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A General Survey of the Theory of the Bethe-Salpeter Equation

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The theory of the Bethe-Salpeter equation is reviewed extensively. The main effort is devoted to describing systematically the theoretical results rather than numerical calculations and applications of the Bethe-Salpeter equation. An almost complete bibliography of the Bethe-Salpeter equation also is presented.

Contents

§1. Introduction

The Bethe-Salpeter (B–S) equation is the most orthodox tool for discussing the relativistic two-body problem in quantum field theory. It was proposed almost twenty years ago, but in the first decade of its history the theoretical study of the B–S equation was not made very intensively because of its mathematical difficulty and the uneasiness about the accuracy of the "ladder approximation". Recently, the B–S equation has acquired a great interest in connection with the Regge-pole theory, the ghost problem, the $O(4)$ symmetry, etc. The ladder approximation provides an interesting theoretical

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model, called the "ladder model", which is manifestly covariant under the Poincaré group. It is important for obtaining qualitative features rather than quantitative results of the relativistic two-body problem.

The purpose of the present article is twofold. First, we review the theory of the B–S equation extensively. We summarize various theoretical results obtained so far, which are believed to be the best known ones. Our main concerns are the theoretical aspects of the bound-state B–S equation; the scattering problem and the applications of the B–S equation are mentioned only briefly. Second, it is attempted to present a rather complete list of the papers concerning the B–S equation. Until now, many authors who worked in the B–S equation wrote their papers without knowing related papers published before. An extensive bibliography will eliminate such an unfortunate situation.

In §§2 and 3, we discuss the general framework of the B–S equation, which is true independently of the model considered. In §§4 and 5, we present some mathematical tools for the investigation of the B–S equation. Sections 6 and 7 are devoted to the scalar-scalar scalar-meson-exchange ladder model. Sections 8, 9 and 10 deal with the three characteristics of the B–S equation: the abnormal solutions, the existence of negative-norm amplitudes and the presence of multiple poles. The spinor-spinor model is considered in §11. In §§12 and 13, we discuss applications of the B–S equation to the Regge-pole theory. Other topics are touched on very briefly in §14.

Throughout the present article, we employ the time-favored metric, that is, for $p_\mu=(p_0, p_1, p_2, p_3)$ p^2 equals $p_0^2-p_1^2-p_2^2-p_3^2$. A 3-vector is indicated by a boldface letter; for instance, $\boldsymbol{p}$ denotes (p_1, p_2, p_3) and $\boldsymbol{p}^2=p_1^2+p_2^2+p_3^2$. Apart from 3-vectors, boldface letters are used for denoting position-space functions; their Fourier transforms are denoted by the same symbols without using boldface.

§2. Derivation of the B-S equation

The B–S equation was proposed by a number of authors. Its first proposal was made without derivation by Nambu (1950),[N27] who wrote down a position-space differential equation in the ladder approximation. The general form of the B–S equation was derived by Salpeter and Bethe*) (1951)[S2] on the basis of the Feynman-graphical consideration. Its field-theoretical foundation was established by Gell-Mann and Low (1951).[G3] The B–S equation was independently proposed also by Schwinger (1951),[S12] who employed the functional-derivative formalism, and by Kita (1952),[K9] who used the S-matrix-theoretical consideration. The derivation based on the energy-plane analyticity was

*) The name, "Bethe-Salpeter" equation, originated from the presentation of their work at a meeting of the American Physical Society [H. A. Bethe and E. E. Salpeter, Phys. Rev. **82** (1951), 309].

presented by Mandelstam (1955)[M4] and reviewed by Lurié, Macfarlane and Takahashi (1965).[L15]

We consider an elastic scattering of two particles a and b. For simplicity of description, we assume that a and b are non-identical scalar particles; modifications to more general cases are straightforward. Let $\boldsymbol{\varphi}_a(x)$ and $\boldsymbol{\varphi}_b(x)$ be the field operators of a and b, respectively, in the Heisenberg representation. The scattering Green's function $\boldsymbol{G}(x_a, x_b;\ y_a, y_b)$ is defined by

$$\boldsymbol{G}(x_a, x_b;\ y_a, y_b) \equiv \langle 0 | \mathrm{T}[\boldsymbol{\varphi}_a(x_a)\boldsymbol{\varphi}_b(x_b)\boldsymbol{\varphi}_a^\dagger(y_a)\boldsymbol{\varphi}_b^\dagger(y_b)] | 0\rangle. \tag{2·1}$$

Here we employ the usual Dirac's bra-ket notation; $|0\rangle$ denotes the true vacuum. The symbols T and † stand for Wick's chronological operator[1] and hermitian conjugation, respectively.

If we expand $\boldsymbol{G}$ into a perturbation series, it is expressed in terms of connected Feynman graphs corresponding to the process $a+b \to a+b$. In order to derive an integral equation for $\boldsymbol{G}$, we rearrange the order of the summation of Feynman integrals. We carry out first the summation in each self-energy part, and next the summation in each $(a+b)$-irreducible part, which is denoted by $\boldsymbol{I}(x_a, x_b;\ y_a, y_b)$ (external propagators are not inclusive), where an $(a+b)$-irreducible part is a part which contains no $(a+b)$-intermediate states in this channel. Then $\boldsymbol{G}$ satisfies

$$\begin{aligned}\boldsymbol{G}(x_a, x_b;\ y_a, y_b) &= \boldsymbol{\Delta}'_{Fa}(x_a - y_a)\boldsymbol{\Delta}'_{Fb}(x_b - y_b) \\ &+ \int d^4z_a \int d^4z_b \int d^4z'_a \int d^4z'_b \boldsymbol{\Delta}'_{Fa}(x_a - z_a)\boldsymbol{\Delta}'_{Fb}(x_b - z_b) \\ &\times \boldsymbol{I}(z_a, z_b;\ z'_a, z'_b)\boldsymbol{G}(z'_a, z'_b;\ y_a, y_b),\end{aligned} \tag{2·2}$$

where $\boldsymbol{\Delta}'_F$ denotes the modified Feynman propagator. The meaning of (2·2) is illustrated in Fig. 1. In (2·2), each quantity should be renormalized if possible.

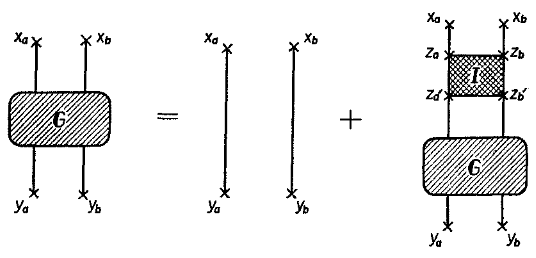

Fig. 1. Graphical representation of the position-space B–S equation (2·2).

Now, it is more convenient to rewrite (2·2) into the equation in the momentum space. Because of the translational invariance of the theory, $\boldsymbol{G}$

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and $\boldsymbol{I}$ are the functions of three differences x_a-x_b, $-y_a+y_b$ and $\eta_a(x_a-y_a)+\eta_b(x_b-y_b)$, where η_a and η_b are arbitrary real quantities such that $\eta_a+\eta_b=1$. Let p, q and P be their conjugate momenta, respectively. Then (2·2) is transcribed as

$$[\Delta'_{Fa}(\eta_a P+p)\Delta'_{Fb}(\eta_b P-p)]^{-1}G(p,q;P)$$
$$=\delta^4(p-q)+\int d^4p' I(p,p';P)G(p',q;P), \tag{2·3}$$

where Δ'_F, G and I are the Fourier transforms of $\boldsymbol{\Delta}'_F$, $\boldsymbol{G}$ and $\boldsymbol{I}$, respectively. The graphical illustration of (2·3) is given in Fig. 2.

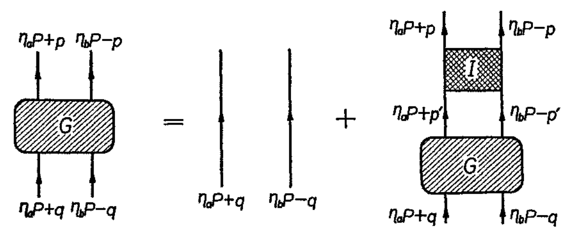

Fig. 2. Graphical representation of the momentum-space B–S equation (2·3).

To simplify the description, we set

$$K(p,q;P)\equiv[\Delta'_{Fa}(\eta_a P+p)\Delta'_{Fb}(\eta_b P-p)]^{-1}\delta^4(p-q) \tag{2·4}$$

and employ the operator notation:

$$\int d^4p' A(p,p')B(p',q)\equiv(AB)(p,q). \tag{2·5}$$

Then (2·3) can be written as

$$KG=1+IG. \tag{2·6}$$

We can formally solve (2·6) as

$$G=(K-I)^{-1}; \tag{2·7}$$

hence

$$GK=1+GI, \tag{2·8}$$

which we can also derive directly. We denote the time-reversal operation by affixing a bar. Then the time-reversed equation of (2·6) can be written as

$$\overline{G}\overline{K}=1+\overline{G}\overline{I}. \tag{2·9}$$

If our theory is invariant under the time reversal (hereafter we always consider such a case), then $\overline{G}=G$, $\overline{K}=K$ and $\overline{I}=I$, and therefore (2·9) is identical with (2·8).

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The invariant squares of momenta are denoted as follows:

$$\begin{aligned} &P^2=s, \quad (p-q)^2=t, \\ &[(\eta_a-\eta_b)P+p+q]^2=u, \\ &(\eta_a P+p)^2=v, \quad (\eta_b P-p)^2=w, \\ &(\eta_a P+q)^2=v_0, \quad (\eta_b P-q)^2=w_0. \end{aligned} \tag{2·10}$$

Let m_a and m_b be the masses of a and b, respectively. Then the mass shells are defined by

$$v=m_a^2, \quad w=m_b^2 \tag{2·11}$$

and

$$v_0=m_a^2, \quad w_0=m_b^2. \tag{2·12}$$

The Feynman amplitude $F(p, P)$ equals the residue of $-(G-K^{-1})$ at (2·12), and the scattering amplitude is defined to be the residue of $G-K^{-1}$ at both (2·11) and (2·12).

In practical considerations, one is mainly concerned with the ladder approximation. In this approximation, Δ'_F is replaced by a free Feynman propagator*)

$$\Delta_F(k, m) \equiv -i(m^2-k^2-i\varepsilon)^{-1}, \tag{2·13}$$

and the integral kernel I contains only a single-particle-exchange contribution, so that I is independent of P. Furthermore, I is proportional to a coupling parameter $\lambda \equiv g_a g_b/(4\pi)^2$, where g_j $(j=a, b)$ denotes the coupling constant between the particle j and the exchanged particle (for $g_a=g_b$, we denote it by g).

In more general cases other than the ladder approximation, the operator $K-I$ is no longer linear with respect to λ. In most literatures, this linearity was retained by regarding λ as an artificially introduced parameter. In the present article, however, in the general case, K and I are regarded as operator-valued, non-linear functions [denoted by $K(\lambda)$ and $I(\lambda)$] of a certain parameter λ as well as of P or s.

Now, we discuss the homogeneous B–S equation for bound states. Let $|B,1\rangle$, $|B,2\rangle$, $\cdots$, $|B,n\rangle$ be degenerate bound states having the 4-momentum P_B with $\boldsymbol{P}_B=\boldsymbol{P}$ and $P_B^2=s_B$. The B–S amplitude for $|B,r\rangle$ and its conjugate are defined to be

$$\begin{aligned} \phi_{Br}(x_a, x_b; P_B) &\equiv \langle 0|\mathrm{T}[\varphi_a(x_a)\varphi_b(x_b)]|B,r\rangle, \\ \bar{\phi}_{Br}(x_a, x_b; P_B) &\equiv \langle B,r|\mathrm{T}[\varphi_a^\dagger(x_a)\varphi_b^\dagger(x_b)]|0\rangle \\ &= \langle 0|\overline{\mathrm{T}}[\varphi_a(x_a)\varphi_b(x_b)]|B,r\rangle^*, \end{aligned} \tag{2·14}$$

*) ε always denotes an infinitesimal positive quantity.

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respectively, where $\overline{\mathrm{T}}$ and $*$ denote the anti-chronological operator and complex conjugation, respectively. Because of the translational invariance of the theory, we can write

$$\begin{aligned}\boldsymbol{\phi}_{Br}(x_a, x_b;\, P_B) &= (2\pi)^{-3/2} e^{-iP_BX} \boldsymbol{\phi}_{Br}(x, P_B),\\ \bar{\boldsymbol{\phi}}_{Br}(x_a, x_b;\, P_B) &= (2\pi)^{-3/2} e^{+iP_BX} \bar{\boldsymbol{\phi}}_{Br}(x, P_B),\end{aligned} \tag{2·15}$$

where

$$\begin{aligned}X &= \eta_a x_a + \eta_b x_b\,,\\ x &= x_a - x_b\,.\end{aligned} \tag{2·16}$$

The reduced amplitude $\boldsymbol{\phi}_{Br}(x, P_B)$ is called also the B–S amplitude.

We insert a complete set of states*) into the middle of the right-hand side of (2·1). Then the contribution to $\boldsymbol{G}(x_a, x_b;\, y_a, y_b)$ from the intermediate states $|B, r\rangle$ $(r=1, 2, \cdots, n)$ may be written as**)

$$\begin{aligned}&\sum_{r=1}^{n}\int d^4P \boldsymbol{\phi}_{Br}(x_a, x_b;\, P)\bar{\boldsymbol{\phi}}_{Br}(y_a, y_b;\, P)\theta(P_0)\delta(P^2 - s_B)\theta(X_0 - Y_0)\\ &= (2\pi)^{-3}\sum_r\int \frac{d^3\boldsymbol{P}}{2\omega_B}\boldsymbol{\phi}_{Br}(x, P_B)\bar{\boldsymbol{\phi}}_{Br}(y, P_B)\\ &\qquad\times\exp[-i\omega_B(X_0 - Y_0) + i\boldsymbol{P}(\boldsymbol{X} - \boldsymbol{Y})]\,\theta(X_0 - Y_0),\end{aligned} \tag{2·17}$$

where

$$\omega_B = (P_B)_0 = (\boldsymbol{P}^2 + s_B)^{1/2} \tag{2·18}$$

and Y and y are defined analogously to (2·16). We employ an identity

$$\theta(z) = -(2\pi i)^{-1}\int dk\, e^{-ikz}(k + i\varepsilon)^{-1}. \tag{2·19}$$

After a transformation $k = P_0 - \omega_B$, (2·17) can be rewritten as

$$i(2\pi)^{-4}\sum_r\int d^4P \boldsymbol{\phi}_{Br}(x, P_B)\bar{\boldsymbol{\phi}}_{Br}(y, P_B)\frac{\exp[-iP(X - Y)]}{2\omega_B(P_0 - \omega_B + i\varepsilon)}\,. \tag{2·20}$$

The Fourier transform of (2·20) reads

$$\frac{i\sum_r \phi_{Br}(p, P)\bar{\phi}_{Br}(q, P)}{2\omega_B(P_0 - \omega_B + i\varepsilon)} \tag{2·21}$$

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*) We here assume that all states have positive norm, but later it will turn out that this assumption may not be true (see §9).

**) There is an objection[L15] to taking $\theta(X_0 - Y_0)$ because each of constituent particles in the initial state should be chronologically earlier than any of those in the final state. Hence one should adopt

$$\theta(\min[(x_a)_0, (x_b)_0] - \max[(y_a)_0, (y_b)_0]) = \theta(X_0 - Y_0 - \tfrac{1}{2}|x_0| - \tfrac{1}{2}|y_0|).$$

This change yields an additional factor $\exp[\tfrac{1}{2}i(P_0 - \omega_B)(|x_0| + |y_0|)]$ to the integrand of (2·20), but the residue at $P_0 = \omega_B$ remains unchanged.

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apart from a term regular at $P_0=\omega_B$. By adding the contribution from the anti-particle states of $|B,r\rangle$, we finally find that $G(p,q;P)$ has a pole term*)

$$\frac{i\sum_{r=1}^{n}\phi_{Br}(p,P_B)\bar{\phi}_{Br}(q,P_B)}{s-s_B+i\varepsilon}\,. \tag{2·22}$$

On substituting (2·22) for the pole term of G in (2·3) or (2·6), we compare the residues at $s=s_B$ of both sides. On account of the linear independence of $\phi_{B1},\cdots,\phi_{Bn}$, we find

$$[\Delta'_{Fa}(\eta_a P_B+p)\Delta'_{Fb}(\eta_b P_B-p)]^{-1}\phi_{Br}(p,P_B)$$
$$=\int d^4p' I(p,p';P_B)\phi_{Br}(p',P_B) \tag{2·23}$$

or

$$K_B\phi_{Br}=I_B\phi_{Br}\,, \tag{2·24}$$

where the subscript B means to put $s=s_B$. Equation (2·23) or (2·24) is usually called the B–S equation. Likewise, from (2·8) we find

$$\bar{\phi}_{Br}K_B=\bar{\phi}_{Br}I_B\,, \tag{2·25}$$

an equation which is identical with the time-reversed one of (2·24):

$$\bar{\phi}_{Br}\overline{K}_B=\bar{\phi}_{Br}\overline{I}_B \tag{2·26}$$

because of the time-reversal invariance ($\overline{K}_B=K_B$ and $\overline{I}_B=I_B$).

As remarked above, it is convenient to regard K and I as functions of a parameter λ. Then the bound-state energy becomes a function of λ, which we denote by $s=s_B(\lambda)$. If $ds_B/d\lambda\neq 0$, as we assume throughout, we can define the inverse function $\lambda=\lambda_B(s)$. Then (2·24) can be rewritten as

$$K(\lambda_B)\phi_{Br}=I(\lambda_B)\phi_{Br}\,. \tag{2·27}$$

This equation can also be derived from the residue of $G(\lambda)$ at $\lambda=\lambda_B$ on the λ plane.

§3. Normalization condition

Since the B–S equation (2·23) is homogeneous, it cannot determine a multiplicative constant of ϕ_{Br}. In order to normalize the B–S amplitude, various methods have been proposed so far.

The normalization condition was first considered by Nishijima (1953, 1954, 1955).[N34)~N36)] He obtained some integral equations for expressions like $\langle A|\mathrm{T}(\varphi\varphi\cdots\varphi^\dagger)|B\rangle$ and proposed a normalization condition**) by calculating

*) More precisely, we should write the numerator (2·22) as $i[\sum_r\phi_{Br}(p,P)\bar{\phi}_{Br}(q,P)]_{s=s_B}$.

**) Unfortunately, his normalization formula is not of convenient form.

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the expectation value of the total charge in a bound state. By means of the Feynman-graphical consideration, Mandelstam (1955)[M4)] found a general rule of calculating any matrix element related to a bound state $|B\rangle$ directly in terms of the corresponding B–S amplitude. In this way he calculated the expectation value of the total charge in $|B\rangle$ and obtained the standard formula of the normalization condition in the ladder approximation [see also, Klein and Zemach (1957)[K12)]].

On the other hand, Allcock (1956)[A2)] proposed a derivation of the normalization condition based on the state-vector normalization $\langle B|B\rangle=1$. Allcock and Hooton (1958)[A3)] verified the equivalence between Allcock's condition and Mandelstam's one.*⁾

The modern way of deriving the normalization condition is based on the Green's function G. Sato (1963)[S4)] proposed a normalization condition in terms of a vertex function which follows from G. It is unpractical, however, because one has to calculate the vertex function. By using the fact that G contains a pole term (2·22), Cutkosky and Leon (1964)[C19)] derived a formula for the normalization condition in compact form and showed its equivalence to the one based on the charge conservation. Furthermore, Nakanishi (1965)[N16)] presented a more convenient formula for the normalization condition by using the double-pole method (see below). Lurié, Macfarlane and Takahashi (1965)[L15)] showed another way of deriving the Cutkosky-Leon condition. Recently, further simple derivations have also been proposed by Arafune (1968)[A7)] and by Llewellyn Smith (1969).[L14)]

There is also a different way of normalizing the B–S amplitude. The method based on the charge conservation cannot in principle be applied to a neutral bound state. To avoid this difficulty, one can make use of the energy-momentum tensor instead of the current. Nambu (1964)[N29)] proposed to find the normalization condition in this way. Predazzi (1965)[P7)] showed the equivalence of all methods of deriving the normalization condition. Nishijima and Singh (1967)[N39)] discussed the derivations based on the charge and the energy-momentum conservation in a more complete way.

We here present some derivations[N15),A7)] of the normalization condition. We suppose that K and I, and hence G, are functions of λ. Then by differentiating (2·7) with respect to λ, one finds

$$\frac{\partial G}{\partial\lambda}=-(K-I)^{-1}\left(\frac{\partial K}{\partial\lambda}-\frac{\partial I}{\partial\lambda}\right)(K-I)^{-1}$$
$$=-G\left(\frac{\partial K}{\partial\lambda}-\frac{\partial I}{\partial\lambda}\right)G. \qquad (3\cdot1)$$

As shown in §2, G has a pole term**⁾

*⁾ See also, Biswas (1958)[B14)] and Green (1960).[G14)]

**⁾ We omit $+i\varepsilon$ by regarding G as an analytic function of s.

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$$i\sum_r \phi_{Br}\bar{\phi}_{Br}/(s-s_B). \tag{3·2}$$

We insert (3·2) into (3·1) and take out the residues of the double pole at $s=s_B$. We then find

$$i\frac{ds_B}{d\lambda}\sum_r \phi_{Br}\bar{\phi}_{Br}=\sum_{r'}\phi_{Br'}\bar{\phi}_{Br'}\left(\frac{\partial K}{\partial\lambda}-\frac{\partial I}{\partial\lambda}\right)_B\sum_r \phi_{Br}\bar{\phi}_{Br}. \tag{3·3}$$

Because of the linear independence of ϕ_{Br}, (3·3) reduces to

$$-i\bar{\phi}_{Br'}\left(\frac{\partial K}{\partial\lambda}-\frac{\partial I}{\partial\lambda}\right)_B\phi_{Br}=\frac{ds_B}{d\lambda}\delta_{rr'}. \tag{3·4}$$

In particular, in the ladder approximation, (3·4) takes a very simple form,

$$i\bar{\phi}_{Br'}K_B\phi_{Br}=\lambda(ds_B/d\lambda)\delta_{rr'}, \tag{3·5}$$

because then

$$\begin{aligned}&\partial K/\partial\lambda=0,\\ &\lambda\partial I/\partial\lambda=I.\end{aligned} \tag{3·6}$$

The normalization condition (3·4) or (3·5) is practically very convenient because the normalization integral can be calculated in a covariant way.

We can eliminate the explicit λ-dependence of (3·4) at the sacrifice of covariance. By using*)

$$\frac{\partial G}{\partial s}=-G\left(\frac{\partial K}{\partial s}-\frac{\partial I}{\partial s}\right)G \tag{3·7}$$

instead of (3·1), we obtain

$$i\bar{\phi}_{Br'}\left(\frac{\partial K}{\partial s}-\frac{\partial I}{\partial s}\right)_B\phi_{Br}=\delta_{rr'} \tag{3·8}$$

in the same way as in the above. In particular, in the ladder approximation, (3·8) reduces to

$$i\bar{\phi}_{Br'}(\partial K/\partial s)_B\phi_{Br}=\delta_{rr'}, \tag{3·9}$$

a result which is equivalent to Mandelstam's original formula. In spite of its appearance, (3·8) cannot be calculated covariantly, because we have to choose a particular Lorentz frame in order to carry out the differentiation with respect to s.

The equivalence between (3·4) and (3·8) can be directly demonstrated in the following way. By differentiating (2·24) with respect to λ, we have

*) We here assume the existence of the derivatives with respect to s. We should note, however, that K has a square-root-type singularity at $s=0$ in the unequal-mass case ($m_a\neq m_b$).

