A General Geometric Fourier Transform

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Abstract. The increasing demand for Fourier transforms on geometric algebras has resulted in a large variety. Here we introduce one single straight forward definition of a general geometric Fourier transform covering most versions in the literature. We show which constraints are additionally necessary to obtain certain features like linearity or a shift theorem. As a result, we provide guidelines for the target-oriented design of yet unconsidered transforms that fulfill requirements in a specific application context. Furthermore, the standard theorems do not need to be shown in a slightly different form every time a new geometric Fourier transform is developed since they are proved here once and for all.

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1. Introduction

The Fourier transform by Jean Baptiste Joseph Fourier is an indispensable tool for many fields of mathematics, physics, computer science and engineering. Especially the analysis and solution of differential equations or signal and image processing can not be imagined without it any more. Its kernel consists of the complex exponential function. With the square root of minus one, the imaginary unit i, as part of the argument it is periodic and therefore suitable for the analysis of oscillating systems.

William Kingdon Clifford created the geometric algebras in 1878, [1]. They usually contain continous submanifolds of geometric square roots of minus one [2, 3]. Each multivector has a natural geometric interpretation so the generalization of the Fourier transform to multivector valued functions in the geometric algebras is very reasonable. It helps to interpret the transform, apply it in a target oriented way to the specific underlying problem and allows a new point of view on fluid mechanics.

Application oriented many different definitions of Fourier transforms in geometric algebras were developed. For example the Clifford Fourier transform introduced by Jancewicz [4] and expanded by Ebling and Scheuermann [5] and Hitzer and Mawardi [6] or the one established by Sommen in [7] and re-established by Bülow [8]. Further we have the quaternionic Fourier transform by Ell [9] and later by Bülow [8], the spacetime Fourier transform by Hitzer [10], the Clifford Fourier transform for color images by Batard et al. [11], the Cylindrical Fourier transform by Brackx et al. [12], the transforms by Felsberg [13] or Ell and Sangwine [14, 15]. All these transforms have different interesting properties and deserve to be studied independently from one another. But the analysis of their similarities reveals a lot about their qualities, too. We concentrate on this matter and summarize all of them in one general definition.

Recently there have been very successful approaches by De Bie, Brackx, De Schepper and Sommen to construct Clifford Fourier transforms from operator exponentials and differential equations [16, 17, 18, 19]. The definition presented in this paper does not cover all of them, partly because their closed integral form is not always known or highly complicated, and partly because they can be produced by combinations and functions of our transforms.

We focus on continuous geometric Fourier transforms over flat spaces $\mathbb{R}^{p,q}$ in their integral representation. That way their finite, regular discrete equivalents as used in computational signal and image processing can be intuitively constructed and direct applicability to the existing practical issues and easy numerical manageability are ensured.

2. Definition of the GFT

We examine geometric algebras $C\ell_{p,q}, p+q=n\in\mathbb{N}$ over $\mathbb{R}^{p+q}$ [20] generated by the associative, bilinear geometric product with neutral element 1 satisfying

$$\boldsymbol{e}_j\boldsymbol{e}_k+\boldsymbol{e}_k\boldsymbol{e}_j=\epsilon_j\delta_{jk}, \tag{2.1}$$

for all $j,k\in\{1,...,n\}$ with the Kronecker symbol δ and

$$\epsilon_j=\begin{cases}1 & \forall j=1,...,p,\\ -1 & \forall j=p+1,...,n.\end{cases} \tag{2.2}$$

For the sake of brevity we want to refer to arbitrary multivectors

$$\boldsymbol{A}=\sum_{k=0}^{n}\sum_{1\leq j_1<...<j_k\leq n} a_{j_1...j_k}\boldsymbol{e}_{j_1}...\boldsymbol{e}_{j_k}\in C\ell_{p,q}, \tag{2.3}$$

$a_{j_1...j_k}\in\mathbb{R}$, as

$$\boldsymbol{A}=\sum_{\boldsymbol{j}} a_{\boldsymbol{j}}\boldsymbol{e}_{\boldsymbol{j}}. \tag{2.4}$$

where each of the 2^n multi-indices $\boldsymbol{j} \subseteq \{1,...,n\}$ indicates a basis vector of $C\ell_{p,q}$ by $\boldsymbol{e_j} = \boldsymbol{e}_{j_1}...\boldsymbol{e}_{j_k}$, $1 \leq j_1 < ... < j_k \leq n$, $\boldsymbol{e}_\emptyset = \boldsymbol{e}_0 = 1$ and its associated coefficient $a_{\boldsymbol{j}} = a_{j_1...j_k} \in \mathbb{R}$.

Definition 2.1. The **exponential function** of a multivector $\boldsymbol{A} \in C\ell_{p,q}$ is defined by the power series

$$e^{\boldsymbol{A}} := \sum_{j=0}^{\infty} \frac{\boldsymbol{A}^j}{j!}. \tag{2.5}$$

Lemma 2.2. *For two multivectors $\boldsymbol{AB} = \boldsymbol{BA}$ that commute amongst each other we have*

$$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}} e^{\boldsymbol{B}}. \tag{2.6}$$

Proof. Analogous to the exponent rule of real matrices. □

Notation 2.3. For each geometric algebra $C\ell_{p,q}$ we will write $\mathscr{I}^{p,q} = \{i \in C\ell_{p,q}, i^2 \in \mathbb{R}^-\}$ to denote the real multiples of all geometric square roots of minus one, compare [2] and [3]. We chose the symbol $\mathscr{I}$ to be reminiscent of the imaginary numbers.

Definition 2.4. Let $C\ell_{p,q}$ be a geometric Algebra, $\boldsymbol{A} : \mathbb{R}^m \to C\ell_{p,q}$ be a multivector field and $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^m$ vectors. A **Geometric Fourier Transform** (GFT) $\mathscr{F}_{F_1,F_2}(\boldsymbol{A})$ is defined by two ordered finite sets $F_1 = \{f_1(\boldsymbol{x},\boldsymbol{u}), ..., f_{\boldsymbol{\mu}}(\boldsymbol{x},\boldsymbol{u})\}$, $F_2 = \{f_{\boldsymbol{\mu}+1}(\boldsymbol{x},\boldsymbol{u}), ..., f_{\boldsymbol{\nu}}(\boldsymbol{x},\boldsymbol{u})\}$ of mappings $f_k(\boldsymbol{x},\boldsymbol{u}) : \mathbb{R}^m \times \mathbb{R}^m \to \mathscr{I}^{p,q}, \forall k = 1, ..., \boldsymbol{\nu}$ and the calculation rule

$$\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) := \int_{\mathbb{R}^m} \prod_{f \in F_1} e^{-f(\boldsymbol{x},\boldsymbol{u})} \boldsymbol{A}(\boldsymbol{x}) \prod_{f \in F_2} e^{-f(\boldsymbol{x},\boldsymbol{u})} \, \mathrm{d}^m \boldsymbol{x}. \tag{2.7}$$

This definition combines many Fourier transforms to a single general one. It enables us to proof the well known theorems just dependent on the properties of the chosen mappings.

Example. Depending on the choice of F_1 and F_2 we get already developed transforms.

1. In the case of $\boldsymbol{A} : \mathbb{R}^n \to \mathcal{G}^{n,0}, n = 2 \pmod 4$ or $n = 3 \pmod 4$, we can reproduce the Clifford Fourier transform introduced by Jancewicz [4] for $n = 3$ and expanded by Ebling and Scheuermann [5] for $n = 2$ and Hitzer and Mawardi [6] for $n = 2 \pmod 4$ or $n = 3 \pmod 4$ using the configuration

$$\begin{aligned} F_1 &= \emptyset, \\ F_2 &= \{f_1\}, \\ f_1(\boldsymbol{x},\boldsymbol{u}) &= 2\pi i_n \boldsymbol{x} \cdot \boldsymbol{u}, \end{aligned} \tag{2.8}$$

   with i_n being the pseudoscalar of $G^{n,0}$.

2. Choosing multivector fields $\mathbb{R}^n \to \mathcal{G}^{0,n}$,

$$\begin{aligned} F_1 =& \emptyset, \\ F_2 =& \{f_1, ..., f_n\}, \\ f_k(\boldsymbol{x}, \boldsymbol{u}) =& 2\pi \boldsymbol{e}_k x_k u_k, \forall k = 1, ..., n \end{aligned} \tag{2.9}$$

   we have the Sommen Bülow Clifford Fourier transform from [7, 8].
3. For $\boldsymbol{A} : \mathbb{R}^2 \to \mathcal{G}^{0,2} \approx \mathbb{H}$ the quaternionic Fourier transform [9, 8] is generated by

$$\begin{aligned} F_1 =& \{f_1\}, \\ F_2 =& \{f_2\}, \\ f_1(\boldsymbol{x}, \boldsymbol{u}) =& 2\pi i x_1 u_1, \\ f_2(\boldsymbol{x}, \boldsymbol{u}) =& 2\pi j x_2 u_2. \end{aligned} \tag{2.10}$$

4. Using $\mathcal{G}^{3,1}$ we can build the spacetime respectively the volume-time Fourier transform from [10][1] with the $\mathcal{G}^{3,1}$-pseudoscalar i_4 as follows

$$\begin{aligned} F_1 =& \{f_1\}, \\ F_2 =& \{f_2\}, \\ f_1(\boldsymbol{x}, \boldsymbol{u}) =& \boldsymbol{e}_4 x_4 u_4, \\ f_2(\boldsymbol{x}, \boldsymbol{u}) =& \epsilon_4 \boldsymbol{e}_4 i_4 (x_1 u_1 + x_2 u_2 + x_3 u_3). \end{aligned} \tag{2.11}$$

