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A GENERAL REPRESENTATION FORMULA FOR BOUNDARY VOLTAGE PERTURBATIONS CAUSED BY INTERNAL CONDUCTIVITY INHOMOGENEITIES OF LOW VOLUME FRACTION

Yves Capdeboscq[1] and Michael S. Vogelius[1]

Abstract. We establish an asymptotic representation formula for the steady state voltage perturbations caused by low volume fraction internal conductivity inhomogeneities. This formula generalizes and unifies earlier formulas derived for special geometries and distributions of inhomogeneities.

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1. Introduction and statement of main result

Consider a conducting object which occupies a bounded, smooth domain $\Omega \subset \mathbb{R}^m$. For simplicity we take $\partial\Omega$ to be C^∞, but this assumption could be considerably weakened. Let $\gamma_0(\cdot)$ denote the smooth background conductivity, that is, the conductivity in the absence of any inhomogeneities. We suppose that

$$0 < c_0 \le \gamma_0(x) \le C_0 < \infty, \quad x \in \Omega$$

for some fixed constants c_0 and C_0. For simplicity, we assume that γ_0 is $C^\infty(\bar{\Omega})$, but this latter assumption could also be considerably weakened. The function ψ denotes the imposed boundary current. It suffices that $\psi \in H^{-1/2}(\partial\Omega)$, with $\int_{\partial\Omega} \psi \, \mathrm{d}\sigma = 0$. The background voltage potential, U, is the solution to the boundary value problem

$$\begin{aligned} \nabla \cdot (\gamma_0(x) \nabla U) &= 0 \qquad \text{in } \Omega, \\ \gamma_0(x) \frac{\partial U}{\partial n} &= \psi \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here n denotes the unit outward normal to the domain Ω.

Let ω_ϵ denote a set of "inhomogeneities" inside Ω. The geometric assumptions about the set of "inhomogeneities" are very simple: we suppose the set ω_ϵ is measurable, and separated away from the boundary, (*i.e.*, $\mathrm{dist}(\omega_\epsilon, \partial\Omega) > d_0 > 0$). Most importantly, we suppose that $0 < |\omega_\epsilon|$ gets arbitrarily small, where $|\omega_\epsilon|$ denotes the Lebesgue measure of ω_ϵ. Let $\hat{\gamma}_\epsilon$ denote the conductivity profile in the presence of the inhomogeneities. The

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[1] Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA. e-mail: vogelius@math.rutgers.edu

function $\hat{\gamma}_\epsilon$ is equal to γ_0, except on the set of inhomogeneities; on the set of inhomogeneities we suppose that $\hat{\gamma}_\epsilon$ equals the restriction of some other smooth function, $\gamma_1 \in C^\infty(\overline{\Omega})$, with

$$0 < c_1 \le \gamma_1(x) \le C_1 < \infty, \quad x \in \Omega.$$

In other words

$$\hat{\gamma}_\epsilon(x) = \begin{cases} \gamma_0(x), & x \in \Omega \setminus \omega_\epsilon \\ \gamma_1(x), & x \in \omega_\epsilon. \end{cases} \tag{2}$$

The voltage potential in the presence of the inhomogeneities is denoted $u_\epsilon(x)$. It is the solution to

$$\begin{aligned} \nabla \cdot (\hat{\gamma}_\epsilon(x) \nabla u_\epsilon) &= 0 \qquad \text{in } \Omega\,, \\ \hat{\gamma}_\epsilon(x) \frac{\partial u_\epsilon}{\partial n} &= \psi \qquad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

We normalize both U and u_ϵ by requiring that

$$\int_{\partial\Omega} U \,\mathrm{d}\sigma = 0\,, \quad \text{and} \quad \int_{\partial\Omega} u_\epsilon \mathrm{d}\sigma = 0.$$

We note that the individual voltages U and u_ϵ need not be smooth (or even continuous) on $\partial\Omega$, however, the difference $u_\epsilon - U$ is smooth in a neighborhood of $\partial\Omega$, due to the regularity of γ_0, and the fact that ω_ϵ is strictly interior.

The aim of this paper is to derive a representation formula for (all possible limits of) $(u_\epsilon - U)|_{\partial\Omega}$ as $|\omega_\epsilon| \to 0$. This representation formula, in a most natural way, generalizes and unifies the specific formulas already derived for a finite set of inhomogeneities of small diameter, and for a finite set of inhomogeneities of small thickness (*cf.* [9] and [5]). The exact relation to these formulas (and others) is discussed in detail in a separate section.

Explicit representation formulas for the boundary voltage perturbations caused by internal inhomogeneities are of significant interest from an "imaging point of view". For instance: if one has very detailed knowledge of the "boundary signatures" of internal inhomogeneities, then it becomes possible to design very effective numerical methods to identify the location of these inhomogeneities. We refer the reader to [3, 4, 7] and [13] for examples of numerical methods based on such specific formulas.

Before stating our main theorem we shall make some preliminary observations. Let 1_{ω_ϵ} denote the characteristic function corresponding to the set ω_ϵ, *i.e.*, the function which takes the value 1 on the set and the value 0 outside. Since the family of functions $|\omega_\epsilon|^{-1} 1_{\omega_\epsilon}$ is bounded in $L^1(\Omega)$, it follows from a combination of the Banach–Alaoglu Theorem and the Riesz Representation Theorem that we may find a regular, positive Borel measure μ, and a subsequence ω_{ϵ_n}, with $|\omega_{\epsilon_n}| \to 0$, such that

$$|\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \,\mathrm{d}x \to \mathrm{d}\mu. \tag{4}$$

The convergence refers to the weak* topology of the dual of $C^0(\overline{\Omega})$. More precisely, for any $\phi \in C^0(\overline{\Omega})$

$$|\omega_{\epsilon_n}|^{-1} \int_{\omega_{\epsilon_n}} \phi \,\mathrm{d}x \to \int_\Omega \phi \,\mathrm{d}\mu.$$

The measure μ satisfies $\int_\Omega \mathrm{d}\mu = 1$, so it is indeed a probability measure. Due to the fact that the sets ω_ϵ stay uniformly bounded away from the boundary, there exists a compact set $K_0 \subset \Omega$ which strictly contains ω_ϵ, in the sense that

$$\omega_\epsilon \subset K_0 \subset \Omega, \quad \text{and} \quad \mathrm{dist}(\omega_\epsilon, \Omega \setminus K_0) > \delta_0 > 0. \tag{5}$$

The support of μ lies inside the same compact set K_0. We shall need the so called Neumann function $N(x,y)$ for the operator $\nabla\cdot(\gamma_0\nabla)$. For $y\in\Omega$, $N(\cdot,y)$ is the solution to the boundary value problem

$$\begin{aligned}\nabla_x\cdot(\gamma_0(x)\nabla_x N(x,y)) &= \delta_y \quad \text{in } \Omega\,,\\ \gamma_0(x)\frac{\partial N}{\partial n_x} &= \frac{1}{|\partial\Omega|} \ \text{ on } \partial\Omega.\end{aligned}$$

The function $N(x,y)$ may be extended by continuity to $y\in\overline{\Omega}$. For $y\in\partial\Omega$ the function $N(\cdot,y)$ may also be interpreted as the solution to the boundary value problem

$$\begin{aligned}\nabla_x\cdot(\gamma_0(x)\nabla_x N(x,y)) &= 0 \quad \text{in } \Omega\,,\\ \gamma_0(x)\frac{\partial N}{\partial n_x} &= -\delta_y+\frac{1}{|\partial\Omega|} \ \text{ on } \partial\Omega.\end{aligned}$$

Theorem 1. *Let ω_{ϵ_n} be a sequence of measurable subsets, with $|\omega_{\epsilon_n}|\to 0$, for which (4) and (5) holds. Given any $\psi\in H^{-1/2}(\partial\Omega)$, with $\int_{\partial\Omega}\psi\,\mathrm{d}\sigma=0$, let U and u_{ϵ_n} denote the solutions to (1) and (3), respectively. There exists a subsequence, also denoted ω_{ϵ_n}, and a matrix valued function $M\in L^2(\Omega,\mathrm{d}\mu)$ such that*

$$(u_{\epsilon_n}-U)(y)=|\omega_{\epsilon_n}|\int_\Omega(\gamma_1-\gamma_0)(x)M_{ij}(x)\frac{\partial U}{\partial x_i}\frac{\partial N}{\partial x_j}(x,y)\,\mathrm{d}\mu(x)+o(|\omega_{\epsilon_n}|)\qquad y\in\partial\Omega.$$

