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A GENERAL INTERPOLATION THEOREM OF MARCINKIEWICZ TYPE

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The Marcinkiewicz interpolation theorem has been generalized, on the one hand, by Calderón [2] and Hunt [4] to quasi-linear operators from a couple of Lorentz spaces to another. After Lions and Peetre discussed interpolation of linear operators from a couple of Banach spaces to another, Krée [6] and Peetre-Sparr [7] have succeeded in generalizing the theory to (quasi-) linear operators from a couple of quasi-normed Abelian groups to another. On the other hand, the weak type assumptions at the end points of indices have also been generalized by Calderón [2] in the case of Lebesgue spaces and by De Vore-Riemenschneider-Sharpley [3] in the case of normed spaces. We give here an interpolation theorem which generalizes all of the above results.

1. Real interpolation groups of a couple of quasi-normed Abelian groups. We recall some of the results of Peetre-Sparr [7] (see [1]).

Let X be an Abelian group. A *quasi-norm* on X is by definition a real-valued function $\| \ \|_X$ on X satisfying the following conditions:

$$\|x\|_X \geqq 0\,, \quad \text{and} \quad \|x\|_X = 0 \Leftrightarrow x = 0\,; \tag{1}$$

$$\|-x\|_X = \|x\|_X\,; \tag{2}$$

$$\|x+y\|_X \leqq \kappa(\|x\|_X + \|y\|_X)\,, \tag{3}$$

where κ is a constant independent of x and y. Such a quasi-norm is called a *κ-quasi-norm*. An Abelian group equipped with a quasi-norm is called a *quasi-normed Abelian group*.

If $(\Omega, \mathscr{M}, \mu)$ is a measure space, then for each $0 < p \leqq \infty$ the Lebesgue space $L^p(\Omega)$ is a quasi-normed Abelian group under the κ_p-quasi-norm

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f(s)|^p d\mu(s)\right)^{1/p}, & 0 < p < \infty\,, \\ \operatorname*{ess\,sup}_{s\in\Omega} |f(s)|\,, & p = \infty\,, \end{cases} \tag{4}$$

where

$$(5)\qquad \kappa_p = \begin{cases} 1, & 1 \leqq p \leqq \infty, \\ 2^{(1-p)/p}, & 0 < p < 1. \end{cases}$$

When $(\Omega, \mathscr{M}, \mu)$ is the multiplicative group $(0, \infty)$ with the Haar measure ds/s, we write L_*^p for $L^p(\Omega)$. In this case we also admit ω as an index and define L_*^ω to be the subspace of L_*^∞ of all elements $f(s)$ such that $f(s) \to 0$ essentially as $s \to \infty$ and as $s \to 0$. The norm is the restriction of the norm of L_*^∞. The index ω is defined to be greater than any finite p but we do not define order relation between ω and ∞ to avoid confusion.

We define $\rho > 0$ by $(2\kappa)^\rho = 2$. Then for each κ-quasi-norm $\|\ \|_X$ there is a 1-quasi-norm $\|\ \|_X^*$ such that

$$(6)\qquad \|x\|_X^* \leqq \|x\|_X^\rho \leqq 2\|x\|_X^*.$$

Thus a natural uniform topology is introduced in the quasi-normed Abelian group X by the metric $\|x - y\|_X^*$.

A pair of quasi-normed Abelian groups (X_0, X_1) is said to be *compatible* if there is a Hausdorff topological group $\mathscr{X}$ for which continuous linear injections $i_0: X_0 \to \mathscr{X}$ and $i_1: X_1 \to \mathscr{X}$ are defined.