5. The Clifford Fourier transform for color images by Batard, Berthier and Saint-Jean [11] for $m = 2, n = 4, \boldsymbol{A} : \mathbb{R}^2 \to \mathcal{G}^{4,0}$, a fixed bivector $\boldsymbol{B}$, and the pseudoscalar i can intuitively be written as

$$\begin{aligned} F_1 =& \{f_1\}, \\ F_2 =& \{f_2\}, \\ f_1(\boldsymbol{x}, \boldsymbol{u}) =& \frac{1}{2}(x_1 u_1 + x_2 u_2)(\boldsymbol{B} + i\boldsymbol{B}), \\ f_2(\boldsymbol{x}, \boldsymbol{u}) =& -\frac{1}{2}(x_1 u_1 + x_2 u_2)(\boldsymbol{B} + i\boldsymbol{B}), \end{aligned} \tag{2.12}$$

   but $(\boldsymbol{B} + i\boldsymbol{B})$ does not square to a negative real number, see [2]. The special property that $\boldsymbol{B}$ and $i\boldsymbol{B}$ commute amongst each other allows us

[1] Please note that Hitzer uses a different notation in [10]. His $\boldsymbol{x} = t\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$ corresponds to our $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4$, with $\boldsymbol{e}_0\boldsymbol{e}_0 = \epsilon_0 = -1$ being equivalent to our $\boldsymbol{e}_4\boldsymbol{e}_4 = \epsilon_4 = -1$.

to express the formula using

$$\begin{aligned}
F_1 =& \{f_1, f_2\},\\
F_2 =& \{f_3, f_4\},\\
f_1(\boldsymbol{x}, \boldsymbol{u}) =& \frac{1}{2}(x_1u_1 + x_2u_2)\boldsymbol{B},\\
f_2(\boldsymbol{x}, \boldsymbol{u}) =& \frac{1}{2}(x_1u_1 + x_2u_2)i\boldsymbol{B},\\
f_3(\boldsymbol{x}, \boldsymbol{u}) =& -\frac{1}{2}(x_1u_1 + x_2u_2)\boldsymbol{B},\\
f_4(\boldsymbol{x}, \boldsymbol{u}) =& -\frac{1}{2}(x_1u_1 + x_2u_2)i\boldsymbol{B},
\end{aligned} \tag{2.13}$$

which fulfills the conditions of Definition 2.4.

6. Using $\mathcal{G}^{0,n}$ and

$$\begin{aligned}
F_1 =& \{f_1\},\\
F_2 =& \emptyset,\\
f_1(\boldsymbol{x}, \boldsymbol{u}) =& -\boldsymbol{x} \wedge \boldsymbol{u}
\end{aligned} \tag{2.14}$$

produces the cylindrical Fourier transform as introduced by Brackx, Schepper and Sommen in [12].

3. General Properties

First we proof general properties valid for arbitrary sets F_1, F_2.

Theorem 3.1 (Existence). *The geometric Fourier transform exists for all integrable multivector fields $\boldsymbol{A} \in L_1(\mathbb{R}^n)$.*

Proof. The property

$$f_k^2(\boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^- \tag{3.1}$$

of the mappings f_k for $k = 1, ..., \boldsymbol{\nu}$ leads to

$$\frac{f_k^2(\boldsymbol{x}, \boldsymbol{u})}{|f_k^2(\boldsymbol{x}, \boldsymbol{u})|} = -1 \tag{3.2}$$

for all $f_k(\boldsymbol{x}, \boldsymbol{u}) \neq 0$. So using the decomposition

$$f_k(\boldsymbol{x}, \boldsymbol{u}) = \frac{f_k(\boldsymbol{x}, \boldsymbol{u})}{|f_k(\boldsymbol{x}, \boldsymbol{u})|}|f_k(\boldsymbol{x}, \boldsymbol{u})| \tag{3.3}$$

we can write $\forall j \in \mathbb{N}$

$$f_k^j(\boldsymbol{x}, \boldsymbol{u}) = \begin{cases} (-1)^l |f_k(\boldsymbol{x}, \boldsymbol{u})|^j & \text{for } j = 2l, l \in \mathbb{N}_0 \\ (-1)^l \frac{f_k(\boldsymbol{x}, \boldsymbol{u})}{|f_k(\boldsymbol{x}, \boldsymbol{u})|}|f_k(\boldsymbol{x}, \boldsymbol{u})|^j & \text{for } j = 2l+1, l \in \mathbb{N}_0 \end{cases} \tag{3.4}$$

which results in

$$\begin{aligned} e^{-f_k(\boldsymbol{x},\boldsymbol{u})} =& \sum_{j=0}^{\infty} \frac{\big(-f_k(\boldsymbol{x},\boldsymbol{u})\big)^j}{j!} \\ =& \sum_{j=0}^{\infty} \frac{(-1)^j |f_k(\boldsymbol{x},\boldsymbol{u})|^{2j}}{(2j)!} \\ & - \frac{f_k(\boldsymbol{x},\boldsymbol{u})}{|f_k(\boldsymbol{x},\boldsymbol{u})|} \sum_{j=0}^{\infty} \frac{(-1)^j |f_k(\boldsymbol{x},\boldsymbol{u})|^{2j+1}}{(2j+1)!} \\ =& \cos\big(|f_k(\boldsymbol{x},\boldsymbol{u})|\big) - \frac{f_k(\boldsymbol{x},\boldsymbol{u})}{|f_k(\boldsymbol{x},\boldsymbol{u})|} \sin\big(|f_k(\boldsymbol{x},\boldsymbol{u})|\big) \end{aligned} \tag{3.5}$$

Because of

$$\begin{aligned} |e^{-f_k(\boldsymbol{x},\boldsymbol{u})}| =& \left| \cos\big(|f_k(\boldsymbol{x},\boldsymbol{u})|\big) - \frac{f_k(\boldsymbol{x},\boldsymbol{u})}{|f_k(\boldsymbol{x},\boldsymbol{u})|} \sin\big(|f_k(\boldsymbol{x},\boldsymbol{u})|\big) \right| \\ \leq& \left| \cos\big(|f_k(\boldsymbol{x},\boldsymbol{u})|\big) \right| + \left| \frac{f_k(\boldsymbol{x},\boldsymbol{u})}{|f_k(\boldsymbol{x},\boldsymbol{u})|} \right| \left| \sin\big(|f_k(\boldsymbol{x},\boldsymbol{u})|\big) \right| \\ \leq& 2 \end{aligned} \tag{3.6}$$

the magnitude of the improper integral

$$\begin{aligned} |\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u})| =& \left| \int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(\boldsymbol{x},\boldsymbol{u})} \boldsymbol{A}(\boldsymbol{x}) \prod_{f\in F_2} e^{-f(\boldsymbol{x},\boldsymbol{u})} \,\mathrm{d}^m\boldsymbol{x} \right| \\ \leq& \int_{\mathbb{R}^m} \prod_{f\in F_1} |e^{-f(\boldsymbol{x},\boldsymbol{u})}| |\boldsymbol{A}(\boldsymbol{x})| \prod_{f\in F_2} |e^{-f(\boldsymbol{x},\boldsymbol{u})}| \,\mathrm{d}^m\boldsymbol{x} \\ \leq& \int_{\mathbb{R}^m} \prod_{f\in F_1} 2|\boldsymbol{A}(\boldsymbol{x})| \prod_{f\in F_2} 2 \,\mathrm{d}^m\boldsymbol{x} \\ =& 2^{\boldsymbol{\nu}} \int_{\mathbb{R}^m} |\boldsymbol{A}(\boldsymbol{x})| \,\mathrm{d}^m\boldsymbol{x} \end{aligned} \tag{3.7}$$

is finite and therefore the geometric Fourier transform exists. □

Theorem 3.2 (Scalar linearity). *The geometric Fourier transform is linear with respect to scalar factors. Let* $b, c \in \mathbb{R}$ *and* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} : \mathbb{R}^m \to C\ell_{p,q}$ *be three multivector fields that satisfy* $\boldsymbol{A}(\boldsymbol{x}) = b\boldsymbol{B}(\boldsymbol{x}) + c\boldsymbol{C}(\boldsymbol{x})$*, then*

$$\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) = b\mathscr{F}_{F_1,F_2}(\boldsymbol{B})(\boldsymbol{u}) + c\mathscr{F}_{F_1,F_2}(\boldsymbol{C})(\boldsymbol{u}). \tag{3.8}$$

Proof. The assertion is an easy consequence of the distributivity of the geometric product over addition, the commutativity of scalars and the linearity of the integral. □

4. Bilinearity

All geometric Fourier transforms from the introductory example can also be expressed in terms of a stronger claim. The mappings $f_1, ..., f_{\boldsymbol{\nu}}$, with the first

$\boldsymbol{\mu}$ ones left of the argument function and the $\boldsymbol{\nu}-\boldsymbol{\mu}$ others on the right of it, are all bilinear and therefore take the form

$$\begin{aligned} f_k(\boldsymbol{x},\boldsymbol{u}) &= f_k(\sum_{j=1}^{m} x_j\boldsymbol{e}_j, \sum_{l=1}^{m} u_l\boldsymbol{e}_l) \\ &= \sum_{j,l=1}^{m} x_j f_k(\boldsymbol{e}_j,\boldsymbol{e}_l) u_l \\ &= \boldsymbol{x}^T M_k \boldsymbol{u}, \end{aligned} \tag{4.1}$$

$\forall k = 1,...,\boldsymbol{\nu}$, where $M_k \in (\mathscr{I}^{p,q})^{m\times m}$, $(M_k)_{jl} = f_k(\boldsymbol{e}_j,\boldsymbol{e}_l)$ according to Notation 2.3.