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$$\left(\frac{\partial K}{\partial \lambda}-\frac{\partial I}{\partial \lambda}\right)_B \phi_{Br}+\frac{ds_B}{d\lambda}\left(\frac{\partial K}{\partial s}-\frac{\partial I}{\partial s}\right)_B \phi_{Br}+(K-I)_B\frac{\partial \phi_{Br}}{\partial \lambda}=0. \tag{3·10}$$

Hence*)

$$\bar{\phi}_{Br'}\left(\frac{\partial K}{\partial \lambda}-\frac{\partial I}{\partial \lambda}\right)_B \phi_{Br}=-\frac{ds_B}{d\lambda}\bar{\phi}_{Br'}\left(\frac{\partial K}{\partial s}-\frac{\partial I}{\partial s}\right)_B \phi_{Br}\,. \tag{3·11}$$

Another very simple derivation of the normalization condition is as follows. We suppose that G has a Laurent expansion at $s=s_B$:

$$G=i\frac{\sum_r \phi_{Br}\bar{\phi}_{Br}}{s-s_B}+G_0+(s-s_B)G_1+\cdots. \tag{3·12}$$

Because of (2·6), G_0 satisfies

$$\left(\frac{\partial K}{\partial s}-\frac{\partial I}{\partial s}\right)_B i\sum_r \phi_{Br}\bar{\phi}_{Br}+(K-I)_B G_0=1. \tag{3·13}$$

Hence we have

$$i\bar{\phi}_{Br'}\left(\frac{\partial K}{\partial s}-\frac{\partial I}{\partial s}\right)_B \sum_r \phi_{Br}\bar{\phi}_{Br}=\bar{\phi}_{Br'}\,, \tag{3·14}$$

from which (3·8) follows immediately. If one makes the same consideration in the λ plane, one immediately obtains (3·4).

Finally, we mention the derivation of the normalization condition based on the charge conservation. From the generalized Ward identity[2)]

$$G(P)\Gamma_\mu(P,P)G(P)=-2iP_\mu\cdot\partial G(P)/\partial s, \tag{3·15}$$

where $\Gamma_\mu(P',P)$ denotes the vector vertex function, we can easily obtain

$$\bar{\phi}_{Br'}\Gamma_\mu(P_B,P_B)\phi_{Br}=2(P_B)_\mu\delta_{rr'} \tag{3·16}$$

by comparing the residues of the double pole at $s=s_B$. In the ladder approximation, we know

$$\Gamma_\mu(P,P)=2iP_\mu\cdot\partial K/\partial s, \tag{3·17}$$

and hence (3·16) reduces to (3·9). In order to show the equivalence between (3·8) and (3·16), we have only to substitute (3·7) for $\partial G/\partial s$ in (3·15) and compare the residue of the double pole again. The derivation based on the energy-momentum conservation can be worked out quite analogously by considering a symmetric-tensor vertex function.

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*) This identity can be used also as a formula for calculating $ds_B/d\lambda$ or $d\lambda_B/ds$ [see §7(A)].

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§4. Solid harmonics of little groups

In the case $s>0$, we can always choose the rest frame $\boldsymbol{P}=0$. Under this choice of the frame, $s=0$ implies $P_\mu=0$. For more than ten years after its proposal, the B–S equation was always considered in the rest frame. In order to include also the case in which P_μ is lightlike in a unified way, however, it is desirable to deal with the general Lorentz frame in a systematic way. For this purpose, Nakanishi (1965)[N16] introduced the notion of the solid harmonics of little groups.

The B–S amplitudes $\phi_{Br}(x_a, x_b;\ P_B)$, $(r=1, \cdots, n)$, has to form a finite-dimensional*) representation of the Poincaré group (i.e., the inhomogeneous Lorentz group). The construction of representations of the Poincaré group was fully investigated in a classical paper by Wigner.[3] Since the translation group is abelian, it has a one-dimensional representation e^{-iP_BX}. Any Lorentz transformation should not change this expression in a particular representation of the Poincaré group. Let

$$\mathcal{L}(P)\equiv\{\Lambda\,|\,\Lambda\in\mathcal{L},\ P\Lambda=P\},\qquad (4\cdot 1)$$

where $\mathcal{L}$ denotes the (homogeneous) Lorentz group (including inversions); $\mathcal{L}(P)$ is usually called the little group belonging to P. Then, according to (2·15), $\phi_{Br}(x, P_B)$, and therefore $\phi_{Br}(p, P_B)$, can transform only for the elements of $\mathcal{L}(P_B)$. That is to say, $\{\phi_{Br}(p, P_B)\}$ forms a representation of $\mathcal{L}(P_B)$.

The structure of $\mathcal{L}(P)$ depends on P:

[1] $\mathcal{L}(P)\simeq O(3)$ for P_μ timelike,
[2] $\mathcal{L}(P)\simeq O(2,1)$ for P_μ spacelike,
[3] $\mathcal{L}(P)\simeq O(3,1)$ for $P_\mu=0$,
[4] $\mathcal{L}(P)\simeq E(2)$ for P_μ lightlike,

where $O(m,1)$ denotes the totality of the real, linear transformations of $(x_1, \cdots, x_{m+1})$ leaving the quadratic form $\sum_{j=1}^m x_j^2 - x_{m+1}^2$ invariant, and $E(2)$ stands for the two-dimensional Euclidean group, which consists of all two-dimensional translations and rotations (including reflection).

By generalizing the definition of the ordinary solid harmonics, we define the solid harmonics, $X_l(p)$, of a little group $\mathcal{L}(P)$ in the following way: $X_l(p)$ is an l-th order homogeneous polynomial in p_0, p_1, p_2, p_3 and satisfies**)

*) The possibility of an infinite-dimensional representation is excluded from the standpoint of the conventional quantum field theory. See also §5.

**) It is interesting to note that Yukawa's bilocal field theory[4] postulates

$$(x^2+r^2)\boldsymbol{U}=0,$$
$$x_\mu(\partial/\partial X_\mu)\boldsymbol{U}=0.$$

If $r=0$, they are equivalent to (4·2) and (4·3). The existence of $r\neq 0$ violates the homogenuity, and leads the field $\boldsymbol{U}$ to a mixture of various spin states.[5]

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$$(\partial/\partial p)^2 X_l(p)=0 \tag{4·2}$$

and

$$P_\mu(\partial/\partial p_\mu)X_l(p)=0 \tag{4·3}$$

simultaneously. It is easy to see that the totality of $X_l(p)$ for l fixed spans a space of a finite-dimensional, irreducible representation of $\mathcal{L}(P)$.

Since P_μ is a covariant vector while $\partial/\partial p_\mu$ is a contravariant one, it should be noted that*)

$$P_\mu\frac{\partial}{\partial p_\mu}\equiv P_0\frac{\partial}{\partial p_0}+P_1\frac{\partial}{\partial p_1}+P_2\frac{\partial}{\partial p_2}+P_3\frac{\partial}{\partial p_3}. \tag{4·4}$$

Hence the homogenuity condition of $X_l(p)$ can be expressed as an invariant form

$$p_\mu(\partial/\partial p_\mu)X_l(p)=lX_l(p). \tag{4·5}$$

We first construct some standard forms of $X_l(p)$ in particular Lorentz frames.

[1] $s>0$. We take $P_\mu=(\sqrt{s},0,0,0)$. Then (4·3) implies that $X_l(p)$ is independent of p_0. Hence (4·2) reduces to the Laplace equation, that is, the definition of $X_l(p)$ coincides with that of the ordinary solid harmonics $\mathcal{Y}_{lm}(\boldsymbol{p})$. We may identify $\mathcal{Y}_{lm}(\boldsymbol{p})$ with $|\boldsymbol{p}|^l Y_{lm}(\theta,\varphi)$ (apart from a sign factor), where $|\boldsymbol{p}|$, θ, φ denote the polar coordinates of $\boldsymbol{p}$. It is convenient to express $\mathcal{Y}_{lm}(\boldsymbol{p})$ in terms of Gegenbauer polynomial[6] $C_k^\alpha(z)$ (k stands for its degree):

$$\mathcal{Y}_{lm}(\boldsymbol{p})=\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2}(2|m|-1)!!\,(p_1\pm ip_2)^{|m|}$$
$$\times|\boldsymbol{p}|^{l-|m|}C_{l-|m|}^{|m|+1/2}(p_3/|\boldsymbol{p}|), \tag{4·6}$$
$$(m=-l,\,-l+1,\cdots,l)$$

where the double sign means $m/|m|$ and $(2k-1)!!\equiv\Pi_{j=1}^k(2j-1)$. The normalization constant has been calculated by using the orthogonality property of the Gegenbauer polynomial:

$$\int_{-1}^1 dz(1-z^2)^{\alpha-1/2}C_k^\alpha(z)C_{k'}^\alpha(z)=\frac{\pi\Gamma(2\alpha+k)}{2^{2\alpha-1}k!(\alpha+k)[\Gamma(\alpha)]^2}\delta_{kk'}. \tag{4·7}$$

[2] $s<0$. We take $P_\mu=(0,0,0,\sqrt{-s})$. Then (4·3) becomes $(\partial/\partial p_3)\times X_l(p)=0$. Hence the definition of $X_l(p)$ reduces to that in the case [1] if we replace p_0 by ip_3. Thus we may define the standard solid harmonics by

$$\tilde{\mathcal{Y}}_{lm}(p_1,p_2,p_0)\equiv\mathcal{Y}_{lm}(p_1,p_2,-ip_0). \tag{4·8}$$

*) Unfortunately, the sign of the space part of (4·4) was wrong in the original paper.[N16]

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[3] $P_\mu=0$. In this case, (4·3) becomes trivial, and hence we need three quantum numbers L, l, m to specify the solid harmonics, which we call the Lorentz solid harmonics because the little group is identical with the Lorentz group. The standard Lorentz solid harmonics $\mathcal{Z}_{Llm}(p)$ $(l=0, 1, \cdots, L;$ $m=-l, -l+1, \cdots, l)$ are defined as follows:

$$\mathcal{Z}_{Llm}(p_0, \boldsymbol{p}) \equiv \mathcal{H}_{Llm}(-ip_0, \boldsymbol{p}), \tag{4·9}$$

$$\mathcal{H}_{Llm}(p_4, \boldsymbol{p}) \equiv |\tilde{p}|^L H_{Llm}(\alpha, \theta, \varphi) \tag{4·10}$$

with $|\tilde{p}|^2=p_4^2+\boldsymbol{p}^2$ and $\cos\alpha=p_4/|\tilde{p}|$ $(0\leq\alpha\leq\pi$ if p_4 is real), where $H_{Llm}(\alpha, \theta, \varphi)$ is a four-dimensional spherical harmonic defined by

$$H_{Llm}(\alpha, \theta, \varphi) \equiv A_{Ll}(\sin\alpha)^l C_{L-l}^{l+1}(\cos\alpha)\, Y_{lm}(\theta, \varphi). \tag{4·11}$$

The normalization constant A_{Ll} is determined by the requirement

$$\int d\Omega_4 |H_{Llm}(\alpha, \theta, \varphi)|^2=1, \tag{4·12}$$

where $d\Omega_4$ denotes a four-dimensional solid angle element. By means of (4·7) we find

$$|A_{Ll}|^2=2^{2l+1}(L+1)(L-l)!(l!)^2/\pi(L+l+1)!. \tag{4·13}$$

Of course, there are other choices of Lorentz solid harmonics which form a complete (pseudo-orthonormal) set. For example, the following choice[N23)] is convenient for the discussion in connection with the case [4]:

$$\hat{\mathcal{Z}}_{LMm}(p)=\hat{A}_{LM\overline{M}}(p_1\pm ip_2)^{|m|}(p_3-p_0)^M(p_3+p_0)^{\overline{M}}F(-M, -\overline{M}; -L; p^2/(p_0^2-p_3^2)),$$
$$(|m|+M+\overline{M}=L,\ M\geq 0,\ \overline{M}\geq 0). \tag{4·14}$$

Note that the power series expansion of the hypergeometric function appearing in (4·14) contains only $\min(M, \overline{M})+1$ terms. The normalization constant $\hat{A}_{LM\overline{M}}$ may be determined by a requirement similar to (4·12); one finds[N3)]

$$|\hat{A}_{LM\overline{M}}|^2=L!(L+1)!/2\pi^2 M!\overline{M}!(L-M)!(L-\overline{M})!. \tag{4·15}$$

[4] $s=0$ but $P_\mu\neq 0$. We take $P_\mu=(P_0, 0, 0, P_0)$, $(P_0\neq 0)$. Then (4·3) becomes

$$\left(\frac{\partial}{\partial p_0}+\frac{\partial}{\partial p_3}\right)X_l(p)=0. \tag{4·16}$$

Hence (4·2) reduces to

$$\left[\left(\frac{\partial}{\partial p_1}\right)^2+\left(\frac{\partial}{\partial p_2}\right)^2\right]X_l(p)=0. \tag{4·17}$$

Therefore the standard solid harmonics, which we denote by $\chi_{lm}(p)$, $(|m|\leq l)$, are given by

$$\chi_{lm}(p)=a_{lm}(p_1\pm ip_2)^{|m|}(p_3-p_0)^{l-|m|}. \tag{4·18}$$

Since (4·18) is equivalent to the $\overline{M}=0$ case of (4·14), we may put $a_{lm}=\widehat{A}_{l,l-|m|,0}$ (Euclidean normalization). It should be remarked that the factor p_3-p_0 appearing in (4·18) is expressible in terms of invariants:

$$p_3-p_0=(-v+w)/2P_0\,. \tag{4·19}$$

In the above, we have discussed the solid harmonics of little groups in some particular Lorentz frames. It is not convenient, however, to insist on taking special frames if one wants to consider the interrelation between various cases. Hence we next investigate the solid harmonics in an arbitrary Lorentz frame.

Let $P_\mu^{(0)}=(\sqrt{s},0,0,0)$ and P_μ be an arbitrary 4-vector such that $P^2=s>0$. We introduce a Lorentz transformation Λ through

$$P=P^{(0)}\Lambda,\ (\Lambda\in\mathcal{L}) \tag{4·20}$$

and define

$$q\equiv p\Lambda^{-1}. \tag{4·21}$$

Then we can prove that the solid harmonics of $\mathcal{L}(P)$ are given by

$$\mathcal{Y}_{lm}(p,P)\equiv\mathcal{Y}_{lm}(\boldsymbol{q}). \tag{4·22}$$

In fact, $\mathcal{Y}_{lm}(p,P)$ is an l-th order homogeneous polynomial in p_0, p_1, p_2, p_3, and

$$(\partial/\partial p)^2\mathcal{Y}_{lm}(p,P)=(\partial/\partial q)^2\mathcal{Y}_{lm}(\boldsymbol{q})=0, \tag{4·23}$$

$$\begin{aligned}P(\partial/\partial p)\mathcal{Y}_{lm}(p,P)&=P^{(0)}(\partial/\partial q)\mathcal{Y}_{lm}(\boldsymbol{q})\\&=\sqrt{s}\,(\partial/\partial q_0)\mathcal{Y}_{lm}(\boldsymbol{q})=0.\end{aligned} \tag{4·24}$$

We now analytically continue in s the solid harmonics $\mathcal{Y}_{lm}(p,P)$ multiplied by a certain function of s. Then we can discuss the cases [1], [2] and [4] in a unified way.

For example, let

$$\Lambda=\begin{pmatrix}\alpha & 0 & 0 & \beta\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \beta & 0 & 0 & \alpha\end{pmatrix} \tag{4·25}$$

with

$$\alpha=\frac{a+a^{-1}s}{2\sqrt{s}},\quad \beta=\frac{a-a^{-1}s}{2\sqrt{s}},\quad (a\neq 0) \tag{4·26}$$

so that

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$$P_\mu=\left(\frac{1}{2}(a+a^{-1}s),\,0,\,0,\,\frac{1}{2}(a-a^{-1}s)\right). \tag{4·27}$$

Then

$$q_1=p_1\,,\quad q_2=p_2\,,\quad q_3=\alpha p_3-\beta p_0\,. \tag{4·28}$$

From (4·22) together with (4·6) and (4·28), we find the explicit expression for $\mathcal{Y}_{lm}(p,P)$. We consider the $s\to 0$ limit of $\mathcal{Y}_{lm}(p,P)$ multiplied by $s^{(l-|m|)/2}$ in order to avoid divergence. Then

$$\lim_{s\to 0} s^{(l-|m|)/2}\mathcal{Y}_{lm}(p,P)=\text{const}(p_1\pm ip_2)^{|m|}(p_3-p_0)^{l-|m|}, \tag{4·29}$$

that is, we obtain $\chi_{lm}(p)$ as it should.

Finally, we prove the self-reproducing property and the orthogonality of $\mathcal{Y}_{lm}(p,P)$, which are important in the application to the B–S equation.

First, we note

$$\int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\varphi' f(\cos\omega)\, Y_{lm}(\theta',\varphi')=h\cdot Y_{lm}(\theta,\varphi), \tag{4·30}$$

where $f(z)$ is an arbitrary continuous function, h being a certain constant, and

$$\cos\omega=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi'). \tag{4·31}$$

The proof of (4·30) is done by expanding $f(\cos\omega)$ into a series of the Legendre polynomials $P_l(\cos\omega)$ and by making use of the addition theorem

$$P_l(\cos\omega)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)\,Y_{lm}^*(\theta',\varphi') \tag{4·32}$$

together with the orthogonality of $Y_{lm}(\theta,\varphi)$.

Since $\boldsymbol{pp}'=|\boldsymbol{p}|\,|\boldsymbol{p}'|\cos\omega$, if $f(\boldsymbol{p},\boldsymbol{p}')$ is an arbitrary, sufficiently decreasing, continuous function of scalar products $\boldsymbol{p}^2$, $\boldsymbol{p}'^2$ and $\boldsymbol{pp}'$ alone, we can rewrite (4·30) as

$$\int d^3\boldsymbol{p}' f(\boldsymbol{p},\boldsymbol{p}')\mathcal{Y}_{lm}(\boldsymbol{p}')=h(\boldsymbol{p}^2)\mathcal{Y}_{lm}(\boldsymbol{p}). \tag{4·33}$$

Let $F(p,p',P^{(0)})$ be an arbitrary, sufficiently decreasing, Feynman-type distribution of the invariants formed out of p, p' and $P^{(0)}$. Then (4·33) implies

$$\int d^4p' F(p,p',P^{(0)})\mathcal{Y}_{lm}(\boldsymbol{p}')=H(p,P^{(0)})\mathcal{Y}_{lm}(\boldsymbol{p}), \tag{4·34}$$

because $pP^{(0)}=\sqrt{s}\,p_0$ and $p'P^{(0)}=\sqrt{s}\,p_0'$, where $H(p,P^{(0)})$ depends only on p^2, $pP^{(0)}$ and s. We may merely replace the variables p and p' by q and q', respectively:

$$\int d^4q' F(q,q',P^{(0)})\mathcal{Y}_{lm}(\boldsymbol{q}')=H(q,P^{(0)})\mathcal{Y}_{lm}(\boldsymbol{q}). \tag{4·35}$$

Because of (4·21) and (4·22) we have

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$$\int d^4p' F(p\Lambda^{-1}, p'\Lambda^{-1}, P^{(0)}) \mathcal{Y}_{lm}(p', P) = H(p\Lambda^{-1}, P^{(0)}) \mathcal{Y}_{lm}(p, P). \quad (4\cdot36)$$

The Lorentz invariance of F and H allows us to rewrite (4·36) as

$$\int d^4p' F(p, p', P) \mathcal{Y}_{lm}(p', P) = H(p, P) \mathcal{Y}_{lm}(p, P). \quad (4\cdot37)$$

We have thus obtained the self-reproducing property of $\mathcal{Y}_{lm}(p, P)$; by analytic continuation in s we see that (4·37) holds for any of the cases [1], [2] and [4].

By means of the same technique, we can also prove the orthogonality of $\mathcal{Y}_{lm}(p, P)$,

$$\int d^4p F(p, P) \mathcal{Y}_{lm}(p, P) \mathcal{Y}^*_{l'm'}(p, P) = H(s) \delta_{ll'} \delta_{mm'}, \quad (4\cdot38)$$

where $F(p, P)$ is an arbitrary, sufficiently decreasing, invariant distribution of p and P.

The self-reproducing property and the orthogonality of $\mathcal{Z}_{Llm}(p)$ immediately follow from those of the four-dimensional spherical harmonics.

In order to treat particles with spin in the helicity formalism, one needs to consider the so-called generalized spherical harmonics instead of Y_{lm}. The corresponding generalization of $\mathcal{Y}_{lm}(p, P)$ is important, but it is not yet made.

§5. Wick rotation

Since the B–S kernel contains the singularities of the Feynman propagators, the standard mathematical theorems can hardly be applied directly to the B–S equation. Wick (1954)[W5)] found a method of overcoming this difficulty. Under the stability conditions of the constituent particles and the bound state, he showed that one can bring the contour of the relative energy to its imaginary axis so that the new kernel becomes of Euclidean metric. This procedure is called the "Wick rotation", which is an unhappy name because the word "rotation" gave rise to much confusion (see the end of this section). Kemmer and Salam (1955)[K8)] extended the Wick rotation to the scattering B–S equation in the elastic region. Tiktopoulos (1964)[T4)] showed that it can be transformed into a Euclidean form by considering its on-the-mass-shell iterative solution. Recently, the Wick rotation was reconsidered in order to solve the scattering B–S equation numerically. Schwartz and Zemach (1966)[S10)] discussed it in the position space. Pagnamenta and Taylor (1966)[P2)] and Saenger (1967)[S1)] investigated what singularities remain unremoved*) by the Wick rotation.

*) Graves-Morris (1966)[G10)] and Levine, Tjon and Wright (1966)[L6)] considered the removal of singularities by the method of subtractions. See also Taylor (1963)[T1)] and Broido and Taylor (1969).[B18)]

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We consider a Feynman amplitude

$$\boldsymbol{\phi}(x,P)\equiv\langle 0|\mathrm{T}[\boldsymbol{\varphi}_a(\eta_b x)\boldsymbol{\varphi}_b(-\eta_a x)]|P\rangle, \tag{5·1}$$

and its conjugate

$$\bar{\boldsymbol{\phi}}(x,P)\equiv\langle 0|\overline{\mathrm{T}}[\boldsymbol{\varphi}_a(\eta_b x)\boldsymbol{\varphi}_b(-\eta_a x)]|P\rangle^*, \tag{5·2}$$

where $|P\rangle$ denotes an arbitrary eigenstate of the total 4-momentum. Because of (2·14) and (2·15), if $|P\rangle=|B,r\rangle$ then $\boldsymbol{\phi}(x,P)$ and $\bar{\boldsymbol{\phi}}(x,P)$ coincide with $\boldsymbol{\phi}_{Br}(x,P_B)$ and $\bar{\boldsymbol{\phi}}_{Br}(x,P_B)$, respectively, apart from a constant factor.

Let

$$\begin{aligned}\boldsymbol{f}(x,P)&\equiv\langle 0|\boldsymbol{\varphi}_a(\eta_b x)\boldsymbol{\varphi}_b(-\eta_a x)|P\rangle,\\ \boldsymbol{g}(x,P)&\equiv\langle 0|\boldsymbol{\varphi}_b(-\eta_a x)\boldsymbol{\varphi}_a(\eta_b x)|P\rangle.\end{aligned} \tag{5·3}$$

Then (5·1) and (5·2) are rewritten as

$$\begin{aligned}\boldsymbol{\phi}(x,P)&=\theta(x_0)\boldsymbol{f}(x,P)+\theta(-x_0)\boldsymbol{g}(x,P),\\ \bar{\boldsymbol{\phi}}(x,P)&=\theta(x_0)[\boldsymbol{g}(x,P)]^*+\theta(-x_0)[\boldsymbol{f}(x,P)]^*.\end{aligned} \tag{5·4}$$

By using the definitions

$$\begin{aligned}\boldsymbol{\phi}(x,P)&=(2\pi)^{-4}\int d^4p e^{-ipx}\phi(p,P),\\ \bar{\boldsymbol{\phi}}(x,P)&=(2\pi)^{-4}\int d^4p e^{ipx}\bar{\phi}(p,P),\end{aligned} \tag{5·5}$$

etc. and the identity (2·19), i.e.,

$$\theta(x_0)=-(2\pi i)^{-1}\int d^4k\, e^{-ikx}\delta^3(\boldsymbol{k})(k_0+i\varepsilon)^{-1}, \tag{5·6}$$

(5·4) is transcribed into the momentum space:

$$\begin{aligned}\phi(p,P)&=\frac{-1}{2\pi i}\int dq_0\frac{f(q_0,\boldsymbol{p},P)}{p_0-q_0+i\varepsilon}+\frac{1}{2\pi i}\int dq_0\frac{g(q_0,\boldsymbol{p},P)}{p_0-q_0-i\varepsilon},\\ \bar{\phi}(p,P)&=\frac{-1}{2\pi i}\int dq_0\frac{[f(q_0,\boldsymbol{p},P)]^*}{p_0-q_0+i\varepsilon}+\frac{1}{2\pi i}\int dq_0\frac{[g(q_0,\boldsymbol{p},P)]^*}{p_0-q_0-i\varepsilon}.\end{aligned} \tag{5·7}$$

The formulas (5·7) present the relation between a Feynman amplitude and its conjugate. That is to say, the absorptive part of $\bar{\phi}$ is equal to the complex conjugate of that of ϕ, and the dispersive part of $\bar{\phi}$ is related to the absorptive part of $\bar{\phi}$ in exactly the same way as the dispersive part of ϕ is related to the absorptive part of ϕ.