Let $X = (X_0, X_1)$ be a compatible couple of quasi-normed Abelian groups with κ_0-quasi-norm $\|\ \|_{X_0}$ and κ_1-quasi-norm $\|\ \|_{X_1}$. Then the sum $X_0 + X_1$ in $\mathscr{X}$ is a quasi-normed Abelian group under

$$(7)\qquad \|x\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1};\ x = x_0 + x_1\},$$

which is a κ-quasi-norm with $\kappa = \max\{\kappa_0, \kappa_1\}$. We also define a κ-quasi-norm $L(x, t)$ on $X_0 + X_1$ with a parameter $0 < t < \infty$ by

$$(8)\qquad L(x, t) = L_X(x, t) = \inf\{\|x_0\|_{X_0} + t^{-1}\|x_1\|_{X_1};\ x = x_0 + x_1\}.$$

This is nothing but $K(t^{-1}, x)$ of Peetre-Sparr [7] but more convenient in many respects. When an $x \in X_0 + X_1$ is fixed, $L(x, t)$ is a positive, decreasing and continuous function of t.

If $0 < \theta < 1$ and $0 < q \leqq \infty$ or $q = \omega$, the *real interpolation group* $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ is defined to be the set of all $x \in X_0 + X_1$ such that

$$(9)\qquad \|x\|_{X_{\theta,q}} = \|t^\theta L(x, t)\|_{L_*^q} < \infty.$$

$X_{\theta,q}$ is a quasi-normed Abelian group under the quasi-norm $\|x\|_{X_{\theta,q}}$.

The index $q = \omega$ is often useful. For example, we have $(C^0, C^1)_{\theta,\infty} = \mathrm{Lip}^\theta$ and $(C^0, C^1)_{\theta,\omega} = \mathrm{lip}^\theta$. For other examples see [5], where $\infty -$ is used instead of ω.

If $0 < q \leqq r$ or if $q = \omega$ and $r = \infty$, then we have the continuous inclusion $X_{\theta,q} \subset X_{\theta,r}$. This is an immediate consequence of the following lemma due to Hunt [4].

LEMMA. *Suppose that $f(t)$ is a non-negative and non-increasing function on $(0, \infty)$ and that $0 < \theta < 1$. If $t^\theta f(t)$ belongs to L_*^q, then it belongs to L_*^r for any $r \geqq q$ and*

$$(\theta r)^{1/r} \| t^\theta f(t) \|_{L_*^r} \leqq (\theta q)^{1/q} \| t^\theta f(t) \|_{L_*^q} . \tag{10}$$

If $(\Omega, \mathscr{M}, \mu)$ is a reasonable measure space, then the Lebesgue spaces $L^p(\Omega)$, $p \geqq P$, are continuously imbedded in the Hausdorff topological vector space of all equivalence classes of measurable functions which belong to L^P on each subset of finite measure. Thus $(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is a compatible couple of quasi-normed Abelian groups for all $0 < p_i \leqq \infty$.

For the couple $X = (L^\infty(\Omega), L^p(\Omega))$ with $0 < p < \infty$, Krée [6] and Bergh (see [1] p. 109) show that

$$L_X(f, t) \sim \left(t^{-p} \int_0^{t^p} (f^*(s))^p ds \right)^{1/p} , \tag{11}$$

where $f^*(t)$ is the non-increasing rearrangement of $f(s)$. Hence we have the equivalence of interpolation groups $(L^\infty(\Omega), L^p(\Omega))_{\theta,q}$ and Lorentz spaces $L^{(p/\theta,q)}(\Omega)$ for all $p \leqq q \leqq \infty$ or $q = \omega$. Here the *Lorentz space* $L^{(p,q)}(\Omega)$ is by definition the space of all equivalence classes of measurable functions $f(s)$ such that

$$\| f \|_{L^{(p,q)}(\Omega)} = \| t^{1/p} f^*(t) \|_{L_*^q} < \infty . \tag{12}$$

In fact, suppose that $f \in L^{(p/\theta,q)}(\Omega)$ with $p \leqq q \leqq \infty$ or $q = \omega$. Then we have

$$\begin{aligned} \| f \|_{X_{\theta,q}} &= \| t^\theta L(f, t) \|_{L_*^q} \\ &\sim \left\| t^\theta \left[\int_0^{t^p} (s/t^p)(f^*(s))^p ds/s \right]^{1/p} \right\|_{L_*^q} \\ &= \left\| \left[\int_0^{t^p} (s/t^p)^{1-\theta} (s^{\theta/p} f^*(s))^p ds/s \right]^{1/p} \right\|_{L_*^q} \\ &= p^{-1/q} \left\| \int_0^u (s/u)^{1-\theta} (s^{\theta/p} f^*(s))^p ds/s \right\|_{L_*^{q/p}}^{1/p} . \end{aligned} \tag{13}$$