1. In the Clifford Fourier transform f_1 can be written with

$$M_1 = 2\pi i_n \,\mathrm{Id}\,. \tag{4.2}$$

2. The $\boldsymbol{\nu} = m = n$ mappings $f_k, k = 1,...,n$ of the Bülow Clifford Fourier transform can be expressed using

$$(M_k)_{lj} = \begin{cases} 2\pi \boldsymbol{e}_k & \text{for } k = l = j, \\ 0 & \text{else.} \end{cases} \tag{4.3}$$

3. Similarly the quaternionic Fourier transform is generated using

$$\begin{aligned} (M_1)_{l\iota} &= \begin{cases} 2\pi i & \text{for } l = \iota = 1, \\ 0 & \text{else,} \end{cases} \\ (M_2)_{l\iota} &= \begin{cases} 2\pi j & \text{for } l = \iota = 2, \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{4.4}$$

4. We can build the spacetime Fourier transform with

$$\begin{aligned} (M_1)_{lj} &= \begin{cases} \boldsymbol{e}_4 & \text{for } l = j = 1, \\ 0 & \text{else,} \end{cases} \\ (M_2)_{lj} &= \begin{cases} \epsilon_4 \boldsymbol{e}_4 i_4 & \text{for } l = j \in \{2,3,4\}, \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{4.5}$$

5. The Clifford Fourier transform for color images can be described by

$$\begin{aligned} M_1 &= \frac{1}{2}\boldsymbol{B}\,\mathrm{Id}, \\ M_2 &= \frac{1}{2} i\boldsymbol{B}\,\mathrm{Id}, \\ M_3 &= -\frac{1}{2}\boldsymbol{B}\,\mathrm{Id}, \\ M_4 &= -\frac{1}{2} i\boldsymbol{B}\,\mathrm{Id}\,. \end{aligned} \tag{4.6}$$

6. The cylindrical Fourier transform can also be reproduced with mappings satisfying (4.1) because we can write

$$\begin{aligned} \boldsymbol{x} \wedge \boldsymbol{u} = & \boldsymbol{e}_1\boldsymbol{e}_2 x_1 u_2 - \boldsymbol{e}_1\boldsymbol{e}_2 x_2 u_1 \\ & + \ldots \\ & + \boldsymbol{e}_{m-1}\boldsymbol{e}_m x_{m-1} u_m - \boldsymbol{e}_{m-1}\boldsymbol{e}_m x_m u_{m-1} \end{aligned} \tag{4.7}$$

and set

$$(M_1)_{lj} = \begin{cases} 0 & \text{for } l = j, \\ \boldsymbol{e}_l\boldsymbol{e}_j & \text{else.} \end{cases} \tag{4.8}$$

Theorem 4.1 (Scaling). *Let $0 \neq a \in \mathbb{R}$ be a real number, $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{B}(a\boldsymbol{x})$ two multivector fields and all F_1, F_2 be bilinear mappings then the geometric Fourier transform satisfies*

$$\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) = |a|^{-m} \mathscr{F}_{F_1,F_2}(\boldsymbol{B})\left(\frac{\boldsymbol{u}}{a}\right). \tag{4.9}$$

Proof. A change of coordinates together with the bilinearity proves the assertion by

$$\begin{aligned} \mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) &= \int_{\mathbb{R}^m} \prod_{f \in F} e^{-f(\boldsymbol{x},\boldsymbol{u})} \boldsymbol{B}(a\boldsymbol{x}) \prod_{f \in B} e^{-f(\boldsymbol{x},\boldsymbol{u})} \, \mathrm{d}^m \boldsymbol{x} \\ &\overset{a\boldsymbol{x}=\boldsymbol{y}}{=} \int_{\mathbb{R}^m} \prod_{f \in F} e^{-f(\frac{\boldsymbol{y}}{a},\boldsymbol{u})} \boldsymbol{B}(\boldsymbol{y}) \prod_{f \in B} e^{-f(\frac{\boldsymbol{y}}{a},\boldsymbol{u})} |a|^{-m} \, \mathrm{d}^m \boldsymbol{y} \\ &\overset{f \text{ bilin.}}{=} |a|^{-m} \int_{\mathbb{R}^m} \prod_{f \in F} e^{-f(\boldsymbol{y},\frac{\boldsymbol{u}}{a})} \boldsymbol{B}(\boldsymbol{y}) \prod_{f \in B} e^{-f(\boldsymbol{y},\frac{\boldsymbol{u}}{a})} \, \mathrm{d}^m \boldsymbol{y} \\ &= |a|^{-m} \mathscr{F}_{F_1,F_2}(\boldsymbol{B})\left(\frac{\boldsymbol{u}}{a}\right). \end{aligned} \tag{4.10}$$

□

5. Products with Invertible Factors

To obtain properties of the GFT like linearity with respect to arbitrary multivectors or a shift theorem we will have to change the order of multivectors and products of exponentials. Since the geometric product usually is neither commutative nor anticommutative this is not trivial. In this section we provide useful lemmata that allow a swap if at least one of the factors is invertible. For more information see [20] and [3].

Remark 5.1. Every multiple of a square root of minus one $i \in \mathscr{I}^{p,q}$ is invertible, since from $i^2 = -r, r \in \mathbb{R} \setminus \{0\}$ follows $i^{-1} = -\frac{i}{r}$. Because of that for all $\boldsymbol{u}, \boldsymbol{x} \in \mathbb{R}^m$ a function $f_k(\boldsymbol{x}, \boldsymbol{u}) : \mathbb{R}^m \times \mathbb{R}^m \to \mathscr{I}^{p,q}$ is pointwise invertible.

Definition 5.2. For an invertible multivector $\boldsymbol{B} \in C\ell_{p,q}$ and an arbitrary multivector $\boldsymbol{A} \in C\ell_{p,q}$ we define

$$\begin{aligned} \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})} =& \frac{1}{2}(\boldsymbol{A} + \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}), \\ \boldsymbol{A}_{\boldsymbol{c}^1(\boldsymbol{B})} =& \frac{1}{2}(\boldsymbol{A} - \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}). \end{aligned} \tag{5.1}$$

Lemma 5.3. *Let $\boldsymbol{B} \in C\ell_{p,q}$ be invertible with the unique inverse $\boldsymbol{B}^{-1} = \frac{\bar{\boldsymbol{B}}}{\boldsymbol{B}^2}, \boldsymbol{B}^2 \in \mathbb{R} \setminus \{0\}$. Every multivector $\boldsymbol{A} \in C\ell_{p,q}$ can be expressed unambiguously by the sum of $\boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})} \in C\ell_{p,q}$ that commutes and $\boldsymbol{A}_{\boldsymbol{c}^1(\boldsymbol{B})} \in C\ell_{p,q}$ that anticommutes with respect to $\boldsymbol{B}$. That means*

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})} + \boldsymbol{A}_{\boldsymbol{c}^1(\boldsymbol{B})}, \\ \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})}\boldsymbol{B} &= \boldsymbol{B}\boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})}, \\ \boldsymbol{A}_{\boldsymbol{c}^1(\boldsymbol{B})}\boldsymbol{B} &= -\boldsymbol{B}\boldsymbol{A}_{\boldsymbol{c}^1(\boldsymbol{B})}. \end{aligned} \tag{5.2}$$

Proof. We will only prove the assertion for $\boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})}$.
Existence: With Definition 5.2 we get

$$\begin{aligned} \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})} + \boldsymbol{A}_{\boldsymbol{c}^1(\boldsymbol{B})} =& \frac{1}{2}(\boldsymbol{A} + \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B} + \boldsymbol{A} - \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}) \\ =& \boldsymbol{A} \end{aligned} \tag{5.3}$$

and considering

$$\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B} = \frac{\bar{\boldsymbol{B}}\boldsymbol{A}\boldsymbol{B}}{\boldsymbol{B}^2} = \boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^{-1} \tag{5.4}$$

we also get

$$\begin{aligned} \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})}\boldsymbol{B} =& \frac{1}{2}(\boldsymbol{A} + \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B})\boldsymbol{B} \\ =& \frac{1}{2}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^{-1})\boldsymbol{B} \\ =& \frac{1}{2}(\boldsymbol{A}\boldsymbol{B} + \boldsymbol{B}\boldsymbol{A}) \\ =& \boldsymbol{B}\frac{1}{2}(\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B} + \boldsymbol{A}) \\ =& \boldsymbol{B}\boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})} \end{aligned} \tag{5.5}$$

Uniqueness: From the first claim in (5.2) we get

$$\boldsymbol{A}_{\boldsymbol{c}^1(\boldsymbol{B})} = \boldsymbol{A} - \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})}, \tag{5.6}$$

together with the third one this leads to

$$\begin{aligned} (\boldsymbol{A} - \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})})\boldsymbol{B} =& -\boldsymbol{B}(\boldsymbol{A} - \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})}) \\ \boldsymbol{A}\boldsymbol{B} - \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})}\boldsymbol{B} =& -\boldsymbol{B}\boldsymbol{A} + \boldsymbol{B}\boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})} \\ \boldsymbol{A}\boldsymbol{B} + \boldsymbol{B}\boldsymbol{A} =& \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})}\boldsymbol{B} + \boldsymbol{B}\boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})} \end{aligned} \tag{5.7}$$

and from the second claim finally follows

$$\begin{aligned} \boldsymbol{AB} + \boldsymbol{BA} =& 2\boldsymbol{BA}_{\boldsymbol{c}^0(\boldsymbol{B})} \\ \frac{1}{2}(\boldsymbol{B}^{-1}\boldsymbol{AB} + \boldsymbol{A}) =& \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})}. \end{aligned} \tag{5.8}$$

The derivation of the expression for $\boldsymbol{A}_{\boldsymbol{c}^1(\boldsymbol{B})}$ works analogously. □

Corollary 5.4 (Decomposition w.r.t. commutativity). *Let $\boldsymbol{B} \in C\ell_{p,q}$ be invertible, then $\forall \boldsymbol{A} \in C\ell_{p,q}$*

$$\boldsymbol{BA} = (\boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{B})} - \boldsymbol{A}_{\boldsymbol{c}^1(\boldsymbol{B})})\boldsymbol{B}. \tag{5.9}$$

Definition 5.5. For $d \in \mathbb{N}, \boldsymbol{A} \in C\ell_{p,q}$, the ordered set $B = \{\boldsymbol{B}_1, ..., \boldsymbol{B}_d\}$ of invertible multivectors and any multi-index $\boldsymbol{j} \in \{0,1\}^d$ we define

$$\begin{aligned} \boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overrightarrow{B})} :=& ((\boldsymbol{A}_{\boldsymbol{c}^{j_1}(\boldsymbol{B}_1)})_{\boldsymbol{c}^{j_2}(\boldsymbol{B}_2)} \cdots)_{\boldsymbol{c}^{j_d}(\boldsymbol{B}_d)}, \\ \boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overleftarrow{B})} :=& ((\boldsymbol{A}_{\boldsymbol{c}^{j_d}(\boldsymbol{B}_d)})_{\boldsymbol{c}^{j_{d-1}}(\boldsymbol{B}_{d-1})} \cdots)_{\boldsymbol{c}^{j_1}(\boldsymbol{B}_1)} \end{aligned} \tag{5.10}$$

recursively with $\boldsymbol{c}^0, \boldsymbol{c}^1$ of Definition 5.2.