Our next task is to find the support properties of $f(p,P)$ and $g(p,P)$. We insert a complete set of states $|N\rangle$ into (5·3). We then have*)

*) We may introduce a norm factor into the summation in (5·8) if necessary.

$$\begin{aligned} f(x,P) &= \sum_N \langle 0|\boldsymbol{\varphi}_a(\eta_b x)|N\rangle\langle N|\boldsymbol{\varphi}_b(-\eta_a x)|P\rangle \\ &= \int d^3\boldsymbol{p}_N[2(p_N)_0]^{-1} \sum_{p=-\eta_a P+p_N} \langle 0|\boldsymbol{\varphi}_a(0)|N\rangle\langle N|\boldsymbol{\varphi}_b(0)|P\rangle e^{-ipx}, \end{aligned} \tag{5·8}$$

where p_N denotes the 4-momentum of the state $|N\rangle$. Since the particle a cannot decay into any state spontaneously, we have

$$\langle 0|\boldsymbol{\varphi}_a(0)|N\rangle = 0 \quad \text{unless } p_N^2 \geq m_a^2,\ (p_N)_0 > 0. \tag{5·9}$$

Thus

$$f(p,P) = 0 \quad \text{unless } (\eta_a P+p)^2 \geq m_a^2,\ \eta_a P_0 + p_0 > 0. \tag{5·10}$$

Likewise, the stability condititon of the particle b leads us to

$$g(p,P) = 0 \quad \text{unless } (\eta_b P-p)^2 \geq m_b^2,\ \eta_b P_0 + p_0 > 0. \tag{5·11}$$

That is to say, in (5·7) we have

$$\begin{aligned} f(q_0,\boldsymbol{p},P) &= 0 \quad \text{unless } q_0 \geq \omega_{\min}, \\ g(q_0,\boldsymbol{p},P) &= 0 \quad \text{unless } q_0 \leq \omega_{\max}, \end{aligned} \tag{5·12}$$

where

$$\begin{aligned} \omega_{\min} &\equiv [m_a^2 + (\eta_a\boldsymbol{P}+\boldsymbol{p})^2]^{1/2} - \eta_a P_0, \\ \omega_{\max} &\equiv \eta_b P_0 - [m_b^2 + (\eta_b\boldsymbol{P}-\boldsymbol{p})^2]^{1/2}. \end{aligned} \tag{5·13}$$

If either $\omega_{\min} \leq 0$ or $\omega_{\max} \geq 0$ happens, we have to encounter a displaced pole*) (or cut) in the Wick rotation. To avoid this unpleasant situation, it is necessary and sufficient to have

$$|P_0| < \min(m_a/|\eta_a|,\ m_b/|\eta_b|). \tag{5·14}$$

If we consider the bound-state problem, that is, if $\phi(p,P)$ is identified with $\phi_{Br}(p,P)$, then the stability condition,

$$m_a + m_b > \sqrt{s}, \tag{5·15}$$

of the bound state implies $\omega_{\min} > \omega_{\max}$. Hence in (5·7) there is a gap between two cuts in the p_0 plane. For any value of s satisfying (5·15), we have both $\omega_{\min} > 0$ and $\omega_{\max} < 0$ if $\boldsymbol{P} = 0$ and if we choose

$$\eta_a = m_a/(m_a+m_b),\quad \eta_b = m_b/(m_a+m_b). \tag{5·16}$$

For the scattering problem, however, since

$$\sqrt{s} > m_a + m_b, \tag{5·17}$$

we have no gap, and hence at least one displaced singularity is necessarily encountered.

*) This terminology is due to Dyson.[7]

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Now, we discuss the Wick rotation. First, we consider the bound-state problem in the ladder approximation.*) For simplicity, we suppose that all particles are scalar**) and that P_μ is timelike. Then, in the rest frame, we have

$$[m_a^2+\boldsymbol{p}^2-(\eta_a P_0+p_0)^2]\,[m_b^2+\boldsymbol{p}^2-(\eta_b P_0-p_0)^2]\,\phi_{Br}(p,P)$$
$$=\frac{\lambda_B(s)}{\pi^2 i}\int d^4p'\frac{\phi_{Br}(p',P)}{\mu^2-(p-p')^2-i\varepsilon}\,, \qquad (5\cdot18)$$

where η_a and η_b are given by (5·16) and μ stands for the exchanged meson mass.

The analyticity of $\phi_{Br}(p',P)$ implied by (5·7) together with (5·12) yields

$$\int_C dp_0'\frac{\phi_{Br}(p_0',\boldsymbol{p}',P)}{\mu^2+(\boldsymbol{p}-\boldsymbol{p}')^2-(p_0-p_0')^2-i\varepsilon}=0, \qquad (5\cdot19)$$

where the contour C is shown in Fig. 3. The contribution from the two quarter circles will tend to zero because of the asymptotic behavior implied by (5·7) [see also §7(C)]. Therefore the integral over the real axis can be replaced by that over the imaginary axis plus the contribution from a possible displaced pole which appears if either

$$p_0+[\mu^2+(\boldsymbol{p}-\boldsymbol{p}')^2]^{1/2}<0 \qquad (5\cdot20)$$

or

$$p_0-[\mu^2+(\boldsymbol{p}-\boldsymbol{p}')^2]^{1/2}>0 \qquad (5\cdot21)$$

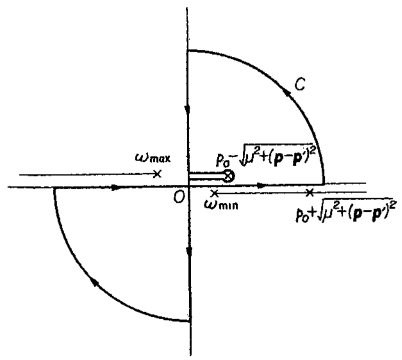

Fig. 3. The contour C in the p_0' plane [in the case (5·21)].

for p_0 real. If we rotate p_0 counterclockwise to the imaginary axis, however, the displaced pole moves into the fourth quadrant [for (5·20)] or into the second one [for (5·21)], whence the p_0' integration becomes that over the imaginary axis alone. Therefore after the above analytic continuation in p_0, which is permissible also in the left-hand side of (5·18) because of (5·7) together with (5·12) again, (5·18) is transformed into

$$[m_a^2+\boldsymbol{p}^2+(p_4-i\eta_a P_0)^2]\,[m_b^2+\boldsymbol{p}^2+(p_4+i\eta_b P_0)^2]\,\tilde{\phi}_{Br}(\tilde{p},P)$$
$$=\frac{\lambda_B(s)}{\pi^2}\int d^4\tilde{p}'\frac{\tilde{\phi}_{Br}(\tilde{p}',P)}{\mu^2+(\tilde{p}-\tilde{p}')^2}\,, \qquad (5\cdot22)$$

---

*) We assume that the analyticity obtained above is not injured by taking the ladder approximation.

**) Extension to the other cases is straightforward, but we have to be careful about the contribution from the two quarter circles in (5·19).

where $\tilde{p}\equiv(\boldsymbol{p}, p_4)$ is a Euclidean 4-vector and $\tilde{\phi}_{Br}(\tilde{p}, P)$ stands for the continued B–S amplitude. We write (5·22) as

$$\widetilde{K}\tilde{\phi}_{Br}=\lambda_B\widetilde{I}\tilde{\phi}_{Br}\,, \tag{5·23}$$

where

$$\widetilde{K}(\tilde{p}, P)\equiv -K(ip_4, \boldsymbol{p}, P). \tag{5·24}$$

In the equal-mass case ($m_a=m_b\equiv m$), by choosing $\eta_a=\eta_b=1/2$, $\widetilde{K}$ becomes particularly simple:

$$\widetilde{K}(\tilde{p}, P)=\left(m^2+\tilde{p}^2-\frac{1}{4}s\right)^2+p_4^2 s, \tag{5·25}$$

that is, it is real and positive definite. Let

$$\hat{\phi}_{Br}\equiv\widetilde{K}^{1/2}\tilde{\phi}_{Br}; \tag{5·26}$$

then (5·23) is rewritten as

$$\widetilde{K}^{-1/2}\widetilde{I}\widetilde{K}^{-1/2}\hat{\phi}_{Br}=\lambda_B^{-1}\hat{\phi}_{Br}\,. \tag{5·27}$$

In our scalar-scalar model, since the operator $\widetilde{K}^{-1/2}\widetilde{I}\widetilde{K}^{-1/2}$ is of the Hilbert-Schmidt type, we can apply standard mathematical theorems to (5·27). For example, all eigenvalues λ_B are discrete and positive, and they do not accumulate at any finite point; the dimension of degeneracy is finite (therefore an infinite-dimensional representation of little group is excluded); the eigenfunctions $\hat{\phi}_{Br}$ form a complete, orthogonal set in the Hilbert space of the square-integrable functions.

Next, we consider the Wick rotation for the scattering problem in the ladder approximation, in which we encounter displaced poles and cuts in general. The Feynman amplitude $F(p, P)$ satisfies

$$[m_a^2-(\eta_a P+p)^2]\,[m_b^2-(\eta_b P-p)^2]\,F(p, P)$$
$$=\frac{\lambda}{\pi^2 i}\left[\frac{1}{\mu^2-(p-q)^2-i\varepsilon}+\int d^4p'\frac{F(p', P)}{\mu^2-(p-p')^2-i\varepsilon}\right]. \tag{5·28}$$

From the physical consideration (or the perturbation expansion), $F(p', P)$ should have the following singularities. It has a sequence of the right-hand singularities located at

$$p_0'=[(m_a+n\mu)^2+\boldsymbol{p}'^2]^{1/2}-\eta_a\sqrt{s}-i\varepsilon,\quad (n=0, 1, 2, \cdots) \tag{5·29}$$

and that of the left-hand ones located at

$$p_0'=\eta_b\sqrt{s}-[(m_b+n'\mu)^2+\boldsymbol{p}'^2]^{1/2}+i\varepsilon,\quad (n'=0, 1, 2, \cdots) \tag{5·30}$$

where those for $n=0$ and for $n'=0$ are poles and all the others are branch

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points. We have also two poles from the propagator of the exchanged meson, but as shown above they are harmless when p_0 is rotated, namely they are removed by the Wick rotation. After the integration over p_0' is carried out, the singularity will remain if and only if one of the right-hand singularities coincides with one of the left-hand ones (then we have a "pinch"). That is to say, after the Wick rotation there still remain the singularities for

$$\sqrt{s}=[(m_a+n\mu)^2+\boldsymbol{p}'^2]^{1/2}+[(m_b+n'\mu)^2+\boldsymbol{p}'^2]^{1/2}. \qquad (n, n'=0, 1, 2, \cdots) \tag{5·31}$$

In particular, in the elastic region $m_a+m_b\leq\sqrt{s}\leq m_a+m_b+\mu$, we encounter only one singularity for

$$\sqrt{s}=\sqrt{m_a^2+\boldsymbol{p}'^2}+\sqrt{m_b^2+\boldsymbol{p}'^2}\,. \tag{5·32}$$

One has to be careful about those unremoved singularities in numerical calculations.

Finally, we make some remarks on the validity of the Wick rotation in order to avoid its possible misuses.

1) One cannot prove the possibility of the Wick rotation for the scattering Green's function (with q_μ arbitrary).

2) The Wick rotation has been verified only in the *physical* region. That is to say, it is applicable if $0\leq s<(m_a+m_b)^2$ for the bound-state problem or if $s\geq(m_a+m_b)^2$ for the scattering problem. Especially, we should not apply the Wick rotation to (5·28) for

$$(m_a-m_b)^2<s<(m_a+m_b)^2, \tag{5·33}$$

for which q_μ is no longer a real Minkowski vector.

3) The Wick rotation has been verified only for the *single* p_0' integral, and hence it is rigorously applicable only to the ladder approximation. The so-called "simultaneous rotation"[W5] of the contours in a *multiple* integral cannot be proved by the use of Cauchy's theorem without making an unjustified transformation of integration variables. If one wants to apply the Wick rotation to a higher-order kernel, one has to investigate its analyticity in several complex variables.

4) The possibility of the Wick rotation in the unphysical region can be shown to some extent if we assume the perturbation-theoretical integral representations (PTIR),[8] which are believed to be valid generally and can be proved in certain cases. The B–S amplitude will be represented as

$$\int_{-1}^{1}dz\int_0^\infty d\gamma\frac{\varphi(z,\gamma;\,p,P)}{[\gamma+\frac{1}{2}(1+z)(m_a^2-v)+\frac{1}{2}(1-z)(m_b^2-w)-i\varepsilon]^2}\,, \tag{5·34}$$

where φ is polynomially dependent on p_μ. We obtain the p_0 analyticity

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necessary for the Wick rotation without encountering any displaced singularity if the denominator function in (5·34) is positive definite (apart from $-i\varepsilon$) at $p_0=0$, that is, if (5·14) holds. Here it should be noted that we have no longer the physical restriction $s\geq 0$. For the scattering Green's function, since the denominator function should be replaced by

$$\gamma+x_1(m_a^2-v)+x_2(m_b^2-w)+x_3(m_a^2-v_0)+x_4(m_b^2-w_0)+x_5(\mu^2-t)-i\varepsilon \qquad (5\cdot 35)$$

with $\gamma\geq 0$, $x_j\geq 0$, $(j=1,\cdots,5)$, $(\sum_j x_j=1)$, the required p_0 analyticity of $G-K^{-1}$ is obtained if (5·14) holds and if $v_0<m_a^2$, $w_0<m_b^2$ and $q_0^2<\mu^2$ for q_μ real.

§6. Wick-Cutkosky model

In this section, we consider the B–S equation (5·18) in the ladder approximation for two scalar particles which exchange massless scalar mesons ($\mu=0$). This model is particularly interesting because it is the only example of the non-trivial, relativistic B–S equation which is exactly solvable even for $P_\mu\neq 0$.

The Wick-Cutkosky model was first investigated by Hayashi and Munakata (1952),[H4] who, however, used a modified integral kernel. The exact solvability of this model was first suggested by Wick (1954).[W5] By means of the Wick rotation (see §5), he showed the existence of discrete energy levels, and proposed the method of an integral representation, by which the eigenvalue problem was reduced to an ordinary differential equation. He further discovered the existence of abnormal solutions (see §8) which have no non-relativistic counterparts. Cutkosky (1954)[C17] continued Wick's analysis and presented a complete set of solutions for $s>0$ explicitly in the equal-mass case and implicitly in the unequal-mass case. To do this, in addition to the integral representation, he introduced the stereographic projection method to the Wick-rotated B–S equation and found the $O(4)$ symmetry*) of the Wick-Cutkosky model for $s\neq 0$. Scarf (1955)[S7] analyzed the weight function, $g_{\kappa n}(z,s)$, of the integral representation in terms of Heun's function[9] and opposed the conclusion of Wick and Cutkosky that as the binding energy goes to zero the eigenvalues of λ for abnormal solutions tend to 1/4 instead of zero, but unfortunately his objection was wrong.[G2),C18] Green (1957)[G12] showed that the B–S equation becomes completely separable by introducing the bipolar coordinates. Nakanishi (1965, 1966)[N16),N20] found some peculiar features (see below) of the solutions for P_μ lightlike. Nakanishi (1967)[N23] also presented complete sets of solutions in the unequal-mass case not only for $s\neq 0$ but also for P_μ lightlike by means of the integral representation (i.e., without using the Wick rotation). Seto (1968, 1969)[S13),S15] refined the

*) See also, Delbourgo, Salam and Strathdee (1967)[D2] and Biswas (1967).[B13]

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stereographic projection method in the unequal-mass case and elegantly obtained all solutions in a unified way. Kyriakopoulos (1968)[K17] investigated the dynamical group of the Wick-Cutkosky model.

Some modified models were investigated by various authors. Sugano and Munakata (1956)[S18] applied the stereographic projection method to the spinor-scalar model. Bastai et al. (1963)[B7),B8] investigated a model corresponding to a potential more singular than the Coulomb potential. Okubo and Feldman (1960)[O4] considered the model in which the constituent particles can annihilate into a meson.

The scattering B–S equation of the Wick-Cutkosky model was first considered by Nishijima (1955),[N37] but he could not find a solution because of the infrared-divergence difficulty of the Feynman amplitude. Okubo and Feldman (1961)[O5] avoided this difficulty by a cutoff and presented an approximate solution near the elastic threshold. Nakanishi (1964)[N12] found an exact solution for P_μ lightlike in compact form by introducing a particular mass into the inhomogeneous term to avoid the infrared divergence. Furthermore, Nakanishi (1964)[N13),N14] obtained the asymptotic expansion as $t \to \infty$ of the solution for s general. Seto (1968)[S14] showed that the scattering Green's function can be obtained exactly by the stereographic projection method. Green and Biswas (1968)[G15] made a similar consideration by means of the bipolar transformation. The Reggeized Wick-Cutkosky model was investigated by a number of authors (see §12).

(A) Eigenvalues

The eigenvalues of the Wick-Cutkosky model are not split by the angular-momentum quantum number l, owing to the special character of the Coulomb force. They are specified by two quantum numbers κ and n; $n(=l+1, l+2, \cdots)$ is the principal quantum number, while $\kappa(=0, 1, \cdots)$ is a new quantum number which has no non-relativistic counterpart. The eigenvalues $\lambda_{\kappa n}(s)$ are determined by an integral equation

$$g_{\kappa n}(z, s) = \frac{\lambda_{\kappa n}(s)}{2n} \int_{-1}^{1} dz' [R(z, z')]^n \frac{g_{\kappa n}(z', s)}{\rho(z', s)}, \tag{6·1}$$

where

$$R(z, z') \equiv \frac{1-z}{1-z'}\theta(z-z') + \frac{1+z}{1+z'}\theta(z'-z), \tag{6·2}$$

$$\rho(z, s) \equiv \frac{1}{2}(1+z)m_a^2 + \frac{1}{2}(1-z)m_b^2 - \frac{1}{4}(1-z^2)s. \tag{6·3}$$

Note*)

$$0 \le R(z, z') \le 1 \quad \text{for } |z| \le 1,\ |z'| \le 1, \tag{6·4}$$

$$\rho(z, s) > 0 \quad \text{for } |z| \le 1,\ s < (m_a + m_b)^2. \tag{6·5}$$

*) We assume $m_a > 0$ and $m_b > 0$.

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In the treatment of the Wick-Cutkosky model, it is convenient to set

$$m_a=1+\varDelta,\ m_b=1-\varDelta,\ (|\varDelta|<1). \tag{6·6}$$

Then (6·3) is rewritten as

$$\rho(z,s)=1+2\varDelta z+\varDelta^2-\frac{1}{4}s(1-z^2). \tag{6·7}$$

Now, by means of an identity

$$D_n(z)[R(z,z')]^n=-2n\delta(z-z'), \tag{6·8}$$

where

$$D_n(z)\equiv(1-z^2)(d/dz)^2+2(n-1)z(d/dz)-n(n-1), \tag{6·9}$$

(6·1) is transformed into a differential equation

$$[D_n(z)+\lambda_{\kappa n}(s)/\rho(z,s)]\,g_{\kappa n}(z,s)=0 \tag{6·10}$$

with boundary conditions

$$g_{\kappa n}(\pm1,s)=0. \tag{6·11}$$

As is easily checked, the eigenvalue problem in the unequal-mass case ($\varDelta\neq0$) is reduced to that in the equal-mass case ($\varDelta=0$) by the Wick-Cutkosky transformation

$$\begin{gathered}\hat{z}=\frac{z+\varDelta}{1+\varDelta z},\qquad \hat{s}=\frac{s-4\varDelta^2}{1-\varDelta^2},\\ \hat{g}_{\kappa n}(\hat{z},\hat{s})=(1-\varDelta\hat{z})^n g_{\kappa n}(z,s),\quad \hat{\lambda}_{\kappa n}(\hat{s})=\frac{\lambda_{\kappa n}(s)}{1-\varDelta^2}\,.\end{gathered} \tag{6·12}$$

Hence, without loss of generality we may confine ourselves to considering the equal-mass case alone,*) as long as we are concerned with the eigenvalues.

We rewrite (6·10) (with $\varDelta=0$**)) into the Strum-Liouville form:

$$\left\{\frac{d}{dz}(1-z^2)^{-n+1}\frac{d}{dz}+(1-z^2)^{-n+1}\left[-n(n-1)+\frac{\lambda_{\kappa n}(s)}{1-\frac{1}{4}s(1-z^2)}\right]\right\}g_{\kappa n}(z,s)=0. \tag{6·13}$$

Unfortunately, nobody has succeeded in finding an analytic expression for $\lambda_{\kappa n}(s)$, but we can obtain its properties in some detail. The quantum number

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*) It is interesting to note that the eigenvalue problem in the equal-mass case for $\hat{s}\neq0$ is further reduced to that in the unequal-mass case for $s=0$ [$\varDelta^2=-\hat{s}/(4-\hat{s})$] by an inverse Wick-Cutkosky transformation. Then we have Heun's equation.[9]

**) All the formulas given below in (A) hold also in the unequal-mass case if we affix the hat (^) to s, λ and $g_{\kappa n}$.

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κ indicates the number of zero points of $g_{\kappa n}(z,s)$ in the open interval $-1<z<+1$, and

$$g_{\kappa n}(-z,s)=(-1)^{\kappa}g_{\kappa n}(z,s). \qquad (6\cdot14)$$

Then it is well known that

$$0<\lambda_{0n}(s)<\lambda_{1n}(s)<\cdots \qquad \text{for } s<4. \qquad (6\cdot15)$$

Furthermore, we have an important property

$$\lambda'_{\kappa n}(s)\equiv(d/ds)\lambda_{\kappa n}(s)<0 \quad \text{for } s<4, \qquad (6\cdot16)$$

which can be proved in the following way. Differentiate $(6\cdot13)$ with respect to s, and integrate the resulting expression multiplied by $g_{\kappa n}(z,s)$ over z from -1 to $+1$. Integrating by parts, we find

$$\int_{-1}^{1}dz\frac{\partial}{\partial s}\left[\frac{\lambda_{\kappa n}(s)}{1-\frac{1}{4}s(1-z^2)}\right]\cdot[g_{\kappa n}(z,s)]^2=0. \qquad (6\cdot17)$$

Therefore

$$\lambda'_{\kappa n}(s)=-\lambda_{\kappa n}(s)\cdot\frac{\displaystyle\int_{-1}^{1}dz[g_{\kappa n}(z,s)]^2\frac{\frac{1}{4}(1-z^2)}{[1-\frac{1}{4}s(1-z^2)]^2}}{\displaystyle\int_{-1}^{1}dz[g_{\kappa n}(z,s)]^2/[1-\frac{1}{4}s(1-z^2)]}<0. \qquad (6\cdot18)$$

According to Mercer's theorem,[10] from $(6\cdot1)$ with $\varDelta=0$ we have

$$\sum_{\kappa=0}^{\infty}\frac{1}{\lambda_{\kappa n}(s)}=\frac{1}{2n}\int_{-1}^{1}\frac{dz}{1-\frac{1}{4}s(1-z^2)}=\frac{4}{n\sqrt{s(4-s)}}\,\mathrm{Tan}^{-1}\sqrt{\frac{s}{4-s}}\,. \qquad (6\cdot19)^{*)}$$

From $(6\cdot19)$ we find

$$\lim_{s\to-\infty}\lambda_{\kappa n}(s)=+\infty. \qquad (6\cdot20)$$

Numerical values of $\lambda_{\kappa n}(s)$ for $0\leq s<4$ and $n+\kappa\leq3$ are given in Cutkosky's paper[C17] (see also Linden[L13]).

For $s=0$, the eigenvalues are more degenerate. In this case, we have exact solutions**)

$$\lambda_{\kappa n}(0)=(\kappa+n)(\kappa+n+1). \qquad (6\cdot21)$$

$$g_{\kappa n}(z,0)=\mathrm{const}(1-z^2)^n C_{\kappa}^{n+1/2}(z); \qquad (6\cdot22)$$

hence from $(6\cdot18)$[N16]

*) The last expression should be used for $0<s<4$.

