A GENERAL METHOD FOR THREE-DIMENSIONAL FILTER COMPUTATION

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ABSTRACT

Application of the Fourier space deconvolution algorithm to three-dimensional reconstruction problems necessitates the computation of a frequency space filter. This calculation, which requires taking the Fourier transform of the system response function, is often lengthy and complex. In addition, it is not always possible to find a closed-form expression, and three-dimensional numerical integration becomes necessary. In this paper, it is shown that for system response functions of the form $d(\theta,\phi)/r^2$ with $d(\theta,\phi)$ an angular function describing the imaging system, the filter computation can in general be reduced to a single integration, which in many cases may be performed analytically. Complete expressions are derived for the general three-dimensional filter, and two examples are given to illustrate the use of such expressions.

1. INTRODUCTION

The reconstruction of a three-dimensional distribution by two-dimensional sectional imaging is a well-known technique in both X-ray and nuclear medicine computed tomography. Consecutive two-dimensional sections are stacked to form a three-dimensional image, with the data for each section being acquired and reconstructed independently of any other section. This approach makes poor use of the available imaging photons in the case of nuclear medicine by rejecting photons whose paths do not lie within a single section. Collimators are used to define the direction of the photons within a section.

Positron-emission tomography, however, eliminates the need for physical collimation by actually measuring the photon direction using coincident detection of the positron annihilation radiation. Thus, positron imaging systems with large angular acceptance, such as dual Anger cameras or dual wire chambers, require a three-dimensional reconstruction algorithm if all photon directions within the detector acceptance are to be used.

Several authors have proposed suitable reconstruction algorithms with particular application to positron tomography. These algorithms are in general based on a Fourier space deconvolution of the imaging-system point response function from a back-projected image. They differ in the calculation of the filter, the three-dimensional Fourier transform of the point response function, a calculation which is dependent on the geometry of the imaging system. The three-dimensional deconvolution approach has been proposed by Chu and Tam (1977) for a large-area, dual detector, stationary positron camera, with a numerical integration to evaluate the filter. The closed-form expression for this filter has been given by Schorr and Townsend (1981), and Colsher (1980) has considered the case of the rotating, dual Anger camera system.

Closed-form expressions have the advantage of easy and efficient computer implementation, with freedom from the potential instabilities inherent in three-dimensional numerical integration. However, their derivation, even when possible, is often lengthy and complex, and strongly dependent on the particular imaging geometry.

In this paper, a general method of filter computation is proposed that involves only a single integration. The method is applicable to imaging systems with a response function of the form $d(\theta,\phi)/r^2$, where $d(\theta,\phi)$ is a function allowing for both the acceptance of the system and any angular factors occurring in the back-projection. In many cases, the remaining integration may be evaluated analytically. Numerical problems are therefore either greatly reduced or completely eliminated.

In the next section, a review of the deconvolution reconstruction method is given, followed by a description of the general method for filter computation in section 3. Application to specific imaging geometries is discussed in section 4.

2. THE RECONSTRUCTION METHOD

Let $a(x,y,z)$ represent the unknown function that is to be reconstructed, in particular a distribution of a positron-emitting radioisotope. Distributions of interest will have compact support S_a [$a(x,y,z) \equiv 0$ outside a bounded and closed region S_a], and be smoothly varying within S_a.

Consider the coordinate system shown in figure 1; $\underline{e} = (e_1, e_2, e_3)$ is a unit vector, ϕ is the angle between the x-axis and the projection of $\underline{e}$ onto the x-y plane, and θ is the angle between $\underline{e}$ and the x-y plane. The components of $\underline{e}$ are given by

$$e_1 = \cos \phi \cos \theta$$
$$e_2 = \sin \phi \cos \theta \qquad (2.1)$$
$$e_3 = \sin \theta .$$

The plane

$$x_0 e_1 + y_0 e_2 + z_0 e_3 = 0 \qquad (2.2)$$

is a plane orthogonal to $\underline{e}$ which passes through the origin. It is called the projection plane in the direction $\underline{e}$. If (u,v) are the Cartesian coordinates of a point in the plane such that $u = y$ and $v = z$ for $\theta = \phi = 0$, then if any point (x_0, y_0, z_0) lies in the plane,

$$x_0 = -u \sin \phi - v \cos \phi \sin \theta$$

$$y_0 = u \cos \phi - v \sin \phi \sin \theta \qquad (2.3)$$

$$z_0 = v \cos \theta .$$

Let (x,y,z) be a point on a line parallel to $\underline{e}$ that intersects the projection plane at the point (u,v). It can be shown that

$$u = y \cos \phi - x \sin \phi$$

$$v = z \cos \theta - (x \cos \phi + y \sin \phi) \sin \theta \qquad (2.4)$$

$$x_0 = x - \sigma e_1, \quad y_0 = y - \sigma e_2, \quad z_0 = z - \sigma e_3 \qquad (2.5)$$

where

$$\sigma = (x \cos \phi + y \sin \phi) \cos \theta + z \sin \theta . \qquad (2.6)$$