Example. Let $\boldsymbol{A} = a_0 + a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_{12}\boldsymbol{e}_{12} \in \mathcal{G}^{2,0}$ then for example

$$\begin{aligned} \boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{e}_1)} =& \frac{1}{2}(\boldsymbol{A} + \boldsymbol{e}_1^{-1}\boldsymbol{A}\boldsymbol{e}_1) \\ =& \frac{1}{2}(\boldsymbol{A} + a_0 + a_1\boldsymbol{e}_1 - a_2\boldsymbol{e}_2 - a_{12}\boldsymbol{e}_{12}) \\ =& a_0 + a_1\boldsymbol{e}_1 \end{aligned} \tag{5.11}$$

and further

$$\begin{aligned} \boldsymbol{A}_{\boldsymbol{c}^{0,0}(\overrightarrow{\boldsymbol{e}_1,\boldsymbol{e}_2})} =& (\boldsymbol{A}_{\boldsymbol{c}^0(\boldsymbol{e}_1)})_{\boldsymbol{c}^0(\boldsymbol{e}_2)} \\ =& (a_0 + a_1\boldsymbol{e}_1)_{\boldsymbol{c}^0(\boldsymbol{e}_2)} = a_0. \end{aligned} \tag{5.12}$$

The computation of the other multi-indices with $d = 2$ works analogously and therefore

$$\begin{aligned} \boldsymbol{A} =& \sum_{\boldsymbol{j}\in\{0,1\}^d} \boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\boldsymbol{e}_1,\boldsymbol{e}_2)} \\ =& \boldsymbol{A}_{\boldsymbol{c}^{00}(\overrightarrow{\boldsymbol{e}_1,\boldsymbol{e}_2})} + \boldsymbol{A}_{\boldsymbol{c}^{01}(\overrightarrow{\boldsymbol{e}_1,\boldsymbol{e}_2})} + \boldsymbol{A}_{\boldsymbol{c}^{10}(\overrightarrow{\boldsymbol{e}_1,\boldsymbol{e}_2})} + \boldsymbol{A}_{\boldsymbol{c}^{11}(\overrightarrow{\boldsymbol{e}_1,\boldsymbol{e}_2})} \\ =& a_0 + a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_{12}\boldsymbol{e}_{12}. \end{aligned} \tag{5.13}$$

Lemma 5.6. *Let $d \in \mathbb{N}, B = \{\boldsymbol{B}_1, ..., \boldsymbol{B}_d\}$ be invertible multivectors and for $\boldsymbol{j} \in \{0,1\}^d$ let $|\boldsymbol{j}| := \sum_{k=1}^d j_k$, then $\forall \boldsymbol{A} \in C\ell_{p,q}$*

$$\begin{aligned} \boldsymbol{A} =& \sum_{\boldsymbol{j}\in\{0,1\}^d} \boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overrightarrow{B})}, \\ \boldsymbol{A}\boldsymbol{B}_1...\boldsymbol{B}_d =& \boldsymbol{B}_1...\boldsymbol{B}_d \sum_{\boldsymbol{j}\in\{0,1\}^d} (-1)^{|\boldsymbol{j}|}\boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overrightarrow{B})}, \\ \boldsymbol{B}_1...\boldsymbol{B}_d\boldsymbol{A} =& \sum_{\boldsymbol{j}\in\{0,1\}^d} (-1)^{|\boldsymbol{j}|}\boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overleftarrow{B})}\boldsymbol{B}_1...\boldsymbol{B}_d. \end{aligned} \tag{5.14}$$

Proof. Apply Lemma 5.3 repeatedly. □

Remark 5.7. The distinction of the two directions can be omitted using the equality

$$\boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overrightarrow{\boldsymbol{B}_1,...,\boldsymbol{B}_d})} = \boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overleftarrow{\boldsymbol{B}_d,...,\boldsymbol{B}_1})}. \tag{5.15}$$

We established it for the sake of notational brevity and will not formulate nor proof every assertion for both directions.

Lemma 5.8. *Let $F = \{f_1(\boldsymbol{x},\boldsymbol{u}),...,f_d(\boldsymbol{x},\boldsymbol{u})\}$ be a set of pointwise invertible functions then the ordered product of their exponentials and an arbitrary multivector $\boldsymbol{A} \in C\ell_{p,q}$ satisfies*

$$\prod_{k=1}^{d} e^{-f_k(\boldsymbol{x},\boldsymbol{u})}\boldsymbol{A} = \sum_{\boldsymbol{j}\in\{0,1\}^d} \boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overleftarrow{F})}(\boldsymbol{x},\boldsymbol{u}) \prod_{k=1}^{d} e^{-(-1)^{j_k} f_k(\boldsymbol{x},\boldsymbol{u})}, \tag{5.16}$$

where $A_{\boldsymbol{c}^{\boldsymbol{j}}(\overleftarrow{F})}(\boldsymbol{x},\boldsymbol{u}) := A_{\boldsymbol{c}^{\boldsymbol{j}}(\overleftarrow{F(\boldsymbol{x},\boldsymbol{u})})}$ is a multivector valued function $\mathbb{R}^m \times \mathbb{R}^m \to C\ell_{p,q}$.

Proof. For all $\boldsymbol{x},\boldsymbol{u} \in \mathbb{R}^m$ the commutation properties of $f_k(\boldsymbol{x},\boldsymbol{u})$ dictate the ones of $e^{-f_k(\boldsymbol{x},\boldsymbol{u})}$ by

$$\begin{aligned} e^{-f_k(\boldsymbol{x},\boldsymbol{u})}\boldsymbol{A} &\overset{\text{Def. 2.1}}{=} \sum_{l=0}^{\infty} \frac{(-f_k(\boldsymbol{x},\boldsymbol{u}))^l}{l!}\boldsymbol{A} \\ &\overset{\text{Lem. 5.3}}{=} \sum_{l=0}^{\infty} \frac{(-f_k(\boldsymbol{x},\boldsymbol{u}))^l}{l!}(\boldsymbol{A}_{\boldsymbol{c}^0(f_k(\boldsymbol{x},\boldsymbol{u}))} + \boldsymbol{A}_{\boldsymbol{c}^1(f_k(\boldsymbol{x},\boldsymbol{u}))}). \end{aligned} \tag{5.17}$$

The shape of this decomposition of $\boldsymbol{A}$ may depend on $\boldsymbol{x}$ and $\boldsymbol{u}$. To stress this fact we will interpret $\boldsymbol{A}_{\boldsymbol{c}^0(f_k(\boldsymbol{x},\boldsymbol{u}))}$ as a multivector function and write $\boldsymbol{A}_{\boldsymbol{c}^0(f_k)}(\boldsymbol{x},\boldsymbol{u})$. According to Lemmma 5.3 we can move $\boldsymbol{A}_{\boldsymbol{c}^0(f_k)}(\boldsymbol{x},\boldsymbol{u})$ through all factors, because it commutes. Analogously swapping $\boldsymbol{A}_{\boldsymbol{c}^1(f_k)}(\boldsymbol{x},\boldsymbol{u})$ will change the sign of each factor because it anticommutes. Hence we get

$$\begin{aligned} &= \boldsymbol{A}_{\boldsymbol{c}^0(f_k)}(\boldsymbol{x},\boldsymbol{u}) \sum_{l=0}^{\infty} \frac{(-f_k(\boldsymbol{x},\boldsymbol{u}))^l}{l!} + \boldsymbol{A}_{\boldsymbol{c}^1(f_k)}(\boldsymbol{x},\boldsymbol{u}) \sum_{l=0}^{\infty} \frac{(f_k(\boldsymbol{x},\boldsymbol{u}))^l}{l!} \\ &= \boldsymbol{A}_{\boldsymbol{c}^0(f_k)}(\boldsymbol{x},\boldsymbol{u})e^{-f_k(\boldsymbol{x},\boldsymbol{u})} + \boldsymbol{A}_{\boldsymbol{c}^1(f_k)}(\boldsymbol{x},\boldsymbol{u})e^{f_k(\boldsymbol{x},\boldsymbol{u})}. \end{aligned} \tag{5.18}$$

Applying this repeatedly to the product we can deduce

$$\begin{aligned}\prod_{k=1}^{d} e^{-f_k(\boldsymbol{x},\boldsymbol{u})}\boldsymbol{A} =& \prod_{k=1}^{d-1} e^{-f_k(\boldsymbol{x},\boldsymbol{u})}(\boldsymbol{A}_{\boldsymbol{c}^0(f_d)}(\boldsymbol{x},\boldsymbol{u})e^{-f_d(\boldsymbol{x},\boldsymbol{u})} \\ &+ \boldsymbol{A}_{\boldsymbol{c}^1(f_d)}(\boldsymbol{x},\boldsymbol{u})e^{f_d(\boldsymbol{x},\boldsymbol{u})}) \\ =& \prod_{k=1}^{d-2} e^{-f_k(\boldsymbol{x},\boldsymbol{u})}(\boldsymbol{A}_{\boldsymbol{c}^{0,0}(\overleftarrow{f_{d-1},f_d})}(\boldsymbol{x},\boldsymbol{u})e^{-f_{d-1}(\boldsymbol{x},\boldsymbol{u})}e^{-f_d(\boldsymbol{x},\boldsymbol{u})} \\ &+ \boldsymbol{A}_{\boldsymbol{c}^{1,0}(\overleftarrow{f_{d-1},f_d})}(\boldsymbol{x},\boldsymbol{u})e^{f_{d-1}(\boldsymbol{x},\boldsymbol{u})}e^{-f_d(\boldsymbol{x},\boldsymbol{u})}) \\ &+ \boldsymbol{A}_{\boldsymbol{c}^{0,1}(\overleftarrow{f_{d-1},f_d})}(\boldsymbol{x},\boldsymbol{u})e^{-f_{d-1}(\boldsymbol{x},\boldsymbol{u})}e^{f_d(\boldsymbol{x},\boldsymbol{u})} \\ &+ \boldsymbol{A}_{\boldsymbol{c}^{1,1}(\overleftarrow{f_{d-1},f_d})}(\boldsymbol{x},\boldsymbol{u})e^{f_{d-1}(\boldsymbol{x},\boldsymbol{u})}e^{f_d(\boldsymbol{x},\boldsymbol{u})}) \\ =& \ldots \\ =& \sum_{\boldsymbol{j}\in\{0,1\}^d} \boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overleftarrow{F})}(\boldsymbol{x},\boldsymbol{u}) \prod_{k=1}^{d} e^{-(-1)^{j_k} f_k(\boldsymbol{x},\boldsymbol{u})}.\end{aligned} \tag{5.19}$$

□

6. Separable GFT

From now on we want to restrict ourselves to an important group of geometric Fourier transforms whose square roots of -1 are independent from the first argument.