**) If one imposes a restriction that no $s<0$ solution should exist because of the stability of the vacuum, from $(6\cdot21)$ one has an upper bound $\lambda=2$ on the value of λ in the equal-mass case.[M2]

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$$\lambda'_{\kappa n}(0) = -\lambda_{\kappa n}(0)\frac{(\kappa+n)(\kappa+n+1)+n^2-1}{2(2\kappa+2n-1)(2\kappa+2n+3)}. \tag{6·23}$$

[In the unequal-mass case, this special degeneracy happens at the pseudo-threshold $s=4\Delta^2$ (see (6·12)) but not at $s=0$.]

Near the elastic threshold ($s\simeq 4$), it is convenient to set $E=2-\sqrt{s}$, where E is the binding energy. For $\kappa=0$, $\lambda_{0n}(s)=O(\sqrt{E})$ and hence

$$\left[1-\frac{1}{4}s(1-z^2)\right]^{-1}\simeq(\pi/\sqrt{E})\delta(z); \tag{6·24}$$

accordingly,[C17)]

$$\lambda_{0n}(s)\simeq(2n/\pi)\sqrt{E}, \tag{6·25}$$

$$g_{0n}(z, s)\simeq\text{const}(1-|z|)^n. \tag{6·26}$$

The formula (6·25) reproduces the n dependence of the energy levels of the hydrogen atom in the non-relativistic theory. On the other hand, for $\kappa\gtrsim 1$ we have[C17)]

$$\lambda_{\kappa n}(s)\simeq\frac{1}{4}+\frac{\pi^2(\kappa-1)^2}{[\log(1-\frac{1}{4}s)]^2}, \tag{6·27}$$

$$g_{\kappa n}(z, s)\simeq(1-z^2)^n z^\nu F\left(\frac{1}{2}(\nu+n+1), \frac{1}{2}(\nu+n); n+1; 1-z^2\right) \quad \text{for } z>0 \tag{6·28}$$

with $\nu\equiv\frac{1}{2}+\sqrt{\frac{1}{4}-\lambda_{\kappa n}}$. Since the solutions with $\kappa\gtrsim 1$ have no non-relativistic limit, they are called abnormal solutions.

(B) Methods of finding solutions

For $4>s>0$, we make an ansatz that the B–S amplitude $\phi_{\kappa nlm}(p, P)$ with $P_\mu=(\sqrt{s}, 0, 0, 0)$ has the following integral representation:

$$\phi_{\kappa nlm}(p, P) = -i\mathcal{Y}_{lm}(\boldsymbol{p})\sum_{j=0}^{n-l-1}\int_{-1}^{1}dz\frac{g^j_{\kappa nl}(z, s)}{[f(z, v, w)-i\varepsilon]^{n-j+2}} \tag{6·29}$$

with

$$f(z, v, w)\equiv\frac{1}{2}(1+z)[(1+\Delta)^2-v]+\frac{1}{2}(1-z)[(1-\Delta)^2-w]. \tag{6·30}$$

On substituting (6·29) in the B–S equation

$$[(1+\Delta)^2-v][(1-\Delta)^2-w]\phi_{\kappa nlm}(p, P) = \frac{\lambda_{\kappa n}(s)}{\pi^2 i}\int d^4p'\frac{\phi_{\kappa nlm}(p', P)}{-(p-p')^2-i\varepsilon}, \tag{6·31}$$

we obtain a system of integral equations for $g^j_{\kappa nl}(z, s)$, which are converted into a system of differential equations

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$$D_{n-j}(z)g^j_{\kappa nl}(z,s)$$
$$=-\lambda_{\kappa n}(s)\sum_{j'=0}^{j}\frac{(n-j+1)!(n-l-1-j')!}{(n-j'+1)!(n-l-1-j)!}\cdot\frac{g^{j'}_{\kappa nl}(z,s)}{[\rho(z,s)]^{j-j'+1}}. \qquad (6\cdot32)$$

The eigenvalues $\lambda_{\kappa n}(s)$ and $g^0_{\kappa nl}(z,s)\equiv\text{const}\, g_{\kappa n}(z,s)$ are determined by the $j=0$ case of (6·32), namely, by (6·10). The other weight functions $g^j_{\kappa nl}(z,s)$ $(j\geq 1)$ are expressed as linear combinations of $(d/dz)^k g_{\kappa n}(z,s)$ $(k=0,1,\cdots,j)$ whose coefficients are certain polynomials in z.

To extend the above method to the case in which P_μ is lightlike, it is not sufficient to replace $\mathcal{Y}_{lm}(\boldsymbol{p})$ in (6·29) by $\chi_{lm}(p)$ because of (4·19). To find a complete set of solutions in this case, we have to introduce the explicit dependence on p_3+p_0.[N23)]

We can more simply obtain the results equivalent to the above by the stereographic projection method. For simplicity, we choose $\eta_a=\eta_b=1/2$, and let $s<4(1-|\varDelta|)^2$; the analytic continuation to $s<4$ is made after we find the solutions. Let

$$\tilde{k}_\mu\equiv\frac{1}{2}(\boldsymbol{P},iP_0),\quad(\tilde{k}^2=s/4) \qquad (6\cdot33)$$

be a Euclidean 4-vector. From (5·22), the Wick-rotated B–S equation reads

$$[(1+\varDelta)^2+(\tilde{p}-i\tilde{k})^2][(1-\varDelta)^2+(\tilde{p}+i\tilde{k})^2]\tilde{\phi}_{\kappa nlm}(\tilde{p},P)$$
$$=\frac{\lambda_{\kappa n}(s)}{\pi^2}\int d^4\tilde{p}'\frac{\tilde{\phi}_{\kappa nlm}(\tilde{p}',P)}{(\tilde{p}-\tilde{p}')^2}. \qquad (6\cdot34)$$

We consider a five-dimensional sphere

$$\xi^2=r^2,\quad(r^2\equiv1+\varDelta^2-\tfrac{1}{4}s) \qquad (6\cdot35)$$

where $\xi=(\xi_1,\cdots,\xi_5)$ is a five-dimensional orthogonal coordinate system such that the ξ_μ axes $(\mu=1,2,3,4)$ coincide with the $\tilde{p}_\mu$ axes. We map the $\tilde{p}$ space onto the sphere as shown in Fig. 4. Then the coordinates of the point on the sphere corresponding to (p_μ) are given by

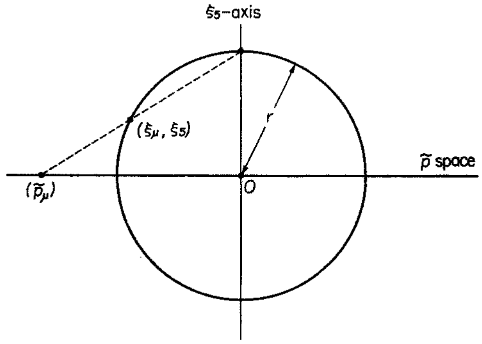

Fig. 4. The stereographic projection of the $\tilde{p}$ space.

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$$\xi_\mu = 2\tilde{p}_\mu \cdot r^2/(\tilde{p}^2+r^2), \quad (\mu=1,2,3,4)$$
$$\xi_5 = r(\tilde{p}^2-r^2)/(\tilde{p}^2+r^2). \tag{6·36}$$

By setting

$$\tilde{\phi}_{\kappa n l m}(\tilde{p}, P) = [(r-\xi_5)/2r]^3 H_{\kappa n l m}(\xi, P), \tag{6·37}$$

(6·34) is transformed into

$$\{r^6-[h(\xi)-\Delta r]^2\} H_{\kappa n l m}(\xi, P)$$
$$= \frac{\lambda_{\kappa n}(s)}{8\pi^2}\int d^5\xi' \frac{\delta(|\xi'|-r)}{1-\xi\xi'/r^2} H_{\kappa n l m}(\xi', P), \tag{6·38}$$

where

$$h(\xi) \equiv ir\cdot\tilde{k}_\mu \xi_\mu + \Delta\xi_5 . \tag{6·39}$$

We introduce a new orthogonal, but complex, coordinate system $\hat{\xi}=(\hat{\xi}_1, \cdots, \hat{\xi}_5)$ such that the $\hat{\xi}_5$ axis has the direction of (irk_μ, Δ) (the origin is left unchanged). The directions of the other axes are rather arbitrary, but we choose them in such a way that $\hat{\xi}_j = \xi_j$ if $\tilde{k}_j=0$.

Now, in the $\hat{\xi}$ coordinate system, the equation clearly exhibits the $O(4)$ symmetry because

$$h(\xi) = [(1-\tfrac{1}{4}s)(\Delta^2-\tfrac{1}{4}s)]^{1/2}\hat{\xi}_5 \tag{6·40}$$

and $\xi\xi'=\hat{\xi}\hat{\xi}'$, provided that the resulting complex contours in $\hat{\xi}$ can be deformed into real ones. Finally, we project the sphere onto the flat space spanned by $\hat{\xi}_1, \cdots, \hat{\xi}_4$. In terms of the coordinates $\tilde{q}_\mu$, which are defined by scaling down the projected values by a factor $r^2/(1+\Delta\hat{\Delta})$, we obtain

$$[(1+\hat{\Delta})^2+\tilde{q}^2][(1-\hat{\Delta})^2+\tilde{q}^2]\hat{\phi}_{\kappa n l m}(\tilde{q})$$
$$=(1-\hat{\Delta}^2)\frac{\lambda_{\kappa n}(\hat{s})}{\pi^2}\int d^4\tilde{q}' \frac{\hat{\phi}_{\kappa n l m}(\tilde{q}')}{(\tilde{q}-\tilde{q}')^2}, \tag{6·41}$$

where

$$\hat{\Delta}^2 \equiv (4\Delta^2-s)/(4-s), \tag{6·42}$$

and $\hat{\phi}_{\kappa n l m}(\tilde{q})$ is defined by the right-hand side of (6·37) with replacement of ξ_5 by $\hat{\xi}_5$. Because (6·41) has the form of the $\tilde{k}_\mu=0$ case of the original equation (6·34), it is easy to solve the former.

The third method may be called the bipolar transformation method. We consider the case $P_\mu=(\sqrt{s}, 0, 0, 0)$ and choose $\eta_a=1/2+2\Delta/s$, $\eta_b=1/2-2\Delta/s$. Let $(|\boldsymbol{p}|, \theta, \varphi)$ be the polar coordinates of $\boldsymbol{p}$, and

$$|\boldsymbol{p}| = \frac{c\sin\beta}{\cos\alpha-\cos\beta},$$
$$p_0 = \frac{c\sin\alpha}{\cos\alpha-\cos\beta}, \tag{6·43}$$

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where

$$c^2 \equiv m_a^2 - \eta_a^2 s = -(4\Delta^2 - s)(4-s)/4s. \tag{6·44}$$

Then it can be shown[G12] that we can find solutions in a separable form,

$$[(1+\Delta)^2 - v][(1-\Delta)^2 - w]\phi_{\kappa nlm}(p, P) = f_{\kappa n}(\alpha) g_{nl}(\beta) Y_{lm}(\theta, \varphi)/|\boldsymbol{p}|. \tag{6·45}$$

(C) B-S amplitudes

By the stereographic projection method with $\eta_a = \eta_b = \frac{1}{2}$ and $P_\mu = (P_0, 0, 0, P_3)$, we obtain a complete set of solutions for all values of s in a unified way:*)

$$\varphi_{\kappa nlm}(p, P) = -iB_{\kappa n}(\hat{s}) \mathcal{Z}_{n-1,l,m}(\hat{p}) \int_{-1}^{1} dz \frac{\hat{g}_{\kappa n}(z, \hat{s})}{[\hat{f}(z, v, w) - i\varepsilon]^{n+2}}, \tag{6·46}$$

where $B_{\kappa n}(\hat{s})$ is a normalization constant, which is determined in §9;

$$\hat{f}(z, v, w) \equiv \frac{1}{2}(1+z)(1-\Delta)[(1+\Delta)^2 - v] + \frac{1}{2}(1-z)(1+\Delta)[(1-\Delta)^2 - w], \tag{6·47}$$

namely,

$$\hat{f}(\hat{z}, v, w) = (1-\Delta^2)(1+\Delta z)^{-1} f(z, v, w), \tag{6·48}$$

and

$$\hat{p}_1 \equiv p_1, \quad \hat{p}_2 \equiv p_2,$$
$$\hat{p}_3 \equiv \frac{\Delta p_3 - \frac{1}{4}(p^2 + r^2) P_3}{(\Delta^2 + \frac{1}{4} P_3^2 r^2)^{1/2}}, \tag{6·49}$$
$$\hat{p}_0 \equiv \frac{4\Delta^2 p_0 - \Delta(p^2 + r^2) P_0 + r^2 (P_3 p_0 - P_0 p_3) P_3}{[(\Delta^2 + \frac{1}{4} P_3^2 r^2)(4\Delta^2 - s)(4-s)]^{1/2}}$$

with $r^2 = 1 + \Delta^2 - \frac{1}{4}s$. Note that $\hat{p}_\mu$ is real if and only if $s < 4\Delta^2$.

In particular, for $s > 0$, by taking the rest frame $P_\mu = (\sqrt{s}, 0, 0, 0)$, (6·49) is reduced to

$$\hat{p}_j = p_j, \quad (j = 1, 2, 3)$$
$$\hat{p}_0 = \frac{4\Delta p_0 - (1 + \Delta^2 - \frac{1}{4}s + p^2)\sqrt{s}}{[(4\Delta^2 - s)(4-s)]^{1/2}}. \tag{6·50}$$

Likewise, (6·49) is simplified by choosing $P_0 = 0$ for $s < 0$ (then $\hat{p}_0 = p_0$) and $P_3 = P_0$ for $s = 0$ [then $\hat{p}_3 - \hat{p}_0$ is proportional to $p_3 - p_0$]; for $P_\mu = 0$ we naturally have $\hat{p}_\mu = p_\mu$. In those Lorentz frames, (6·46) is proportional to $\mathcal{Y}_{lm}(\boldsymbol{p})$ for

*) The transformation from ξ to $\hat{\xi}$ is generally singular at $s=0$, but a particular choice of the $\hat{\xi}_4$ axis eliminates this singularity.[S15]

$s>0$ and $\mathcal{Z}_{n-1,l,m}(p)$ for $P_\mu=0$; it becomes proportional to $\tilde{\mathcal{Y}}_{lm}(p_1, p_2, p_0)$ for $s<0$ if $\mathcal{Z}_{n-1,l,m}(\hat{p})$ is replaced by $\mathcal{Z}_{n-1,l,m}(\hat{p}_1, \hat{p}_2, i\hat{p}_0, i\hat{p}_3)$, and to $\chi_{M+|m|,m}(p)$ for P_μ lightlike if $\mathcal{Z}_{n-1,l,m}(\hat{p})$ is replaced by $\hat{\mathcal{Z}}_{n-1,M,m}(\hat{p})$ [see (4·14)].

At the degeneracy point $s=4\Delta^2$, namely, at $\hat{s}=0$, $\hat{p}_0$ becomes singular, and hence only the most singular part in (6·46) becomes significant. Since

$$\mathcal{Z}_{Llm}(p)=A_{Ll}\cdot\mathcal{Y}_{lm}(\boldsymbol{p})(\sqrt{p^2})^{L-l}C_{L-l}^{l+1}(p_0/\sqrt{p^2}), \tag{6·51}$$

we find

$$\mathcal{Z}_{n-1,l,m}(\hat{p})\sim\mathcal{Y}_{lm}(\hat{p}_1, \hat{p}_2, \hat{p}_3)\hat{p}_0^{n-l-1}. \tag{6·52}$$

First, we consider the unequal-mass case ($\Delta\neq0$); then we can use (6·50), whence

$$\mathcal{Z}_{n-1,l,m}(\hat{p})\sim\hat{s}^{-(n-l-1)/2}\mathcal{Y}_{lm}(\boldsymbol{p})[\alpha(v,w)]^{n-l-1} \tag{6·53}$$

with

$$\alpha(v,w)\equiv-2\Delta p_0+\Delta(1+p^2)=(\partial/\partial z)\hat{f}(z,v,w). \tag{6·54}$$

Therefore, it is easy to show, by using (6·22) and Rodrigues' formula,[6] that all the solutions at $s=4\Delta^2$ are written as

$$-iB'_{\kappa+n,l}\mathcal{Y}_{lm}(\boldsymbol{p})\int_{-1}^{1}dz\frac{(1-z^2)^{l+1}C_{\kappa+n-l-1}^{l+3/2}(z)}{[\hat{f}(z,v,w)-i\varepsilon]^{l+3}}, \tag{6·55}$$

that is, they are specified by only three quantum numbers $\kappa+n$, l and m. Some solutions are missing at $s=4\Delta^2$; the reason for this peculiar phenomenon is discussed in §10.

Next, we consider the equal-mass case ($\Delta=0$) for P_μ lightlike. In this case, (6·52) reduces to

$$\mathcal{Z}_{n-1,l,m}(\hat{p})\sim s^{-(n-l-1)/2}\mathcal{Y}_{lm}\Big(p_1, p_2, -\frac{1}{2}(p^2+1)\Big)(p_3-p_0)^{n-l-1}, \tag{6·56}$$

but since

$$P_0(p_3-p_0)=-(v-w)/2=(\partial/\partial z)f(z,v,w), \tag{6·57}$$

the solutions are written as*)

$$-iB^{(0)\prime}_{\kappa+n,l,m}(p_1\pm ip_2)^{|m|}R^{l-|m|}C_{l-|m|}^{|m|+1/2}((p^2+1)/R)$$
$$\times\int_{-1}^{1}dz\frac{(1-z^2)^{l+1}C_{\kappa+n-l-1}^{l+3/2}(z)}{[f(z,v,w)-i\varepsilon]^{l+3}} \tag{6·58}$$

with

$$R^2\equiv(p^2-1)^2-4(p_3^2-p_0^2). \tag{6·59}$$

Again some solutions are missing. It is interesting to note that the $s\to0$

*) The solutions for $|m|\leq l-2$ were overlooked previously.[N24)]

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limits in a moving frame of (6·46) with (6·50) for $\Delta=0$ yield only the $|m|=l$ and $|m|=l-1$ cases of (6·58).

The $\Delta=0$ and $P_\mu=0$ case is extremely simple because then (6·38) is invariant under $O(5)$. We find that a complete set of solutions are given by

$$-iB^{(0)}_{NLl}\mathcal{Z}_{Llm}(p)(1-p^2-i\varepsilon)^{-L-3}C^{L+3/2}_{N-L-1}((1+p^2)/(1-p^2-i\varepsilon)), \quad (6\cdot60)$$
$$(N-1\geqslant L\geqslant l\geqslant |m|)$$

where $N=\kappa+n$ and $L=\kappa+l$, but this identification of quantum numbers is not strict, that is, the solutions given in (6·60) are not necessarily the $s\to0$ limits of the solutions for $P_\mu=(\sqrt{s},0,0,0)$.

Finally, some remarks on the solutions are in order.

1) There are qualitative differences between the equal-mass case and the unequal-mass case. This is because the latter has a finite interval between two singular points $s=0$ and $s=4\Delta^2$.

2) The Wick-Cutkosky model exhibits the $O(4)$ symmetry just as in the non-relativistic hydrogen atom.[11] Because of the additional freedom of the relativistic problem, however, we have encountered the differential equation (6·10), about which group theory can say nothing.

3) In the Wick-Cutkosky model, the B–S amplitudes are infrared divergent on the mass shell $[v=(1+\Delta)^2$ and $w=(1-\Delta)^2]$. To avoid this difficulty, it is convenient to modify the definition of the mass shell. In the equal-mass case, $v=w$ seems to be the natural extension of the mass shell. For $v=w$, $f(z,v,w)$ becomes independent of z. In the unequal-mass case, therefore, we define the mass shell by

$$(1-\Delta)[(1+\Delta)^2-v]=(1+\Delta)[(1-\Delta)^2-w], \quad (6\cdot61)$$

so that $\hat{f}(z,v,w)$ may become independent of z. It is interesting to note that (6·61) is satisfied by

$$v=(1+\Delta+\delta)^2, \quad w=(1-\Delta+\delta)^2 \quad (6\cdot62)$$

if we neglect the order of δ^2. Under the above definition of the mass shell, all the solution with κ odd vanish on the mass shell because of (6·14).

4) In the equal-mass case (or in the unequal-mass case for $s\leqq0$), we can define the p_0-parity Π (the sign change under $p_0\to-p_0$) for certain solutions:

$\Pi=(-1)^\kappa$ for the $s>0$ solutions proportional to $\mathcal{Y}_{lm}(\boldsymbol{p})$;
$\Pi=(-1)^{L-l}$ for the $P_\mu=0$ solutions proportional to $\mathcal{Z}_{Llm}(p)$;
$\Pi=(-1)^{l-m}$ for the $s<0$ solutions proportional to $\widetilde{\mathcal{Y}}_{lm}(p_1,p_2,p_0)$.

This notion can be extended to other models.

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§7. Scalar-scalar ladder model

In this section, we consider the B–S equation in the ladder approximation for two scalar particles which exchange scalar mesons having mass μ. If $\mu=0$ then this model reduces to the Wick-Cutkosky model, but there are some qualitative differences between the $\mu=0$ case and the $\mu\neq 0$ case.

Following Wick's suggestion,[W5)] Wanders (1956, 1957)[W1),W2)] first introduced a two-variable integral representation (PTIR) for the invariant B–S amplitude and discussed the non-relativistic limit by means of the equation for its weight function. Ida and Maki (1961)[I1)] showed that all invariant B–S amplitudes in the equal-mass case have the above integral representation by using Mercer's theorem.[10)] Sato (1963)[S4)] solved the equation for the weight function by means of Fredholm theory in the $l=0$ and $l=1$ cases, and simplified the derivation of the non-relativistic limit. The partial-wave B–S equation was considered by Nakanishi (1963),[N9)] who extended the above-mentioned results to the case of general l. Kramer and Meetz (1966)[K13)] analyzed the equation for the weight function in the $l=0$ case in a somewhat different way.

For $P_\mu=0$, the above integral representation reduces to a single dispersion integral, and the spectral function is obtained by solving a Volterra equation.[N5,)C1),S4)] Nakanishi (1960, 1963)[N6)~N8)] noticed that a similar analysis would possible for $s\leq 0$ by using a double dispersion representation.

Bassetto, Ciccariello and Tonin (1965)[B6)] showed that some solutions for P_μ lightlike can be obtained from those for $P_\mu=0$ in terms of a power series of P_0. Nakanishi (1968)[N24)] proved the non-existence of the lightlike solutions with $|m|<l$ in the equal-mass case. Naito (1968)[N1)] demonstrated the one-to-one correspondence between the solutions for $P_\mu=0$ with $|m|=l$ and those for P_μ lightlike.

Ciafaloni and Menotti (1965)[C7)] made operator analysis for the Wick-rotated B–S equation in the equal-mass case [see also Ciafaloni (1967)[C10)]]. They investigated the properties of $\lambda_B(s)$ as a function of s, and found a formula for $\lambda_B'(0)$, which was generalized to the unequal-mass case by Chung and Wright (1967).[C6)] Naito and Nakanishi (1969)[N4)] proved the reality of $\lambda_B(s)$ in the unequal-mass case.

In the position space, the Wick-rotated B–S equation becomes a partial differential equation, which is converted into an integral equation by the operation of the free Green's function $\widetilde{\boldsymbol{K}}^{-1}$. The asymptotic behavior and other properties of $\widetilde{\boldsymbol{K}}^{-1}$ were investigated by Wick (1954),[W5)] Swift and Lee (1964),[S22),*)] Ciafaloni and Menotti (1965),[C7)] Keam (1966),[K6)] and others. Honferkamp (1968)[H7)] converted the position-space B–S equation into two Volterra integral equations to obtain solutions. Arafune[A8)] made use of the Wick-rotated position-space B–S equation in order to prove some qualitative properties of the eigenvalues.

*) The asymptotic behavior given in their paper is erroneous.

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Gourdin and Tran Thanh Van (1959)[G8] developed an approximation method to the B–S equation by expanding it in terms of four-dimensional spherical harmonics (see §11). Kawaguchi (1965)[K5] considered some approximate Fredholm solutions for eigenvalues.

Cosenza, Sertorio and Toller (1964)[C15] investigated the models of non-Fredholm type such as the vector-meson-exchange model.