2.1 The projection operator

The projection operator P for any function $a(x,y,z)$ on S_a is defined by

$$Pa[u,v,\theta,\phi] = p_a(u,v,\theta,\phi) = \int_{-\infty}^{\infty} a(x_0-\tau e_1, y_0-\tau e_2, z_0-\tau e_3) \, d\tau , \qquad (2.7)$$

where the relationship between (u,v) and (x_0,y_0,z_0) is given by equations (2.3). For given $\underline{e}$, the function $p_a(u,v,\theta,\phi)$ is a two-dimensional projection of $a(x,y,z)$ onto the plane given by equation (2.2), and it is easy to see that

$$p_a(u,v,\theta,\phi) = p_a(u,v,\theta+\pi,\phi) . \qquad (2.8)$$

Thus, it is sufficient to limit the subsequent discussion to the (θ,ϕ) region defined by

$$E_{\theta,\phi} = \left\{ (\theta,\phi): \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \ -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \right\} .$$

2.2 The back-projection operator

Let $\Omega \subset E_{\theta,\phi}$ be some region, and let $d = d(\theta,\phi)$ be a function with support Ω. Consider the function $p = p(u,v,\theta,\phi)$ on $R_2 \times E_{\theta,\phi}$ for which (u,v) and points (x,y,z) on the projection line through (u,v) are related by equation (2.4). Define the back-projection operation B_d by:

$$B_d p\big[x,y,z\big] = b_p(x,y,z) = \iint\limits_{\Omega} d(\theta,\phi)p(u,v,\theta,\phi)\ d\omega\ , \tag{2.9}$$

where $d\omega = \cos\theta\ d\theta\ d\phi$. The function $d(\theta,\phi)$ is termed the detector function because its support Ω is defined by the angular acceptance of the imaging system, and its value must include any angular weighting factors that occur in the back-projection process. Some specific examples will be discussed in section 4. The function $b_p(x,y,z)$ in equation (2.9) is called the back-projection of $p(u,v,\theta,\phi)$.

Suppose $p_a(u,v,\theta,\phi)$ are the set of projections of the function $a(x,y,z)$ as defined by equation (2.7) measured along projection directions $\underline{e}$ within Ω by a detection system with detector function $d(\theta,\phi)$. Using equations (2.9), (2.7), and (2.5), the result of the back-projection of these projections is a function denoted by $a_b(x,y,z)$ and given by

$$a_b(x,y,z) = \iint\limits_{\Omega} d(\theta,\phi)\int_{-\infty}^{\infty} a(x-te_1,\ y-te_2,\ z-te_3)\ dt\ d\omega\ , \tag{2.10}$$

where $t = \tau + \sigma$ and $dt = d\tau$. This may be written formally as

$$a_b = B_d Pa\ . \tag{2.11}$$

2.3 The reconstruction problem

The reconstruction problem in positron-emission tomography may be stated as follows: consider an unknown distribution of positron-emitting radioactivity $a(x,y,z)$ and a positron camera with the capability of measuring projections $p_a(u,v,\theta,\phi)$ of $a(x,y,z)$ within an angular range Ω, where (u,v) and (x,y,z) are related by equations (2.4); determine the unknown function $a(x,y,z)$, given the detector function $d(\theta,\phi)$.

The first step in the solution to be adopted here is to obtain the back-projection $a_b(x,y,z)$, using equation (2.9), from the measured projections $p_a(u,v,\theta,\phi)$ and the known detector function $d(\theta,\phi)$. Equation (2.10) represents an integral equation of the first kind in the unknown function $a(x,y,z)$ which may be solved, as will be shown below, by Fourier transformation.