Here we changed variable as $t^p = u$. Since the integral in the norm is the convolution on $(0, \infty)$ of the integrable function

$$h(u) = \begin{cases} 0, & 0 < u < 1 , \\ u^{\theta-1}, & u \geqq 1 , \end{cases}$$

and $(s^{\theta/p} f^*(s))^p \in L_*^{q/p}$, where $q/p \geqq 1$, the right hand side is bounded from above by

$$p^{-1/q}(1-\theta)^{-1/p}\|s^{\theta/p}f^*(s)\|_{L_*^q}.$$

On the other hand, since $f^*(s)$ is non-increasing, the right hand side of (13) is bounded from below by

$$p^{-1/q}\left\|(f^*(u))^p\int_0^u (s/u)^{1-\theta}s^\theta\,ds/s\right\|_{L_*^{q/p}}^{1/p} = p^{-1/q}\|u^{\theta/p}f^*(u)\|_{L_*^q}.$$

Hence it follows that every $f\in(L^\infty(\Omega), L^p(\Omega))_{\theta,q}$ belongs to $L^{(p/\theta,q)}(\Omega)$ and that two quasi-norms are equivalent.

2. The general interpolation theorem. We assume from now on that $X=(X_0, X_1)$ and $Y=(Y_0, Y_1)$ are compatible couples of quasi-normed Abelian groups and that T is an operator defined on a subset $D(T)$ of X_0+X_1 and with values in Y_0+Y_1.

DEFINITION 1. Let ξ_0, ξ_1, η_0 and $\eta_1\in[0, 1]$ with $\xi_0<\xi_1$ and $\eta_0\neq\eta_1$ and let r_0 and $r_1\in(0, \infty)$. Then T is said to be of *generalized weak type* $((\xi_0, r_0), \eta_0; (\xi_1, r_1), \eta_1)$ if there is a constant $M<\infty$ independent of $x\in D(T)$ such that

$$L_Y(Tx, t)\leqq M\left\{t^{-\eta_0}\left[\int_{t^\gamma}^\infty (s^{\xi_0}L_X(x, s))^{r_0}ds/s\right]^{1/r_0} + t^{-\eta_1}\left[\int_0^{t^\gamma} (s^{\xi_1}L_X(x, s))^{r_1}ds/s\right]^{1/r_1}\right\}, \tag{14}$$

where

$$\gamma=(\eta_1-\eta_0)/(\xi_1-\xi_0). \tag{15}$$

The generalized weak type $(p_1, q_1; p_2, q_2)$ of De Vore-Riemenschneider-Sharpley [3] is our generalized weak type $((1/p_1, 1), 1/q_1); ((1/p_2, 1), 1/q_2)$.

We do not assume any kind of linearity of T. The main result of the present article is the following.

THEOREM 1. *Suppose that T is an operator of generalized weak type $((\xi_0, r_0), \eta_0; (\xi_1, r_1), \eta_1)$. Then for any $0<\theta<1$ and $0<q\leqq r\leqq\infty$ or $0<q\leqq r\leqq\omega$ there is a constant $C<\infty$ such that*

$$\|Tx\|_{Y_{\eta,r}}\leqq C\|x\|_{X_{\xi,q}},\quad x\in D(T)\cap X_{\xi,q}, \tag{16}$$

where

$$\xi=(1-\theta)\xi_0+\theta\xi_1,\quad \eta=(1-\theta)\eta_0+\theta\eta_1. \tag{17}$$