The values of the function $M(\cdot)$ are symmetric, positive definite matrices in the sense that

$$\begin{gathered}M_{ij}(x)=M_{ji}(x),\quad \text{and}\\ \min\left\{1,\frac{\gamma_0(x)}{\gamma_1(x)}\right\}\ |\xi|^2\le M_{ij}(x)\xi_i\xi_j\le\max\left\{1,\frac{\gamma_0(x)}{\gamma_1(x)}\right\}\ |\xi|^2,\\ \xi\in\mathbb{R}^m,\quad \mu \text{ almost everywhere in the set } \{x:\ \gamma_0(x)\neq\gamma_1(x)\}\cdot\end{gathered}\tag{6}$$

The subsequence ω_{ϵ_n} and the matrix valued function $M\in L^2(\Omega,\mathrm{d}\mu)$ are independent of the boundary flux ψ. The term $o(|\omega_{\epsilon_n}|)$ is such that $\|o(|\omega_{\epsilon_n}|)\|_{L^\infty(\partial\Omega)}/|\omega_{\epsilon_n}|$ converges to 0 for any fixed $\psi\in H^{-1/2}$, and uniformly on $\{\psi:\ \int_{\partial\Omega}\psi\,\mathrm{d}\sigma=0,\ \|\psi\|_{L^2(\partial\Omega)}\le 1\}$.

Remark 1.

The variational formulations of the problems (1) and (3) yield

$$\int_\Omega\gamma_0\nabla(U-u_\epsilon)\cdot\nabla v\,\mathrm{d}x=\int_{\omega_\epsilon}(\gamma_1-\gamma_0)\nabla u_\epsilon\cdot\nabla v\,\mathrm{d}x,\tag{7}$$

for any $v\in H^1(\Omega)$. Let y be a fixed point on $\partial\Omega$, and let $v_m\in C^1(\overline{\Omega})$ be a sequence that converges to $N(\cdot,y)$ in $W^{1,1}(\Omega)$, and in $C^1(K_0)$ (K_0 being as in (5)). Using the fact that $U-u_\epsilon$ is smooth near $\partial\Omega$, and the fact that $\omega_\epsilon\subset K_0$, we may now, by insertion of v_m into (7), and passage to the limit, conclude that

$$\int_\Omega\gamma_0\nabla(U-u_\epsilon)\cdot\nabla_x N(x,y)\,\mathrm{d}x=\int_{\omega_\epsilon}(\gamma_1-\gamma_0)\nabla u_\epsilon\cdot\nabla_x N(x,y)\,\mathrm{d}x.$$

After integration by parts this yields

$$\begin{aligned}(u_\epsilon-U)(y)&=\int_{\omega_\epsilon}(\gamma_1-\gamma_0)(x)\nabla u_\epsilon\cdot\nabla_x N(x,y)\,\mathrm{d}x\\ &=|\omega_\epsilon|\int_\Omega(\gamma_1-\gamma_0)(x)|\omega_\epsilon|^{-1}1_{\omega_\epsilon}\nabla u_\epsilon\cdot\nabla_x N(x,y)\,\mathrm{d}x.\end{aligned}\tag{8}$$

Theorem 1 characterizes all possible limit points for the integral

$$\int_\Omega (\gamma_1 - \gamma_0)(x)|\omega_\epsilon|^{-1} 1_{\omega_\epsilon} \nabla u_\epsilon \cdot \nabla_x N(x,y) \mathrm{d}x, \quad \text{as} \quad |\omega_\epsilon| \to 0.$$

Note that the functions u_ϵ converge to U in $H^1(\Omega)$, and thus ∇u_ϵ converge to ∇U in $L^2(\Omega)$; it is the fact that these gradients do not converge in $L^\infty(\Omega)$ which makes Theorem 1 non trivial, and which accounts for the introduction of the polarization tensor M. The calculation of all possible limit points of the above integral shows a lot of similarity to the calculation of limiting (effective) energy expressions by the technique of H-convergence. At the center of our calculation is a variation of the compensated compactness technique developed by Murat and Tartar [14].

Remark 2.

We note that the asymptotic formula in Theorem 1 is actually valid for all y in $\overline{\Omega} \setminus K_0$, and not just for y on $\partial\Omega$. The remainder term in the asymptotic formula in Theorem 1 is not $o(|\omega_\epsilon|)$ uniformly with respect to the ellipticity constants c_i and C_i. Take for example $0 < c_0 < C_0 < \infty$ to be fixed, but let c_1 approach 0, or let C_1 approach ∞. In this case it is easy to see that there exist ω_ϵ, with $|\omega_\epsilon| \to 0$ for which u_ϵ converge to a limit different from the background potential U, *i.e.*, the remainder term is not even $o(1)$ uniformly in c_1 and C_1. The bounds established for the polarization tensor M are optimal, they are "achieved" for instance by inhomogeneities in the shape of thin "sheets". For the inverse conductivity problem these polarization tensor bounds immediately lead to optimal (small volume) inhomogeneity size estimates in terms of a single (integral) boundary measurement, see [8]. Related size estimates have been derived, without any assumption of smallness, in [1] and [12].

As formulated here, Theorem 1 applies only to isotropic conductivities γ_0 and γ_1. The representation part immediately generalizes to anisotropic γ's, with the corresponding asymptotic formula reading

$$(u_{\epsilon_n} - U)(y) = |\omega_{\epsilon_n}| \int_\Omega M_{ij}(x)(\gamma_1 - \gamma_0)_{ik}(x) \frac{\partial U}{\partial x_k} \frac{\partial N}{\partial x_j}(x,y)\, \mathrm{d}\mu(x) + o(|\omega_{\epsilon_n}|) \quad y \in \partial\Omega \text{ (or } y \in \overline{\Omega} \setminus K_0).$$

Remark 3.

Suppose the background conductivity γ_0 is a constant, and let $\Phi(x,y)$ denote the standard "free-space" Green's function for the operator $\nabla \cdot (\gamma_0 \nabla \quad) = \gamma_0 \Delta$

$$\Phi(x,y) = \frac{1}{2\pi\gamma_0} \log|x-y|, \quad m = 2,$$
$$\Phi(x,y) = \frac{1}{(2-m)A_m\gamma_0}|x-y|^{2-m}, \quad m \geq 3.$$

The constant A_m is the area of the unit sphere in $\mathbb{R}^m$. Straightforward integration by parts shows that

$$\frac{\partial N}{\partial x_j}(x,z) = \gamma_0 \frac{\partial}{\partial x_j} \int_{\partial\Omega} N(x,y) \frac{\partial \Phi}{\partial n_y}(y,z)\, \mathrm{d}\sigma_y + \frac{\partial \Phi}{\partial x_j}(x,z)$$

$(x,z)\in\Omega\times\Omega$, $x\neq z$. Based on the asymptotic formula in Theorem 1 we now calculate

$$\begin{aligned}\gamma_0\int_{\partial\Omega}(u_{\epsilon_n}-U)(y)\frac{\partial\Phi}{\partial n_y}(y,z)\,\mathrm{d}\sigma_y &= |\omega_{\epsilon_n}|\int_\Omega(\gamma_1(x)-\gamma_0)M_{ij}(x)\frac{\partial U}{\partial x_i}\gamma_0\\ &\quad\times\frac{\partial}{\partial x_j}\left(\int_{\partial\Omega}N(x,y)\frac{\partial\Phi}{\partial n_y}(y,z)\,\mathrm{d}\sigma_y\right)\mathrm{d}\mu(x)+o(|\omega_{\epsilon_n}|)\\ &= |\omega_{\epsilon_n}|\int_\Omega(\gamma_1(x)-\gamma_0)M_{ij}(x)\frac{\partial U}{\partial x_i}\frac{\partial N}{\partial x_j}(x,z)\,\mathrm{d}\mu(x)\\ &\quad-|\omega_{\epsilon_n}|\int_\Omega(\gamma_1(x)-\gamma_0)M_{ij}(x)\frac{\partial U}{\partial x_i}\frac{\partial\Phi}{\partial x_j}(x,z)\,\mathrm{d}\mu(x)+o(|\omega_{\epsilon_n}|)\\ &= (u_{\epsilon_n}-U)(z)-|\omega_{\epsilon_n}|\int_\Omega(\gamma_1(x)-\gamma_0)M_{ij}(x)\frac{\partial U}{\partial x_i}\frac{\partial\Phi}{\partial x_j}(x,z)\,\mathrm{d}\mu(x)+o(|\omega_{\epsilon_n}|)\end{aligned}$$

for any $z\in\Omega\setminus K_0$. By rearranging terms we get

$$(u_{\epsilon_n}-U)(z)-\gamma_0\int_{\partial\Omega}(u_{\epsilon_n}-U)(y)\frac{\partial\Phi}{\partial n_y}(y,z)\,\mathrm{d}\sigma_y=|\omega_{\epsilon_n}|\int_\Omega(\gamma_1(x)-\gamma_0)M_{ij}(x)\frac{\partial U}{\partial x_i}\frac{\partial\Phi}{\partial x_j}(x,z)\mathrm{d}\mu(x)+o(|\omega_{\epsilon_n}|),$$