Recall that the Fourier transform in three dimensions, denoted by $\mathcal{F}_3$, is defined by

$$\mathcal{F}_3 \ s\big[k_x, k_y, k_z\big] = S(k_x, k_y, k_z)$$

$$= \iiint\limits_{\mathbb{R}_3} s(x,y,z) \ \exp \ (-2\pi i(xk_x + yk_y + zk_z)) \ dx \ dy \ dz \quad (2.12)$$

for any function s for which equation (2.12) exists. Applying this tranformation to both sides of equation (2.10) yields

$$A_b(k_x, k_y, k_z) = \iint\limits_{\Omega} d(\theta,\phi) \int_{-\infty}^{\infty} \iiint\limits_{\mathbb{R}_3} a(x-te_1, \ y-te_2, \ z-te_3)$$

$$\times \exp \ (-2\pi i(xk_x + yk_y + zk_z)) \ dx \ dy \ dz \ dt \ d\omega \ . \quad (2.13)$$

By setting $\xi = x - te_1$, $\eta = y - te_2$, $\zeta = z - te_3$, this reduces to

$$A_b(k_x, k_y, k_z) = \tilde{H}(k_x, k_y, k_z) \ A(k_x, k_y, k_z) \quad (2.14)$$

where

$$\tilde{H}(k_x, k_y, k_z) = \iint\limits_{\Omega} d(\theta,\phi)\delta(e_1 k_x + e_2 k_y + e_3 k_z) \ d\omega \ , \quad (2.15)$$

using the Dirac δ-function representation $\delta(x) = \int_{-\infty}^{\infty} \exp \ (-2\pi i x t) \ dt$ [see for example Gel'fand and Shilov (1964), p. 359]. The functions A_b and A are the Fourier transforms of a_b and a, respectively.

Let $V \subset \mathbb{R}_3$ be the support of $\tilde{H}(k_x, k_y, k_z)$ and let

$$G(k_x, k_y, k_z) = \begin{cases} \dfrac{1}{\tilde{H}(k_x, k_y, k_z)} & \text{for } (k_x, k_y, k_z) \in V \\[2mm] 0 & \text{otherwise} \end{cases} \quad (2.16)$$

It then follows from equations (2.14) and (2.16) that

$$A(k_x, k_y, k_z) = G(k_x, k_y, k_z) \ A_b(k_x, k_y, k_z) \quad (2.17)$$

for all $(k_x, k_y, k_z) \in V$. As defined previously, $a(x,y,z)$ is a function with compact support S_a, and thus its Fourier transform $A(k_x, k_y, k_z)$ must be an entire

function in each of k_x, k_y, k_z. Clearly, since equation (2.17) defines A on the set V, if $V \neq \mathbb{R}_3$, the function A is known only on V but is completely determined in $\mathbb{R}_3$ by analytic continuation. The case $V \neq \mathbb{R}_3$ has been termed the limited-angle reconstruction method. Techniques to estimate A outside V by, for example, an iterative method (Papoulis 1975, Jeavons et al. 1981) have not proved very successful in practice . For $V = \mathbb{R}_3$, the reconstruction is exact.

Applying the inverse Fourier transform to equation (2.17) yields the solution

$$a_f(x,y,z) = \iiint\limits_V G(k_x,k_y,k_z)A_b(k_x,k_y,k_z) \exp\left(2\pi i(xk_x+yk_y+zk_z)\right) dk_x \, dk_y \, dk_z \quad (2.18)$$

with $a_f = a$ for the exact reconstruction $(V \equiv \mathbb{R}_3)$ and $a_f \simeq a$ in the limited-angle case. The closeness of a_f to the exact solution depends strongly on the extent of the region of unmeasured frequencies, i.e. the $(k_x,k_y,k_z) \notin V$.

Using the convolution theorem of the three-dimensional Fourier transform in equation (2.14) results in

$$a_b(x,y,z) = \iiint\limits_{\mathbb{R}_3} \tilde{h}(x-\xi, \, y-\eta, \, z-\zeta)a(\xi,\eta,\zeta) \, d\xi \, d\eta \, d\zeta \, , \quad (2.19)$$

where the function $\tilde{h}(x,y,z)$, the inverse Fourier transform of $\tilde{H}(k_x,k_y,k_z)$, is the system point-response function for the back-projection $B_d p_\delta$ of projections p_δ of a point source $\delta(x,y,z)$. It can immediately be seen from equation (2.15) that, in polar coordinates, $\tilde{h}(x,y,z)$ may be written

$$h(r,\theta,\phi) = \frac{d(\theta,\phi)}{r^2} \qquad \text{for } -\infty < r < \infty \, , \, (\theta,\phi) \in \Omega \quad (2.20)$$

with (see figure 1):

$$x = r \cos \phi \cos \theta$$
$$y = r \sin \phi \cos \theta \qquad\qquad\qquad (2.21)$$
$$z = r \sin \theta$$

$$-\infty < r < \infty \, , \, -\pi/2 \leq \theta \leq \pi/2 \, , \, -\pi/2 \leq \phi \leq \pi/2 \, .$$