PROOF. Because of (10) it suffices to prove (16) only when $q=r$. First we consider the case where $q=r\geqq\max\{r_0, r_1\}$. We have by (14)

$$\|Tx\|_{Y_{\eta,q}}\leqq\kappa_q M\left\{\left\|t^{\eta-\eta_0}\left[\int_{t^\gamma}^\infty (s^{\xi_0}L(x, s))^{r_0}ds/s\right]^{1/r_0}\right\|_{L_*^q}\right.$$

$$+ \left\| t^{\eta-\eta_1} \left[\int_0^{t^\gamma} (s^{\xi_1} L(x, s))^{r_1} ds/s \right]^{1/r_1} \right\|_{L_*^q} \Bigg\}$$

$$= \kappa_q |\gamma|^{-1/q} M \left\{ \left\| \int_u^\infty (s/u)^{(\xi_0-\xi)r_0} (s^\xi L(x, s))^{r_0} ds/s \right\|_{L_*^{q/r_0}}^{1/r_0} \right.$$

$$\left. + \left\| \int_0^u (s/u)^{(\xi_1-\xi)r_1} (s^\xi L(x, s))^{r_1} \right\|_{L_*^{q/r_1}}^{1/r_1} \right\}$$

$$\leqq \kappa_q |\gamma|^{-1/q} M(((\xi - \xi_0)r_0)^{-1/r_0} + ((\xi_1 - \xi)r_1)^{-1/r_1}) \| x \|_{X_{\xi,q}} .$$

The theorem in the general case is reduced to the above by the following.

PROPOSITION 1. *If T is of generalized weak type $((\xi_0, r_0), \eta_0; (\xi_1, r_1), \eta_1)$, then it is of generalized weak type $((\xi_0, q_0), \eta_0; (\xi_1, q_1), \eta_1)$ for any $0 < q_0 \leqq r_0$ and $0 < q_1 \leqq r_1$.*

PROOF. Since $L(x, s)$ is decreasing in s, we have by Lemma

$$\left[\int_0^{t^\gamma} (s^{\xi_1} L(x, s))^{r_1} ds/s \right]^{1/r_1} \leqq (\xi_1 r_1)^{-1/r_1} (\xi_1 q_1)^{1/q_1} \left[\int_0^{t^\gamma} (s^{\xi_1} L(x, s))^{q_1} ds/s \right]^{1/q_1} .$$

Similarly we have

$$\left[\int_{t^\gamma}^\infty (s^{\xi_0} L(x, s))^{r_0} ds/s \right]^{1/r_0}$$

$$\leqq (\xi_0 r_0)^{-1/r_0} (\xi_0 q_0)^{1/q_0} \left\{ \int_{t^\gamma}^\infty (s^{\xi_0} L(x, s))^{q_0} ds/s + \int_0^{t^\gamma} (s^{\xi_0} L(x, t^\gamma))^{q_0} ds/s \right\}^{1/q_0}$$

$$\leqq \kappa_{q_0} (\xi_0 r_0)^{-1/r_0} (\xi_0 q_0)^{1/q_0} \left\{ \left[\int_{t^\gamma}^\infty (s^{\xi_0} L(x, s))^{q_0} ds/s \right]^{1/q_0} + (\xi_0 q_0)^{-1/q_0} t^{\gamma\xi_0} L(x, t^\gamma) \right\} .$$

For the second term we have

$$t^{-\eta_0+\gamma\xi_0} L(x, t^\gamma) = (\xi_1 q_1)^{1/q_1} t^{-\eta_1} \left[\int_0^{t^\gamma} (s^{\xi_1} L(x, t^\gamma))^{q_1} ds/s \right]^{1/q_1} .$$

Thus the right hand side of (14) is bounded by a constant times

$$t^{-\eta_0} \left[\int_{t^\gamma}^\infty (s^{\xi_0} L(x, s))^{q_0} ds/s \right]^{1/q_0} + t^{-\eta_1} \left[\int_0^{t^\gamma} (s^{\xi_1} L(x, s))^{q_1} ds/s \right]^{1/q_1} .$$

3. The Holmstedt theorem for quasi-linear operators.

T is assumed as above to be an operator from $D(T) \subset X_0 + X$ into $Y_0 + Y_1$.

DEFINITION 2. T is said to be *quasi-linear* if $x + y$ belongs to $D(T)$ whenever x and y belong to $D(T)$ and if there are constants k and c independent of x and y such that

$$(18) \qquad L_Y(T(x + y), t) \leqq k(L_Y(Tx, ct) + L_Y(Ty, ct)) .$$

If T is linear, then clearly (18) holds with $k = \kappa_Y$ and $c = 1$.