Definition 6.1. We call a GFT **left (right) separable**, if

$$f_l = |f_l(\boldsymbol{x},\boldsymbol{u})| i_l(\boldsymbol{u}), \tag{6.1}$$

$\forall l = 1,...,\boldsymbol{\mu}$, $(l = \boldsymbol{\mu}+1,...,\boldsymbol{\nu})$, where $|f_l(\boldsymbol{x},\boldsymbol{u})| : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a real function and $i_l : \mathbb{R}^m \to \mathscr{I}^{p,q}$ a function that does not depend on $\boldsymbol{x}$.

Example. The first five transforms from the introductory example are separable, while the cylindrical transform (vi) can not be expressed in the way of (6.1) except for the two dimensional case.

We have seen in the proof of Lemma 5.8 that the decomposition of a constant multivector $\boldsymbol{A}$ with respect to a product of exponentials generally results in multivector valued functions $\boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(F)}(\boldsymbol{x},\boldsymbol{u})$ of $\boldsymbol{x}$ and $\boldsymbol{u}$. Separability guarantees independence from $\boldsymbol{x}$ and therefore allows separation from the integral.

Corollary 6.2 (Decomposition independent from $\boldsymbol{x}$). *Consider a set of functions $F = \{f_1(\boldsymbol{x},\boldsymbol{u}),...,f_d(\boldsymbol{x},\boldsymbol{u})\}$ satisfying condition (6.1) then the ordered product of their exponentials and an arbitrary multivector $\boldsymbol{A} \in C\ell_{p,q}$ satisfies*

$$\prod_{k=1}^{d} e^{-f_k(\boldsymbol{x},\boldsymbol{u})}\boldsymbol{A} = \sum_{\boldsymbol{j}\in\{0,1\}^d} \boldsymbol{A}_{\boldsymbol{c}^{\boldsymbol{j}}(\overleftarrow{F})}(\boldsymbol{u}) \prod_{k=1}^{d} e^{-(-1)^{j_k} f_k(\boldsymbol{x},\boldsymbol{u})}. \tag{6.2}$$

Remark 6.3. If a GFT can be expressed as in (6.1) but with multiples of square roots of -1 $i_k \in \mathscr{I}^{p,q}$, which are independent from $\boldsymbol{x}$ and $\boldsymbol{u}$, the parts $\boldsymbol{A}_{\boldsymbol{c}^j(\overleftarrow{F})}$ of $\boldsymbol{A}$ will be constants. Note that the first five GFTs from the reference example satisfy this stronger condition, too.

Definition 6.4. For a set of functions $F = \{f_1(\boldsymbol{x},\boldsymbol{u}), ..., f_d(\boldsymbol{x},\boldsymbol{u})\}$ and a multi-index $\boldsymbol{j} \in \{0,1\}^d$, we define the set of functions $F(\boldsymbol{j})$ by

$$F(\boldsymbol{j}) := \{(-1)^{j_1} f_1(\boldsymbol{x},\boldsymbol{u}), ..., (-1)^{j_d} f_d(\boldsymbol{x},\boldsymbol{u})\}. \tag{6.3}$$

Theorem 6.5 (Left and right products). *Let $\boldsymbol{C} \in C\ell_{p,q}$ and $\boldsymbol{A},\boldsymbol{B} : \mathbb{R}^m \to C\ell_{p,q}$ be two multivector fields with $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{B}(\boldsymbol{x})$ then a left separable geometric Fourier transform obeys*

$$\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) = \sum_{\boldsymbol{j}\in\{0,1\}^{\boldsymbol{\mu}}} \boldsymbol{C}_{\boldsymbol{c}^j(\overleftarrow{F_1})}(\boldsymbol{u})\mathscr{F}_{F_1(\boldsymbol{j}),F_2}(\boldsymbol{B})(\boldsymbol{u}). \tag{6.4}$$

If $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x})\boldsymbol{C}$ we analogously get

$$\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) = \sum_{\boldsymbol{k}\in\{0,1\}^{(\boldsymbol{\nu}-\boldsymbol{\mu})}} \mathscr{F}_{F_1,F_2(\boldsymbol{k})}(\boldsymbol{B})(\boldsymbol{u})\boldsymbol{C}_{\boldsymbol{c}^{\boldsymbol{k}}(\overrightarrow{F_2})}(\boldsymbol{u}) \tag{6.5}$$

for a right separable GFT.

Proof. We restrict ourselves to the proof of the first assertion.

$$\begin{aligned}
\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) &= \int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(\boldsymbol{x},\boldsymbol{u})}\boldsymbol{C}\boldsymbol{B}(\boldsymbol{x}) \prod_{f\in F_2} e^{-f(\boldsymbol{x},\boldsymbol{u})}\,\mathrm{d}^m\boldsymbol{x} \\
&\overset{\text{Lem. 5.8}}{=} \int_{\mathbb{R}^m} \Big(\sum_{\boldsymbol{j}\in\{0,1\}^{\boldsymbol{\mu}}} \boldsymbol{C}_{\boldsymbol{c}^j(\overleftarrow{F_1})}(\boldsymbol{u}) \prod_{l=1}^{\boldsymbol{\mu}} e^{-(-1)^{j_l} f_l(\boldsymbol{x},\boldsymbol{u})}\Big) \\
&\qquad \boldsymbol{B}(\boldsymbol{x}) \prod_{f\in F_2} e^{-f(\boldsymbol{x},\boldsymbol{u})}\,\mathrm{d}^m\boldsymbol{x} \\
&= \sum_{\boldsymbol{j}\in\{0,1\}^{\boldsymbol{\mu}}} \boldsymbol{C}_{\boldsymbol{c}^j(\overleftarrow{F_1})}(\boldsymbol{u}) \int_{\mathbb{R}^m} \prod_{l=1}^{\boldsymbol{\mu}} e^{-(-1)^{j_l} f_l(\boldsymbol{x},\boldsymbol{u})} \\
&\qquad \boldsymbol{B}(\boldsymbol{x}) \prod_{f\in F_2} e^{-f(\boldsymbol{x},\boldsymbol{u})}\,\mathrm{d}^m\boldsymbol{x} \\
&= \sum_{\boldsymbol{j}\in\{0,1\}^{\boldsymbol{\mu}}} \boldsymbol{C}_{\boldsymbol{c}^j(\overleftarrow{F_1})}(\boldsymbol{u})\mathscr{F}_{F_1(\boldsymbol{j}),F_2}(\boldsymbol{B})(\boldsymbol{u})
\end{aligned} \tag{6.6}$$

The second one follows in the same way. ∎

Corollary 6.6 (Uniform constants). *Let the claims from Theorem* 6.5 *hold. If the constant $\boldsymbol{C}$ satisfies $\boldsymbol{C} = \boldsymbol{C}_{\boldsymbol{c}^j(\overleftarrow{F_1})}(\boldsymbol{u})$ for a multi-index $\boldsymbol{j} \in \{0,1\}^{\boldsymbol{\mu}}$ then the theorem simplifies to*

$$\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) = \boldsymbol{C}\mathscr{F}_{F_1(\boldsymbol{j}),F_2}(\boldsymbol{B})(\boldsymbol{u}) \tag{6.7}$$

for $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{B}(\boldsymbol{x})$ respectively

$$\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) = \mathscr{F}_{F_1,F_2(\boldsymbol{k})}(\boldsymbol{B})(\boldsymbol{u})\boldsymbol{C} \tag{6.8}$$

for $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x})\boldsymbol{C}$ *and* $\boldsymbol{C} = \boldsymbol{C}_{\boldsymbol{c}^{\boldsymbol{k}}(\overrightarrow{F_2})}(\boldsymbol{u})$ *for a multi-index* $\boldsymbol{k} \in \{0,1\}^{(\boldsymbol{\nu}-\boldsymbol{\mu})}$.[2]

Corollary 6.7 (Left and right linearity). *The geometric Fourier transform is left (respectively right) linear if* F_1 *(respectively* F_2*) only consists of functions* f_k *with values in the center of* $C\ell_{p,q}$*, that means* $\forall \boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^m, \forall \boldsymbol{A} \in C\ell_{p,q}$: $\boldsymbol{A} f_k(\boldsymbol{x}, \boldsymbol{u}) = f_k(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{A}$*.*

Remark 6.8. Note that for empty sets F_1 (or F_2) necessarily all elements satisfy commutativity and therefore the condition in corollary 6.7.

The different appearances of Theorem 6.5 are summarized in Table 1 and Table 2.