Implementation of this reconstruction algorithm [equation (2.14)] is there-fore based on the knowledge of $\tilde{H}(k_x,k_y,k_z)$, which is defined by:

$$\tilde{H}(k_x,k_y,k_z) = \iiint_{\mathbb{R}_3} \tilde{h}(x,y,z) \exp(-2\pi i(xk_x+yk_y+zk_z)) \, dx \, dy \, dz \qquad (2.22)$$

for all $(k_x,k_y,k_z) \in V$. This algorithm has also been proposed by other authors (Chu and Tam 1977, Colsher 1980), and in each case it has been necessary to evaluate equation (2.22) for the particular angular acceptance region Ω of the imaging system. In the next section it is shown that, under the general condition:

$$d(-\theta,\phi) = d(\theta,-\phi) = d(\theta,\phi) \qquad (2.23)$$

equation (2.22) may be reduced to a single integration.

3. GENERAL EVALUATION OF $\tilde{H}(k_x,k_y,k_z)$

Transforming equation (2.22) in polar coordinates [equations (2.21)] and sub-stituting for $h(r,\theta,\phi)$ from equation (2.19) yields:

$$H(R,\Theta,\Phi) = \iint_{\Omega} d(\theta,\phi) \cos\theta \, d\theta \, d\phi \int_{-\infty}^{\infty} \exp\{-2\pi i r R(\cos(\Phi-\phi)\cos\theta\cos\Theta + \sin\theta\sin\Theta)\} \, dr,$$
$$(3.1)$$

where

$$k_x = R \cos\Phi \cos\Theta$$
$$k_y = R \sin\Phi \cos\Theta \qquad (3.2)$$
$$k_z = R \sin\Theta$$

and $-\infty < R < \infty$, $-\pi/2 \le \Theta \le \pi/2$, $-\pi/2 \le \Phi \le \pi/2$. It should be noted that the inte-grals on the right-hand side of equation (3.1), and other integrals of this type, are to be considered within the theory of generalized functions (Gel'fand and Shilov 1964). Note also that symmetry with respect to integration in the variable r implies

$$H(R,\Theta,\Phi) = H(-R,\Theta,\Phi) \qquad (3.3)$$

for any $d(\theta,\phi)$. Again using $\delta(x) = \int_{-\infty}^{\infty} e^{-2\pi i sx} \, ds$, and setting $s = r|R|$, equation (3.1) becomes

$$H(R,\Theta,\Phi) = \frac{D(\Theta,\Phi)}{|R|} \qquad (3.4)$$

with

$$D(\Theta,\Phi) = \iint\limits_{\Omega} \delta\big(\cos\,(\Phi-\phi)\,\cos\,\theta\,\cos\,\Theta + \sin\,\theta\,\sin\,\Theta\big)\,d(\theta,\phi)\,d\omega \qquad (3.5)$$

for $-\pi/2 \leq \Theta \leq \pi/2$, $-\pi/2 \leq \Phi \leq \pi/2$.

Let $\Omega \subset \Omega_0 \subset E_{\theta,\phi}$, where

$$\Omega_0 = \{(\theta,\phi): -\theta_0 \leq \theta \leq \theta_0;\ -\phi_0 \leq \phi \leq \phi_0\} \qquad (3.6)$$

for $0 < \theta_0 < \pi/2$ and $0 < \phi_0 < \pi/2$.

Then, from equations (2.23), (3.5) (replacing Ω by Ω_0) and (3.6), it can be seen that

$$D(-\Theta,\Phi) = D(\Theta,-\Phi) = D(\Theta,\Phi)\ . \qquad (3.7)$$

Hence, with these conditions, the evaluation of equation (3.5) may be confined to the Fourier space region:

$$0 \leq \Theta \leq \pi/2\ ,\quad 0 \leq \Phi \leq \pi/2\ .$$

It will be seen later that $\Theta = 0$ and $\Theta = \pi/2$ are special cases and must be treated separately. Thus, with $-\theta_0 \leq \theta \leq \theta_0$ for $0 < \theta_0 < \pi/2$ and $0 < \Theta < \pi/2$, the product $\cos\,\theta\,\cos\,\Theta$ is greater than zero. Using

$$\delta\big(f(x)g(x)\big) = \frac{1}{f(x)}\,\delta(g(x)),\ f(x) > 0 \qquad (3.8)$$