Krée [6] calls an operator T with $D(T) = X_0 + X_1$ quasi-linear if there are constants k_0 and k_1 such that for any $x_0 \in X_0$ and $x_1 \in X_1$ there are $y_0 \in Y_0$ and $y_1 \in Y_1$ satisfying

$$T(x_0 + x_1) = y_0 + y_1 \text{ and } \|y_i\|_{Y_i} \leqq k_i \|x_i\|_{X_i} . \tag{19}$$

This implies

$$L_Y(Tx, t) \leqq kL_X(x, t), \quad x \in X_0 + X_1 , \tag{20}$$

with $k = \max\{k_0, k_1\}$. Hence it follows that $T: X_{\theta,q} \to Y_{\theta,q}$ is bounded.

We consider, however, operators T whose restrictions $T: X_i \to Y_i$ are not necessarily bounded.

DEFINITION 3. Let $\xi, \eta \in [0, 1]$ and $r \in (0, \infty]$. T is said to be of *generalized weak type* $((\xi, r), \eta)$ if there exists a constant $M < \infty$ such that

$$\| Tx \|_{Y_{\eta,\infty}} \leqq M \| x \|_{X_{\xi,r}}, \quad x \in D(T) \cap X_{\xi,r} . \tag{21}$$

If $\xi = 0$ or 1 (resp. $\eta = 0$ or 1), then we replace $X_{\xi,r}$ by X_ξ (resp. $Y_{\eta,\infty}$ by Y_η).

If T is of generalized weak type $((\xi, r), \eta)$, then it is clearly of generalized weak type $((\xi, q), \eta)$ for any $0 < q \leqq r$.

The following theorem is due to Holmstedt [8] when T is linear.

THEOREM 2. *Let ξ_0, ξ_1, η_0 and $\eta_1 \in [0, 1]$ with $\xi_0 < \xi_1$ and $\eta_0 \neq \eta_1$ and let r_0 and $r_1 \in (0, \infty)$. If a quasi-linear operator T is simultaneously of generalized weak type $((\xi_0, r_0), \eta_0)$ and $((\xi_1, r_1), \eta_1)$, and if there is a constant a such that for every $x \in D(T)$ and $0 < t < \infty$ there are $x_0 \in D(T) \cap X_0$ and $x_1 \in D(T) \cap X_1$ satisfying $x = x_0 + x_1$ and*

$$\| x_0 \|_{X_0} + t^{-1} \| x_1 \|_{X_1} \leqq aL_X(x, t) , \tag{22}$$

then T is of generalized weak type $((\xi_0, r_0), \eta_0;\ (\xi_1, r_1), \eta_1)$ and, in particular, the conclusion of Theorem 1 holds.

PROOF. Let x be an arbitrary element in $D(T)$. If we replace a by a larger number, we can find a piecewise constant functions $x_0(t) \in D(T) \cap X_0$ and $x_1(t) \in D(T) \cap X_1$ such that

$$\| x_0(t) \|_{X_0} + t^{-1} \| x_1(t) \|_{X_1} \leqq aL_X(x, t) , \quad 0 < t < \infty . \tag{23}$$

Then applying (18) to $x = x_0(t^\gamma)$ and $y = x_1(t^\gamma)$, we have

$$\begin{aligned} L_Y(Tx, t) &\leqq kL_Y(Tx_0(t^\gamma), ct) + kL_Y(Tx_1(t^\gamma), ct) \\ &\leqq kM_0(ct)^{-\eta_0} \| x_0(t^\gamma) \|_{X_{\xi_0,r_0}} + kM_1(ct)^{-\eta_1} \| x_1(t^\gamma) \|_{X_{\xi_1,r_1}} . \end{aligned} \tag{24}$$

The modifications necessary in the cases $\xi_i = 0, 1$ or $\eta_i = 0, 1$ would be obvious.

