xi}^T\boldsymbol{M}\boldsymbol{\xi} + (\boldsymbol{q}^T\boldsymbol{\xi})^2 + \delta\boldsymbol{b}^T\boldsymbol{\xi}.$$

Each of the terms on the right-hand side is nonnegative, so they are all zeros. Since $\boldsymbol{0} < \boldsymbol{b}$ and $\boldsymbol{0} \leq \boldsymbol{\xi} \neq \boldsymbol{0}$, it follows that $\boldsymbol{b}^T\boldsymbol{\xi} > 0$. Hence δ must be zero. Therefore we obtain

$$\boldsymbol{\xi} \geq \boldsymbol{0},\ \boldsymbol{M}\boldsymbol{\xi} \geq \boldsymbol{0},\ \boldsymbol{\xi}^T\boldsymbol{M}\boldsymbol{\xi} = 0 \text{ and } \boldsymbol{q}^T\boldsymbol{\xi} = 0.$$

By Lemma 4.6, we see that LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has no strictly positive feasible solutions. This contradicts the assumption (ii). ∎

It is known that LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has a solution, which can be computed by Lemke's method, under the assumption (i) above and

(ii)' LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has a feasible solution.

The assumption (ii)' is weaker than (ii) in the theorem. The combination of assumptions (i) and (ii)' is not sufficient to ensure the boundedness of S'. When S' is unbounded, either LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has no feasible solutions or the solution set of LCP$[\boldsymbol{M}, \boldsymbol{q}]$ is unbounded. In the remainder of this section, we will investigate these two cases in detail.

We consider a sequence $\{(\boldsymbol{x}^p, \boldsymbol{y}^p, \theta^p)\} \subset S'$. By the definition of S', each $(\boldsymbol{x}^p, \boldsymbol{y}^p, \theta^p)$ satisfies

$$\begin{aligned} &\boldsymbol{y}^p = (1-\theta^p)\{\boldsymbol{M}\boldsymbol{x}^p + (1+\boldsymbol{q}^T\boldsymbol{x}^p)\boldsymbol{q}\} + \theta^p\boldsymbol{b}, && (17)\\ &(\boldsymbol{x}^p, \boldsymbol{y}^p) \geq \boldsymbol{0}, \\ &x_i^p y_i^p = \theta^p a_i\ (i = 1, \dots, n). && (18) \end{aligned}$$

It follows from the relations above that

$$\begin{aligned} \boldsymbol{e}^T\boldsymbol{a} &\geq \theta^p\boldsymbol{e}^T\boldsymbol{a} \\ &= (\boldsymbol{x}^p)^T\boldsymbol{y}^p \\ &= (1-\theta^p)(\boldsymbol{x}^p)^T\boldsymbol{M}\boldsymbol{x}^p + (1-\theta^p)(1+\boldsymbol{q}^T\boldsymbol{x}^p)\boldsymbol{q}^T\boldsymbol{x}^p + \theta^p\boldsymbol{b}^T\boldsymbol{x}^p. \end{aligned}$$

Each term on the last equality satisfies

$$
\begin{aligned}
&(1-\theta^p)(\boldsymbol{x}^p)^T \boldsymbol{M}\boldsymbol{x}^p \geq 0, && \text{(since } \boldsymbol{M} \text{ is copositive-plus)}\\
&(1-\theta^p)(1+\boldsymbol{q}^T\boldsymbol{x}^p)\boldsymbol{q}^T\boldsymbol{x}^p \geq -\frac{1-\theta^p}{4} \geq -\frac{1}{4}, &&\\
&\theta^p \boldsymbol{b}^T\boldsymbol{x}^p \geq 0. &&
\end{aligned}
$$

Hence

$$
\theta^p \boldsymbol{e}^T\boldsymbol{a} + \frac{1-\theta^p}{4} \geq (1-\theta^p)(\boldsymbol{x}^p)^T\boldsymbol{M}\boldsymbol{x}^p, \tag{19}
$$

$$
\theta^p \boldsymbol{e}^T\boldsymbol{a} \geq (1-\theta^p)(1+\boldsymbol{q}^T\boldsymbol{x}^p)\boldsymbol{q}^T\boldsymbol{x}^p, \tag{20}
$$

$$
\theta^p \boldsymbol{e}^T\boldsymbol{a} + \frac{1-\theta^p}{4} \geq \theta^p\boldsymbol{b}^T\boldsymbol{x}^p. \tag{21}
$$

Assume now that $\|(\boldsymbol{x}^p, \boldsymbol{y}^p)\|$ tends to infinity as p tends to infinity. We see from (17) that $\|\boldsymbol{x}^p\|$ tends to infinity with p, hence also $\boldsymbol{b}^T\boldsymbol{x}^p$ tends to infinity with p. Thus, by (21),

$$
\lim_{p\to\infty} \theta^p = 0 \ .
$$

We know by this relation and (20) that the sequence $\{\boldsymbol{q}^T\boldsymbol{x}^p\}$ is bounded and that every limit point of the sequence lies in $[-1, 0]$.