For the scattering problem, there appeared a number of mutually unrelated papers. Gourdin (1958)[G6] formulated a method of calculating scattering phase shifts (see §11). Okubo and Feldman (1960)[O5] introduced an integral representation for the Feynman amplitude. Wanders (1960)[W3] proved the Mandelstam representation for ladder graphs by using the scattering B–S equation. The Regge behavior in the t channel was investigated by Lee and Sawyer (1962)[L3] and many others (see §12). Mattioli (1968)[M9] discussed the convergence of the ladder series as a series of Feynman distributions. Choudhury (1968)[C4] reduced the partial-wave scattering B–S equation to an infinite system of algebraic equations.

Since the $\mu \neq 0$ ladder model is not solvable in closed form, a number of numerical calculations have been made so far. Until recently, only the equal-mass case was considered for simplicity. After some bound-state calculations of Vosko (1960)[V2] and Schwartz (1965),[S9] Schwartz and Zemach (1966)[S10] made rather extensive numerical calculations*) of eigenvalues and of the scattering problem in the elastic region by a variational method. Haymaker (1967)[H5),H6] calculated the scattering phase shifts below the elastic threshold and extrapolated them to the physical region. Levine, Wright and Tjon (1967)[L7),L8] calculated the scattering phase shifts in the inelastic region by the subtraction method, and also discussed the effect of the self-energy correction. Cohen, Pagnamenta and Taylor (1967)[C13] proposed a method of finding an approximate solution. Ladányi (1969)[L2] calculated phase shifts by a least-squares method. On the other hand, further numerical analyses of the bound-state problem were made by Pagnamenta (1968),[P3] Ladányi (1968)[L1] and by Linden and Mitter (1969);[L11] the unequal-mass case was calculated by Linden (1969).[L12]

(A) Eigenvalues

Since the eigenvalue problem cannot be reduced to a one-dimensional equation except for $P_\mu = 0$, it is quite difficult to investigate detailed properties of the eignvalues $\lambda_B(s)$ analytically. From the Wick-rotated equation (5·22), however, we can prove the discreteness of the eigenvalues and their reality and positive definiteness.

In the unequal-mass case, the reality of $\lambda_B(s)$ for $s > 0$ is non-trivial

*) Their results were compared with those of a relativistic wave equation by Son and Sucher (1967)[S17] and with those of the N/D method by Vasavada (1968).[V1]

because then $\widetilde{K}$ is complex. The proof*) goes as follows.[N4)] We integrate (5·22) over $\tilde{p}_\mu$ after multiplying it by $[(\tilde{\phi}_{Br}(\boldsymbol{p}, -p_4, P)]^*$. We can show that the complex conjugate equation is identical with the original one except for $\lambda_B(s)$, by transforming p_4 and p_4' into $-p_4$ and $-p_4'$, respectively. Hence we have $\lambda_B^*(s)=\lambda_B(s)$. The proof for $\mathrm{Re}\lambda_B(s)>0$ is as follows.[A8)] We integrate (5·22) over $\tilde{p}_\mu$ after multiplying it by $[\tilde{\phi}_{Br}(\tilde{p}, P)]^*$. Because of $\mathrm{Re}\widetilde{K}>0$, $\mathrm{Re}\lambda_B(s)>0$ follows from the positive definiteness of the right-hand side, which can easily be shown by transforming the integral into the position-space one.

It is important to know the sign of $\lambda_B'(s)$ in connection with the norm of $|B\rangle$ (see §9). Furthermore, if $\lambda_B'(s)$ happens to vanish, the inverse function $s=s_B(\lambda)$ will be singular at such a point. One expects $\lambda_B'(s)<0$ for any value of s at least in the equal-mass case, but at present we have no general proof of this conjecture. In the ladder model, because of (3·6) and $\partial I/\partial s=0$, (3·11) implies

$$\frac{1}{\lambda_B}\frac{d\lambda_B}{ds}=\frac{\bar{\phi}_{Br}(\partial K/\partial s)\phi_{Br}}{\bar{\phi}_{Br}K\phi_{Br}}. \tag{7·1}$$

We make the Wick rotation in the right-hand side. By means of (5·7) it is easy to show that in the rest frame $P_\mu=(\sqrt{s}, 0, 0, 0)$ the Wick-rotated form of $\bar{\phi}_{Br}(p, P)$ equals $-[\tilde{\phi}_{Br}(\boldsymbol{p}, -p_4, P)]^*$, whence in the equal-mass case (with $\eta_a=\eta_b=1/2$) it is equal to $[\tilde{\phi}_{Br}(\boldsymbol{p}, p_4, P)]^*$ apart from a sign factor (see §9(A)). Hence, on account of (5·25), (7·1) is rewritten as

$$\frac{\lambda_B'(s)}{\lambda_B(s)}=-\frac{\int d^4\tilde{p}\,\frac{1}{2}[m^2+\tilde{p}^2-2p_4^2+\frac{1}{4}s]\,|\tilde{\phi}_{Br}(\tilde{p}, P)|^2}{\int d^4\tilde{p}\,[(m^2+\tilde{p}^2-\frac{1}{4}s)^2+p_4^2 s]\,|\tilde{\phi}_{Br}(\tilde{p}, P)|^2}. \tag{7·2}$$

The troublemaker is $-p_4^2$ in the integrand of the numerator. We can, however, prove $\lambda_B'(s)<0$ for $0\leq s<4m^2$ only for the partial-wave ground states by making use of the Wick-rotated position-space B–S equation.[A8)]

Near $s=0$, it is convenient to specify the eigenvalues $\lambda_B(s)$ by three quantum numbers ν, L and l, where L denotes the four-dimensional angular momentum for $P_\mu=0$ and ν is an additional quantum number; $\lambda_{\nu Ll}(s)$ becomes independent of l at $s=0$ due to the $O(3,1)$ symmetry. By means of (7·2) and

$$\int d\Omega_4(1-2\cos^2\alpha)\,|H_{Llm}(\alpha, \theta, \varphi)|^2=l(l+1)/L(L+2), \tag{7·3}$$

where H_{Llm} is defined by (4·11), we can easily show that[C7),**)]

*) Note added in proof: The proof is incomplete.

**) Here (and also hereafter) we assume that $\lambda_{\nu L}(0)\neq\lambda_{\nu' L'}(0)$ unless $\nu=\nu'$ and $L=L'$. Therefore (7·4) does not hold in the equal-mass Wick-Cutkosky model [see (6·23)].

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$$\lambda'_{\nu Ll}(0)=a_{\nu L}+b_{\nu L}\cdot l(l+1) \tag{7·4}$$

with

$$a_{\nu L}<0, \qquad b_{\nu L}<0. \tag{7·5}$$

In the unequal-mass case, (7·1) is no longer true at $s=0$ because of the explicit dependence of K on $\sqrt{s}$. We can still, however, prove (7·4),[C6] but without (7·5).

In quite an analogous way to the reasoning about $d\lambda_B/ds$, we can show that[O3]

$$\frac{1}{\lambda_B}\frac{\partial\lambda_B}{\partial\mu}=-\frac{\int d^4\tilde{p}'\int d^4\tilde{p}\,[\tilde{\phi}_{Br}(\tilde{p}',P)]^*\{(\partial/\partial\mu)\,[\mu^2+(\tilde{p}-\tilde{p}')^2]^{-1}\}\tilde{\phi}_{Br}(\tilde{p},P)}{\int d^4\tilde{p}'\int d^4\tilde{p}\,[\tilde{\phi}_{Br}(\tilde{p}',P)]^*[\mu^2+(\tilde{p}-\tilde{p}')^2]^{-1}\tilde{\phi}_{Br}(\tilde{p},P)}>0$$

$$\text{for } \mu\geq 0 \tag{7·6}$$ *)

in the equal-mass case, provided that $\partial\tilde{\phi}_{Br}/\partial\mu$ is well defined.**) Thus the eigenvalues of the $\mu\neq 0$ model are always larger than the corresponding ones of the Wick-Cutkosky model. In particular, abnormal solutions will not appear for $\lambda_B<1/4$ also in the $\mu\neq 0$ model. Furthermore, we can in general show[N26]

$$\lim_{s\to-\infty}\lambda_B(s)=+\infty. \tag{7·7}$$

As an analytic function of s, $\lambda_B(s)$ is conjectured to be holomorphic except for a cut $s\geqslant(m_1+m_2)^2$. It is expected further that an unsubtracted dispersion relation holds for $1/\lambda_B(s)$ and that the spectral function is positive semi-definite at least in the equal-mass case.[C7]

(B) Wick-rotated equations

We can in principle solve the Wick-rotated equation (5·22) by the modified Fredholm theory.***) Since it is trivial, however, to carry out the angular integrations, we can convert (5·22) into a partial-wave B-S equation.

For $P_0=\sqrt{s}>0$, on setting****)

$$\tilde{\phi}_{\nu Llm}(\tilde{p},P)=Y_{lm}(\theta,\varphi)\,|\boldsymbol{p}|^{-1}\psi_{\nu Ll}(|\boldsymbol{p}|,p_4;\,s), \quad (L\geq l\geq|m|) \tag{7·8}$$

we find

---

*) Transform the integrals into those in the position space to show the positive definiteness.

**) We can prove that $\lambda_B(s)$ is continuous in μ in the equal-mass case (and also in the unequal-mass case for $s\leq 0$),[12] but the continuity of $\tilde{\phi}_{Br}$ in μ is not evident.

***) Since the trace of the kernel is divergent in this case, we have to avoid its use. Smithies[13] presented the Fredholm resolvent without using the trace of the kernel.

****) We assume that the solutions are specified by the four quantum numbers ν, L, l, m, where L is the four-dimensional angular momentum quantum number for $P_\mu=0$, and $L=l+\kappa$ in the equal-mass case.

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$$[m_a^2+|\boldsymbol{p}|^2+(p_4-i\eta_a\sqrt{s})^2]\,[m_b^2+|\boldsymbol{p}|^2+(p_4+i\eta_b\sqrt{s})^2]\,\psi_{\nu Ll}(|\boldsymbol{p}|,p_4;s)$$
$$=\frac{2}{\pi}\lambda_{\nu Ll}(s)\int_0^\infty d|\boldsymbol{p}'|\int_{-\infty}^\infty dp_4' Q_l\left(\frac{\mu^2+|\boldsymbol{p}|^2+|\boldsymbol{p}'|^2+(p_4-p_4')^2}{2|\boldsymbol{p}||\boldsymbol{p}'|}\right)\psi_{\nu Ll}(|\boldsymbol{p}'|,p_4';s), \tag{7·9}$$

where $Q_l(z)$ denotes the Legendre function of the second kind, which behaves like z^{-l-1} as $z\to\infty$ and is related to that of the first kind through

$$Q_l(z)=\frac{1}{2}\int_{-1}^1 d\zeta\frac{P_l(\zeta)}{z-\zeta}. \tag{7·10}$$

The trace of the kernel of (7·9) is finite, whence we can apply the classical Fredholm theory to it. Indeed, the trace $\sigma_l(s)$ of the kernel is given by a convergent integral,

$$\sigma_l(s)=\frac{2}{\pi}\int_0^\infty d|\boldsymbol{p}|\int_{-\infty}^\infty dp_4\frac{Q_l(1+\mu^2/2|\boldsymbol{p}|^2)}{[m_a^2+|\boldsymbol{p}|^2+(p_4-i\eta_a\sqrt{s})^2]\,[m_b^2+|\boldsymbol{p}|^2+(p_4+i\eta_b\sqrt{s})^2]}. \tag{7·11}$$

After some manipulation,[N9] we can show that

$$\sigma_l(s)=\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\frac{x_3^l\delta(1-x_1-x_2-x_3)}{x_1m_a^2+x_2m_b^2+x_3\mu^2-x_1x_3m_a^2-x_2x_3m_b^2-x_1x_2s}. \tag{7·12}$$

Note that apart from x_3^l, (7·12) is exactly the Feynman-parametric integral corresponding to the triangle graph on the mass shell.

In the equal-mass case (with $\eta_a=\eta_b=1/2$), (7·9) can be transformed into an integral equation of the Hilbert-Schmidt type. Hence Mercer's theorem[10] gives us

$$\sum_{\nu,L}[\lambda_{\nu Ll}(s)]^{-1}=\sigma_l(s) \tag{7·13}$$

with $\lambda_{\nu Ll}(s)>0$. Furthermore, the amplitude $\psi_{\nu Ll}(|\boldsymbol{p}|,p_4;s)$ is symmetric or antisymmetric under the sign change of p_4. Hence we can easily calculate the sums $\sigma_l^+(s)$ and $\sigma_l^-(s)$ of $[\lambda_{\nu Ll}(s)]^{-1}$ over the symmetric solutions only and over the antisymmetric ones only, respectively.[11] We naturally have

$$\sigma_l^+(s)>\sigma_l^-(s)>0, \tag{7·14}$$

a result which definitely shows the existence of the antisymmetric solutions.

For $P_\mu=0$, on setting

$$\tilde{\phi}_{\nu Llm}(\tilde{p},0)=H_{Llm}(\alpha,\theta,\varphi)(\tilde{p}^2)^{-1/2}f_{\nu L}(\tilde{p}^2), \tag{7·15}$$

we find

$$(m_a^2+\alpha)(m_b^2+\alpha)f_{\nu L}(\alpha)=\frac{\lambda_{\nu L}(0)}{L+1}\int_0^\infty d\alpha'[h(\alpha,\alpha')]^{L+1}f_{\nu L}(\alpha'), \tag{7·16}$$

where

$$h(\alpha,\alpha')\equiv\{\alpha+\alpha'+\mu^2-[(\alpha+\alpha'+\mu^2)^2-4\alpha\alpha']^{1/2}\}/2\sqrt{\alpha\alpha'}\,. \qquad (7\cdot17)$$

Since (7·16) is rather manageable, we may investigate the solutions for $s>0$ in a power series of s, that is, starting from (7·16) we successively solve certain inhomogeneous equations whose kernel is identical with that of (7·16); the expansion coefficients of $\lambda_{\nu Ll}(s)$ are determined by the solvability conditions of those equations. In particular, in the equal-mass case (with $\eta_a=\eta_b=1/2$) we find $\tilde{\phi}_{\nu Llm}\to(-1)^{L-l}\tilde{\phi}_{\nu Llm}$ for $s>0$ under $p_4\to-p_4$, and hence the B–S amplitudes with $L-l$ odd vanish on the mass shell because $p_4=0$ there.

Finally, we consider the non-relativistic limit of the B–S equation. If the light velocity c is not put equal to unity, the Wick-rotated B–S equation in the equal-mass case (with $\eta_a=\eta_b=1/2$) reads[67]

$$[(\boldsymbol{p}^2+p_4^2c^{-2}+mE-\tfrac{1}{4}E^2c^{-2})^2+p_4^2(2mc^2-E)^2c^{-4}]\tilde{\phi}_{Br}(\boldsymbol{p},p_4/c;\,\boldsymbol{0},P_0/c)$$
$$=\frac{m^2c\lambda_B}{\pi^2}\int d^3\boldsymbol{p}'\int\frac{dp_4'}{c}\cdot\frac{\tilde{\phi}_{Br}(\boldsymbol{p}',p_4'/c;\,\boldsymbol{0},P_0/c)}{\hat{\mu}^2+(\boldsymbol{p}-\boldsymbol{p}')^2+(p_4-p_4')^2c^{-2}}, \qquad (7\cdot18)$$

where $E\equiv2mc^2-\sqrt{s}$ denotes the binding energy, and λ_B and $\hat{\mu}$ are the interaction strength and the interaction range, respectively. Keeping m, E, λ_B and $\hat{\mu}$ finite, we consider the $c\to\infty$ limit. We divide (7·18) by $\widetilde{K}$ (the quantity in the square bracket), which tends to

$$(\boldsymbol{p}^2+mE)^2+4m^2p_4^2 \qquad (7\cdot19)$$

as $c\to\infty$. By integrating the resulting equation over p_4/c and making $c\to\infty$, we obtain

$$(\boldsymbol{p}^2/m+E)\hat{\phi}_{Br}(\boldsymbol{p},E)=\frac{\lambda_B}{2\pi}\int d^3\boldsymbol{p}'\frac{\hat{\phi}_{Br}(\boldsymbol{p}',E)}{\hat{\mu}^2+(\boldsymbol{p}-\boldsymbol{p}')^2}, \qquad (7\cdot20)$$

where

$$\hat{\phi}_{Br}(\boldsymbol{p},E)\equiv\lim_{c\to\infty}\int_{-\infty}^{\infty}dp_4\tilde{\phi}_{Br}(\boldsymbol{p},p_4;\,\boldsymbol{0},2mc-Ec^{-1}). \qquad (7\cdot21)$$

The non-relativistic limit (7·20) is equivalent to the Schrödinger equation

$$[-\tfrac{1}{2}(m/2)^{-1}(\partial/\partial\boldsymbol{x})^2-\pi\lambda_B|\boldsymbol{x}|^{-1}\exp(-\hat{\mu}|\boldsymbol{x}|)]\hat{\boldsymbol{\phi}}_{Br}(\boldsymbol{x},E)=(-E)\hat{\boldsymbol{\phi}}_{Br}(\boldsymbol{x},E) \qquad (7\cdot22)$$

with a boundary condition. For abnormal solutions, the wave functions corresponding to (7·21) should vanish identically.

(C) Integral representations

It is convenient to express the B–S amplitudes in terms of the perturbation-theoretical integral representation (PTIR).[8),*)]

*) The two-variable PTIR or its equivalent for the vertex function was proposed by Deser, Gilbert and Sudarshan,[14] Faĭnberg[15] and Ida,[16] independently, on the basis of the axiomatic field theory, but unfortunately their derivations were wrong. Its correctness, however, was proved to all orders in perturbation theory by Nakanishi.[17]

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For $s>0$, in the rest frame $P_\mu=(\sqrt{s},0,0,0)$, the B–S amplitude is written as*)

$$\phi_{\nu LlM}(p,P)=-i\mathcal{Y}_{lm}(\boldsymbol{p})\int_{-1}^{1}dz\int_{-0}^{\infty}d\alpha\frac{\varphi_{\nu Ll}^{[h]}(z,\alpha;s)}{[f(z,\alpha;v,w)-i\varepsilon]^{h+2}} \tag{7·23}$$

with

$$f(z,\alpha;v,w)\equiv\alpha+\frac{1}{2}(1+z)(m_a^2-v)+\frac{1}{2}(1-z)(m_b^2-w); \tag{7·24}$$

h is an arbitrary non-negative integer, and

$$\varphi_{\nu Ll}^{[h]}(z,\alpha;s)=(h+1)!\left(\int_{-0}^{\alpha}d\alpha\right)^h\varphi_{\nu Ll}^{[0]}(z,\alpha;s). \tag{7·25}$$

We insert (7·23) into the B–S equation (5·18), choosing h in such a way that the integration over p'_μ converges. We obtain an integral equation for the weight function $\varphi_{\nu Ll}^{[h]}$, which is reduced to that for $\varphi_{\nu Ll}^{[0]}$:

$$\varphi_{\nu Ll}^{[0]}(z,\alpha;s)=A_{\nu Ll}(s)\delta(\alpha)-\lambda_{\nu Ll}(s)\int_{-1}^{1}dz'\int_{0}^{\infty}d\alpha' \times H_l(z,\alpha;z',\alpha';s)\varphi_{\nu Ll}^{[0]}(z',\alpha';s), \tag{7·26}$$

where

$$H_l(z,\alpha;z',\alpha';s) \equiv\frac{R(z,z')}{2\alpha}\int_0^1 dx x'(1-x)\delta(x(1-x)\alpha-R(z,z')g(z',\alpha',x,s)) \tag{7·27}$$

with

$$g(z,\alpha,x,s)\equiv(1-x)\alpha+(1-x)^2\rho(z,s)+x\mu^2, \tag{7·28}$$

and $R(z,z')$ and $\rho(z,s)$ are given by (6·2) and (6·3), respectively. The eigenvalues $\lambda_{\nu Ll}(s)$ are determined by

$$A_{\nu Ll}(s)\equiv\frac{1}{2}\lambda_{\nu Ll}(s)\int_{-1}^{1}dz\int_0^\infty d\alpha\int_0^1 dx\frac{x'(1-x)}{g(z,\alpha,x,s)}\varphi_{\nu Ll}^{[0]}(z,\alpha,s), \tag{7·29}$$

because $\varphi_{\nu Ll}^{[0]}$ is proportional to $A_{\nu Ll}$.

The weight function $\varphi_{\nu Ll}^{[0]}$ satisfies the boundary conditions

$$\varphi_{\nu Ll}^{[0]}(\pm1,\alpha;s)=0, \tag{7·30}$$

which can easily be proved by means of (7·26), (7·27) and (7·29). Furthermore, we have[N9)]

$$\varphi_{\nu Ll}^{[0]}(z,\alpha;s)=0 \qquad \text{for } \alpha<0, \tag{7·31}$$

*) The trace of the kernel of the equation for $\varphi_{\nu Ll}^{[h]}$ equals (7·12), whence all solutions (for which the Wick rotation is possible) are represented as (7·23) at least in the equal-mass case because of Mercer's theorem.

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$$\lim_{\alpha\to+\infty}\varphi_{\nu Ll}^{[h]}(z,\alpha;\,s)=0 \qquad \text{for } h=0,1,\cdots,l+1. \tag{7·32}$$

From (7·32) we can show[A8)]

$$\phi_{\nu Llm}(p,P)=O((p^2)^{-3-l/2}) \text{ as } p^2\to\infty. \tag{7·33}$$

It is elementary to carry out the integration over x in (7·27). We find

$$H_l(z,\alpha;\,z',\alpha';\,s)=\frac{[x_1^l(1-x_1)+x_2^l(1-x_2)]\,\theta(\alpha R^{-1}-\alpha'-\mu^2-2\mu\sqrt{\alpha'+\rho})}{2\alpha[(\alpha R^{-1}-\alpha'-\mu^2)^2-4\mu^2(\alpha'+\rho)]^{1/2}} \tag{7·34}$$

with $R\equiv R(z,z')$ and $\rho\equiv\rho(z',s)$, and x_1 and x_2 are two roots of the equation

$$(\alpha R^{-1}+\rho)x^2+(\mu^2-\alpha R^{-1}-\alpha'-2\rho)x+\alpha'+\rho=0; \tag{7·35}$$

$0<x_j<1$ $(j=1,2)$ in the support of H_l. As is seen from (7·34), H_l has an inverse square-root-type singularity at the boundary of its support, but we can prove[S4),N9)] that its second iterated kernel $H_l^{(2)}(z,\alpha;\,z',\alpha';\,s)$ can be transformed into a bounded kernel belonging to a finite region. Hence (7·26) is solved by means of the Fredholm theory. If $A_{\nu Ll}(s)=0$, (7·26) is solvable only if the Fredholm determinant vanishes; hence we can suppose that $A_{\nu Ll}(s)$ is proportional to the Fredholm determinant. Then (7·29) determines the eigenvalues even if $A_{\nu Ll}(s)=0$.

For $s<0$, the above analysis can be repeated by replacing $\mathcal{Y}_{lm}(\boldsymbol{p})$ in (7·23) by $\tilde{\mathcal{Y}}_{lm}(p_1,p_2,p_0)$.