Now, in case $\xi_0 > 0$ we have

$$t^{-\eta_0} \| x_0(t^\gamma) \|_{X_{\xi_0, r_0}} \leqq \kappa_{r_0} t^{-\eta_0} \left\{ \left[\int_{t^\gamma}^\infty (s^{\xi_0} L(x_0(t^\gamma), s))^{r_0} ds/s \right]^{1/r_0} + \left[\int_0^{t^\gamma} (s^{\xi_0} L(x_0(t^\gamma), s))^{r_0} ds/s \right]^{1/r_0} \right\} .$$

Here we have

$$t^{-\eta_0} \left[\int_{t^\gamma}^\infty (s^{\xi_0} L(x_0(t^\gamma), s))^{r_0} ds/s \right]^{1/r_0} \leqq \kappa_X t^{-\eta_0} \left[\int_{t^\gamma}^\infty (s^{\xi_0} (L(x, s) + L(x_1(t^\gamma), s))^{r_0} ds/s \right]^{1/r_0}$$
$$\leqq \kappa_X \kappa_{r_0} \left\{ t^{-\eta_0} \left[\int_{t^\gamma}^\infty (s^{\xi_0} L(x, s))^{r_0} ds/s \right]^{1/r_0} + t^{-\eta_0} \left[\int_{t^\gamma}^\infty (s^{\xi_0} L(x_1(t^\gamma), s))^{r_0} ds/s \right]^{1/r_0} \right\} .$$

Since $L(x_1(t^\gamma), s) \leqq s^{-1} \| x_1(t^\gamma) \|_{X_1}$,

$$t^{-\eta_0} \left[\int_{t^\gamma}^\infty (s^{\xi_0} L(x_1(t^\gamma), s))^{r_0} ds/s \right]^{1/r_0}$$
$$\leqq ((1-\xi_0) r_0)^{-1/r_0} t^{-\eta_0 + \gamma(\xi_0 - 1)} \| x_1(t^\gamma) \|_{X_1}$$
$$\leqq a((1-\xi_0) r_0)^{-1/r_0} t^{-\eta_1 + \gamma \xi_1} L(x, t^\gamma)$$
$$\leqq a((1-\xi_0) r_0)^{-1/r_0} (\xi_1 r_1)^{1/r_1} t^{-\eta_1} \left[\int_0^{t^\gamma} (s^{\xi_1} L(x, s))^{r_1} ds/s \right]^{1/r_1} .$$

Here we employed the fact that $L(x, s)$ is decreasing.

Similarly we have

$$t^{-\eta_0} \left[\int_0^{t^\gamma} (s^{\xi_0} L(x_0(t^\gamma), s))^{r_0} ds/s \right]^{1/r_0}$$
$$\leqq t^{-\eta_0} \left[\int_0^{t^\gamma} (s^{\xi_0} \| x_0(t^\gamma) \|_{X_0})^{r_0} ds/s \right]^{1/r_0}$$
$$\leqq a(\xi_0 r_0)^{-1/r_0} t^{-\eta_0 + \gamma \xi_0} L(x, t^\gamma)$$
$$\leqq a(\xi_0 r_0)^{-1/r_0} (\xi_1 r_1)^{1/r_1} t^{-\eta_1} \left[\int_0^{t^\gamma} (s^{\xi_1} L(x, s))^{r_1} ds/s \right]^{1/r_1} .$$

In case $\xi_0 = 0$ we have

$$t^{-\eta_0} \| x_0(t^\gamma) \|_{X_0} \leqq a t^{-\eta_0} L(x, t^\gamma)$$
$$\leqq a(\xi_1 r_1)^{1/r_1} t^{-\eta_1} \left[\int_0^{t^\gamma} (s^{\xi_1} L(x, s))^{r_1} ds/s \right]^{1/r_1} .$$

Thus the first term of the right hand side of (24) is bounded by a constant multiple of the right hand side of (14).

The second term of (24) is estimated similarly. We employ the inequality

$$t^{\gamma\xi_0}L(x, t^\gamma) \leqq ((1-\xi_0)r_0)^{1/r_0}\left[\int_{t^\gamma}^\infty (s^{\xi_0}L(x, s))^{r_0}ds/s\right]^{1/r_0},$$

which is obtained from the fact that $sL(x, s)$ is increasing.