	GFT	$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{B}(\boldsymbol{x})$
1.	Clifford	$\mathscr{F}_{f_1} = \boldsymbol{C}\mathscr{F}_{f_1}$
2.	Bülow	$\mathscr{F}_{f_1,\dots,f_n} = \boldsymbol{C}\mathscr{F}_{f_1,\dots,f_n}$
3.	Quaternionic	$\mathscr{F}_{f_1,f_2} = \boldsymbol{C}_{\boldsymbol{c}^0(i)}\mathscr{F}_{f_1,f_2} + \boldsymbol{C}_{\boldsymbol{c}^1(i)}\mathscr{F}_{-f_1,f_2}$
4.	Spacetime	$\mathscr{F}_{f_1,f_2} = \boldsymbol{C}_{\boldsymbol{c}^0(\boldsymbol{e}_4)}\mathscr{F}_{f_1,f_2} + \boldsymbol{C}_{\boldsymbol{c}^1(\boldsymbol{e}_4)}\mathscr{F}_{-f_1,f_2}$
5.	Color Image	$\mathscr{F}_{f_1,f_2,f_3,f_4} = \boldsymbol{C}_{\boldsymbol{c}^{00}(\overleftarrow{\boldsymbol{B},i\boldsymbol{B}})}\mathscr{F}_{f_1,f_2,f_3,f_4}$
		$+\boldsymbol{C}_{\boldsymbol{c}^{10}(\overleftarrow{\boldsymbol{B},i\boldsymbol{B}})}\mathscr{F}_{-f_1,f_2,f_3,f_4}$
		$+\boldsymbol{C}_{\boldsymbol{c}^{01}(\overleftarrow{\boldsymbol{B},i\boldsymbol{B}})}\mathscr{F}_{f_1,-f_2,f_3,f_4}$
		$+\boldsymbol{C}_{\boldsymbol{c}^{11}(\overleftarrow{\boldsymbol{B},i\boldsymbol{B}})}\mathscr{F}_{-f_1,-f_2,f_3,f_4}$
6.	Cylindrical $n = 2$	$\mathscr{F}_{f_1} = \boldsymbol{C}_{\boldsymbol{c}^0(\boldsymbol{e}_{12})}\mathscr{F}_{f_1} + \boldsymbol{C}_{\boldsymbol{c}^1(\boldsymbol{e}_{12})}\mathscr{F}_{-f_1}$
	Cylindrical $n \neq 2$	-

Table 1. Theorem 6.5 (Left products) applied to the GFTs of the first example enumerated in the same order. Notations: on the l.h.s. $\mathscr{F}_{F_1,F_2} = \mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u})$, on the r.h.s $\mathscr{F}_{F_1',F_2'} = \mathscr{F}_{F_1',F_2'}(\boldsymbol{B})(\boldsymbol{u})$

We have seen how to change the order of a multivector and a product of exponentials in the previous section. To get a shift theorem we will have to separate sums appearing in the exponent and sort the resulting exponentials with respect to the summands. Note that corollary 6.2 can be applied in two ways here, because exponentials appear on both sides.

[2] Corrolary 6.6 follows directly from $(\boldsymbol{C}_{\boldsymbol{c}^{\boldsymbol{j}}(\overleftarrow{F_1})})_{\boldsymbol{c}^{\boldsymbol{k}}(\overleftarrow{F_1})} = 0$ for all $\boldsymbol{k} \neq \boldsymbol{j}$ because no non-zero component of $\boldsymbol{C}$ can commute and anticommute with respect to a function in F_1.

	GFT	$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x})\boldsymbol{C}$
1.	Clif. $n = 2 \pmod 4$	$\mathscr{F}_{f_1} = \mathscr{F}_{f_1}\boldsymbol{C}_{\boldsymbol{c}^0(i)} + \mathscr{F}_{-f_1}\boldsymbol{C}_{\boldsymbol{c}^1(i)}$
	Clif. $n = 3 \pmod 4$	$\mathscr{F}_{f_1} = \mathscr{F}_{f_1}\boldsymbol{C}$
2.	Bülow	$\mathscr{F}_{f_1,...,f_n}$
		$= \sum_{\boldsymbol{k}\in\{0,1\}^n} \mathscr{F}_{(-1)^{k_1}f_1,...,(-1)^{k_n}f_n}\boldsymbol{C}_{\boldsymbol{c}^{\boldsymbol{k}}(\overrightarrow{f_1,...,f_n})}$
3.	Quaternionic	$\mathscr{F}_{f_1,f_2} = \mathscr{F}_{f_1,f_2}\boldsymbol{C}_{\boldsymbol{c}^0(j)} + \mathscr{F}_{f_1,-f_2}\boldsymbol{C}_{\boldsymbol{c}^1(j)}$
4.	Spacetime	$\mathscr{F}_{f_1,f_2} = \mathscr{F}_{f_1,f_2}\boldsymbol{C}_{\boldsymbol{c}^0(\boldsymbol{e}_4 i_4)} + \mathscr{F}_{f_1,-f_2}\boldsymbol{C}_{\boldsymbol{c}^1(\boldsymbol{e}_4 i_4)}$
5.	Color Image	$\mathscr{F}_{f_1,f_2,f_3,f_4} = \mathscr{F}_{f_1,f_2,f_3,f_4}\boldsymbol{C}_{\boldsymbol{c}^{00}(\overrightarrow{\boldsymbol{B},i\boldsymbol{B}})}$
		$+\mathscr{F}_{f_1,f_2,-f_3,f_4}\boldsymbol{C}_{\boldsymbol{c}^{10}(\overrightarrow{\boldsymbol{B},i\boldsymbol{B}})}$
		$+\mathscr{F}_{f_1,f_2,f_3,-f_4}\boldsymbol{C}_{\boldsymbol{c}^{01}(\overrightarrow{\boldsymbol{B},i\boldsymbol{B}})}$
		$+\mathscr{F}_{f_1,f_2,-f_3,-f_4}\boldsymbol{C}_{\boldsymbol{c}^{11}(\overrightarrow{\boldsymbol{B},i\boldsymbol{B}})}$
6.	Cylindrical	$\mathscr{F}_{f_1} = \mathscr{F}_{f_1}\boldsymbol{C}$

Table 2. Theorem 6.5 (Right products) applied to the GFTs of the first example, enumerated in the same order. Notations: on the l.h.s. $\mathscr{F}_{F_1,F_2} = \mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u})$, on the r.h.s $\mathscr{F}_{F_1',F_2'} = \mathscr{F}_{F_1',F_2'}(\boldsymbol{B})(\boldsymbol{u})$

Not every factor will need to be swapped with every other one. So, to keep things short, we will make use of the notation $\boldsymbol{c}^{(J)_l}(f_1,...,f_l,0,...,0)$ for $l \in \{1,...,d\}$ instead of distinguishing between differently sized multi-indices for every l that appears. The zeros at the end substitutionary indicate real numbers. They commute with every multivector. That implies, that for the last $d-l$ factors no swap and therefore no separation needs to be made. It would also be possible to use the notation $\boldsymbol{c}^{(J)_l}(f_1,...,f_{l-1},0,...,0)$ for $l \in \{1,...,d\}$, because every function commutes with itself. We chose the other one where no exceptional treatment of f_1 is necessary. But please note that the multivectors $(J)_l$ indicating the commutative and anticommutative parts will all have zeros from l to d and therefore form a strictly triangular matrix.

Lemma 6.9. *Let a set of functions $F = \{f_1(\boldsymbol{x},\boldsymbol{u}),...,f_d(\boldsymbol{x},\boldsymbol{u})\}$ fulfill (*6.1*) and be linear with respect to $\boldsymbol{x}$. Further let $J \in \{0,1\}^{d\times d}$ be a strictly lower triangular matrix, that is associated column by column with a multi-index $\boldsymbol{j} \in \{0,1\}^d$ by $\forall k = 1,...,d : (\sum_{l=1}^{d} J_{l,k}) \bmod 2 = j_k$, with $(J)_l$ being its l-th* *row, then*

$$\begin{aligned}&\prod_{l=1}^{d} e^{-f_l(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{u})}\\&=\sum_{\boldsymbol{j}\in\{0,1\}^d}\sum_{\substack{J\in\{0,1\}^{d\times d},\\ \sum_{l=1}^{d}(J)_l \bmod 2=\boldsymbol{j}}}\prod_{l=1}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{(J)_l}(\overleftarrow{f_1,\dots,f_l,0,\dots,0})}\prod_{l=1}^{d} e^{-(-1)^{j_l}f_l(\boldsymbol{y},\boldsymbol{u})}\end{aligned}\tag{6.9}$$

or alternatively with strictly upper triangular matrices J

$$\begin{aligned}&\prod_{l=1}^{d} e^{-f_l(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{u})}\\&=\sum_{\boldsymbol{j}\in\{0,1\}^d}\sum_{\substack{J\in\{0,1\}^{d\times d},\\ \sum_{l=1}^{d}(J)_l \bmod 2=\boldsymbol{j}}}\prod_{l=1}^{d} e^{-(-1)^{j_l}f_l(\boldsymbol{x},\boldsymbol{u})}\prod_{l=1}^{d} e^{-f_l(\boldsymbol{y},\boldsymbol{u})}_{\boldsymbol{c}^{(J)_l}(\overrightarrow{0,\dots,0,f_l,\dots,f_d})}.\end{aligned}\tag{6.10}$$

We do not explicitly indicate the dependence of the partition on $\boldsymbol{u}$ as in corollary 6.2, because the functions in the exponents already contain this dependence. Please note that the decomposition is pointwise.