[see Gel'fand and Shilov (1964)], equation (3.5) may be written:

$$D(\Theta,\Phi) = \frac{1}{\cos\,\Theta}\int_{\Phi-\phi_0}^{\Phi+\phi_0}\int_{-\theta_0}^{\theta_0}\delta(\cos\,\phi + \tan\,\theta\,\tan\,\Theta)\,d(\theta,\Phi-\phi)\,d\theta\,d\phi\ . \qquad (3.9)$$

Setting $t = \cos\,\phi + \tan\,\theta\,\tan\,\Theta$, $\theta = \tan^{-1}\{(t-\cos\,\phi)/\tan\,\Theta\}$ equation (3.9) becomes

$$D(\Theta,\Phi) = \frac{\tan\,\Theta}{\cos\,\Theta}\int_{\Phi-\phi_0}^{\Phi+\phi_0}\int_{t_1}^{t_2}\delta(t)\,\frac{d\{\tan^{-1}[(t-\cos\,\phi)/\tan\,\Theta],\,\Phi-\phi\}}{\tan^2\,\Theta + (t-\cos\,\phi)^2}\,dt\,d\phi\ , \qquad (3.10)$$

where

$$t_1 = \cos\,\phi - \tan\,\theta_0\,\tan\,\Theta$$

$$t_2 = \cos\,\phi + \tan\,\theta_0\,\tan\,\Theta\ .$$

Since

$$\int_a^b \delta(t)f(t)\,dt = \frac{1}{2}\{\text{sign (b)} - \text{sign (a)}\}\,f(0) \qquad (3.11)$$

equation (3.10) may be written

$$D(\Theta,\Phi) = \int_{\Phi-\phi_0}^{\Phi+\phi_0} s(\phi;\theta_0,\Theta) \, g(\phi;\theta,\Phi) \, d\phi \qquad (3.12)$$

for $0 < \Theta < \pi/2$, $0 \le \Phi \le \pi/2$,

where the functions s and g are given by

$$s(\phi;\theta_0,\Theta) = \frac{1}{2}\{\text{sign}\,(\cos\,\phi + \tan\,\theta_0\,\tan\,\Theta) - \text{sign}\,(\cos\,\phi - \tan\,\theta_0\,\tan\,\Theta)\} \qquad (3.13)$$

$$g(\phi;\theta,\Phi) = \frac{\tan\,\Theta}{\cos\,\Theta} \frac{d\left[\tan^{-1}(\cos\,\phi/\tan\,\Theta),\Phi-\phi\right]}{\tan^2\,\Theta + \cos^2\,\phi} , \qquad (3.14)$$

since $d(-\theta,\phi) = d(\theta,\phi)$. Equation (3.1) is thus reduced to a single integration, equation (3.12), with the functions s and g given by equations (3.13) and (3.14). Computer implementation of equation (3.12) may be made more efficient by a detailed analysis of equation (3.13). The function $s(\phi;\theta_0,\Theta)$, with a value of +1, -1 or 0, has the effect of segmenting Fourier space into at most four regions as follows:

$$D(\Theta,\Phi) = 0 \qquad\qquad \begin{array}{l} 0 < \Theta < \pi/2 - \theta_0 \;, \\[4pt] 0 \le \Phi < \ell_1 - \phi_0 \end{array} \qquad (3.15)$$

$$D(\Theta,\Phi) = \int_{\Phi-\phi_0}^{\Phi+\phi_0} g(\phi;\theta,\Phi) \, d\phi \qquad \begin{array}{l} \pi/2 - \theta_0 \le \Theta < \pi/2 \\[4pt] 0 \le \Phi \le \pi/2 \end{array} \qquad (3.16)$$

$$D(\Theta,\Phi) = \int_{a}^{b} g(\phi;\theta,\Phi) \, d\phi \qquad \begin{array}{l} 0 < \Theta < \pi/2 - \theta_0 \\[4pt] |\ell_1 - \phi_0| \le \Phi \le \pi/2 \end{array} \qquad (3.17)$$