4. **Applications.** First we prove the reiteration theorem of Peetre-Sparr [7] as an application of Theorem 2.

THEOREM 3. *Suppose that* $X = (X_0, X_1)$ *and* $Y = (Y_0, Y_1)$ *are compatible couples of quasi-normed Abelian groups and that* $0 \leqq \theta_0 < \theta_1 \leqq 1$. *Let* $0 < \eta < 1$ *and* $0 < q \leqq \infty$ *or* $q = \omega$ *be arbitrary numbers and let*

$$\theta = (1-\eta)\theta_0 + \eta\theta_1 . \tag{25}$$

(1) *If* $Y_i \subset X_{\theta_i,\infty}$, $i = 0, 1$, *then*

$$Y_{\eta,q} \subset X_{\theta,q} ; \tag{26}$$

(2) *If* $X_{\theta_i,q_i} \subset Y_i$, $i = 0, 1$, *for some* $0 < q_i \leqq \omega$ *or* ∞, *then*

$$X_{\theta,q} \subset Y_{\eta,q} ; \tag{27}$$

(3) *If* $X_{\theta_i,q_i} \subset Y_i \subset X_{\theta_i,\infty}$, $i = 0, 1$, *for some* $0 < q_i \leqq \omega$ *or* ∞, *then*

$$Y_{\eta,q} = X_{\theta,q} . \tag{28}$$

Here the inclusion $A \subset B$ means that the quasi-normed Abelian group A is included in the quasi-normed Abelian group B and there exists a constant M such that

$$\|a\|_B \leqq M\|a\|_A , \quad a \in A ,$$

and $A = B$ means that A and B are the same Abelian group with equivalent quasi-norms.

If $\theta_0 = 0$ (resp. $\theta_1 = 1$), then $X_{\theta_0,\infty}$ and X_{θ_0,q_0} (resp. $X_{\theta_1,\infty}$ and X_{θ_1,q_1}) should be replaced by X_0 (resp. X_1).

PROOF. (1) Define the operator T: $Y_0 + Y_1 \to X_0 + X_1$ by

$$T(y_0 + y_1) = y_0 + y_1 , \quad y_i \in Y_i .$$

This is a linear injective operator of generalized weak types $((0, *), \theta_0)$ and $((1, *), \theta_1)$ simultaneously. Hence it follows from Theorem 2 that the identity operator T: $Y_{\eta,q} \to X_{\theta,q}$ is bounded.

(2) In this case the identity operator T: $X_{\theta_0,q_0} + X_{\theta_1,q} \to Y_0 + Y_1$ is linear and simultaneously of generalized weak type $((\theta_0, q_0), 0)$ and $((\theta_1, q_1), 1)$. Hence T: $X_{\theta,q} \to Y_{\eta,q}$ is bounded.

Let $(\Omega, \mathscr{M}, \mu)$ be a measure space. As we have shown in § 1,

$$(L^{\infty}(\Omega),\ L^{p}(\Omega))_{\theta,r} = L^{(p/\theta,r)}(\Omega)$$

for any $r \geqq p$. Since p can be chosen arbitrarily small, the reiteration theorem verifies the following.

PROPOSITION 2. *Let $0 < p_1 < p_0 \leqq \infty$ and $q_0,\ q_1 \in (0, \infty] \cup \{\omega\}$. Then for any $0 < \theta < 1$ and $0 < r \leqq \infty$ or $r = \omega$ we have*

$$(L^{(p_0,q_0)}(\Omega),\ L^{(p_1,q_1)}(\Omega))_{\theta,r} = L^{(p,r)}(\Omega)\ , \tag{29}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\ . \tag{30}$$

Lastly we show that the interpolation theorem of Calderón [2] and Hunt [4] is a consequence of Theorem 2.