$z\in\Omega\setminus K_0$, and by letting z tend to a point on $\partial\Omega$ we now obtain

$$(u_{\epsilon_n}-U)(z)-2\gamma_0\int_{\partial\Omega}(u_{\epsilon_n}-U)(y)\frac{\partial\Phi}{\partial n_y}(y,z)\,\mathrm{d}\sigma_y=2|\omega_{\epsilon_n}|\int_\Omega(\gamma_1(x)-\gamma_0)M_{ij}(x)\frac{\partial U}{\partial x_i}\frac{\partial\Phi}{\partial x_j}(x,z)\,\mathrm{d}\mu(x)+o(|\omega_{\epsilon_n}|),$$

$z\in\partial\Omega$, as an alternate asymptotic formula relating boundary data of $(u_{\epsilon_n}-U)$ to data characterizing the location of the internal inhomogeneities. The integral on the left-hand side should be interpreted as a standard double layer potential.

2. Preliminary convergence estimates

In this section we shall examine exactly how the u_ϵ converge to U. As mentioned earlier this convergence does not take place in $W^{1,\infty}(\Omega)$, however, it does take place in $H^1(\Omega)$, as well as in $C^{0,\beta}(\overline{\Omega})$, for some $\beta>0$. We shall consider functions that are defined slightly more generally than u_ϵ and U. Given $F\in H^{-1}(\Omega)$ (here interpreted as the dual of $H^1(\Omega)$) and $f\in H^{-1/2}(\partial\Omega)$, with $\int_\Omega F\,\mathrm{d}x=\int_{\partial\Omega}f\,\mathrm{d}\sigma$, let V and v_ϵ denote the (variational) solutions to

$$\begin{aligned}\nabla\cdot(\gamma_0(x)\nabla V)&=F\quad\text{in }\Omega,\\ \gamma_0(x)\frac{\partial V}{\partial n}&=f\quad\text{on }\partial\Omega,\end{aligned}\tag{9}$$

and

$$\begin{aligned}\nabla\cdot(\hat\gamma_\epsilon(x)\nabla v_\epsilon)&=F\quad\text{in }\Omega,\\ \hat\gamma_\epsilon(x)\frac{\partial v_\epsilon}{\partial n}&=f\quad\text{on }\partial\Omega,\end{aligned}\tag{10}$$

respectively. The functions V and v_ϵ are normalized by $\int_{\partial\Omega}V\,\mathrm{d}\sigma=0$ and $\int_{\partial\Omega}v_\epsilon\,\mathrm{d}\sigma=0$.

Lemma 1. *Let V and v_ϵ be as introduced above, let $K_0\subset\Omega$ be a compact set that strictly contains all ω_ϵ, as in (5), and let α be any positive number. There exists a constant C such that*

$$\|v_\epsilon-V\|_{H^1(\Omega)}\le C|\omega_\epsilon|^{1/2}\left(\|F\|_{C^{0,\alpha}(K_0)}+\|F\|_{H^{-1}(\Omega)}+\|f\|_{H^{-1/2}(\partial\Omega)}\right).$$

Furthermore, given any $\eta > 0$, there exists a constant C_η such that

$$\|v_\epsilon - V\|_{L^2(\Omega)} \le C_\eta |\omega_\epsilon|^{\frac{1}{2}+\frac{1}{m^*}-\eta} \left(\|F\|_{C^{0,\alpha}(K_0)} + \|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)}\right).$$

The integer m^* is defined by $m^* = \max\{m, 2\}$, where m is the dimension of the ambient space.

Proof. By simple manipulation of the variational formulations of (9) and (10), and the use of interior estimates for V (*cf.* [11], Cor. 6.3 and Th. 8.24)

$$\begin{aligned}
\left|\int_\Omega \hat{\gamma}_\epsilon \nabla(v_\epsilon - V)\cdot\nabla w \,\mathrm{d}x\right| &= \left|\int_\Omega (\gamma_0 - \hat{\gamma}_\epsilon)\nabla V\cdot\nabla w\,\mathrm{d}x\right| \\
&\le C|\omega_\epsilon|^{1/2}\|\nabla V\|_{L^\infty(\omega_\epsilon)}\|\nabla w\|_{L^2(\Omega)} \\
&\le C|\omega_\epsilon|^{1/2}\left(\|F\|_{C^{0,\alpha}(K_0)} + \|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)}\right)\|\nabla w\|_{L^2(\Omega)},
\end{aligned}$$

so that $\|v_\epsilon - V\|_{H^1(\Omega)} \le C|\omega_\epsilon|^{1/2}\left(\|F\|_{C^{0,\alpha}(K_0)} + \|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)}\right)$, as asserted by the first statement in this lemma. We also have

$$\int_\Omega \gamma_0\nabla(v_\epsilon - V)\cdot\nabla w\,\mathrm{d}x = \int_\Omega (\gamma_0 - \hat{\gamma}_\epsilon)\nabla v_\epsilon\cdot\nabla w\,\mathrm{d}x, \quad w \in H^1(\Omega). \tag{11}$$

Select w as the solution to

$$\begin{aligned}
\nabla\cdot(\gamma_0\nabla w) &= V - v_\epsilon \quad \text{in } \Omega, \\
\gamma_0\frac{\partial w}{\partial n} &= \frac{1}{|\partial\Omega|}\int_\Omega (V - v_\epsilon)\,\mathrm{d}x \quad \text{on } \partial\Omega,
\end{aligned}$$

normalized by $\int_{\partial\Omega} w\,\mathrm{d}\sigma = 0$. Elliptic estimates show that $\|w\|_{H^2(\Omega)} \le C\|v_\epsilon - V\|_{L^2(\Omega)}$, and after insertion of this w into (11) we now obtain

$$\begin{aligned}
\int_\Omega (v_\epsilon - V)^2\,\mathrm{d}x &= \int_\Omega \gamma_0\nabla(v_\epsilon - V)\cdot\nabla w\,\mathrm{d}x \\
&= \left|\int_\Omega (\gamma_0 - \hat{\gamma}_\epsilon)\nabla v_\epsilon\cdot\nabla w\,\mathrm{d}x\right| \\
&\le C\left(\int_{\omega_\epsilon}|\nabla v_\epsilon|^q\,\mathrm{d}x\right)^{1/q}\left(\int_\Omega |\nabla w|^p\,\mathrm{d}x\right)^{1/p} \\
&\le C_q\left(\int_{\omega_\epsilon}|\nabla v_\epsilon|^q\,\mathrm{d}x\right)^{1/q}\|w\|_{H^2(\Omega)} \\
&\le C_q\left(\int_{\omega_\epsilon}|\nabla v_\epsilon|^q\,\mathrm{d}x\right)^{1/q}\|v_\epsilon - V\|_{L^2(\Omega)},
\end{aligned} \tag{12}$$

provided p and q are related by $\frac{1}{q} + \frac{1}{p} = 1$, and provided we require that $q > \frac{2m^*}{m^*+2}$ (so that $1 < p < \frac{2m^*}{m^*-2}$, and therefore, by Sobolev's Imbedding Theorem $\left(\int_\Omega |\nabla w|^p\,\mathrm{d}x\right)^{1/p} \le C_p\|w\|_{H^2(\Omega)}$, *cf.* [11], p. 155). For any $1 < q < 2$