$$D(\Theta,\Phi) = \int_{\Phi-\phi_0}^{-\ell_1} g(\phi;\theta,\Phi) \, d\phi + \int_{\ell_1}^{\Phi+\phi_0} g(\phi;\theta,\Phi) \, d\phi \qquad \begin{array}{l} 0 < \Theta < \pi/2 - \theta_0 \\[4pt] 0 \le \Phi < \phi_0 - \ell_1 \end{array} \qquad (3.18)$$

with

$$a = \max\{\Phi - \phi_0, \, \ell_1\}$$

$$b = \min\{\Phi + \phi_0, \, \ell_2\}$$

and

$$\ell_1 = \cos^{-1}(\tan\,\theta_0\,\tan\,\Theta)$$

$$\ell_2 = \cos^{-1}(-\tan\,\theta_0\,\tan\,\Theta) \ .$$

The special cases $D(0,\Phi)$ and $D(\pi/2,\Phi)$ may be computed from equations (3.5), (3.8), and (3.11). It can be shown that

$$D(0,\Phi) = \begin{cases} 0 & 0 \le \Phi < \pi/2 - \phi_0 \\ 2\int_0^{\theta_0} d(\theta,\pi/2-\Phi)\ d\theta & \pi/2 - \phi_0 \le \Phi \le \pi/2 \end{cases} \tag{3.19}$$

$$D(\pi/2,\Phi) = 2\int_0^{\phi_0} d(0,\phi)\ d\phi \qquad 0 \le \Phi \le \pi/2 \tag{3.20}$$

In summary, equations (3.15) to (3.20) may be used to compute $D(\Theta,\Phi)$ for a given detector function $d(\theta,\phi)$. The deconvolution filter required in equation (2.14) is then given immediately by equation (3.4). The remaining one-dimensional integrations over finite intervals may be performed either numerically, or, if $d(\theta,\phi)$ is of a suitable form, analytically.

It can be seen from equation (3.15) that for $\phi_0 = \pi/2$, i.e. full rotation, the two-dimensional region in which $D(\Theta,\Phi) = 0$ reduces to a line even for $\theta_0 < \pi/2$ and an exact reconstruction is possible. If both $\phi_0 < \pi/2$ and $\theta_0 < \pi/2$ only the limited angle approximation to $a(x,y,z)$, as discussed in the previous section, may be found using this method.

In the next section, to illustrate the use of these equations for the evaluation of $H(k_x,k_y,k_z)$, two interesting imaging geometries will be considered.

4.   SPECIFIC DETECTOR FUNCTIONS

Suppose the detector function $d(\theta,\phi)$ is of the form

$$d_n(\theta,\phi) = \cos^n\theta \cos^n\phi \qquad \text{for } n = 0, 1 . \tag{4.1}$$

The filters corresponding to these two detector functions may be evaluated directly using equations (3.15) to (3.20).

4.1 $n = 0$

In this case, the detector function is a constant, independent of θ,ϕ. Suppose $0 < \theta < \theta_0 \le \pi/2$ and $\phi_0 = \pi/2$, i.e. large-area detectors rotated through $180°$, such

as the Anger camera system considered by Colsher (1980). Let $0 < \Theta < \pi/2$, so that g_0, the g-function for $n = 0$, from equation (3.14) is independent of Φ and given by

$$g_0(\phi;\Theta) = \frac{\tan \Theta}{\cos \Theta} \frac{1}{\tan^2 \Theta + \cos^2 \phi} .$$ (4.2)

It may be verified that

$$\int_\alpha^\beta g_0(\phi;\Theta) \, d\phi = \begin{cases} \tan^{-1} \dfrac{\cos \alpha}{\sin \Theta \sin \alpha} - \tan^{-1} \dfrac{\cos \beta}{\sin \Theta \sin \beta} , & \begin{array}{c} 0 \leq \alpha \leq \beta \leq \pi \\ \text{or} \\ -\pi \leq \alpha \leq \beta \leq 0 \end{array} \\[4ex] \pi + \tan^{-1} \dfrac{\cos \alpha}{\sin \Theta \sin \alpha} - \tan^{-1} \dfrac{\cos \beta}{\sin \Theta \sin \beta} , & -\pi \leq \alpha < 0 < \beta \leq \pi . \end{cases}$$ (4.3)