Assuming -1 is a limit point of $\{\boldsymbol{q}^T\boldsymbol{x}^p\}$, we will show that LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has no feasible solutions. For simplicity of notation, we further assume that $\{\boldsymbol{q}^T\boldsymbol{x}^p\}$ itself converges to -1. Since $\lim_{p\to\infty}\theta^p = 0$, it follows from (18) that for each i, at least one of x_i^p and y_i^p converges to zero as p tends to infinity. Let

$$
I_0 = \{i : \lim_{p\to\infty} x_i^p = 0\ , \quad I_+ = \{i : 1 \leq i \leq n,\ i \notin I_0\}, \tag{22}
$$

$$
J_0 = \{j : \lim_{p\to\infty} y_j^p = 0\ , \quad J_+ = \{j : 1 \leq j \leq n,\ j \notin J_0\}. \tag{23}
$$

Then $I_0 \cup J_0 = \{1, \ldots, n\}$ and $I_+ \cap J_+ = \emptyset$. Let $\boldsymbol{I}_j$ and $\boldsymbol{M}_i$ denote the j'th column of the identity and the i'th column of $\boldsymbol{M}$, respectively. Define the set

$$
A = \left\{ \sum_{j\in J_+} \begin{pmatrix} \boldsymbol{I}_j \\ 0 \end{pmatrix} \eta_j - \sum_{i\in I_+} \begin{pmatrix} \boldsymbol{M}_i \\ -q_i \end{pmatrix} \xi_i \ :\ \xi_i \geq 0\ (i \in I_+),\ \eta_j \geq 0\ (j \in J_+) \right\}.
$$

By (17), we see that the vector

$$
-\sum_{j\in J_0} \begin{pmatrix} \boldsymbol{I}_j \\ 0 \end{pmatrix} y_j^p + \sum_{i\in I_0} \begin{pmatrix} \boldsymbol{M}_i \\ 0 \end{pmatrix} (1-\theta^p)x_i^p + \begin{pmatrix} (1-\theta^p)(1+\boldsymbol{q}^T\boldsymbol{x}^p)\boldsymbol{q} + \theta^p\boldsymbol{b} \\ (1-\theta^p)\sum_{i\in I_+} q_i x_i^p \end{pmatrix}
$$

is in A. Note that the vector converges to $\begin{pmatrix} \mathbf{0} \\ -1 \end{pmatrix}$ as $p \to \infty$, which belongs to A since A is closed. Therefore, there exist $\xi_i \geq 0$ $(i \in I_+)$ and $\eta_j \geq 0$ $(j \in J_+)$ such that

$$\sum_{j \in J_+} \begin{pmatrix} \boldsymbol{I}_j \\ 0 \end{pmatrix} \eta_j - \sum_{i \in I_+} \begin{pmatrix} \boldsymbol{M}_i \\ -q_i \end{pmatrix} \xi_i = \begin{pmatrix} \mathbf{0} \\ -1 \end{pmatrix}.$$

Letting $\xi_i = 0$ $(i \in I_0)$, we obtain the vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^T$ such that

$$\boldsymbol{\xi} \geq \mathbf{0}, \ \ \boldsymbol{M}\boldsymbol{\xi} \geq \mathbf{0}, \ \ \boldsymbol{\xi}^T \boldsymbol{M} \boldsymbol{\xi} = \mathbf{0} \text{ and } \boldsymbol{q}^T \boldsymbol{\xi} = -1.$$

Hence, by Lemma 4.6, LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has no feasible solutions.

Thus, we have shown that if -1 is a limit point of $\{\boldsymbol{q}^T \boldsymbol{x}^p\}$, then LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has no feasible solutions. This implies that if LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has a feasible solution then we can take an $\epsilon > 0$ such that for all sufficiently large p, $1 + \boldsymbol{q}^T \boldsymbol{x}^p \geq \epsilon$. Therefore, for all sufficiently large p, we may regard

$$(\hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p) = \left(\frac{(1-\theta^p)\boldsymbol{x}^p}{1 + \boldsymbol{q}^T \boldsymbol{x}^p}, \frac{\boldsymbol{y}^p}{1 + \boldsymbol{q}^T \boldsymbol{x}^p} \right)$$

as an approximate solution of LCP$[\boldsymbol{M}, \boldsymbol{q}]$ because it satisfies

$$\begin{aligned}
&\hat{\boldsymbol{y}}^p = \boldsymbol{M}\hat{\boldsymbol{x}}^p + (1-\theta^p)\boldsymbol{q} + \frac{\theta^p}{1 + \boldsymbol{q}^T \boldsymbol{x}^p}\boldsymbol{b}, \\
&\lim_{p \to \infty} \frac{\theta^p}{1 + \boldsymbol{q}^T \boldsymbol{x}^p}\boldsymbol{b} = \mathbf{0} \\
&(\hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p) \geq \mathbf{0}, \\
&\lim_{p \to \infty} \hat{x}_i^p \hat{y}_i^p = 0 \quad (i = 1, \ldots, n).
\end{aligned}$$