DEFINITION 4. Let $(\Omega, \mathscr{M}, \mu)$ and $(\Omega', \mathscr{M}', \mu')$ be two measure spaces and let T be an operator with the domain $D(T)$ in the space of (equivalence classes of) measurable functions on Ω and the range in the space of (equivalence classes of) measurable functions on Ω'. T is said to be *quasi-linear* if $f + g \in D(T)$ whenever f and $g \in D(T)$ and if there exists a constant K independent of f and g such that

$$|T(f+g)| \leqq K(|Tf| + |Tg|)\ , \quad \text{a.e.} \tag{31}$$

THEOREM 4. *Let T be a quasi-linear operator from the domain $D(T)$ of measurable functions on Ω into the space of measurable functions on Ω' and let $p_0,\ p_1,\ q_0,\ q_1 \in (0, \infty]$ with $p_1 < p_0$ and $q_0 \neq q_1$. If for each $f(s) \in D(T)$ and $m > 0$ the truncations*

$$f_0(s) = \begin{cases} f(s)\ , & |f(s)| \leqq m\ , \\ \dfrac{f(s)}{|f(s)|}m\ , & |f(s)| > m\ , \end{cases} \tag{32}$$

$$f_1(s) = \begin{cases} 0\ , & |f(s)| \leqq m\ , \\ f(s) - \dfrac{f(s)}{|f(s)|}m\ , & |f(s)| > m\ , \end{cases} \tag{33}$$

belong to $D(T)$ and if there are constants $M_0,\ M_1,\ r_0,\ r_1 > 0$ such that

$$\|Tf\|_{L^{(q_0,\infty)}(\Omega')} \leqq M_0 \|f\|_{L^{(p_0,r_0)}(\Omega)}\ , \tag{34}$$

$$\|Tf\|_{L^{(q_1,\infty)}(\Omega')} \leqq M_1 \|f\|_{L^{(p_1,r_1)}(\Omega)} \tag{35}$$

for all $f \in D(T)$, then for every $0 < \theta < 1$ and $0 < r \leqq \infty$ or $r = \omega$ there is a constant M such that

(36) $$\|Tf\|_{L^{(q,r)}(\Omega')} \leqq M\|f\|_{L^{(p,r)}(\Omega)}$$

for all $f \in D(T)$, *where*

(37) $$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

PROOF. Let $0 < P < \min\{p_0, p_1\}$ and $0 < Q < \min\{q_0, q_1\}$ and regard T as an operator from the couple $X = (L^\infty(\Omega), L^P(\Omega))$ into the couple $Y = (L^\infty(\Omega'), L^Q(\Omega'))$.

The quasi-linearity condition (31) implies

$$(T(f+g))^*(t) \leqq K\{(Tf)^*(t/2) + (Tg)^*(t/2)\}.$$

Since $L_Y(h, t) \sim \left[t^{-Q}\int_0^{tQ} (h^*(s))^Q ds\right]^{1/Q}$, it follows that T is quasi-linear in the sense of Definition 2.

In view of Proposition 2, conditions (34) and (35) say that T is simultaneously of generalized weak type $((P/p_0, r_0), Q/q_0)$ and $((P/p_1, r_1), Q/q_1)$.

Lastly, since the infimum $L_X(f, t) = \inf\{\|f_0\|_{L^\infty(\Omega)} + t^{-1}\|f_1\|_{L^P(\Omega)}; f = f_0 + f_1\}$ is attained by some truncations (32) and (33) for each t, every $f \in D(T) \cap (L^\infty(\Omega) + L^P(\Omega))$ has a decomposition $f = f_0 + f_1$ with $f_0 \in D(T) \cap L^\infty(\Omega)$ and $f_1 \in D(T) \cap L^P(\Omega)$ such that

$$\|f_0\|_{L^\infty(\Omega)} + t^{-1}\|f_1\|_{L^P(\Omega)} = L_X(f, t).$$

Hence it follows from Theorems 1 and 2 that there exists a constant $C < \infty$ such that

$$\|Tf\|_{Y_{Q/q,r}} \leqq C\|f\|_{X_{P/p,r}}, \quad f \in D(T) \cap X_{P/p,r}.$$

Since $X_{P/p,r} = L^{(p,r)}(\Omega)$ and $Y_{Q/q,r} = L^{(q,r)}(\Omega')$ by Proposition 2, we have (36).

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