For $P_\mu=0$, PTIR reduces to a single dispersion representation, namely,

$$\phi_{\nu Llm}(p,0)=-i\mathcal{Z}_{Llm}(p)\int_0^\infty d\gamma\frac{\varphi_{\nu L}^{[h]}(\gamma)}{(\gamma-p^2-i\varepsilon)^{h+1}} \tag{7·36}$$

with

$$\varphi_{\nu L}^{[h]}(\gamma)=0 \quad \text{unless } \gamma\geq\min(m_a^2,m_b^2), \tag{7·37}$$

and $\varphi_{\nu L}^{[h]}(\gamma)$ is related to $\varphi_{\nu L}^{[0]}(\gamma)$ in an analogous way to (7·25). For simplicity, we consider the case $m_a<m_b<m_a+\mu$. Then it is easy to see that

$$\varphi_{\nu L}^{[0]}(\gamma)=a_{\nu L}\delta(\gamma-m_a^2)+b_{\nu L}\delta(\gamma-m_b^2)+\psi_{\nu L}(\gamma) \tag{7·38}$$

with

$$\psi_{\nu L}(\gamma)=0 \quad \text{unless } \gamma\geq(m_a+\mu)^2. \tag{7·39}$$

Then the B–S equation reduces to

$$\psi_{\nu L}(\gamma)=\lambda_{\nu L}[a_{\nu L}K_L(\gamma,m_a^2)+b_{\nu L}K_L(\gamma,m_b^2)+\int_0^\infty d\gamma' K_L(\gamma,\gamma')\psi_{\nu L}(\gamma')] \tag{7·40}$$

with

$$K_L(\gamma,\gamma')\equiv(\gamma-m_a^2)^{-1}(\gamma-m_b^2)^{-1}\int_0^1 dx\,x^L\theta(x(1-x)\gamma-(1-x)\gamma'-x\mu^2), \tag{7·41}$$

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and

$$a_{\nu L}=F_{\nu L}(m_a^2), \qquad b_{\nu L}=-F_{\nu L}(m_b^2) \tag{7·42}$$

with

$$F_{\nu L}(\beta)\equiv\frac{\lambda_{\nu L}}{m_b^2-m_a^2}\int_0^\infty d\gamma\,\varphi_{\nu L}^{[1]}(\gamma)\int_0^1 dx\frac{x^L(1-x)}{(1-x)\gamma+x\mu^2-x(1-x)\beta}\,. \tag{7·43}$$

It should be noted that the integral in (7·41) is, apart from x^L, equal to the absorptive part of the second-order self-energy graph, whence

$$K_L(\gamma,\gamma')=0 \quad \text{unless } \gamma\geq(\sqrt{\gamma'}+\mu)^2. \tag{7·44}$$

Because of $\mu\neq0$, therefore, $\psi_{\nu L}(\gamma)$ for any γ finite can be obtained exactly by a finite-order iteration. Furthermore, since (4·40) is of Volterra type because of (7·44), the iterative solution allows term-by-term integrations over γ. Since $\psi_{\nu L}(\gamma)$ is expressed as a linear combination of $a_{\nu L}$ and $b_{\nu L}$, (7·42) leads us to an eigenvalue problem of a 2×2 matrix.

If $m_a=m_b$, one of the δ functions in (7·38) should be replaced by δ'. If $m_b>m_a+\mu$, $(\gamma-m_b^2)^{-1}$ in (7·41) should be understand as Cauchy's principal value. In the latter case, as $\mu\to0$ the solutions seem to tend to those of the unequal-mass Wick-Cutkosky model rather smoothly with identifications

$$\gamma=\frac{1}{2}(1+z)m_a^2+\frac{1}{2}(1-z)m_b^2 \tag{7·45}$$

and $h-1=n=L+1$.

As $s\to0$ (7·23) should tend to (7·36) by definition. Since $\mathcal{Z}_{Llm}(p)$ contains p_0^{L-l} and $p_0\simeq(v-w)/2\sqrt{s}$ near $s=0$, (7·23) can tend to (7·36) only if

$$\int_{-1}^1 dz\,\varphi_{\nu Ll}^{[h]}(z,\alpha;\,s)=O(s^{(L-l)/2}), \tag{7·46}$$

as is seen by comparing their absorptive parts. In particular, from (7·26) we have

$$A_{\nu Ll}(s)=O(s^{(L-l)/2}). \tag{7·47}$$

Now, we consider the solutions for $P_\mu=(P_0,0,0,P_0)$, which are given by

$$-i(p_1\pm ip_2)^{|m|}\int_{-1}^1 dz\int_{-0}^\infty d\alpha\frac{\varphi_{\nu L|m|}^{[h]}(z,\alpha,0)}{[f(z,\alpha;\,v,w)-i\varepsilon]^{h+2}}\,. \tag{7·48}$$

We can prove, in the equal-mass case,*) that all solutions are represented as in (7·48), that is, lightlike solutions exist only for $m=\pm l$ and those for $|m|<l$ are missing (see §10). This result corresponds to the well-known

*) This restriction is due to some technical reasons. The statement is expected to remain valid also in the unequal-mass case.

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fact that a massless particle having a spin $l>0$ has only two polarizations independently of l. If we consider (7·23) in a moving frame, its $s\to 0$ limit is proportional to $(p_1\pm ip_2)^{|m|}(p_3-p_0)^{l-|m|}$ as shown in (4·29), but because

$$P_0(p_3-p_0)=-(v-w)/2=(\partial/\partial z)f(z,\alpha;\,v,w), \tag{7·49}$$

it can be rewritten as

$$\mathrm{const}(p_1\pm ip_2)^{|m|}\int_{-1}^{1}dz\int_{-0}^{\infty}d\alpha\frac{(\partial/\partial z)^{l-|m|}\varphi^{[h]}_{\nu Ll}(z,\alpha;\,0)}{[f(z,\alpha;\,v,w)-i\varepsilon]^{h-l+|m|+2}}\,. \tag{7·50}$$

Since we can show[N24)]

$$(\partial/\partial z)^{l-|m|}\varphi^{[h]}_{\nu Ll}(z,\alpha;\,0)=\mathrm{const}(\partial/\partial\alpha)^{l-|m|}\varphi^{[h]}_{\nu L|m|}(z,\alpha;\,0) \tag{7·51}$$

by means of (7·26) and (7·47), (7·50) reduces to (7·48).

Finally, we briefly mention the double dispersion approach to the B–S equation. For $s\leq 0$, the vertex function is proved to be holomorphic in a topological product of two cut planes of v and w.[18)] Hence it is natural to assume that the invariant B–S amplitudes for $s\leq 0$ have a double dispersion representation:*)

$$\phi_{\nu L00}(p,P)=-i\int_0^{\infty}dv'\int_0^{\infty}dw'\frac{\sigma_{\nu L}(v',w')}{(v'-v-i\varepsilon)(w'-w-i\varepsilon)}\,. \tag{7·52}$$

For $\mu\neq 0$, the spectral function $\sigma_{\nu L}(v',w')$ can be written as

$$\begin{aligned}\sigma_{\nu L}(v',w')&=\sigma^0_{\nu L}\delta(v'-m_a^2)\delta(w'-m_b^2)\\&+\sigma^a_{\nu L}(v')\theta(v'-(m_a+\mu)^2)\delta(w'-m_b^2)\\&+\sigma^b_{\nu L}(w')\theta(w'-(m_b+\mu)^2)\delta(v'-m_a^2)\\&+\sigma^{ab}_{\nu L}(v',w')\theta(v'-(m_a+\mu)^2)\theta(w'-(m_b+\mu)^2).\end{aligned} \tag{7·53}$$

The B–S equation is converted into simultaneous equations of $\sigma^0_{\nu L}$, $\sigma^a_{\nu L}$, $\sigma^b_{\nu L}$ and $\sigma^{ab}_{\nu L}$; $\sigma^{ab}_{\nu L}$ is solved in the Neumann series in terms of the others, but unfortunately the equations of $\sigma^a_{\nu L}$ and $\sigma^b_{\nu L}$ are not of Volterra type.

§8. Abnormal solutions

Abnormal solutions are the solutions which have no counterparts in the non-relativistic potential theory. Their appearance is intimately related to the extra freedom, relative time or relative energy, of the B–S equation. Indeed, abnormal solutions disappear if the retardation effect of the interaction is neglected.

Wick (1954)[W5)] and Cutkosky (1954)[C17)] discovered the existence of ab-

*) This is verified explicitly in the Wick-Cutkosky model.[N8)]

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normal solutions in the Wick-Cutkosky model. Green and Biswas (1957)[G13] and Biswas (1958)[B13] proposed to identify abnormal solutions with strange particles, but of course this identification is not adequate. On the other hand, Scarf and Umezawa (1958)[S8] tried to exclude abnormal solutions by the non-normalizability due to the degeneracy of their eigenvalues at zero binding energy in the Wick-Cutkosky model, but this criterion is not correct because the degeneracy does not occur in the $\mu\neq 0$ model. Ohnuki, Takao and Umezawa (1960)[O1] showed that the B–S equation has abnormal solutions even in the static model and that they do not correspond to the eigenstates of the original Hamiltonian. Mugibayashi (1961)[M12] showed that abnormal solutions appear even in the exact B–S equation for the static model, contrary to Wick's expectation that the appearance of abnormal solutions would be owing to the inadequacy of the ladder approximation. Watanabe (1960),[W4] Ida and Maki (1961)[I1] and Nakanishi (1964, 1965)[N14),N16] tried to distinguish abnormal solutions from normal ones, but no clear-cut difference was found in analyticity, Regge behavior and normalization. Ohnuki and Watanabe (1965)[O3] pointed out that abnormal solutions in the Wick-Cutkosky model correspond to the solutions of the Schrödinger equation with an "abnormal potential", and suggested to exclude abnormal solutions as unphysical ones. On the contrary, Naito (1968)[N2] proved, in the relativistic $\mu\neq 0$ ladder model, that certain abnormal solutions do not vanish on the mass shell, that is to say, they contribute to the S-matrix.*) Bui-Duy (1968, 1969)[B20),B21] investigated abnormal solutions in a special model which has no spatial freedom and in the static model.**)

(A) Static model

We consider the static model in which two fixed nucleons a and b having mass m exchange scalar mesons having mass μ. The B–S equation in the ladder approximation reads

$$(m-i\partial/\partial t_a)(m-i\partial/\partial t_b)\boldsymbol{\phi}_\kappa(x_a, x_b, E_\kappa)$$
$$=g^2\boldsymbol{\Delta}_F(x_a-x_b, \mu^2)\boldsymbol{\phi}_\kappa(x_a, x_b, E_\kappa), \qquad (8\cdot1)$$

where t_a and t_b are the 0-th components of x_a and x_b, respectively, and g and E_κ denote the coupling constant and the energy of the bound state $|\kappa\rangle$, respectively. Let $x=x_a-x_b$, $T=\frac{1}{2}(t_a+t_b)$ and

$$\boldsymbol{\phi}_\kappa(x_a, x_b, E)=e^{-iET}\boldsymbol{\phi}_\kappa(x, E); \qquad (8\cdot2)$$

then (8·1) reduces to

*) This property is evident in the Wick-Cutkosky model if we admit to employ the mass-shell definition (6·61).

**) A conceptual error seems to be involved in his definition of abnormal solutions.

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$$\left[(\partial/\partial x_0)^2+\left(m-\frac{1}{2}E_\kappa\right)^2\right]\boldsymbol{\phi}_\kappa(x, E_\kappa)=g^2\boldsymbol{\varDelta}_F(x, \mu^2)\boldsymbol{\phi}_\kappa(x, E_\kappa). \tag{8·3}$$

Since (8·3) does not involve $\partial/\partial\boldsymbol{x}$, $\boldsymbol{x}$ can be regarded as parameters. From the definition of the B–S amplitude (cf. §5), $\boldsymbol{\phi}_\kappa$ can be represented as

$$\boldsymbol{\phi}_\kappa(x_0, \boldsymbol{x}, E)=\theta(x_0)\int_{m-E/2}^{\infty} d\omega\varphi_\kappa(\omega, \boldsymbol{x}, E)e^{-i\omega x_0} + \theta(-x_0)\int_{-\infty}^{-m+E/2} d\omega\varphi_\kappa(\omega, \boldsymbol{x}, E)e^{-i\omega x_0}. \tag{8·4}$$

Since any solution to (8·3) is either symmetric or antisymmetric in x_0, we may confine ourselves to considering $x_0>0$ only. Then, on account of (8·4) with $E<2m$, we can analytically continue (8·3) in x_0 and bring x_0 into the negative imaginary axis. Writing $|\boldsymbol{x}|=r$, $x_0=-iry$, $g^2/4\pi^2=\lambda$ and $r(m-\frac{1}{2}E_\kappa)=\alpha_\kappa$, we have

$$[-(d/dy)^2+\lambda \boldsymbol{V}(y, \mu r)]\boldsymbol{f}_\kappa(y;\ E_\kappa, \mu r)=-\alpha_\kappa^2\boldsymbol{f}_\kappa(y;\ E_\kappa, \mu r), \tag{8·5}$$

where $\boldsymbol{f}_\kappa$ denotes the continued B–S amplitude and

$$\boldsymbol{V}(y, \mu r)\equiv -\frac{\mu r}{\sqrt{1+y^2}}K_1(\mu r\sqrt{1+y^2}), \tag{8·6}$$

$K_1(z)$ being a modified Bessel function. Though (8·5) has been derived only for $y>0$, it is natural to extend it to $y<0$ in such a way that $\boldsymbol{f}_\kappa(y;\ E_\kappa, \mu r)$ is either symmetric or antisymmetric in y. Then (8·4) leads to the boundary conditions

$$\lim_{y\to\pm\infty}\boldsymbol{f}_\kappa(y;\ E_\kappa, \mu r)=0. \tag{8·7}$$

Thus the problem is closely analogous to a one-dimensional Schrödinger equation.

For $\mu\neq 0$, since $\boldsymbol{V}(y, \mu r)$ asymptotically behaves like $|y|^{-3/2}\exp(-\mu r|y|)$ for $|y|$ large, (8·5) with (8·7) has a finite number of discrete eigenvalues for any value of $\lambda>0$. For $\mu=0$, we have

$$\lim_{\mu\to 0}\boldsymbol{V}(y, \mu r)=-1/(1+y^2). \tag{8·8}$$

In this case, it can be shown[01)] that infinitely many eigenvalues of λ tend to 1/4 as the binding energy goes to zero just as in the Wick-Cutkosky model.

For λ infinitesimal, (8·5) with (8·7) has only one solution, which is labeled as $\kappa=0$. Let $z=\alpha_0 y$; then (8·5) for $\kappa=0$ is rewritten as

$$\left[-\left(\frac{d}{dz}\right)^2+\frac{\lambda}{\alpha_0}\cdot\frac{1}{\alpha_0}\boldsymbol{V}\left(\frac{z}{\alpha_0}, \mu r\right)\right]\boldsymbol{f}_0=-\boldsymbol{f}_0. \tag{8·9}$$

Since $\boldsymbol{\alpha}_0$ should also be infinitesimal, we can approximately replace (8·9) by

$$[(d/dz)^2+\pi e^{-\mu r}\lambda\alpha_0^{-1}\delta(z)]\boldsymbol{f}_0=\boldsymbol{f}_0. \tag{8·10}$$

The solution to (8·10) is evidently given by

$$\boldsymbol{f}_0=\text{const } e^{-|z|} \tag{8·11}$$

with

$$\pi e^{-\mu r}\lambda\alpha_0^{-1}=2, \tag{8·12}$$

namely

$$E_0=2m-(g^2/4\pi)r^{-1}e^{-\mu r}. \tag{8·13}$$

This value is exactly the eigenvalue of the total Hamiltonian of the static model. Any other solutions to (8·5), which are abnormal solutions ($\kappa>0$), do not correspond to eigenstates of the total Hamiltonian.

The above analysis can be extended to the case of the exact B–S equation, which is expressed in compact form in the static model.[M12)] The exact B–S equation can be written as

$$[-(d/dy)^2+\boldsymbol{U}(y;\ \mu r,\lambda)]\boldsymbol{f}_\kappa(y;E_\kappa,\mu r)=-\alpha_\kappa^2\boldsymbol{f}_\kappa(y;\ E_\kappa,\mu r) \tag{8·14}$$

with (8·7), where

$$\boldsymbol{U}(y;\ \mu r,\lambda)\equiv\lambda h'(y,\mu r)-\lambda^2\pi e^{-\mu r}\cdot h(y,\mu r)+\lambda^2[h(y,\mu r)]^2 \tag{8·15}$$

with

$$h(y,\mu r)\equiv\int_0^\infty dk\frac{k\sin kr}{\mu^2+k^2}\exp[-\sqrt{\mu^2+k^2}ry]. \tag{8·16}$$

Of course, the first term of the right-hand side in (8·15) coincides with $\lambda\boldsymbol{V}(y;\ \mu r)$.

For $\mu=0$, (8·16) reduces to

$$h(y,0)=\pi/2-\mathrm{Tan}^{-1}y, \tag{8·17}$$

whence

$$\boldsymbol{U}(y;\ 0,\lambda)=-\lambda(1+y^2)^{-1}+\lambda^2[-(\pi/2)^2+(\mathrm{Tan}^{-1}y)^2]<0. \tag{8·18}$$

In this case, we can find an exact solution[M12)]

$$\boldsymbol{f}_0(y;\ E_0,0)=\exp[-\lambda\int^y\mathrm{Tan}^{-1}y'dy'] \tag{8·19}$$

with

$$\alpha_0=(\pi/2)\lambda, \tag{8·20}$$

in accord with (8·12) for $\mu=0$. Contrary to the case of the ladder approximation, infinitely many abnormal solutions ($\kappa>0$) exist even for $\lambda>0$ infinitesimal, because $\boldsymbol{U}(y;\ 0,\lambda)$ behaves like $-\lambda^2\pi/|y|$ for $|y|$ large. For $\mu\neq0$, qualitative features are the same as those in the ladder approximation.

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Thus the exact B–S equation of the static model has abnormal solutions which do not correspond to the eigenstates of the total Hamiltonian.

From the above analysis, one might suppose that the introduction of the relative-time freedom would always imply the appearance of abnormal solutions. This is not the case, however. If we consider the B–S equation for a bound state of a V particle and an N particle in the Lee model[19] (in this model one has three quantum fields V, N and θ and an interaction $V \rightleftarrows N + \theta$), then we can show[M12] that it has no abnormal solutions. The reason is explained as follows. Since the locations of the two constituent particles are fixed, we can label them as a and b according to their positions. Each of the particles a and b can emit a θ particle only after it has absorbed a θ particle. Therefore the proper time t_a of a cannot elapse independently of t_b, that is to say, the relative time $t=t_a-t_b$ cannot become arbitrarily large in its magnitude as long as a and b are bound. Thus this freedom is, in effect, *freezed* in the Lee model.

(B) Physical reality of abnormal solutions

Abnormal solutions are of course existent in relativistic models. One might suppose that their appearance would be a defect of the *homogeneous* B–S equation, that is, abnormal solutions would not appear as the residues of poles in the scattering Green's function. This expectation is denied explicitly in the Wick-Cutkosky model[N16),S14] and for $P_\mu=0$ in the equal-mass case of the $\mu \neq 0$ model. Furthermore, in the latter case, by means of a power series expansion in s, we can show that this situation is continued to $s>0$ and that abnormal solutions with the even p_0 parity do not vanish on the mass shell if $L-l$ is identified with κ.[N2] Thus in relativistic theories, there is no reason to reject abnormal solutions as unphysical ones.

Since abnormal solutions are definitely unphysical in the static model, it appears quite difficult to reconcile the above results to each other. Furthermore, since the abnormal solutions in the Wick-Cutkosky model are closely related to those of the static model,[O3] the above dilemma seems to be very serious. One of the ways out of this dilemma would be to suppose that some poles of the S-matrix could be unphysical, that is, they might not correspond to external particles.[20] Such an assumption will, however, violate the unitarity of the S-matrix as is seen from a double scattering shown in Fig. 5; if this process has an unphysical pole $(s-s_B+i\varepsilon)^{-1}$ and if B is not observed

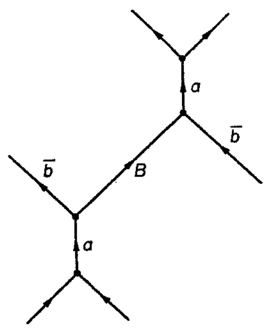

Fig. 5. A double scattering involving a bound state B in the intermediate state ($\bar{b}$ denotes the antiparticle of b).

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as a real particle, the absorptive part of this scattering amplitude, which contains $\delta(s-s_B)$, cannot be expressed in terms of products of the scattering amplitudes of subprocesses.

The following explanation provides a more reasonable resolution of the dilemma. Suppose that abnormal solutions are physical at least if they contribute to the S-matrix. The reason why they are unphysical in the static model is that since the scattering problem cannot be dealt with in the static model discussed above, a superselection rule operates between the normal solution and each of abnormal solutions. The latter belongs to a different world, which has an abnormal potential, and does not interact the normal world. In this sense, abnormal solutions are unphysical. In the relativistic theory, however, the superselection rule no longer holds, and normal and abnormal solutions are mutually transmutable. Thus abnormal solutions should be regarded as physical ones.

The following two examples may support the above explanation.

(1) The Dirac equation has negative-energy solutions, which are physical because they can be understood in terms of antiparticle states. In the non-relativistic approximation, however, the Dirac equation reduces to two decoupled Schrödinger equations. The negative-energy one of the latter is unphysical in the non-relativistic theory.

(2) Heisenberg's S-matrix[21] in potential theory can have the poles which do not correspond to the solutions of the Schrödinger equation.[22] They are unphysical poles in this sense. They correspond however, to thresholds in the crossed channel,[23] and therefore they are physically meaningful in the relativistic theory.

There may be a number of objections to regarding abnormal solutions as physical ones. One might ask why abnormal solutions are not observed in the hydrogen atom. The reason is of course attributed to the smallness of the fine structure constant. In the relativistic ladder models, no abnormal solutions appear if $g^2/4\pi<\pi$,[*] where g denotes the coupling constant.

As seen in the discussion of the Lee model, abnormal solutions seem to come out mainly from the large values of the relative time. Accordingly, one might suspect that the constituent particles would have a timelike separation in the states of abnormal solutions, which should therefore be acausal bound states. This objection is of a matter of taste because the point concerns the internal structure of the states. The non-relativistic "common sense" does not necessarily remain valid in the relativistic quantum field theory.

[*] The exact B-S equation of the static model seems to be pathological in this respect.

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§9. Norm and normalization constants

It is in principle straightforward to calculate the normalization constant of a B–S amplitude, if it is known, according to the normalization condition presented in §3. The important implication of the normalization condition, however, had never been clarified for ten years after Mandelstam's proposal of it.

Nakanishi (1965)[N16),N17)] explicitly calculated the normalization integrals in some special cases of the equal-mass Wick-Cutkosky model, and discovered that certain B–S amplitudes have negative or zero norm.*) Ohnuki and Watanabe (1965)[O2),O3)] and Ciafaloni and Menotti (1965)[C7)] independently proved that all B–S amplitudes of the odd p_0 parity have negative norm in the equal-mass ladder model, provided that $\lambda_B'(s)<0$. Nakanishi (1966)[N20)] investigated the unequal-mass Wick-Cutkosky model and found that the norm of certain solutions changes at the pseudothreshold $s=4\Delta^2$. Seto (1968, 1969)[S13),S15)] made Nakanishi's results more precise by using the solutions obtained by the stereographic projection method (see §6). Ciafaloni (1967)[C9)] investigated the norm of the B–S amplitudes in the cutoff spinor-spinor ladder model (see §11).

(A) General consideration

The normalization condition (3·5) for ladder models reads

$$i\int d^4p\bar{\phi}_{Br}(p,P)K(p,P)\phi_{Br}(p,P)=\epsilon_{Br}(P)\lambda_B(s)/\lambda_B'(s), \qquad (9\cdot1)$$

where we have inserted a norm factor $\epsilon_{Br}(P)$, which takes values not only $+1$ but also -1 and 0, because $\phi_{Br}(p,P)$ can have negative or zero norm; in that case, $|B,r\rangle$ should be regarded as a negative-norm or zero-norm state. Correspondingly, $\epsilon_{Br}(P)$ should be inserted into the numerator of (3·2) for $\epsilon_{Br}(P)=\pm1$, but the pole term of G is no longer of such a form as (3·2) for $\epsilon_{Br}(P)=0$.**)

We can generally discuss the cases in which $\phi_{Br}(p,P)$ has a definite p_0 parity $\Pi_{Br}(P)=\pm1$ [i.e., the equal-mass case for $P_\mu=(\sqrt{s},0,0,0)$, $(s>0)$, with $\eta_a=\eta_b=1/2$ and the unequal-mass (including equal-mass) case for $P_\mu=(0,0,0,\sqrt{-s})$, $(s\leq0)$; see §6(C)]. From (5·7), the Wick-rotated amplitudes are represented as

$$\tilde{\phi}_{Br}(\tilde{p},P)=\frac{-1}{2\pi i}\int_{\omega_{\min}}^{\infty}dq_0\frac{f_{Br}(q_0,\boldsymbol{p},P)}{ip_4-q_0}+\frac{1}{2\pi i}\int_{-\infty}^{\omega_{\max}}dq_0\frac{g_{Br}(q_0,\boldsymbol{p},P)}{ip_4-q_0},$$

*) Predazzi (1965)[P7)] made a check of Nakanishi's results by using an inadequate approximation.