$$\begin{aligned}
\|\nabla v_\epsilon\|_{L^q(\omega_\epsilon)} &\le \|\nabla(v_\epsilon - V)\|_{L^q(\omega_\epsilon)} + \|\nabla V\|_{L^q(\omega_\epsilon)} \\
&\le \left(\int_{\omega_\epsilon} 1\,\mathrm{d}x\right)^s \|\nabla(v_\epsilon - V)\|_{L^2(\omega_\epsilon)} + |\omega_\epsilon|^{1/q}\|\nabla V\|_{L^\infty(\omega_\epsilon)} \\
&\le C\left(|\omega_\epsilon|^{(s+1/2)} + |\omega_\epsilon|^{1/q}\right)\left(\|F\|_{C^{0,\alpha}(K_0)} + \|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)}\right)
\end{aligned} \tag{13}$$

with $s = \frac{1}{q} - \frac{1}{2}$. A combination of (12) and (13) yields

$$\begin{aligned}\|v_\epsilon - V\|_{L^2(\Omega)} &\leq C_q \left(\int_{\omega_\epsilon} |\nabla v_\epsilon|^q \, \mathrm{d}x \right)^{1/q} \\ &\leq C_q |\omega_\epsilon|^{1/q} \left(\|F\|_{C^{0,\alpha}(K_0)} + \|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} \right),\end{aligned}$$

for any $\frac{2m^*}{m^*+2} < q < 2$. We note that $\frac{1}{q}$ approaches $\frac{m^*+2}{2m^*} = \frac{1}{m^*} + \frac{1}{2}$ from below as q approaches $\frac{2m^*}{m^*+2}$ from above. The previous estimate now immediately implies that, given any $\eta > 0$, there exists a constant C_η such that

$$\|v_\epsilon - V\|_{L^2(\Omega)} \leq C_\eta |\omega_\epsilon|^{\frac{1}{2} + \frac{1}{m^*} - \eta} \left(\|F\|_{C^{0,\alpha}(K_0)} + \|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} \right),$$

the second statement of this lemma. $\square$

Remark 4.

Let K_0 be a compact subset of Ω that strictly contains all ω_ϵ, in the sense of (5). A combination of the L^2-estimate in Lemma 1 with the interior estimate (of De Giorgi–Nash–Moser type) found in [11] (Th. 8.24), yields

$$\|v_\epsilon - V\|_{C^{0,\beta}(\overline{\Omega})} \leq C_\eta |\omega_\epsilon|^{\frac{1}{m} - \eta} \left(\|F\|_{C^{0,\beta}(K_0)} + \|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} \right),$$

for some $\beta > 0$. For this estimate we have also used the fact that $\nabla \cdot (\gamma_0 \nabla(v_\epsilon - V)) = 0$ away from ω_ϵ, and the fact that $\frac{\partial}{\partial n}(v_\epsilon - V) = 0$ on $\partial\Omega$, to ensure that the L^2-norm of $v_\epsilon - V$ "bounds" the $C^{0,\beta}$ norm (appropriately) away from ω_ϵ.

3. Proof of main result

We shall use the notation $V^{(j)}$ and $v_\epsilon^{(j)}$ for the solutions to the problems (9) and (10) in the special case when $F = \frac{\partial \gamma_0}{\partial x_j}$, $f = \gamma_0 n_j$, n_j being the j'th coordinate of the outward normal vector to $\partial\Omega$. Notice that $V^{(j)}$ is given by a simple formula: $V^{(j)} = x_j - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} x_j \, \mathrm{d}\sigma$. Due to Lemma 1 we may estimate

$$\begin{aligned}\left\| |\omega_\epsilon|^{-1} 1_{\omega_\epsilon} \nabla v_\epsilon^{(j)} \right\|_{L^1(\Omega)} &= \int_{\omega_\epsilon} |\omega_\epsilon|^{-1} \left| \nabla v_\epsilon^{(j)} \right| \mathrm{d}x \\ &\leq \int_{\omega_\epsilon} |\omega_\epsilon|^{-1} \left| \nabla (v_\epsilon^{(j)} - V^{(j)}) \right| \mathrm{d}x + \int_{\omega_\epsilon} |\omega_\epsilon|^{-1} \left| \nabla V^{(j)} \right| \mathrm{d}x \\ &\leq |\omega_\epsilon|^{-1} \left(\int_{\omega_\epsilon} 1 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega \left| \nabla (v_\epsilon^{(j)} - V^{(j)}) \right|^2 \mathrm{d}x \right)^{1/2} + 1 \\ &\leq C. \qquad (14)\end{aligned}$$

By extracting a subsequence, also referred to as ω_{ϵ_n}, from the sequence given in Theorem 1, we may thus suppose that

$$\begin{aligned}|\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \, \mathrm{d}x \to \mathrm{d}\mu, \quad \text{and,} \\ |\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \frac{\partial}{\partial x_i} v_{\epsilon_n}^{(j)} \, \mathrm{d}x \to \mathrm{d}\mathcal{M}_{ij}.\end{aligned}$$

The convergence in both cases refers to the weak* topology of the dual of $C^0(\overline{\Omega})$, and $\mathcal{M}_{ij}$ (as well as μ) are regular Borel measures with support inside K_0. Let $\phi \in C^0(\overline{\Omega})$, then by the very definition of the measure $\mathcal{M}_{ij}$

$$
\begin{aligned}
\left|\int_\Omega \phi \,\mathrm{d}\mathcal{M}_{ij}\right| &= \left|\lim |\omega_{\epsilon_n}|^{-1} \int_\Omega 1_{\omega_{\epsilon_n}} \frac{\partial}{\partial x_i} v^{(j)}_{\epsilon_n} \phi \,\mathrm{d}x\right| \\
&\leq \underline{\lim}\, |\omega_{\epsilon_n}|^{-1} \int_\Omega 1_{\omega_{\epsilon_n}} \left|\frac{\partial}{\partial x_i}(v^{(j)}_{\epsilon_n} - V^{(j)})\right| |\phi| \,\mathrm{d}x \\
&\quad + \lim |\omega_{\epsilon_n}|^{-1} \int_\Omega 1_{\omega_{\epsilon_n}} \left|\frac{\partial}{\partial x_i} V^{(j)}\right| |\phi| \,\mathrm{d}x \\
&\leq \underline{\lim}\, |\omega_{\epsilon_n}|^{-1/2} \left(\int_\Omega \left|\frac{\partial}{\partial x_i}(v^{(j)}_{\epsilon_n} - V^{(j)})\right|^2 \mathrm{d}x\right)^{1/2} \left(\int_\Omega |\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} |\phi|^2 \,\mathrm{d}x\right)^{1/2} \\
&\quad + \int_\Omega \left|\frac{\partial}{\partial x_i} V^{(j)}\right| |\phi| \,\mathrm{d}\mu \\
&\leq C \left(\int_\Omega |\phi|^2 \,\mathrm{d}\mu\right)^{1/2}.
\end{aligned}
$$

As a consequence of this estimate it follows that the functional

$$
\phi \to \int_\Omega \phi \,\mathrm{d}\mathcal{M}_{ij}
$$

may be extended to a bounded linear functional on $L^2(\Omega, \mathrm{d}\mu)$. Therefore, by Riesz's Representation Theorem, it is given by

$$
\int_\Omega \phi \,\mathrm{d}\mathcal{M}_{ij} = \int_\Omega \phi M_{ij} \,\mathrm{d}\mu,
$$

for some function $M_{ij} \in L^2(\Omega, \mathrm{d}\mu)$. In other words

$$
|\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \frac{\partial}{\partial x_i} v^{(j)}_{\epsilon_n} \,\mathrm{d}x \to \mathrm{d}\mathcal{M}_{ij} = M_{ij} \,\mathrm{d}\mu. \tag{15}
$$

The following central lemma establishes the constitutive relationship between $\lim\ |\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \frac{\partial}{\partial x_j} u_{\epsilon_n} \,\mathrm{d}x$ and the gradient of the background potential. Its proof is based on a variation of the clever "integration by parts technique" originally developed by Murat and Tartar in the context of H-convergence (the Div–Curl Lemma) *cf.* [14].