The remaining integration may therefore be performed analytically and the appropriate filter written down immediately from equations (3.15) to (3.20) with equations (4.2) and (4.3). From equation (3.15), since $\phi_0 = \pi/2$ there is no two-dimensional region for which $D(\Theta,\Phi) = 0$ and an exact reconstruction is possible. From equations (3.16) and (3.17), after some manipulation:

$$D_0(\Theta) = \begin{cases} \pi & \pi/2 - \theta_0 \leq \Theta < \pi/2 \\[2ex] 2 \tan^{-1} \dfrac{\sin \theta_0}{\sqrt{\cos^2 \Theta - \sin^2 \theta_0}} & 0 < \Theta < \pi/2 - \theta_0 \end{cases}$$ (4.4)

for $0 \leq \Phi \leq \pi/2$. Using equation (3.4) and the identity

$$\tan^{-1} \frac{\sin \theta_0}{\sqrt{\cos^2 \Theta - \sin^2 \theta_0}} = \sin^{-1} \frac{\sin \theta_0}{\cos \Theta} ,$$

the result is essentially the filter derived by Colsher (1980) for a rotating positron camera. It may easily be verified that the result agrees with Colsher also at $\Theta = 0$ and $\Theta = \pi/2$ using equations (3.19) and (3.20). This work therefore generalizes Colsher's result, valid only for $\phi_0 = \pi/2$, and shows that, as expected, for $\phi_0 < \pi/2$ only an approximate reconstruction of $a(x,y,z)$ will be possible. Equation (4.3) is valid also in the case of $\phi_0 < \pi/2$.

4.2 $\underline{n = 1}$

The detector function, from equation (4.1), is

$$d_1(\theta,\phi) = \cos \theta \cos \phi , \text{ within } \Omega$$

$$= \cos \theta'$$

where θ' is the angle between the unit vector $\underline{e}$ and the x-axis, the axis perpendicular to the detectors. This is the angular function discussed by Chu and Tam (1977), which arises when the back-projected image is formed by the intersection of positron annihilation event lines with a set of yz planes parallel to the stationary detectors. When the detectors remain stationary, the filter is the one given by Schorr and Townsend (1981). Suppose, however, that the detectors are rotated through an angle $\phi' \leq \pi/4$, while keeping the back-projection planes fixed in the no-rotation position. The back-projection process remains the same, but the filter must be modified to include the data from the additional angular positions.

The new filter is computed as follows: let g_1 be the g-function from equation (3.14) for the detector function $d_1(\theta,\phi)$. Then, for $0 < \Theta < \pi/2$,

$$g_1(\phi;\Theta,\Phi) = \frac{\tan \Theta}{\cos \Theta} \ \frac{\cos\left[\tan^{-1}(\cos \phi/\tan \Theta)\right] \cos(\Phi-\phi)}{\tan^2 \Theta + \cos^2 \phi} \ . \tag{4.5}$$

Using the identity

$$\tan \Theta \sin(\tan^{-1} \frac{\cos \phi}{\tan \Theta}) = \cos \phi \cos(\tan^{-1} \frac{\cos \phi}{\tan \Theta}) \ ,$$

it may be verified by differentiation that

$$\int g_1(\phi;\Theta,\Phi) \ d\phi = \sin \Theta \sin \phi \cos \Phi \cos (\tan^{-1} \frac{\cos \phi}{\tan \Theta}) - \frac{\sin \Phi}{\cos \Theta} \sin (\tan^{-1} \frac{\cos \phi}{\tan \Theta}) \tag{4.6}$$

$$0 < \Theta < \pi/2 \ , \quad 0 \leq \Phi \leq \pi/2$$

for all ϕ. Expressions for $D_1(\Theta,\Phi)$ may then be written down using equations (3.15) to (3.18) with equation (4.6). Finally, from equations (3.19) and (3.20),

$$D_1(0,\Phi) = \begin{cases} 0 & 0 \leq \Phi < \pi/2 - \phi_0 \\[2ex] 2 \sin \theta_0 \sin \Phi & \pi/2 - \phi_0 \leq \Phi \leq \pi/2 \end{cases} \tag{4.7}$$

$$D_1(\pi/2,\Phi) = 2 \sin \phi_0 \qquad 0 \leq \Phi \leq \pi/2 \tag{4.8}$$

The filter $H_1(R,\Theta,\Phi)$ is obtained using equation (3.4). The use of this filter is discussed elsewhere (Townsend et al. 1982).