**) In general, zero-norm B–S amplitudes do not satisfy the orthogonality condition. The normalization condition in this case was discussed by Arafune (1968).[A7)]

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$$\tilde{\phi}_{Br}(\tilde{p}, P) = \frac{-1}{2\pi i}\int_{\omega_{\min}}^{\infty} dq_0 \frac{[f_{Br}(q_0, \boldsymbol{p}, P)]^*}{ip_4 - q_0} + \frac{1}{2\pi i}\int_{-\infty}^{\omega_{\max}} dq_0 \frac{[g_{Br}(q_0, \boldsymbol{p}, P)]^*}{ip_4 - q_0}. \tag{9·2}$$

Hence

$$\tilde{\bar{\phi}}_{Br}(\tilde{p}, P) = -[\tilde{\phi}_{Br}(\boldsymbol{p}, -p_4, P)]^* = -\Pi_{Br}(P)[\tilde{\phi}_{Br}(\tilde{p}, P)]^*. \tag{9·3}$$

Therefore, on account of (5·24), the Wick-rotated form of (9·1) becomes

$$\Pi_{Br}(P)\int d^4\tilde{p}\widetilde{K}(\tilde{p},P)\,|\tilde{\phi}_{Br}(\tilde{p}, P)|^2 = -\epsilon_{Br}(P)\lambda_B(s)/\lambda_B'(s). \tag{9·4}$$

The integral in (9·4) is positive because $\widetilde{K}(\tilde{p}, P) > 0$ [cf. (5·25) for $s>0$]. Since $\lambda_B(s)>0$, therefore, if $\lambda_B'(s)<0$ is proved [see (6·16) and §7 (A)],*) we obtain

$$\epsilon_{Br}(P) = \Pi_{Br}(P). \tag{9·5}$$

Thus the B–S amplitudes of the odd p_0 parity have negative norm.

For $s<0$ and for $P_\mu=0$, the existence of negative-norm solutions has a group-theoretical reason. Since $O(2,1)$ and $O(3,1)$ are non-compact groups, they have no non-trivial, unitary representations of a finite dimension. Indeed, the p_0 parities in those cases are simply determined by the solid harmonics $\widetilde{\mathcal{Y}}_{lm}(p_1, p_2, p_0)$ and $\mathcal{Z}_{Llm}(p)$. For example, for $P_\mu=0$ (9·3) essentially reduces to

$$-i\mathcal{Z}_{Llm}^*(ip_4, \boldsymbol{p}) = -[-i\mathcal{Z}_{Llm}(-ip_4, \boldsymbol{p})]^* = -(-1)^{L-l}[-i\mathcal{Z}_{Llm}(ip_4, \boldsymbol{p})]^*, \tag{9·6}$$

because p^2 is invariant under $p_0 \to -p_0$. Thus the norm factor for $P_\mu=0$ is $(-1)^{L-l}$, and likewise that for $P_\mu=(0,0,0,\sqrt{-s})$ is $(-1)^{l-m}$.

For $P_\mu=(P_0, 0, 0, P_0)$, $(P_0\neq 0)$, the p_0 parity is no longer well defined. When the lightlike B–S amplitude is proportional to $(p_1 \pm ip_2)^{|m|}(p_3-p_0)^{l-|m|}$, its conjugate is proportional to $(p_1 \mp ip_2)^{|m|}(p_3-p_0)^{l-|m|}$. Therefore, their product involves a factor $(p_3-p_0)^{2(l-|m|)}$, which becomes $(p_3-ip_4)^{2(l-|m|)}$ when the Wick rotation is performed. Let θ be the polar angle in the (p_3, p_4) plane. Then the left-hand side of (9·1), when Wick-rotated, involves a subintegral

$$\int_0^{2\pi} d\theta\, e^{-2i(l-|m|)\theta}\,\varphi(e^{-i\theta}), \tag{9·7}$$

where $\varphi(\zeta)$ is a certain function holomorphic in $|\zeta|<1+\varepsilon$, $(\varepsilon>0)$, if $|P_0|<\min(m_a/|\eta_a|, m_b/|\eta_b|)$.**) By expanding $\varphi(\zeta)$ in powers of ζ, we find that (9·7) vanishes if $|m|<l$. Hence (9·1) implies that the lightlike solutions have zero norm unless $m=\pm l$. The implication of this result is clarified in §10

*) In the static model, we can easily prove $\lambda_B'(s)<0$,[93] and hence (9·5) is true.

**) This is because then (7·24) cannot vanish there. Cf. (5·14).

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(B) Wick-Cutkosky model

In the Wick-Cutkosky model, since we know the exact solutions, we can calculate the normalization integrals more explicitly. The conjugate, $\bar{\phi}_{\kappa nlm}(p, P)$, of (6·46) is obtained simply by replacing $B_{\kappa n}(\hat{s})\mathcal{Z}_{n-1,l,m}(\hat{p})$ by its complex conjugate $B^*_{\kappa n}(\hat{s})[\mathcal{Z}_{n-1,l,m}(\hat{p})]^*$. For $s<4\Delta^2$, both $\hat{p}_\mu$ and $q_\mu \equiv r^{-2}(1+\Delta\hat{\Delta})\hat{p}_\mu$ [cf. §6(B)] are real. In this case, it is convenient to rewrite (9·1) in terms of q_μ. After some manipulation, we find

$$-I_{\kappa nl}(\hat{s}) = (1-\Delta^2)\epsilon_{\kappa nlm}(P)\lambda_{\kappa n}(\hat{s})[d\lambda_{\kappa n}(\hat{s})/d\hat{s}]^{-1}, \tag{9·8}$$

where

$$I_{\kappa nl}(\hat{s}) \equiv i\left(1-\frac{1}{4}s\right)^{-2n-3}\int d^4q\bar{\hat{\phi}}_{\kappa nlm}(q,P)[(1+\hat{\Delta})^2-q^2] \times [(1-\hat{\Delta})^2-q^2]\hat{\phi}_{\kappa nlm}(q,P) \tag{9·9}$$

with

$$\hat{\Delta}^2 \equiv (4\Delta^2-s)/(4-s) = -\hat{s}/(4-\hat{s}), \tag{9·10}$$

$$\hat{\phi}_{\kappa nlm}(q,P) \equiv -iB_{\kappa n}(\hat{s})\mathcal{Z}_{n-1,l,m}(q)\int_{-1}^{1} dz \frac{\hat{g}_{\kappa n}(z,\hat{s})}{[\hat{f}_\Delta(z,q^2)-i\varepsilon]^{n+2}}, \tag{9·11}$$

$$\hat{f}_\Delta(z,q^2) \equiv \tfrac{1}{2}(1+z)(1-\hat{\Delta})[(1+\hat{\Delta})^2-q^2] + \tfrac{1}{2}(1-z)(1+\hat{\Delta})[(1-\hat{\Delta})^2-q^2]. \tag{9·12}$$

By means of the Wick rotation, from (9·6) we find

$$(-1)^{n-l-1}I_{\kappa nl}(\hat{s}) > 0, \tag{9·13}$$

namely,

$$\epsilon_{\kappa nlm}(P) = (-1)^{n-l-1} \tag{9·14}$$

for $s<4\Delta^2$. This result is of course dependent on the choice of the Lorentz solid harmonics. For $s<0$, it is natural to employ $\mathcal{Z}_{n-1,lm}(\hat{p}_1, \hat{p}_2, i\hat{p}_0, i\hat{p}_3)$ instead of $\mathcal{Z}_{n-1,l,m}(\hat{p})$ [cf. §6(C)]; then we have

$$\epsilon_{\kappa nlm}(P) = (-1)^{l-m} \tag{9·15}$$

instead of (9·14). Likewise, for $P_\mu = (P_0, 0, 0, P_0)$, if we replace $\mathcal{Z}_{n-1,l,m}(\hat{p})$ by $\hat{\mathcal{Z}}_{n-1,M,m}(\hat{p})$ defined in (4·14), we have

$$\begin{aligned}\epsilon_{\kappa nMm}(P) &= 1 \text{ for } M=\bar{M},\\ &= 0 \text{ for } M\neq\bar{M},\end{aligned} \tag{9·16}$$

where $\bar{M} = n-M-|m|-1$.

Near $s=4\Delta^2$, we can explicitly carry out the integration in (9·9). The Feynman parametrization yields

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$$I_{\kappa n l}(\hat{s})=\frac{(-1)^{n-l-1}}{2(n+1)^2(1-\Delta^2)^{2n+3}}|B_{\kappa n}(\hat{s})|^2 J_{\kappa n}(\hat{s}), \tag{9·17}$$

where

$$J_{\kappa n}(\hat{s})\equiv\int_0^1 dx\, x^{n+1}(1-x)^{n+1}\int_{-1}^1 dz\hat{g}_{\kappa n}(z,\hat{s})\int_{-1}^1 dz'\hat{g}_{\kappa n}(z',\hat{s}) \times\left[\frac{\partial^2}{\partial\beta\partial\alpha}\{(\alpha+\beta)^2-\alpha\beta\hat{s}\}^{-n-1}\right]_{\beta=1-\alpha} \tag{9·18}$$

with

$$\alpha\equiv\frac{1}{2}[(1+z)x+(1-z')(1-x)]. \tag{9·19}$$

We can calculate $J_{\kappa n}(\hat{s})$ near $\hat{s}=0$ by expanding $\hat{g}_{\kappa n}(z,\hat{s})$ and $\{(\alpha+\beta)^2-\alpha\beta\hat{s}\}^{-n-1}$ in powers of $\hat{s}$ to order $\hat{s}^\kappa$; this calculation is rather involved.[N17)] We find

$$J_{\kappa n}(\hat{s})=(-1)^\kappa c_{\kappa n}\hat{s}^\kappa+O(\hat{s}^{\kappa+1}), \tag{9·20}$$

where

$$c_{\kappa n}\equiv\frac{2^{2\kappa+4n+2}[(\kappa+n)!]^4[(\kappa+n+1)!]^2[(\kappa+2n)!]^3}{\kappa![(2n)!]^2(2\kappa+2n)![(2\kappa+2n+1)!]^3}>0, \tag{9·21}$$

provided that we normalize $\hat{g}_{\kappa n}(z,\hat{s})$ by

$$\hat{g}_{\kappa n}(z,0)=(1-z^2)^n C_\kappa^{n+1/2}(z). \tag{9·22}$$

From (9·8), (6·23), (9·17), (9·20) and (9·14), we have

$$|B_{\kappa n}(\hat{s})|^2=\frac{(n+1)^2[(\kappa+n)(\kappa+n+1)+n^2-1]}{(2\kappa+2n-1)(2\kappa+2n+3)c_{\kappa n}}(1-\Delta^2)^{2n+4}|\hat{s}|^{-\kappa}[1+O(\hat{s})]. \tag{9·23}$$

For $4>s>4\Delta^2$, we can no longer use (9·9) because q_μ is a complex vector there, whence we have to consider the original integral in (9·1). As is seen from (6·49), $\hat{p}_1$, $\hat{p}_2$ and $\hat{p}_3$ are real, while $\hat{p}_0$ is purely imaginary. Therefore,

$$[\mathcal{Z}_{n-1,l,m}(\hat{p})]^*\equiv\mathcal{Z}^*_{n-1,l,m}(\hat{p}^*)=(-1)^{n-l-1}\mathcal{Z}^*_{n-1,l,m}(\hat{p}). \tag{9·24}$$

Except for this extra factor $(-1)^{n-l-1}$, the normalization integral is analytically continued from $\hat{s}>0$ to $\hat{s}<0$. Hence we should have

$$I_{\kappa n l}(\hat{s})=\frac{|B_{\kappa n}(\hat{s})|^2 J_{\kappa n}(\hat{s})}{2(n+1)^2(1-\Delta^2)^{2n+3}} \tag{9·25}$$

instead of (9·17). Then (9·20) (with $\hat{s}>0$) implies that for $s>4\Delta^2$

$$\epsilon_{\kappa n l m}(P)=(-1)^\kappa \tag{9·26}$$

at least near $s=4\Delta^2$; (9·23) remains true also for $s>4\Delta^2$. Undoubtedly, (9·26) remains true in $4>s>4\Delta^2$ because there is no singular point there;

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indeed, in the equal-mass case, (9·26) is of course true because of (9·5).

As seen from (9·20), the normalization integral for $s=4\Delta^2$ vanishes unless $\kappa=0$. Therefore, (6·55) and (6·58) have zero norm unless $\kappa=0$ and $n=l+1$ [cf. (6·53) and (6·56)]. It is straightforward to obtain

$$|B'_{l+1,l}|^2=|A_{ll}B_{0,l+1}(0)|^2=b_l(1-\Delta^2)^{2l+6}, \tag{9·27}$$

$$|B^{(0)\prime}_{l+1,l,m}|^2=\frac{(2l+1)(l-|m|)!\,[(2|m|-1)!!]^2}{\pi 2^{2l-2|m|+2}(l+|m|)!}\,b_l, \tag{9·28}$$

where

$$b_l\equiv\frac{(l+2)(2l+3)\,[(2l+3)!]^2}{\pi 2^{2l+4}(2l+5)\,[(l+1)!]^4}\,. \tag{9·29}$$

The normalization constant of (6·60) is directly calculated:[N16)]

$$|B^{(0)}_{NLl}|^2=\frac{4(2N-1)(2N+1)(2N+3)(N-L-1)!\,[(2L+2)!]^2}{[N(N+1)+(N-L+l)^2-1]\,(N+L+1)!\,[(L+1)!]^2} \tag{9·30}$$

together with

$$\epsilon_{NLlm}(0)=(-1)^{L-l}. \tag{9·31}$$

Finally, we note[N17)] that the normalization constant of the normal solution behaves like $E^{(2n+3)/4}$, where E denotes the binding energy.

(C) Ghost problem

As discussed above, the appearance of the negative-norm B–S amplitudes is quite a common phenomenon in the B–S formalism. Their existence almost inevitably leads us to the introduction of negative-norm states, namely, ghosts. We may call them the B–S ghosts in order to distinguish them from the ghosts[24)] which appear in the Lee model. Contrary to the Lee-model ghosts, the B–S ghosts are caused by the manifest relativistic covariance of the theory, but not related to any pathological features at small distances. It is therefore quite plausible that the B–S ghosts appear even in any new field theory (involving a fundamental length, say) as long as the manifest relativistic covariance of the theory is retained.

The existence of the B–S ghosts of course contradicts an axiom of the quantum field theory. For example, therefore, we can no longer prove the positive definiteness of the Lehmann's spectral function[25)] for the one-particle modified propagator. Furthermore, the unitarity of the S-matrix is not obvious. Since the B–S ghosts are caused by the relativistic covariance, they have some resemblances to the scalar photons in the manifestly covariant quantum electrodynamics which was formulated by Gupta[26)] and Bleuler.[27)] Since the unitarity of the S-matrix is not violated in the latter, we conjecture that it is not violated also in the B–S formalism. Indeed, as far as we have investigated, for $(m_a+m_b)^2>s>(m_a-m_b)^2$ the B–S ghosts always vanish identically

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on the mass shell, that is, they do not contribute to the S-matrix. For $s<(m_a-m_b)^2$ they do not necessarily vanish [see (9·14)] on the mass shell, but, in this case, since either a or b becomes unstable, it may not be adequate to use $K(p, P)$ in the B–S equation. A more crucial test of the unitarity of the S-matrix is to investigate the transition probability of a process whose final state involves a B–S ghost, e.g., $a+b\to B+\gamma$, where γ may be a photon, say. This calculation is not worked out as yet.

§10. Multiple poles of the scattering Green's function

As seen in §§6 and 7, some B–S amplitudes happen to become missing when their eigenvalues become degenerate. Since the scattering Green's function $G(s, \lambda)$ should be, however, an analytic function of s and λ, it is impossible that any singularity in s cannot suddenly disappear at a particular value λ_0 of λ. Thus in such a case simple poles of $G(s, \lambda)$, whose eigenvalue trajectories coincide at $\lambda=\lambda_0$, should yield multiple poles there.

The existence of multiple poles in s in the B–S formalism was discovered by Nakanishi (1965).[N18] He derived the generalized B–S equations for the residues of multiple poles, and applied to the equal-mass Wick-Cutkosky model for $s=0$. Nakanishi (1966)[N20] extended this analysis to the unequal-mass Wick-Cutkosky model for $s=4\Delta^2$. Nakanishi (1968)[N24] and Naito (1969)[N3] showed the existence of multiple poles in the $\mu\neq 0$ ladder model. Arafune (1968)[A7] discussed the normalization properties of the generalized B–S amplitudes. Nakanishi (1969)[N25),N26] developed a general theory of multiple poles synthesized out of coinciding simple poles, together with applications to concrete models, and showed that the generalized B–S amplitudes can be expressed in terms of the ordinary B–S amplitudes.

(A) Coinciding simple poles

We suppose that the scattering Green's function $G(s, \lambda)$ has M simple poles, whose trajectories $s=s_m(\lambda)$,*) $(m=1, \cdots, M)$, become coincident at $\lambda=\lambda_0$, that is,**)

$$G(s, \lambda)=\sum_{m=1}^{M}\frac{iR_m(\lambda)}{s-s_m(\lambda)}+\widehat{G}(s, \lambda), \quad (\lambda\neq\lambda_0) \tag{10·1}$$

with

$$s_m(\lambda_0)=s_0, \quad (m=1, \cdots, M) \tag{10·2}$$

where $\widehat{G}(s, \lambda)$ is non-singular near $s=s_0$ and $\lambda=\lambda_0$. We assume that $s=s_m(\lambda)$ is an analytic function of λ and that

*) For simplicity of notation, we write m instead of B_m.

**) In this section, s is regarded as a complex variable, and hence $+i\varepsilon$ is omitted in the denominator.

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$$s_m' \equiv s_m'(\lambda_0) \neq 0. \qquad (m=1, \cdots, M) \tag{10·3}$$

In order to yield a multiple pole, it is necessary, but not sufficient as seen below, that the residues $R_m(\lambda)$ have a pole at $\lambda=\lambda_0$. Let N be the maximal integer such that

$$\lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)^N R_m(\lambda) \neq 0 \tag{10·4}$$

for some m.

Because of (10·4), the pole term of (10·1) is expanded into

$$\sum_{m=1}^{M} \frac{iR_m(\lambda)}{s-s_m(\lambda)} = i\sum_{k=0}^{N} \frac{(\lambda-\lambda_0)^{-N+k}}{k!} \left[\left(\frac{\partial}{\partial\lambda}\right)^k \sum_{m=1}^{M} \frac{(\lambda-\lambda_0)^N R_m(\lambda)}{s-s_m(\lambda)}\right]_{\lambda=\lambda_0} + O(\lambda-\lambda_0). \tag{10·5}$$

We assume that $G(s, \lambda)$ has no s-independent poles in λ (at least at $\lambda=\lambda_0$). This is indeed the case in the scalar-scalar scalar-meson-exchange ladder model because all eigenvalues tend to infinity as $s\to-\infty$ [see (6·20) and (7·7)]. It is evident that (10·5) has no fixed pole at $\lambda=\lambda_0$ if and only if

$$\frac{1}{k!}\left[\left(\frac{\partial}{\partial\lambda}\right)^k \sum_{m=1}^{M} \frac{(\lambda-\lambda_0)^N R_m(\lambda)}{s-s_m(\lambda)}\right]_{\lambda=\lambda_0} = 0. \quad (k=0, 1, \cdots, N-1) \tag{10·6}$$

In order to calculate (10·6), it is convenient to introduce the following functionals of $s_m(\lambda)$:[N25)]

$$h_{m,k}^{(j)} \equiv \lim_{\lambda\to\lambda_0} \frac{1}{j!}\left(\frac{d}{d\lambda}\right)^j \left[\frac{s_m(\lambda)-s_0}{\lambda-\lambda_0}\right]^{-k-j}$$

$$= \lim_{s\to s_0} \frac{1}{j!}\left(\frac{d}{ds}\right)^j \lambda_m'(s) \left[\frac{\lambda_m(s)-\lambda_0}{s-s_0}\right]^{k-1}. \tag{10·7}$$

Since

$$\frac{1}{k!}\left[\left(\frac{d}{d\lambda}\right)^k \frac{1}{s-s_m(\lambda)}\right]_{\lambda=\lambda_0} = \sum_{j=0}^{k} \frac{h_{m,-k}^{(j)}}{(s-s_0)^{k-j+1}} \tag{10·8}$$

as is easily shown by rewriting the expansion of $[s-s_m(\lambda)]^{-1}$ in powers of $s_m(\lambda)-s_0$ into that in powers of $\lambda-\lambda_0$, we have

$$\frac{1}{k!}\left[\left(\frac{\partial}{\partial\lambda}\right)^k \sum_{m=1}^{M} \frac{(\lambda-\lambda_0)^N R_m(\lambda)}{s-s_m(\lambda)}\right]_{\lambda=\lambda_0} = \sum_{m=1}^{M}\sum_{l=0}^{k}\sum_{j=0}^{k-l} \frac{h_{m,-k+l}^{(j)}}{(s-s_0)^{k-l-j+1}} R_m^{(l)}, \tag{10·9}$$

where

$$R_m^{(l)} \equiv \lim_{\lambda\to\lambda_0} \frac{1}{l!}\left(\frac{\partial}{\partial\lambda}\right)^l (\lambda-\lambda_0)^N R_m(\lambda). \tag{10·10}$$

Therefore, (10·6) becomes

$$\sum_{m=1}^{M}\sum_{l=0}^{n} h_{m,-k+l}^{(n-l)} R_m^{(l)} = 0, \quad (0\leq n\leq k\leq N-1) \tag{10·11}$$

and (10·1) tends to

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$$G(s, \lambda_0)=\sum_{n=0}^{N}\frac{iR^{[n]}}{(s-s_0)^{N-n+1}}+\widehat{G}(s, \lambda_0), \tag{10·12}$$

where

$$R^{[n]}\equiv\sum_{m=1}^{M}\sum_{l=0}^{n} h_{m,-N+l}^{(n-l)}R_m^{(l)}. \quad (n=0, 1, \cdots, N) \tag{10·13}$$

In particular, since

$$h_{m,-k}^{(0)}=s_m'^k, \tag{10·14}$$

(10·11) and (10·13) imply

$$\sum_{m=1}^{M} s_m'^k R_m^{(0)}=\delta_{kN}R^{[0]}. \quad (k=0, 1, \cdots, N) \tag{10·15}$$

If $M<N+1$, then the Vandermonde determinant,

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ s_1' & s_2' & \cdots & s_M' \\ \cdots & \cdots & \cdots & \cdots \\ s_1'^{M-1} & s_2'^{M-1} & \cdots & s_M'^{M-1} \end{vmatrix} = \sum_{M\geq m>l\geq 1}(s_m'-s_l'), \tag{10·16}$$

has to vanish because otherwise we have $R_m^{(0)}=0$, $(m=1, \cdots, M)$, in contradiction with (10·4). Thus $M<N+1$ is possible only when some of s_m' coincide. On the other hand, if $M>N+1$, then we expect that in most cases it is possible to select $N+1$ poles from M ones without changing the essential features.

The most important case is $M=N+1$. In this case, from (10·15) and (10·16) with $M=N+1$, we have the following important result: A necessary and sufficient condition for $R^{[0]}\neq 0$, i.e., for the existence of a multiple pole of order $N+1$ is that $s_1', \cdots, s_{N+1}'$ are different from each other. We can further prove,[N25)] by using (10·13) and (10·7), that if $s_1', \cdots, s_{N+1}'$ are mutually equal then

$$R^{[n]}=0, \quad (n=0, 1, \cdots, [(N-1)/2]) \tag{10·17}$$

where $[k]$ denotes the greatest integer not exceeding k, and that if the Taylor expansions of $s_1'(\lambda), \cdots, s_{N+1}'(\lambda)$ at $\lambda=\lambda_0$ are common to order $(\lambda-\lambda_0)^N$ then*)

$$R^{[n]}=0, \quad (n=0, 1, \cdots, N-1) \tag{10·18}$$

that is, the multiple pole is absent at $s=s_0$ in spite of the singular behavior of $R_m(\lambda)$ at $\lambda=\lambda_0$. A typical example of (10·18) is provided by the unequal-mass Wick-Cutkosky model for $s=0$.

(B) Generalized B-S amplitudes

The scattering Green's function $G(s, \lambda)$ satisfies

*) Of course $R^{[N]}$ cannot vanish even if $s_1'(\lambda)=\cdots=s_{N+1}'(\lambda)$.