Lemma 2. *Let U and u_ϵ denote the solutions to (1) and (3) for some $\psi \in H^{-1/2}(\Omega)$, with $\int_{\partial\Omega} \psi \,\mathrm{d}\sigma = 0$. Let ω_{ϵ_n}, with $|\omega_{\epsilon_n}| \to 0$, be a sequence for which (4), (5) and (15) hold. Then $(\gamma_1 - \gamma_0)|\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \frac{\partial}{\partial x_j} u_{\epsilon_n} \,\mathrm{d}x$ is convergent in the weak* topology of the dual of $C^0(\overline{\Omega})$, with*

$$
\lim\ (\gamma_1 - \gamma_0)|\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \frac{\partial}{\partial x_j} u_{\epsilon_n} \,\mathrm{d}x = (\gamma_1 - \gamma_0) M_{ij} \frac{\partial U}{\partial x_i} \,\mathrm{d}\mu. \tag{16}
$$

Proof. It suffices to prove that we may extract a subsequence such that

$$
(\gamma_1 - \gamma_0)|\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \frac{\partial}{\partial x_j} u_{\epsilon_n} \,\mathrm{d}x
$$

converges to the limit given by the right-hand side in (16). The fact that the limit is independent of the particular subsequence then guarantees that the entire sequence will be convergent. We may repeat the argument which led to (14), in order to conclude that

$$\| \, |\omega_\epsilon|^{-1} 1_{\omega_\epsilon} \nabla u_\epsilon \, \|_{L^1(\Omega)} \leq C \|\psi\|_{H^{-1/2}(\partial\Omega)},$$

so that, upon extraction of a subsequence

$$|\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \frac{\partial}{\partial x_j} u_{\epsilon_n} \, \mathrm{d}x \to \mathrm{d}\nu_j$$

in the weak* topology of the dual of $C^0(\overline{\Omega})$. In order to complete the proof of this lemma we must show that

$$\int_\Omega \phi(\gamma_1 - \gamma_0) \, \mathrm{d}\nu_j = \int_\Omega \phi(\gamma_1 - \gamma_0) \frac{\partial U}{\partial x_i} \, \mathrm{d}\mathcal{M}_{ij}, \tag{17}$$

for all ϕ sufficiently smooth (*e.g.* $\phi \in C^1(\overline{\Omega})$). We first observe that

$$\int_\Omega \hat{\gamma}_\epsilon \nabla(u_\epsilon - U) \cdot \nabla(v_\epsilon^{(j)} \phi) \mathrm{d}x = \int_\Omega (\gamma_0 - \hat{\gamma}_\epsilon) \nabla U \cdot \nabla(v_\epsilon^{(j)} \phi) \mathrm{d}x, \tag{18}$$

and

$$\int_\Omega \gamma_0 \nabla(u_\epsilon - U) \cdot \nabla(V^{(j)} \phi) \, \mathrm{d}x = \int_\Omega (\gamma_0 - \hat{\gamma}_\epsilon) \nabla u_\epsilon \cdot \nabla(V^{(j)} \phi) \, \mathrm{d}x. \tag{19}$$

We then calculate

$$\begin{aligned}
\int_\Omega \hat{\gamma}_\epsilon \nabla(u_\epsilon - U) \cdot \nabla(v_\epsilon^{(j)} \phi) \, \mathrm{d}x &= \int_\Omega \hat{\gamma}_\epsilon \nabla(u_\epsilon - U) \cdot \left(\nabla v_\epsilon^{(j)}\right) \phi \, \mathrm{d}x + \int_\Omega \hat{\gamma}_\epsilon \nabla(u_\epsilon - U) \cdot (\nabla \phi) v_\epsilon^{(j)} \, \mathrm{d}x \\
&= \int_\Omega \hat{\gamma}_\epsilon \nabla(u_\epsilon - U) \cdot \left(\nabla v_\epsilon^{(j)}\right) \phi \, \mathrm{d}x + \int_\Omega \hat{\gamma}_\epsilon \nabla(u_\epsilon - U) \cdot (\nabla \phi) V^{(j)} \, \mathrm{d}x \\
&\quad + O\left(\|u_\epsilon - U\|_{H^1(\Omega)} \|v_\epsilon^{(j)} - V^{(j)}\|_{L^2(\Omega)}\right) \\
&= \int_\Omega \hat{\gamma}_\epsilon \nabla(u_\epsilon - U) \cdot \left(\nabla v_\epsilon^{(j)}\right) \phi \, \mathrm{d}x + \int_\Omega \gamma_0 \nabla(u_\epsilon - U) \cdot (\nabla \phi) V^{(j)} \, \mathrm{d}x \\
&\quad + \int_\Omega (\hat{\gamma}_\epsilon - \gamma_0) \nabla(u_\epsilon - U) \cdot (\nabla \phi) V^{(j)} \, \mathrm{d}x + o\left(|\omega_\epsilon|\right) \\
&= -\int_\Omega \hat{\gamma}_\epsilon (u_\epsilon - U) \nabla v_\epsilon^{(j)} \cdot \nabla \phi \, \mathrm{d}x - \int_\Omega (u_\epsilon - U) \frac{\partial \gamma_0}{\partial x_j} \phi \, \mathrm{d}x \\
&\quad + \int_{\partial\Omega} (u_\epsilon - U) \gamma_0 n_j \phi \, \mathrm{d}\sigma + \int_\Omega \gamma_0 \nabla(u_\epsilon - U) \cdot (\nabla \phi) V^{(j)} \, \mathrm{d}x \\
&\quad + \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \nabla(u_\epsilon - U) \cdot (\nabla \phi) V^{(j)} \, \mathrm{d}x + o\left(|\omega_\epsilon|\right) \\
&= -\int_\Omega \hat{\gamma}_\epsilon (u_\epsilon - U) \nabla V^{(j)} \cdot \nabla \phi \, \mathrm{d}x - \int_\Omega (u_\epsilon - U) \frac{\partial \gamma_0}{\partial x_j} \phi \, \mathrm{d}x \\
&\quad + \int_{\partial\Omega} (u_\epsilon - U) \gamma_0 n_j \phi \, \mathrm{d}\sigma + \int_\Omega \gamma_0 \nabla(u_\epsilon - U) \cdot (\nabla \phi) V^{(j)} \, \mathrm{d}x \\
&\quad + \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \nabla(u_\epsilon - U) \cdot (\nabla \phi) V^{(j)} \, \mathrm{d}x + o\left(|\omega_\epsilon|\right).
\end{aligned} \tag{20}$$

Here we have used Lemma 1 to estimate the difference $v_\epsilon^{(j)} - V^{(j)}$, as well as the difference $u_\epsilon - U$. We also calculate

$$\begin{aligned}\int_\Omega \gamma_0 \nabla(u_\epsilon - U)\cdot\nabla(V^{(j)}\phi)\,\mathrm{d}x &= \int_\Omega \gamma_0\nabla(u_\epsilon - U)\cdot\left(\nabla V^{(j)}\right)\phi\,\mathrm{d}x + \int_\Omega \gamma_0\nabla(u_\epsilon - U)\cdot(\nabla\phi)V^{(j)}\,\mathrm{d}x\\ &= -\int_\Omega \gamma_0(u_\epsilon - U)\nabla V^{(j)}\cdot\nabla\phi\,\mathrm{d}x - \int_\Omega (u_\epsilon - U)\frac{\partial\gamma_0}{\partial x_j}\phi\,\mathrm{d}x \\ &\quad+\int_{\partial\Omega}(u_\epsilon - U)\gamma_0 n_j\phi\,\mathrm{d}\sigma + \int_\Omega \gamma_0\nabla(u_\epsilon - U)\cdot(\nabla\phi)V^{(j)}\,\mathrm{d}x.\end{aligned} \tag{21}$$