Proof. We will only prove the first assertion. The second one follows analogously by applying corollary 6.2 the other way around.

$$\begin{aligned}\prod_{l=1}^{d} e^{-f_l(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{u})} &\overset{F\text{ lin.}}{=}\prod_{l=1}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})-f_l(\boldsymbol{y},\boldsymbol{u})}\\ &\overset{\text{Lem. 2.2}}{=}\prod_{l=1}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})}e^{-f_l(\boldsymbol{y},\boldsymbol{u})}\\ &=e^{-f_1(\boldsymbol{x},\boldsymbol{u})}e^{-f_1(\boldsymbol{y},\boldsymbol{u})}\prod_{l=2}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})}e^{-f_l(\boldsymbol{y},\boldsymbol{u})}\\ &\overset{\text{cor. 6.2}}{=}e^{-f_1(\boldsymbol{x},\boldsymbol{u})}\big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^0(f_1)}e^{-f_1(\boldsymbol{y},\boldsymbol{u})}e^{-f_2(\boldsymbol{y},\boldsymbol{u})}\\ &\quad+e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^1(f_1)}e^{f_1(\boldsymbol{y},\boldsymbol{u})}e^{-f_2(\boldsymbol{y},\boldsymbol{u})}\big)\prod_{l=3}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})}e^{-f_l(\boldsymbol{y},\boldsymbol{u})}\end{aligned}\tag{6.11}$$

$$
\begin{aligned}
\overset{\text{cor. 6.2}}{=} & e^{-f_1(\boldsymbol{x},\boldsymbol{u})}\big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^0(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{00}(\overleftarrow{f_1,f_2})} e^{-f_1(\boldsymbol{y},\boldsymbol{u})} e^{-f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^0(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{01}(\overleftarrow{f_1,f_2})} e^{-f_1(\boldsymbol{y},\boldsymbol{u})} e^{f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^0(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{10}(\overleftarrow{f_1,f_2})} e^{f_1(\boldsymbol{y},\boldsymbol{u})} e^{-f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^0(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{11}(\overleftarrow{f_1,f_2})} e^{f_1(\boldsymbol{y},\boldsymbol{u})} e^{f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^1(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{00}(\overleftarrow{f_1,f_2})} e^{f_1(\boldsymbol{y},\boldsymbol{u})} e^{-f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^1(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{01}(\overleftarrow{f_1,f_2})} e^{f_1(\boldsymbol{y},\boldsymbol{u})} e^{f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^1(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{10}(\overleftarrow{f_1,f_2})} e^{-f_1(\boldsymbol{y},\boldsymbol{u})} e^{-f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^1(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{11}(\overleftarrow{f_1,f_2})} e^{-f_1(\boldsymbol{y},\boldsymbol{u})} e^{f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})}\big) \\
& \prod_{l=4}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})} e^{-f_l(\boldsymbol{y},\boldsymbol{u})}
\end{aligned}
\tag{6.12}
$$

There are only 2^δ ways of distributing the signs of δ exponents, so some of the summands can be combined.

$$
\begin{aligned}
= & e^{-f_1(\boldsymbol{x},\boldsymbol{u})}\big(\big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^0(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{00}(\overleftarrow{f_1,f_2})} + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^1(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{10}(\overleftarrow{f_1,f_2})}\big) \\
& e^{-f_1(\boldsymbol{y},\boldsymbol{u})} e^{-f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + \big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^0(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{01}(\overleftarrow{f_1,f_2})} + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^1(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{11}(\overleftarrow{f_1,f_2})}\big) \\
& e^{-f_1(\boldsymbol{y},\boldsymbol{u})} e^{f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + \big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^0(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{10}(\overleftarrow{f_1,f_2})} + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^1(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{00}(\overleftarrow{f_1,f_2})}\big) e^{f_1(\boldsymbol{y},\boldsymbol{u})} \\
& e^{-f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
& + \big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^0(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{11}(\overleftarrow{f_1,f_2})} + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^1(f_1)} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{01}(\overleftarrow{f_1,f_2})}\big) e^{f_1(\boldsymbol{y},\boldsymbol{u})} \\
& e^{f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})}\big) \prod_{l=4}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})} e^{-f_l(\boldsymbol{y},\boldsymbol{u})}
\end{aligned}
\tag{6.13}
$$

To get a compact notation we expand all multi-indices by adding zeros until they have the same length. Note that the last non zero argument in terms like $\boldsymbol{c}^{000}(\overleftarrow{f_1,0,0})$ always coincides with the exponent of the corresponding factor. Because of that it will always commute and could as well be replaced by a zero, too.

$$
\begin{aligned}
&= e^{-f_1(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{000}(\overleftarrow{f_1,0,0})} \\
&\Big(\big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{000}(\overleftarrow{f_1,f_2,0})} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{000}(\overleftarrow{f_1,f_2,f_3})} + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{100}(\overleftarrow{f_1,f_2,0})} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{100}(\overleftarrow{f_1,f_2,f_3})} \big) \\
&e^{-f_1(\boldsymbol{y},\boldsymbol{u})} e^{-f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
&+ \big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{000}(\overleftarrow{f_1,f_2,0})} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{010}(\overleftarrow{f_1,f_2,f_3})} + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{100}(\overleftarrow{f_1,f_2,0})} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{110}(\overleftarrow{f_1,f_2,f_3})} \big) \\
&e^{-f_1(\boldsymbol{y},\boldsymbol{u})} e^{f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
&+ \big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{000}(\overleftarrow{f_1,f_2,0})} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{100}(\overleftarrow{f_1,f_2,f_3})} + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{100}(\overleftarrow{f_1,f_2,0})} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{000}(\overleftarrow{f_1,f_2,f_3})} \big) \\
&e^{f_1(\boldsymbol{y},\boldsymbol{u})} e^{-f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \\
&+ \big(e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{000}(\overleftarrow{f_1,f_2,0})} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{110}(\overleftarrow{f_1,f_2,f_3})} + e^{-f_2(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{100}(\overleftarrow{f_1,f_2,0})} e^{-f_3(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{010}(\overleftarrow{f_1,f_2,f_3})} \big) \\
&e^{f_1(\boldsymbol{y},\boldsymbol{u})} e^{f_2(\boldsymbol{y},\boldsymbol{u})} e^{-f_3(\boldsymbol{y},\boldsymbol{u})} \Big) \\
&\prod_{l=4}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})} e^{-f_l(\boldsymbol{y},\boldsymbol{u})}
\end{aligned}
\tag{6.14}
$$

For $\delta = 3$ we look at all strictly lower triangular matrices $J \in \{0,1\}^{\delta \times \delta}$ with the property

$$
\forall k = 1, ..., \delta : (\sum_{l=1}^{\delta} (J)_{l,k}) \bmod 2 = j_k. \tag{6.15}
$$

That means the l-th row $(J)_l$ of J contains a multi-index $(J)_l \in \{0,1\}^{\delta}$, with the last $\delta - l - 1$ entries being zero and the k-th column sum being even when $j_k = 0$ and being odd when $j_k = 1$. For example the first multi-index is $\boldsymbol{j} = (0,0,0)$. There are only two different strictly lower triangular matrices that have columns summing up to even numbers:

$$
J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.16}
$$

Their first row contains the multi-index that belongs to $e^{-f_1(\boldsymbol{x},\boldsymbol{u})}$, the second one belongs to $e^{-f_2(\boldsymbol{x},\boldsymbol{u})}$ and so on. So the Summands with exactly these multi-indices are the ones assigned to the product of exponentials whose signs are invariant during the reordering. With this notation and all $J \in \{0,1\}^{3\times 3}$ that satisfy the property (6.15) we can write

$$
\begin{aligned}
\prod_{l=1}^{d} e^{-f_l(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{u})} = &\sum_{\boldsymbol{j}\in\{0,1\}^3} \sum_{J} \prod_{l=1}^{3} e^{-f_l(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{(J)_l}(\overleftarrow{f_1,\ldots,f_l,0,\ldots,0})} \prod_{l=1}^{3} e^{-(-1)^{j_l} f_l(\boldsymbol{y},\boldsymbol{u})} \\
&\prod_{l=4}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})} e^{-f_l(\boldsymbol{y},\boldsymbol{u})}.
\end{aligned}
\tag{6.17}
$$

Using mathematical induction with matrices $J \in \{0,1\}^{\delta\times\delta}$ like introduced above for growing δ and corollary 6.2 repeatedly until we reach $\delta = d$ we get

$$= \sum_{\boldsymbol{j}\in\{0,1\}^d} \sum_J \prod_{l=1}^{d} e^{-f_l(\boldsymbol{x},\boldsymbol{u})}_{\boldsymbol{c}^{(J)_l}(\overleftarrow{f_1,\dots,f_l,0,\dots,0})} \prod_{l=1}^{d} e^{-(-1)^{j_l} f_l(\boldsymbol{y},\boldsymbol{u})}. \tag{6.18}$$

□

Remark 6.10. The number of actually appearing summands is usually much smaller than in Theorem 6.11. It is determined by the amount of distinct strictly lower (upper) triangular matrices J with entries being either zero or one, particularly

$$2^{\frac{d(d-1)}{2}}. \tag{6.19}$$

Theorem 6.11 (Shift). *Let $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x} - \boldsymbol{x}_0)$ be multivector fields, F_1, F_2 be linear with respect to $\boldsymbol{x}$, $\boldsymbol{j} \in \{0,1\}^{\boldsymbol{\mu}}, \boldsymbol{k} \in \{0,1\}^{(\boldsymbol{\nu}-\boldsymbol{\mu})}$ be multi-indices and $F_1(\boldsymbol{j}), F_2(\boldsymbol{k})$ as introduced in Definition* 6.4*, then a separable GFT suffices*

$$\begin{aligned}\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) = &\sum_{\boldsymbol{j},\boldsymbol{k}} \sum_{J,K} \prod_{l=1}^{\boldsymbol{\mu}} e^{-f_l(\boldsymbol{x}_0,\boldsymbol{u})}_{\boldsymbol{c}^{(J)_l}(\overleftarrow{f_1,\dots,f_l,0,\dots,0})} \mathscr{F}_{F_1(\boldsymbol{j}),F_2(\boldsymbol{k})}(\boldsymbol{B})(\boldsymbol{u}) \\ &\prod_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}} e^{-f_l(\boldsymbol{x}_0,\boldsymbol{u})}_{\boldsymbol{c}^{(K)_{l-\boldsymbol{\mu}}}(\overrightarrow{0,\dots,0,f_l,\dots,f_{\boldsymbol{\nu}}})},\end{aligned} \tag{6.20}$$

where $J \in \{0,1\}^{\boldsymbol{\mu}\times\boldsymbol{\mu}}$ and $K \in \{0,1\}^{(\boldsymbol{\nu}-\boldsymbol{\mu})\times(\boldsymbol{\nu}-\boldsymbol{\mu})}$ are the strictly lower, respectively upper, triangular matrices with rows $(J)_l, (K)_{l-\boldsymbol{\mu}}$ summing up to $(\sum_{l=1}^{\boldsymbol{\mu}}(J)_l) \bmod 2 = \boldsymbol{j}$ respectively $(\sum_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}}(K)_{l-\boldsymbol{\mu}}) \bmod 2 = \boldsymbol{k}$ as in Lemma 6.9*.*