More precisely, if we define the index sets I_0 and J_0 as in (22) and (23), we can similarly prove that LCP$[\boldsymbol{M}, \boldsymbol{q}]$ has a solution $(\boldsymbol{x}, \boldsymbol{y})$ satisfying $x_i = 0$ $(i \in I_0)$ and $y_j = 0$ $(j \in J_0)$.

5. Concluding remarks

(A) The system (7) can be partitioned into two subsystems:

$$\boldsymbol{X}\boldsymbol{y} = \theta \boldsymbol{a} \ \text{ and } \ (\boldsymbol{x}, \boldsymbol{y}) \geq \mathbf{0}, \tag{24}$$

and

$$\boldsymbol{y} = (1-\theta)\boldsymbol{f}(\boldsymbol{x}) + \theta \boldsymbol{b}.$$

It was shown in [20] that (24) is closely related to the logarithmic barrier function method. Consider the problem:

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{x}^T\boldsymbol{y} - \theta \sum_{i=1}^{n} a_i \log x_i y_i \\ \text{subject to} & (\boldsymbol{x}, \boldsymbol{y}) \geq \mathbf{0}. \end{array}$$

It is easily seen that $(\boldsymbol{x}, \boldsymbol{y})$ is a global minimum solution of the problem if and only if it satisfies (24). This implies that if (7) has a solution, then $(\boldsymbol{x}, \boldsymbol{y})$ is a solution of (7) if and only if it is a global minimum solution of the problem:

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{x}^T\boldsymbol{y} - \theta \sum_{i=1}^{n} a_i \log x_i y_i \\ \text{subject to} & \boldsymbol{y} = (1-\theta)\boldsymbol{f}(\boldsymbol{x}) + \theta \boldsymbol{b}, \\ & (\boldsymbol{x}, \boldsymbol{y}) \geq \mathbf{0}. \end{array}$$

(B) The reader may be interested in extending the framework presented so far. Recall that the system

$$\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, \theta) \equiv (1-\theta)\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}) + \theta \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0} \text{ and } (\boldsymbol{x}, \boldsymbol{y}, \theta) \in R^{2n}_{+} \times [0,1] \qquad (25)$$

with the parameter θ decreasing from 1 to 0 serves as a continuous deformation from the artificial system

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}) \equiv \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} - \boldsymbol{a} \\ \boldsymbol{y} - \boldsymbol{b} \end{pmatrix} = \mathbf{0} \ \text{ and } (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{+},$$

which has a known unique solution, into

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) \equiv \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} \\ \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \end{pmatrix} = \mathbf{0} \ \text{ and } (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{+},$$

which is equivalent to CP[$\boldsymbol{f}$]. As a natural extension, we may replace the mapping $\boldsymbol{G}$ above by

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} - \boldsymbol{a} \\ \boldsymbol{y} - \boldsymbol{g}(\boldsymbol{x}) \end{pmatrix},$$

where $\boldsymbol{g} : R^n \to R^n$. To ensure the uniqueness of the solution of the resulting artificial system

$$\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, 1) \equiv \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}) \equiv \begin{pmatrix} \boldsymbol{X}\boldsymbol{y} - \boldsymbol{a} \\ \boldsymbol{y} - \boldsymbol{g}(\boldsymbol{x}) \end{pmatrix} = \mathbf{0} \ \text{ and } (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{+} \qquad (26)$$

and the boundedness of the set S of solutions $(\boldsymbol{x}, \boldsymbol{y}, \theta)$ of (25) with $\theta > 0$, we need to impose appropriate assumptions on the mapping $\boldsymbol{g}$.

Such an extension is especially useful when we deal with the problem LCP[$\boldsymbol{M}, \boldsymbol{q}$] associated with a bimatrix game [23], where $\boldsymbol{M}$ and $\boldsymbol{q}$ are of the form

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{A} \\ \boldsymbol{B}^T & \boldsymbol{O} \end{bmatrix} \quad \text{and} \quad \boldsymbol{q} = -\boldsymbol{e} = -(1, \ldots, 1)^T \in R^n.$$

Let $\boldsymbol{a} \geq \boldsymbol{0}$, and

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{e} \ .$$

Then we can easily verify that (26) has a unique solution and that the set S of all solutions $(\boldsymbol{x}, \boldsymbol{y}, \theta)$ of (25) is bounded.

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