5. CONCLUSION

The reconstruction of fully three-dimensional images requires a suitable algorithm. Extension of the back-projection and Fourier space deconvolution algorithm to three-dimensions is one possibility that has been successfully implemented. However, the evaluation of the three-dimensional Fourier transform of the imaging system response function can be lengthy and complex. In this paper, a method has been given that greatly simplifies the calculation by reducing the Fourier transform to, at worst, a one-dimensional integration over a finite interval that must be performed numerically. Depending on the imaging-system geometry, expressed in a general detector function $d(\theta, \phi)$, the final integration may be solved analytically. The resulting closed-form expressions, although appearing lengthy, are easily implemented on a computer.

The method has been shown to agree with the work of Colsher (1980) for a rotating positron camera.

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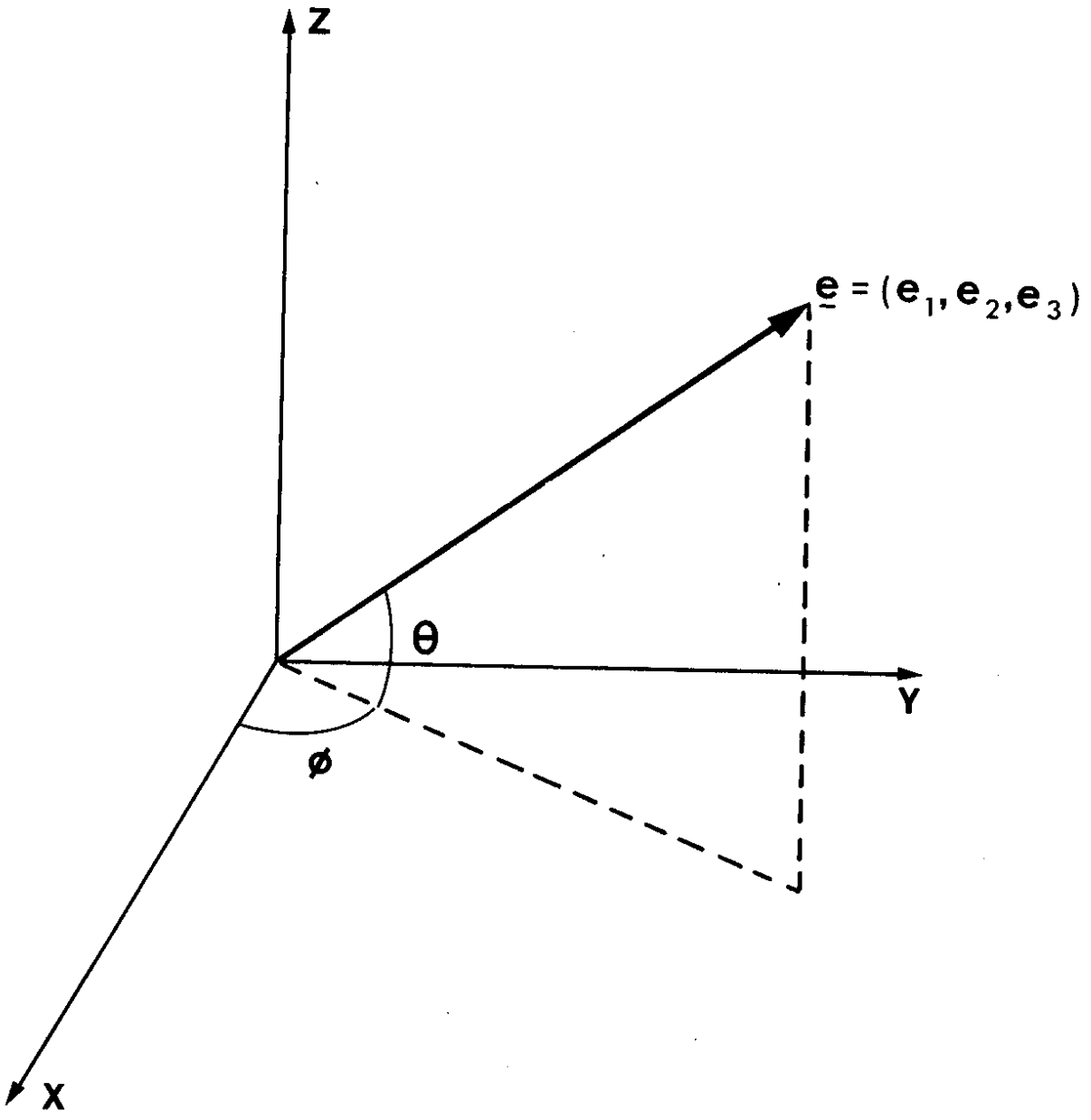
Fig. 1 Coordinate system