A direct combination of (20) and (21) (and Lemma 1) gives

$$\int_\Omega \hat{\gamma}_\epsilon\nabla(u_\epsilon - U)\cdot\nabla(v_\epsilon^{(j)}\phi)\,\mathrm{d}x = \int_\Omega \gamma_0\nabla(u_\epsilon - U)\cdot\nabla(V^{(j)}\phi)\,\mathrm{d}x + \int_{\omega_\epsilon}(\gamma_1-\gamma_0)\nabla(u_\epsilon - U)\cdot(\nabla\phi)V^{(j)}\,\mathrm{d}x + o\left(|\omega_\epsilon|\right),$$

so that, due to (18) and (19),

$$\begin{aligned}\int_{\omega_\epsilon}(\gamma_0-\gamma_1)\nabla U\cdot\nabla(v_\epsilon^{(j)}\phi)\,\mathrm{d}x &= \int_{\omega_\epsilon}(\gamma_0-\gamma_1)\nabla u_\epsilon\cdot\nabla(V^{(j)}\phi)\,\mathrm{d}x - \int_{\omega_\epsilon}(\gamma_0-\gamma_1)\nabla u_\epsilon\cdot(\nabla\phi)V^{(j)}\,\mathrm{d}x\\ &\quad+\int_{\omega_\epsilon}(\gamma_0-\gamma_1)\nabla U\cdot(\nabla\phi)V^{(j)}\,\mathrm{d}x + o\left(|\omega_\epsilon|\right)\\ &= \int_{\omega_\epsilon}(\gamma_0-\gamma_1)\nabla u_\epsilon\cdot\nabla V^{(j)}\;\phi\,\mathrm{d}x + \int_{\omega_\epsilon}(\gamma_0-\gamma_1)\nabla U\cdot(\nabla\phi)v_\epsilon^{(j)}\,\mathrm{d}x\\ &\quad+O\left(\|V^{(j)}-v_\epsilon^{(j)}\|_{L^2(\Omega)}|\omega_\epsilon|^{1/2}\|\nabla U\|_{L^\infty(\omega_\epsilon)}\right) + o\left(|\omega_\epsilon|\right)\\ &= \int_{\omega_\epsilon}(\gamma_0-\gamma_1)\nabla u_\epsilon\cdot\nabla V^{(j)}\;\phi\,\mathrm{d}x + \int_{\omega_\epsilon}(\gamma_0-\gamma_1)\nabla U\cdot(\nabla\phi)v_\epsilon^{(j)}\,\mathrm{d}x + o\left(|\omega_\epsilon|\right).\end{aligned}$$

After rearrangement and a rescaling this yields

$$\int_\Omega(\gamma_0-\gamma_1)\nabla U\;\cdot|\omega_\epsilon|^{-1}1_{\omega_\epsilon}\nabla v_\epsilon^{(j)}\;\phi\,\mathrm{d}x = \int_\Omega(\gamma_0-\gamma_1)|\omega_\epsilon|^{-1}1_{\omega_\epsilon}\nabla u_\epsilon\cdot\nabla V^{(j)}\;\phi\,\mathrm{d}x + o(1). \tag{22}$$

Passing to the limit along the subsequence ω_{ϵ_n} (using that ∇U is smooth inside Ω, and that $\mathrm{d}\mathcal{M}_{ij}$ has compact support) we now obtain

$$\int_\Omega \phi(\gamma_0-\gamma_1)\frac{\partial}{\partial x_i}U\,\mathrm{d}\mathcal{M}_{ij} = \int_\Omega \phi(\gamma_0-\gamma_1)\frac{\partial}{\partial x_i}V^{(j)}\,\mathrm{d}\nu_i = \int_\Omega \phi(\gamma_0-\gamma_1)\,\mathrm{d}\nu_j,$$

which is the desired identity (17). This completes the proof of Lemma 2. □

We are presently ready for:

Proof of Theorem 1. Let ω_{ϵ_n} be a subsequence for which (4), (5) and (15) hold. Clearly such a subsequence exists, and it is completely independent of the boundary flux ψ. We recall the identity (8), which asserts that

$$(u_{\epsilon_n} - U)(y) = |\omega_{\epsilon_n}|\int_\Omega(\gamma_1-\gamma_0)(x)|\omega_{\epsilon_n}|^{-1}1_{\omega_{\epsilon_n}}\nabla u_{\epsilon_n}\cdot\nabla_x N(x,y)\mathrm{d}x, \quad y\in\partial\Omega.$$

Let $K_0 \subset \Omega$ denote a compact set that strictly contains the sets ω_{ϵ_n}. Given any $y \in \partial\Omega$, it is possible to find a vector valued function $\phi_y \in C^0(\overline{\Omega})$ such that

$$\phi_y(x) = \nabla_x N(x,y), \quad \forall x \in K_0.$$

Using Lemma 2 we now get

$$\begin{aligned}
(u_{\epsilon_n} - U)(y) &= |\omega_{\epsilon_n}| \int_\Omega (\gamma_1 - \gamma_0)(x) |\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \nabla u_{\epsilon_n} \cdot \phi_y(x)\,\mathrm{d}x \\
&= |\omega_{\epsilon_n}| \int_\Omega (\gamma_1 - \gamma_0)(x) |\omega_{\epsilon_n}|^{-1} 1_{\omega_{\epsilon_n}} \frac{\partial}{\partial x_j} u_{\epsilon_n} \left(\phi_y(x)\right)_j \,\mathrm{d}x \\
&= |\omega_{\epsilon_n}| \int_\Omega (\gamma_1 - \gamma_0) M_{ij} \frac{\partial U}{\partial x_i} \left(\phi_y\right)_j \,\mathrm{d}\mu + o(|\omega_{\epsilon_n}|) \\
&= |\omega_{\epsilon_n}| \int_\Omega (\gamma_1 - \gamma_0) M_{ij} \frac{\partial U}{\partial x_i} \frac{\partial N}{\partial x_j}(x,y)\mathrm{d}\mu(x) + o(|\omega_{\epsilon_n}|),
\end{aligned}$$

which verifies the asymptotic statement in Theorem 1. By (equi-)continuity and compactness it follows immediately that $\|o(|\omega_{\epsilon_n}|)\|_{L^\infty(\partial\Omega)}/|\omega_{\epsilon_n}| \to 0$ for any fixed $\psi \in H^{-1/2}(\partial\Omega)$, and uniformly on $\{\psi : \int_{\partial\Omega} \psi\,\mathrm{d}\sigma = 0, \|\psi\|_{L^2(\partial\Omega)} \leq 1\}$. In the following section we show that the tensor M_{ij} has the stated symmetry- and positivity properties.

□

4. Properties of the polarization tensor

The identity (22) immediately extends to the case when U and u_ϵ are replaced by V and v_ϵ, satisfying (9) and (10) ($F \in C^{0,\alpha}(K_0)$). In particular we may insert $V = V^{(i)}$, and $v_\epsilon = v_\epsilon^{(i)}$, to arrive at

$$\int_\Omega (\gamma_0 - \gamma_1) \nabla V^{(i)} \cdot |\omega_\epsilon|^{-1} 1_{\omega_\epsilon} \nabla v_\epsilon^{(j)} \; \phi\,\mathrm{d}x = \int_\Omega (\gamma_0 - \gamma_1) |\omega_\epsilon|^{-1} 1_{\omega_\epsilon} \nabla v_\epsilon^{(i)} \cdot \nabla V^{(j)} \; \phi\,\mathrm{d}x + o(1).$$

Passing to the limit along the subsequence ω_{ϵ_n}, using the limiting relationship (15), we now obtain

$$\begin{aligned}
\int_\Omega (\gamma_0 - \gamma_1) M_{ij} \; \phi\,\mathrm{d}\mu &= \int_\Omega (\gamma_0 - \gamma_1) \frac{\partial V^{(i)}}{\partial x_k} M_{kj} \; \phi\,\mathrm{d}\mu \\
&= \int_\Omega (\gamma_0 - \gamma_1) M_{ki} \frac{\partial V^{(j)}}{\partial x_k} \; \phi\,\mathrm{d}\mu \\
&= \int_\Omega (\gamma_0 - \gamma_1) M_{ji}\, \phi\,\mathrm{d}\mu,
\end{aligned}$$

which verifies the symmetry of M, in the sense of (6). To verify the bounds in Theorem 1 we calculate