Proof. First we put the transfomed function down to $\boldsymbol{B}(\boldsymbol{y})$ using a change of coordinates.

$$\begin{aligned}\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) &= \int_{\mathbb{R}^m} \prod_{l=1}^{\boldsymbol{\mu}} e^{-f_l(\boldsymbol{x},\boldsymbol{u})} \boldsymbol{A}(\boldsymbol{x}) \prod_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}} e^{-f_l(\boldsymbol{x},\boldsymbol{u})} \,\mathrm{d}^m\boldsymbol{x} \\ &= \int_{\mathbb{R}^m} \prod_{l=1}^{\boldsymbol{\mu}} e^{-f_l(\boldsymbol{x},\boldsymbol{u})} \boldsymbol{B}(\boldsymbol{x}-\boldsymbol{x}_0) \prod_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}} e^{-f_l(\boldsymbol{x},\boldsymbol{u})} \,\mathrm{d}^m\boldsymbol{x} \\ &\overset{\boldsymbol{y}=\boldsymbol{x}-\boldsymbol{x}_0}{=} \int_{\mathbb{R}^m} \prod_{l=1}^{\boldsymbol{\mu}} e^{-f_l(\boldsymbol{y}+\boldsymbol{x}_0,\boldsymbol{u})} \boldsymbol{B}(\boldsymbol{y}) \prod_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}} e^{-f_l(\boldsymbol{y}+\boldsymbol{x}_0,\boldsymbol{u})} \,\mathrm{d}^m\boldsymbol{y}\end{aligned} \tag{6.21}$$

Now we separate and sort the factors with the above Lemma 6.9.

$$\begin{aligned}
&\overset{\text{Lem. 6.9}}{=} \int_{\mathbb{R}^m} \sum_{\boldsymbol{j}\in\{0,1\}^{\boldsymbol{\mu}}} \sum_{\substack{J\in\{0,1\}^{\boldsymbol{\mu}\times\boldsymbol{\mu}} \\ \sum(J)_l \bmod 2=\boldsymbol{j}}} \\
&\qquad \prod_{l=1}^{\boldsymbol{\mu}} e^{-f_l(\boldsymbol{x}_0,\boldsymbol{u})}_{\boldsymbol{c}^{(J)_l}\overleftarrow{(f_1,\dots,f_l,0,\dots,0)}} \prod_{l=1}^{\boldsymbol{\mu}} e^{-(-1)^{j_l} f_l(\boldsymbol{y},\boldsymbol{u})} \boldsymbol{B}(\boldsymbol{y}) \\
&\qquad \sum_{\boldsymbol{k}\in\{0,1\}^{(\boldsymbol{\nu}-\boldsymbol{\mu})}} \sum_{\substack{K\in\{0,1\}^{(\boldsymbol{\nu}-\boldsymbol{\mu})\times(\boldsymbol{\nu}-\boldsymbol{\mu})} \\ \sum(K)_l \bmod 2=\boldsymbol{k}}} \\
&\qquad \prod_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}} e^{-(-1)^{k_{l-\boldsymbol{\mu}}} f_l(\boldsymbol{y},\boldsymbol{u})} \prod_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}} e^{-f_l(\boldsymbol{x}_0,\boldsymbol{u})}_{\boldsymbol{c}^{(K)_{l-\boldsymbol{\mu}}}\overrightarrow{(0,\dots,0,f_l,\dots,f_{\boldsymbol{\nu}})}} \,\mathrm{d}^m\boldsymbol{y} \\
&= \sum_{\boldsymbol{j},\boldsymbol{k}} \sum_{J,K} \prod_{l=1}^{\boldsymbol{\mu}} e^{-f_l(\boldsymbol{x}_0,\boldsymbol{u})}_{\boldsymbol{c}^{(J)_l}\overleftarrow{(f_1,\dots,f_l,0,\dots,0)}} \\
&\qquad \mathscr{F}_{F_1(\boldsymbol{j}),F_2(\boldsymbol{k})}(\boldsymbol{B})(\boldsymbol{u}) \prod_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}} e^{-f_l(\boldsymbol{x}_0,\boldsymbol{u})}_{\boldsymbol{c}^{(K)_{l-\boldsymbol{\mu}}}\overrightarrow{(0,\dots,0,f_l,\dots,f_{\boldsymbol{\nu}})}}
\end{aligned} \tag{6.22}$$

□

Corollary 6.12 (Shift). *Let $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x}-\boldsymbol{x}_0)$ be multivector fields, F_1 and F_2 each consist of mutually commutative functions*[3] *being linear with respect to $\boldsymbol{x}$, then the GFT obeys*

$$\mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u}) = \prod_{l=1}^{\boldsymbol{\mu}} e^{-f_l(\boldsymbol{x}_0,\boldsymbol{u})} \mathscr{F}_{F_1,F_2}(\boldsymbol{B})(\boldsymbol{u}) \prod_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}} e^{-f_l(\boldsymbol{x}_0,\boldsymbol{u})}. \tag{6.23}$$

Remark 6.13. For sets F_1, F_2 that each consist of less than two functions the condition of corollary 6.12 is necessarily satisfied, compare e.g. reference examples 1,3 and 4.

The specific forms, our standard examples take, are summarized in Table 3. As expected they are often shorter than what could be expected from Remark 6.10.

7. Conclusions and Outlook

For multivector fields over $\mathbb{R}^{p,q}$ with values in any geometric algebra $G^{p,q}$ we have successfully defined a general geometric Fourier transform. It comprehends all popular Fourier transforms from current literature in the introductory example. Its existence, independent from the specific choice of functions F_1, F_2, could be proved for all integrable multivector fields, see Theorem 3.1. Theorem 3.2 shows that our geometric Fourier transform is generally linear

[3] Cross commutativity between F_1 and F_2 is not necessary.

	GFT	$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x}-\boldsymbol{x}_0)$
1.	Clifford	$\mathscr{F}_{f_1} = \mathscr{F}_{f_1} e^{-2\pi i \boldsymbol{x}_0\cdot\boldsymbol{u}}$
2.	Bülow	$\mathscr{F}_{f_1,\ldots,f_n} = \sum_{\boldsymbol{k}\in\{0,1\}^n} \sum_K \mathscr{F}_{(-1)^{k_1} f_1,\ldots,(-1)^{k_n} f_n}$ $\prod_{l=1}^n e^{-2\pi x_{0k} u_k}_{\boldsymbol{c}^{(K)_l}(\overrightarrow{0,\ldots,0,f_l,\ldots,f_n})}$
3.	Quaternionic	$\mathscr{F}_{f_1,f_2} = e^{-2\pi i x_{01} u_1} \mathscr{F}_{f_1,f_2} e^{-2\pi j x_{02} u_2}$
4.	Spacetime	$\mathscr{F}_{f_1,f_2} = e^{-\boldsymbol{e}_4 x_{04} u_4} \mathscr{F}_{f_1,f_2} e^{-\epsilon_4 \boldsymbol{e}_4 i_4 (x_1u_1+x_2u_2+x_3u_3)}$
5.	Color Image	$\mathscr{F}_{f_1,f_2,f_3,f_4} = e^{-\frac{1}{2}(x_{01}u_1+x_{02}u_2)(\boldsymbol{B}+i\boldsymbol{B})} \mathscr{F}_{f_1,f_2,f_3,f_4}$ $e^{\frac{1}{2}(x_{01}u_1+x_{02}u_2)(\boldsymbol{B}+i\boldsymbol{B})}$
6.	Cyl. $n=2$	$\mathscr{F}_{f_1} = e^{\boldsymbol{x}_0\wedge\boldsymbol{u}} \mathscr{F}_{f_1}$
	Cyl. $n\neq 2$	-

TABLE 3. Theorem 6.11 (Shift) applied to the GFTs of the first example, enumerated in the same order. Notations: on the l.h.s. $\mathscr{F}_{F_1,F_2} = \mathscr{F}_{F_1,F_2}(\boldsymbol{A})(\boldsymbol{u})$, on the r.h.s $\mathscr{F}_{F_1',F_2'} = \mathscr{F}_{F_1',F_2'}(\boldsymbol{B})(\boldsymbol{u})$, in the second row K represents all strictly upper triangular matrices $\in \{0,1\}^{n\times n}$ with rows $(K)_{l-\boldsymbol{\mu}}$ summing up to $(\sum_{l=\boldsymbol{\mu}+1}^{\boldsymbol{\nu}} (K)_{l-\boldsymbol{\mu}}) \bmod 2 = \boldsymbol{k}$. The simplified shape of the color image FT results from the commutativity of $\boldsymbol{B}$ and $i\boldsymbol{B}$ and application of Lemma 2.2.

over the field of real numbers. All transforms from the reference example consist of bilinear F_1 and F_2. We proved that this property is sufficient to ensure the scaling property of Theorem 4.1.

If a general geometric Fourier transform is separable as introduced in Definition 6.1, then Theorem 6.5 (Left and right products) guarantees that constant factors can be separated from the vector field to be transformed. As a consequence general linearity is achieved by choosing F_1, F_2 with values in the center of the geometric algebra $C\ell_{p,q}$, compare Corollary 6.7. All examples except for the cylindrical Fourier transform [12] satisfy this claim.

Under the condition of linearity with respect to the first argument of the functions of the sets F_1 and F_2 additionally to the just mentioned separability, we also proved a shift property (Theorem 6.11).

In future papers we are going to state the necessary constraints for a generalized convolution theorem, invertibility, derivation theorem and we will examine how simplifications can be achieved based on symmetry properties of the multivector fields to be transformed. We will also construct generalized geometric Fourier transforms in a broad sense from combinations of the ones introduced in this paper and from decomposition into their sine and cosine parts which will also cover the vector and bivector Fourier transforms of [18]. It would further be of interest to extend our approach to Fourier transforms defined on spheres or other non-Euclidean manifolds, to functions in the Schwartz space and to square-integrable functions.

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