$$\begin{aligned}
\xi_i\xi_j \int_\Omega (\gamma_1-\gamma_0)|\omega_\epsilon|^{-1}1_{\omega_\epsilon}\nabla v_\epsilon^{(j)}\cdot\nabla V^{(i)}\ \phi\,\mathrm{d}x &= \xi_i\xi_j|\omega_\epsilon|^{-1}\int_{\omega_\epsilon}(\gamma_1-\gamma_0)\nabla V^{(j)}\cdot\nabla V^{(i)}\ \phi\,\mathrm{d}x\\
&\quad +\xi_i\xi_j|\omega_\epsilon|^{-1}\int_\Omega(\hat\gamma_\epsilon-\gamma_0)\nabla\left[(v_\epsilon^{(j)}-V^{(j)})\phi\right]\cdot\nabla V^{(i)}\,\mathrm{d}x\\
&\quad -\xi_i\xi_j|\omega_\epsilon|^{-1}\int_{\omega_\epsilon}(\gamma_1-\gamma_0)(v_\epsilon^{(j)}-V^{(j)})\nabla\phi\cdot\nabla V^{(i)}\,\mathrm{d}x\\
&= \xi_i\xi_j|\omega_\epsilon|^{-1}\int_{\omega_\epsilon}(\gamma_1-\gamma_0)\nabla V^{(j)}\cdot\nabla V^{(i)}\ \phi\,\mathrm{d}x\\
&\quad +\xi_i\xi_j|\omega_\epsilon|^{-1}\int_\Omega(\hat\gamma_\epsilon-\gamma_0)\nabla\left[(v_\epsilon^{(j)}-V^{(j)})\phi\right]\cdot\nabla V^{(i)}\,\mathrm{d}x\ +o(1)\\
&= \xi_i\xi_j|\omega_\epsilon|^{-1}\int_{\omega_\epsilon}(\gamma_1-\gamma_0)\nabla V^{(j)}\cdot\nabla V^{(i)}\ \phi\,\mathrm{d}x\\
&\quad +\xi_i\xi_j|\omega_\epsilon|^{-1}\int_\Omega\hat\gamma_\epsilon\nabla\left[(v_\epsilon^{(j)}-V^{(j)})\phi\right]\cdot\nabla(V^{(i)}-v_\epsilon^{(i)})\,\mathrm{d}x\\
&\quad +\xi_i\xi_j|\omega_\epsilon|^{-1}\int_\Omega\hat\gamma_\epsilon\nabla\left[(v_\epsilon^{(j)}-V^{(j)})\phi\right]\cdot\nabla v_\epsilon^{(i)}\,\mathrm{d}x\\
&\quad -\xi_i\xi_j|\omega_\epsilon|^{-1}\int_\Omega\gamma_0\nabla\left[(v_\epsilon^{(j)}-V^{(j)})\phi\right]\cdot\nabla V^{(i)}\,\mathrm{d}x+o(1)\\
&= \xi_i\xi_j|\omega_\epsilon|^{-1}\Bigg(\int_{\omega_\epsilon}(\gamma_1-\gamma_0)\nabla V^{(j)}\cdot\nabla V^{(i)}\ \phi\,\mathrm{d}x\\
&\quad -\int_\Omega\hat\gamma_\epsilon\nabla(V^{(j)}-v_\epsilon^{(j)})\cdot\nabla(V^{(i)}-v_\epsilon^{(i)})\ \phi\,\mathrm{d}x\Bigg)+o(1). \qquad (23)
\end{aligned}$$

We introduce the notation

$$V=V^{(i)}\xi_i=\left(x_i-\frac{1}{|\partial\Omega|}\int_{\partial\Omega}x_i\,\mathrm{d}\sigma\right)\xi_i,\quad \text{and}\quad v_\epsilon=v_\epsilon^{(i)}\xi_i.$$

A combination of the estimate (23) with the limiting relationships, (4) and (15), that define the measure μ and the tensor M, now yields

$$\begin{aligned}
\int_\Omega(\gamma_1-\gamma_0)M_{ij}\xi_i\xi_j\ \phi\,\mathrm{d}\mu &= |\omega_{\epsilon_n}|^{-1}\int_{\omega_{\epsilon_n}}(\gamma_1-\gamma_0)|\nabla V|^2\ \phi\,\mathrm{d}x \qquad (24)\\
&\quad -|\omega_{\epsilon_n}|^{-1}\int_\Omega\hat\gamma_{\epsilon_n}|\nabla\left(V-v_{\epsilon_n}\right)|^2\ \phi\,\mathrm{d}x+o(1),
\end{aligned}$$

for any $\phi\in C^1(\overline{\Omega})$ (and the subsequence ω_{ϵ_n}). We shall make use of the following estimate concerning the second term of the right-hand side.

Lemma 3. *Let V and v_ϵ be as introduced above. For any fixed $\phi\in C^1(\overline{\Omega})$, $\phi\ge 0$,*

$$|\omega_\epsilon|^{-1}\int_\Omega\hat\gamma_\epsilon|\nabla\left(V-v_\epsilon\right)|^2\ \phi\,\mathrm{d}x\le|\omega_\epsilon|^{-1}\int_{\omega_\epsilon}\frac{(\gamma_1-\gamma_0)^2}{\gamma_1}|\nabla V|^2\ \phi\,\mathrm{d}x+o(1).$$

Proof of Lemma 3. From (23) it follows immediately that

$$\begin{aligned}|\omega_\epsilon|^{-1}\int_\Omega \hat{\gamma}_\epsilon|\nabla(V-v_\epsilon)|^2\;\phi\,\mathrm{d}x &= |\omega_\epsilon|^{-1}\int_{\omega_\epsilon}(\gamma_1-\gamma_0)|\nabla V|^2\;\phi\,\mathrm{d}x\\ &\quad -|\omega_\epsilon|^{-1}\int_\Omega(\gamma_1-\gamma_0)1_{\omega_\epsilon}\nabla v_\epsilon\cdot\nabla V\;\phi\,\mathrm{d}x+o(1)\\ &= |\omega_\epsilon|^{-1}\int_{\omega_\epsilon}(\gamma_1-\gamma_0)\nabla(V-v_\epsilon)\cdot\nabla V\;\phi\,\mathrm{d}x+o(1),\end{aligned}$$

and thus

$$\begin{aligned}|\omega_\epsilon|^{-1}\int_\Omega \hat{\gamma}_\epsilon|\nabla(V-v_\epsilon)|^2\;\phi\,\mathrm{d}x &\le |\omega_\epsilon|^{-1}\left(\int_{\omega_\epsilon}\hat{\gamma}_\epsilon|\nabla(V-v_\epsilon)|^2\;\phi\,\mathrm{d}x\right)^{1/2}\\ &\quad\times\left(\int_{\omega_\epsilon}\frac{(\gamma_1-\gamma_0)^2}{\gamma_1}|\nabla V|^2\;\phi\,\mathrm{d}x\right)^{1/2}+o(1),\end{aligned}$$

for any $\phi\in C^1(\overline{\Omega})$, $\phi\ge 0$. A combination of this with the fact that $a^2<ab+c\Rightarrow a^2<b^2+2c$ for a,b and c positive, gives the desired estimate. □

We are now ready to complete the proof of the inequalities for the tensor M, as stated in Theorem 1. According to (24) we have

$$\int_\Omega(\gamma_1-\gamma_0)M_{ij}\xi_i\xi_j\;\phi\,\mathrm{d}\mu\le|\omega_{\epsilon_n}|^{-1}\int_{\omega_{\epsilon_n}}(\gamma_1-\gamma_0)|\nabla V|^2\;\phi\,\mathrm{d}x+o(1),$$

and according to (24), and the estimate in Lemma 3, we also have

$$\begin{aligned}\int_\Omega(\gamma_1-\gamma_0)M_{ij}\xi_i\xi_j\;\phi\,\mathrm{d}\mu &\ge |\omega_{\epsilon_n}|^{-1}\int_{\omega_{\epsilon_n}}(\gamma_1-\gamma_0)|\nabla V|^2\;\phi\,\mathrm{d}x\\ &\quad -|\omega_{\epsilon_n}|^{-1}\int_{\omega_{\epsilon_n}}\frac{(\gamma_1-\gamma_0)^2}{\gamma_1}|\nabla V|^2\;\phi\,\mathrm{d}x+o(1)\\ &= |\omega_{\epsilon_n}|^{-1}\int_{\omega_{\epsilon_n}}(\gamma_1-\gamma_0)\frac{\gamma_0}{\gamma_1}|\nabla V|^2\;\phi\,\mathrm{d}x+o(1),\end{aligned}$$

for any $\phi\in C^1(\overline{\Omega})$, $\phi\ge 0$. After passage to the limit along the subsequence ω_{ϵ_n} a combination of these two inequalities shows that

$$(\gamma_1-\gamma_0)(x)\frac{\gamma_0}{\gamma_1}(x)|\xi|^2\le(\gamma_1-\gamma_0)(x)M_{ij}(x)\xi_i\xi_j\le(\gamma_1-\gamma_0)(x)|\xi|^2,\quad \xi\in\mathbb{R}^m,$$

μ almost everywhere in Ω (here we use that the rationals are dense in $\mathbb{R}^m$, that the terms involved are continuous in ξ, and that a countable union of sets of measure zero again has measure zero). By cancellation of the common factor $(\gamma_1-\gamma_0)(x)$ we conclude that

$$\min\left\{1,\frac{\gamma_0}{\gamma_1}(x)\right\}|\xi|^2\le M_{ij}(x)\xi_i\xi_j\le\max\left\{1,\frac{\gamma_0}{\gamma_1}(x)\right\}|\xi|^2,\quad \xi\in\mathbb{R}^m,$$

μ almost everywhere in the set $\{x:\ \gamma_0(x)\ne\gamma_1(x)\ \}$, as stated in Theorem 1.

5. Some particular cases

Two particular cases that have already been studied, and for which very specific information has been derived about the measure μ and the polarization tensor $M_{ij}(x)$ concern (1) *a finite collection of well separated, diametrically small inhomogeneities* and (2) *a finite collection of well separated, thin inhomogeneities.* In the first case $\omega_\epsilon = \cup_{l=1}^{K} \mathbf{z}_l + \epsilon B_l$, where $\mathbf{z}_l \in \Omega$, $l = 1, \ldots, K$, is a set of K distinct points, and each $B_l \subset \mathbb{R}^m$ is a bounded, smooth domain containing the origin. In the second case $\omega_\epsilon = \cup_{l=1}^{K} \omega_\epsilon^l$, where each ω_ϵ^l has the form $\omega_\epsilon^l = \{x' + \eta n(x') : x' \in \sigma_l, \ |\eta| < \epsilon\}$; $\sigma_l \subset \mathbb{R}^m$, $l = 1, \ldots, K$, is a set of nonintersecting smooth surfaces, and $n(x')$ denotes a smooth, unit, normal vector field to σ_l. Since we suppose $\mathbf{z}_l$, B_l and σ_l are fixed, no extraction of a subsequence is necessary.

For the voltage potential corresponding to a finite collection of well separated (interior) inhomogeneities one obtains (*cf.* [10] and [9])

$$\begin{aligned}(u_\epsilon - U)(y) &= \epsilon^m \sum_{l=1}^{K} (\gamma_1 - \gamma_0) M_{ij}^{(l)} \frac{\partial U}{\partial x_i}(\mathbf{z}_l) \frac{\partial N}{\partial x_j}(\mathbf{z}_l, y) + O\left(\epsilon^{m+\frac{1}{2}}\right) \\ &= |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0)(x) M_{ij}(x) \frac{\partial U}{\partial x_i}(x) \frac{\partial N}{\partial x_j}(x, y)\, \mathrm{d}\mu + o(|\omega_\epsilon|),\end{aligned}$$

with

$$\mu = \frac{1}{\sum |B_l|} \sum_{l=1}^{K} |B_l| \ \delta_{\mathbf{z}_l} \quad \text{and} \quad M_{ij}(z_l) = \frac{1}{|B_l|} M_{ij}^{(l)} = \frac{1}{|B_l|} \int_{B_l} \frac{\partial}{\partial z_i} \phi_j(z)\, \mathrm{d}z.$$

Here ϕ_j $(m \geq 2)$ denotes the solution to

$$\begin{aligned}&\nabla_z \cdot (\gamma(z) \nabla_z \phi_j) = 0 \quad \text{in} \ \ \mathbb{R}^m, \\ &\phi_j(z) - z_j \to 0 \quad \text{as} \ \ |z| \to \infty.\end{aligned}$$

To include the case $m = 1$, the correct condition to impose is $\nabla_z \phi_j(z) - e_j \to 0$ as $|z| \to \infty$. The function $\gamma(z)$ is the rescaled conductivity, given by $\gamma(z) = \gamma_1$ for $z \in B_l$, $\gamma(z) = \gamma_0$ for $z \in \mathbb{R}^m \setminus B_l$ (supposing for simplicity that γ_0 and γ_1 are constants). Higher order terms of the expansion have been derived in [2].

For the voltage potential corresponding to a finite collection of well separated, thin inhomogeneities one obtains (*cf.* [5] and [6])

$$\begin{aligned}(u_\epsilon - U)(y) &= 2\epsilon \sum_{l=1}^{K} \int_{\sigma_l} (\gamma_1 - \gamma_0)(x) M_{ij}^{(l)}(x) \frac{\partial U}{\partial x_i}(x) \frac{\partial N}{\partial x_j}(x, y)\, \mathrm{d}\sigma_x + o(\epsilon) \\ &= |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0)(x) M_{ij}(x) \frac{\partial U}{\partial x_i}(x) \frac{\partial N}{\partial x_j}(x, y)\, \mathrm{d}\mu + o(|\omega_\epsilon|).\end{aligned}$$

Here $\mu = \frac{1}{\sum A(\sigma_l)} \sum_{l=1}^{K} \ \delta_{\sigma_l}$, with δ_{σ_l} being the "Dirac measure" supported on σ_l, and $A(\sigma_l)$ being the "area" of σ_l. $M_{ij}(x)$, $x \in \sigma_l$, is a positive definite symmetric matrix whose first $n - 1$ eigenvectors form a basis for the tangent space to σ_l, and whose last eigenvector is the normal. The eigenvalue corresponding to the normal direction is γ_0/γ_1, the eigenvalues corresponding to the tangential directions are all equal to 1. Notice that these eigenvalues are extreme, in the sense that they (simultaneously) "achieve" the bounds established in Theorem 1.

Without giving any details of the analysis we shall describe one additional special case of our general formula, namely that corresponding to a set of inhomogeneities in the form of a "very fine scale" periodic array of small balls. The periodic array has period ϵ, and the balls are centered in those period cells that fall inside some smooth subdomain $\omega \subset\subset \Omega$. Each ball has radius $\epsilon^{(1+d)}$ for some $d > 0$. The conductivity, as before, equals γ_0 outside the balls, and equals γ_1 inside the balls. As $\epsilon \to 0$ the volume fraction of balls (inside ω) approaches $\beta = c_m \epsilon^{md}$, so the total volume of the inhomogeneities approaches $c_m \epsilon^{md} |\omega|$. The wellknown Maxwell–Claussius–Mossotti formula asserts that this low volume fraction array of balls (to order $\beta = c_m \epsilon^{md}$) behaves like an effective medium with conductivity $\gamma_0 + (D_\epsilon - \gamma_0) 1_\omega$, where the constant D_ϵ is given by

$$\frac{D_\epsilon - \gamma_0}{D_\epsilon + (m-1)\gamma_0} = \beta \frac{\gamma_1 - \gamma_0}{\gamma_1 + (m-1)\gamma_0}.$$

For $y \in \partial\Omega$ we may now (essentially by means of a small amplitude perturbation formula) derive that

$$\begin{aligned}(u_\epsilon - U)(y) &= \int_\omega (D_\epsilon - \gamma_0) \nabla U \nabla_x N(x,y)\,\mathrm{d}x + o\left(\epsilon^{md}\right) \\ &= c_m \epsilon^{md} \int_\omega (\gamma_1 - \gamma_0) \frac{m\gamma_0}{\gamma_1 + (m-1)\gamma_0} \nabla U \nabla_x N(x,y)\,\mathrm{d}x + o\left(\epsilon^{md}\right) \\ &= |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij} \frac{\partial U}{\partial x_i} \frac{\partial N}{\partial x_j}(x,y)\,\mathrm{d}\mu + o(|\omega_\epsilon|),\end{aligned}$$

where M is the tensor $M_{ij} = \frac{m\gamma_0}{\gamma_1 + (m-1)\gamma_0} \delta_{ij}$, and μ is the standard Lebesque measure, restricted to ω, and normalized by $\frac{1}{|\omega|}$.

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