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$$H(s,\lambda)G(s,\lambda)=1, \tag{10·19}$$

where

$$H(s,\lambda)\equiv K(s,\lambda)-I(s,\lambda). \tag{10·20}$$

As discussed in §2, (10·19) and (10·1) yield the B–S equations

$$H(s_m(\lambda),\lambda)R_m(\lambda)=0. \tag{10·21}$$

Let

$$H^{(l)}\equiv\frac{1}{l!}\left[\left(\frac{\partial}{\partial s}\right)^{l}H(s,\lambda_0)\right]_{s=s_0}. \tag{10·22}$$

Then (10·19) and (10·12) yield the "generalized B–S equations"

$$\sum_{l=0}^{n}H^{(n-l)}R^{[l]}=0. \quad (n=0,1,\cdots,N) \tag{10·23}$$

Now, $R_m(\lambda)$ can be written as

$$R_m(\lambda)=\sum_r R_{mr}(\lambda) \tag{10·24}$$

with

$$R_{mr}(\lambda)=\epsilon_{mr}(\lambda)\phi_{mr}(\lambda)\bar{\phi}_{mr}(\lambda), \tag{10·25}$$

where $\epsilon_{mr}(\lambda)$ is a norm factor (see §9). Since in the ordinary situation the r-dependence of the B–S amplitudes is purely determined by their solid harmonics, we can deal with each set of the partial residues $R_{1r}(\lambda),\cdots,R_{Mr}(\lambda)$ separately. That is to say, we can use (10·10), (10·11), (10·13), (10·15), (10·21) and (10·23) by affixing a subscript r to each residue.

We assume that $s_1(\lambda),\cdots,s_M(\lambda)$ are real in a real neighborhood of $\lambda=\lambda_0$. Then it is evident from (10·13) together with (10·25) and (10·10) that $R_r^{[n]}$ is time-reversal invariant and that $R_r^{[n]}$ is separable.*) Furthermore, for $M=N+1$ (we hereafter consider this case only), (10·15) implies that $R_{mr}^{(0)}$ is proportional to $R_r^{[0]}$. Accordingly, $R_r^{[0]}$ is factorizable as

$$R_r^{[0]}=\phi_r^{[0]}\bar{\psi}_r^{[0]}, \tag{10·26}$$

where $\psi_r^{[0]}$ is proportional to $\phi_r^{[0]}$ because of time-reversal invariance. Then we can inductively show by means of (10·23)[N18] that $R_r^{[n]}$ is expressed as

$$R_r^{[n]}=\sum_{k=0}^{n}c_r^{(k)}\sum_{l=0}^{n-k}\phi_r^{[n-k-l]}\bar{\phi}_r^{[l]}, \quad (n=0,1,\cdots,N) \tag{10·27}$$

where $c_r^{(0)},\cdots,c_r^{(N)}$ are undetermined real constants. We call $\phi_r^{[0]},\cdots,\phi_r^{[N]}$ the "generalized B–S amplitudes",**) which satisfy the generalized B–S equations

*) A function $A(p,q)$ is separable if it can be expressed as $A(p,q)=\Sigma_k f_k(p)_k g(q)$, where the summation goes over a finite number of terms.

**) The states corresponding to $\phi_r^{[n]}$, $(n>0)$, are not the states in the ordinary sense but are called multipole ghosts, an example of which was first constructed by Heisenberg[28] in the Lee model.

$$\sum_{l=0}^{n} H^{(n-l)}\phi_r^{[l]}=0. \quad (n=0,1,\cdots,N) \tag{10·28}$$

The generalized B–S amplitudes $\phi_r^{[n]}$ can be expressed in terms of the ordinary ones $\phi_{mr}(\lambda)$. Taking (10·4) into account, we rewrite (10·25) as

$$R_{mr}(\lambda)=\epsilon_r\xi_m[s_m(\lambda)-s_0]^{-N}\varphi_{mr}(\lambda)\bar{\varphi}_{mr}(\lambda), \tag{10·29}$$

where $\epsilon_r=\pm 1$ is the r-dependent part of $\epsilon_{mr}(\lambda)$ and ξ_m is a real constant, which is appropriately chosen later. The unnormalized B–S amplitude $\varphi_{mr}(\lambda)$ is assumed to be expanded into

$$\varphi_{mr}(\lambda)=\sum_{j=0}^{N}[s_m(\lambda)-s_0]^j\varphi_{mr}^{(j)}+O((\lambda-\lambda_0)^{N+1}). \tag{10·30}$$

The sign of $\epsilon_r\xi_m[s_m(\lambda)-s_0]^{-N}$ equals the norm factor $\epsilon_{mr}(\lambda)$, whence it *changes* at $s_m(\lambda)=s_0$ if N is odd. From (10·15) and (10·29) we have

$$\sum_{m=1}^{N+1} s_m'^{-l}\,\xi_m\cdot\epsilon_r\varphi_{mr}^{(0)}\bar{\varphi}_{mr}^{(0)}=\delta_{l0}R_r^{[0]}. \quad (l=0,1,\cdots,N) \tag{10·31}$$

If we choose ξ_{mr} in such a way that

$$\sum_{m=1}^{N+1} s_m'^{-l}\xi_m=\delta_{l0}, \quad (l=0,1,\cdots,N) \tag{10·32}$$

then we find

$$R_r^{[0]}=\epsilon_r\varphi_{mr}^{(0)}\bar{\varphi}_{mr}^{(0)}, \tag{10·33}$$

that is,

$$\phi_r^{[0]}=\varphi_{1r}^{(0)}=\cdots=\varphi_{N+1,r}^{(0)} \tag{10·34}$$

with $c_r^{(0)}=\epsilon_r$. Since $\phi_r^{[0]}$ satisfies the ordinary B–S equation, (10·34) tells us that $N+1$ B–S amplitudes $\phi_{mr}(\lambda)$, $(m=1,\cdots,N+1)$, apart from the normalization constant, tend to a common B–S amplitude $\phi_r^{[0]}$ as $\lambda\to\lambda_0$.

By using (10·11), and (10·13), we can further analyze the relationship between $\phi_r^{[n]}$ and $\varphi_{mr}(\lambda)$. We conjecture, and can prove for $n\leq 2$,[N25)] that

$$\phi_r^{[n]}=\sum_{m=1}^{N+1}\xi_m\varphi_{mr}^{(n)}, \quad (n=0,1,\cdots,N) \tag{10·35}$$

if the constants $c_r^{(k)}$ are chosen appropriately, where

$$\xi_m=\frac{\prod_{n\neq m} s_n'^{-1}}{\prod_{n\neq m}(s_n'^{-1}-s_m'^{-1})}=\frac{\prod_{n\neq m}\lambda_n'}{\prod_{n\neq m}(\lambda_n'-\lambda_m')} \tag{10·36}$$

according to (10·32).

For $n=1$, it is straightforward to check that $\phi_r^{[1]}$ given by (10·35) satisfies the generalized B–S equation of the first order,

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$$H^{(1)}\phi_r^{[0]}+H^{(0)}\phi_r^{[1]}=0. \tag{10·37}$$

In fact, by differentiating the B–S equation

$$H(s,\lambda_m(s))\varphi_{mr}(\lambda_m(s))=0 \tag{10·38}$$

with respect to s and setting $s=s_0$, we find

$$(H^{(1)}+s_m'^{-1}[\partial H/\partial\lambda]_{s=s_0})\phi_r^{[0]}+H^{(0)}\varphi_{mr}^{(1)}=0 \tag{10·39}$$

on account of (10·34); hence on summing (10·39) over m after multiplying it by ξ_m, we obtain (10·37) by means of (10·32).

Since $H^{(0)}\phi_r^{[0]}=0$, (10·37) is solvable if and only if

$$i\bar{\phi}_{r'}^{[0]}H^{(1)}\phi_r^{[0]}=0 \qquad \text{for any } r'. \tag{10·40}$$

In particular, for $r'=r$, (10·40) implies that $\phi_r^{[0]}$ has zero norm [see (3·8)]. Thus the existence of zero-norm B–S amplitudes is a necessary condition (at least in the case $M=N+1$) for the existence of multiple poles of the scattering Green's function.

(C) Examples

In the Wick-Cutkosky model, multiple poles appear at the pseudothreshold $s=4\Delta^2$ (including the equal-mass case). From (6·46) together with (6·49) or (6·50) and (9·23), we see that the partial residue $R_{\kappa nlm}\equiv\epsilon_{\kappa nlm}\phi_{\kappa nlm}\bar{\phi}_{\kappa nlm}$ behaves like

$$R_{\kappa nlm}\sim(s-4\Delta^2)^{-(\kappa+n-l-1)} \tag{10·41}$$

near $s=4\Delta^2$. For fixed values of $\kappa+n$, l and m, $N+1$ eigenvalues $\lambda_{\kappa n}(s)$ corresponding to $\kappa=0,1,\cdots,N$ become degenerate at $s=4\Delta^2$, where $N=\kappa+n-l-1$. Furthermore, (6·23) shows that $\lambda'_{0,N+l+1}$, $\lambda'_{1,N+l}$, $\cdots$, $\lambda'_{N,l+1}$ are different from each other. Therefore, according to the general theory described above, at $s=4\Delta^2$ we have a multiple pole of order $N+1=\kappa+n-l$ for each $(\kappa+n,l,m)$. The generalized B–S amplitudes can, in principle, be calculated from (6·46) according to (10·35). They can also be obtained by solving the generalized B–S equation (10·28).[N18)]

Multiple poles are generally present at $s=0$ because the extra degeneracy of the eigenvalues happens due to the $O(3,1)$ symmetry for $P_\mu=0$.*) For definiteness, we consider the scalar-scalar $\mu\neq0$ ladder model. As stated in §7(A), the eigenvalues $\lambda_{\nu Ll}(0)$ are independent of l. We assume that $\lambda_{\nu Ll}(0)\neq\lambda_{\nu'L'l'}(0)$ unless $\nu=\nu'$ and $L=L'$. For fixed values of ν, L and m, therefore, we have $L-|m|+1$ normalized B–S amplitudes

*) The unequal-mass Wick-Cutkosky model is exceptional because the $O(3,1)$ symmetry exists even for $s\neq0$.

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$$\phi_{\nu Llm}(p,P) = -iB_{\nu Ll}(s)\mathcal{Y}_{lm}(p,P)f_{\nu Ll}(v,w,s),$$
$$(l=|m|,\ |m|+1,\cdots,L) \tag{10·42}$$

whose eigenvalues become degenerate at $s=0$, where $\mathcal{Y}_{lm}(p,P)$ is defined by (4·22). The normalization constant $B_{\nu Ll}(s)$ is defined in such a way that $f_{\nu Ll}(v,w,0)$ is finite but does not vanish identically. Then, as discussed in §7(C), in order for (10·42) to tend to

$$\phi_{\nu Llm}(p,0) = -iB'_{\nu Ll}\mathcal{Z}_{Llm}(p)f_{\nu L}(p^2) \tag{10·43}$$

as $s\to 0$ with $\boldsymbol{P}=0$ (hence $P_0\to 0$), $B_{\nu Ll}(s)$ has to behave like $s^{-(L-l)/2}$ near $s=0$. Furthermore, as shown in (4·29), $\mathcal{Y}_{lm}(p,P)$ is of order $s^{-(l-|m|)/2}$ near $s=0$ in a moving frame. Thus the partial residue $R_{\nu Llm}\equiv\epsilon_{\nu Llm}\phi_{\nu Llm}\bar{\phi}_{\nu Llm}$ behaves like

$$R_{\nu Llm}\sim s^{-(L-|m|)} \tag{10·44}$$

near $s=0$. According to the general theory, therefore, at $s=0$ we have a multiple pole of order $L-|m|+1$ for each (ν,L,m), as long as $\lambda'_{\nu Ll}(0)\neq\lambda'_{\nu Ll'}(0)$ unless $l=l'$.

In the Lorentz frame (4·27), from (4·22) with (4·6) we find

$$R^{(0)}_{\nu Llm} = s'^{-L+|m|}_{\nu Ll}\frac{(l+|m|)!}{(l-|m|)!(|m|!)^2}a^{-2|m|}(p_1\pm ip_2)^{|m|}(p_3-p_0)^{l-|m|}$$
$$(q_1\mp iq_2)^{|m|}(q_3-q_0)^{l-|m|}A_{\nu Ll}(p,q,P), \tag{10·45}$$

where $A_{\nu Ll}$ is a quantity independent of m. On the other hand, from (10·15) we have

$$R^{(0)}_{\nu Llm} = R^{[0]}_{\nu Lm}\Big/\prod_{j=|m|,\,j\neq l}^{L}(s'_{\nu Ll}-s'_{\nu Lj}). \tag{10·46}$$

Therefore

$$\frac{R^{(0)}_{\nu,L,L,m}R^{(0)}_{\nu,L,L-1,m+1}}{R^{(0)}_{\nu,L,L,m+1}R^{(0)}_{\nu,L,L-1,m}} = \frac{(L+m)(L-m-1)s'_{\nu,L,L-1}}{(L+m+1)(L-m)s'_{\nu LL}} = \frac{s'_{\nu,L,L-1}-s'_{\nu Lm}}{s'_{\nu LL}-s'_{\nu Lm}} \tag{10·47}$$

for $0\leq m\leq L-1$. From (10·47) we have

$$\lambda'_{\nu Ll} = (1/2L)[-(L+l)(L-l-1)\lambda'_{\nu LL}+(L+l+1)(L-l)\lambda'_{\nu,L,L-1}], \tag{10·48}$$

a result which is equivalent to (7·4). That is to say, the consistency condition for multiple poles corresponding to different values of m exactly reproduces the perturbation formula (7·4) for eigenvalues which is a consequence of the $O(3,1)$ symmetry at $P_\mu=0$.

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§11. Spinor-spinor model

In this section, we consider the B–S equation for a system of two spinor particles which exchange bosons having spin 0 or 1. Almost all practically important two-body systems, e.g., hydrogen atom, positronium and deuteron, belong to this case. In spite of its importance, much is not known about the solutions because the B–S amplitude consists of many components and because the Wick-rotated kernel is not of Fredholm type.

In the early stage of its research, the spinor-spinor B–S equation was approximately solved for small binding energies. The non-relativistic limit of the B–S equation was discussed by Salpeter and Bethe (1951)[S2] and Hayashi and Munakata (1952).[H4] The relativistic effects on hydrogen atom was investigated by Salpeter (1952),[S3] Brown (1952),[B19] Arnowitt (1953)[A9] and Newcomb and Salpeter (1955).[N32] The hyperfine structure of positronium was calculated by Karplus and Klein (1952),[K1] Fulton and Karplus (1954)[F6] and Fulton and Martin (1954).[F7] Those results are important as the experimental verification of the B–S equation. The two-nucleon system was considered by Lévy (1952)[L9] and Klein (1953)[K11] in the instantaneous approximation. In order to eliminate the negative-energy components, they employed a perturbation expansion in powers of $\mu/2m$, but Arnowitt and Gasiorowitz (1954)[A10] criticized the adequacy of this procedure. The general formal solutions in the instantaneous approximation were obtained by Green and Biswas (1957)[G13] and later by Reinfelds (1962).[R1] Kawaguchi (1961)[K4] considered approximate solutions by neglecting recoil effects.

The fully covariant B–S equation in the ladder approximation was first investigated by Goldstein (1953).[G4] He found that the pseudoscalar part of the B–S amplitude decouples from the remainders when $m_a = m_b$ and $P_\mu = 0$, and obtained the explicit solution for $\mu = 0$, which exists for any positive value of λ. In order to eliminate this continuous spectrum, Goldstein introduced a cutoff, and proposed to take out a particular value of λ such that the solution would become independent of the cutoff. A similar standpoint was taken by McCarthy and Green (1954).[M10] However, Green (1955)[G11] pointed out an error committed by Goldstein, and it became hopeless to extract discrete solutions in the pseudoscalar equation. Mandelstam (1955, 1956)[M4),M5] investigated the physical acceptability of the solution, and concluded that $4\pi\lambda$ should be smaller than $\pi/6$. Non-Goldstein exact solutions for $P_\mu = 0$, $m_a = m_b$ and $\mu = 0$ were investigated by a number of authors. Biswas and Green (1956)[B12] obtained some position-space solutions for spinless meson exchange. Bastai, Bertocchi, Furlan and Tonin (1963)[B8] found a very simple solution to the scalar-vector part for the vector-coupling theory. Kummer (1964)[K14] discovered a discrete set of solutions to the axialvector-tensor part for vector coupling.

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On the other hand, Scarf and Umezawa (1958)[S8] pointed out that the spinor-spinor B–S equation reduces to the scalar-scalar one if the coupling is a parity-violating one, $1+\gamma_5$. In the analysis of the cutoff Fermi-Yang model,[29),*)] Baumann and Thirring (1960)[B9] used the single dispersion representation for the $P_\mu=0$, $\mu\neq0$ spinless bound-state amplitudes [see also, Baumann, Freund and Thirring (1960),[B10] Nakanishi (1965)[N15] and Munakata and Aotsuka (1968)[M14]]. Tiktopoulos (1965)[T7] proved the existence of a discrete spectrum for $4\pi\lambda<1/2$ in the $\mu=0$ case, and showed that some of the B–S amplitudes indeed tend to the non-relativistic wave functions.

Many authors tried to reduce the spinor-spinor B–S equation to a more convenient form on the basis of group-theoretical analysis. Gourdin (1958, 1959)[G6),G7)] decomposed the B–S amplitude into four 3-scalars and four 3-vectors, and expanded them in terms of (scalar and vector) four-dimensional spherical harmonics. After carrying out the angular integrations, he obtained three decoupled systems of one-dimensional integral equations suitable for numerical computation [see also, Gourdin and Tran Thanh Van (1960)[G9] and Swift and Lee (1963)[S21]]. Günther (1964)[G16] made separation of angular variables in the position-space B–S equation for $m_a=m_b$ and $\mu=0$. Keam (1968)[K7] made an elegant group-theoretical analysis of the position-space B–S equation for $P_\mu=0$. The $P_\mu=0$ amplitude was investigated also in connection with the classification of the Lorentz (or Toller) poles[30] by Mueller (1968)[M11] and Ito (1969).[I3]

Gürsey, Lee and Nauenberg (1964)[G17] utilized the B–S equation for the derivation of the $SU(3)$ mass formula. Daboul and Delbourgo (1966)[D1] also considered departures from a higher symmetry called $\widetilde{U}(12)$. Delbourgo, Salam and Strathdee (1967)[D2] discussed an approximate $O(5)$ symmetry of the B–S equation for $P_\mu=0$, $m_a=m_b$ and $\mu=0$. Ciafaloni (1967)[C9] investigated symmetry and normalization properties in the cutoff B–S equation. Barbieri, Ciafaloni and Menotti (1968)[B4] analyzed the gauge-non-invariance property of the ladder approximation.

Accurate numerical computations of the B–S equation were made by Narayanaswamy and Pagnamenta (1968).[N30),N31)] They found strong cutoff dependence of solutions. Ito, Mizouchi, Murota, Nakano, Noda and Tanaka (1967)[I2] made a detailed numerical analysis of the nucleon-nucleon scattering [see also, Murota, Noda and Tanaka (1969)[M15]].

We employ the following definitions of the γ matrices:

*) The case of a factorizable kernel (chain model) was investigated by Katsumori (1954),[K2] Maki (1956),[M3] Polubarinov (1958)[P6] and others. Yamamoto (1959)[Y2] replaced the ladder-model kernel by a factorizable one as an approximation.

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$$\begin{aligned}
&\gamma_0=\gamma^0=(\gamma_0)^\dagger,\quad \gamma_k=-\gamma^k=-(\gamma_k)^\dagger,\quad (k=1,2,3)\\
&\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu},\quad \gamma_\mu\gamma^\mu=4,\\
&\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3=-\gamma_5^\dagger,\quad \gamma_\mu\gamma_5=-\gamma_5\gamma_\mu,\\
&(\gamma_0)^2=-(\gamma_k)^2=-(\gamma_5)^2=1,\quad (k=1,2,3)\\
&C^{-1}\gamma^\mu C=-(\gamma^\mu)^T,
\end{aligned} \tag{11·1}$$

where C stands for the charge conjugation matrix, and the symbols † and T denote hermitian conjugation and transposition, respectively.

The fermion-antifermion B–S equation in the ladder approximation reads*)

$$\begin{aligned}
&[m_a-\gamma^\mu(\eta_a P_\mu+p_\mu)]\phi(p,P)[m_b+\gamma^\nu(\eta_b P_\nu-p_\nu)]\\
&\quad=\frac{\lambda}{\pi^2 i}\int d^4p'\frac{-g_{\mu\nu}}{\mu^2-(p-p')^2-i\varepsilon}\Gamma^\mu\phi(p',P)\Gamma^\nu
\end{aligned} \tag{11·2}$$

with

$$\begin{aligned}
\Gamma^\mu&=(0,0,0,1) && \text{for scalar coupling,}\\
&=(0,0,0,\gamma_5) && \text{for pseudoscalar coupling,}\\
&=\gamma^\mu && \text{for vector coupling,}\\
&=i\gamma_5\gamma^\mu && \text{for axialvector coupling,}
\end{aligned} \tag{11·3}$$

where we have suppressed subscripts B, r. The B–S amplitude $\phi(p,P)$ is expressed as a 4×4 matrix. Likewise, the fermion-fermion B–S equation reads

$$\begin{aligned}
&[m_a-\gamma^\mu(\eta_a P_\mu+p_\mu)]\phi(p,P)[m_b-(\gamma^\nu)^T(\eta_b P_\nu-p_\nu)]\\
&\quad=\frac{\lambda}{\pi^2 i}\int d^4p'\frac{-g_{\mu\nu}}{\mu^2-(p-p')^2-i\varepsilon}\Gamma^\mu\phi(p',P)(\Gamma^\nu)^T.
\end{aligned} \tag{11·4}$$

Let

$$\phi^C(p,P)\equiv\phi(p,P)C^{-1}; \tag{11·5}$$

then (11·4) is rewritten as

$$\begin{aligned}
&[m_a-\gamma^\mu(\eta_a P_\mu+p_\mu)]\phi^C(p,P)[m_b+\gamma^\nu(\eta_b P_\nu-p_\nu)]\\
&\quad=\frac{\lambda}{\pi^2 i}\int d^4p'\frac{-g_{\mu\nu}}{\mu^2-(p-p')^2-i\varepsilon}\Gamma^\mu\phi^C(p',P)\hat{\Gamma}^\nu
\end{aligned} \tag{11·6}$$

with

$$\begin{aligned}
\hat{\Gamma}^\nu&=-\Gamma^\nu && \text{for vector coupling,}\\
&=\Gamma^\nu && \text{otherwise.}
\end{aligned} \tag{11·7}$$

---

*) We neglect the gradient-dependent term in the propagator of the spin 1 meson.

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Thus, apart from the sign of λ for vector coupling, we have only to consider (11·2) without loss of generality.

For $P_\mu=0$ and $m_a=m_b=1$ with $\eta_a=\eta_b=1/2$, it is convenient to decompose the B–S amplitude in terms of the γ matrices in the following way:

$$\phi(p,0)\gamma_5=\phi^P(p)+\gamma^\mu\phi_\mu^A(p)+\frac{1}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\phi_{\mu\nu}^T(p)$$
$$+\gamma^\mu\gamma_5\phi_\mu^V(p)+\gamma_5\phi^S(p) \qquad (11\cdot8)$$

with $\phi_{\mu\nu}^T=-\phi_{\nu\mu}^T$. On inserting (11·8) into (11·2), we find

$$(1-p^2)\phi^P=\lambda^P I[\phi^P], \qquad (11\cdot9)$$

$$(1+p^2)\phi_\mu^A-2p_\mu p^\nu\phi_\nu^A+4p^\nu\phi_{\mu\nu}^T=\lambda^A I[\phi_\mu^A], \qquad (11\cdot10)$$

$$(1-p^2)\phi_{\mu\nu}^T+2(p_\nu p^\sigma\phi_{\mu\sigma}^T-p_\mu p^\sigma\phi_{\nu\sigma}^T)$$
$$-(p_\mu\phi_\nu^A-p_\nu\phi_\mu^A)=\lambda^T I[\phi_{\mu\nu}^T]; \qquad (11\cdot11)$$

$$(1-p^2)\phi_\mu^V+2p_\mu p^\nu\phi_\nu^V-2p_\mu\phi^S=\lambda^V I[\phi_\mu^V], \qquad (11\cdot12)$$

$$(1+p^2)\phi^S-2p^\mu\phi_\mu^V=\lambda^S I[\phi^S], \qquad (11\cdot13)$$

where

$$I[f]\equiv\frac{1}{\pi^2 i}\int d^4p'\frac{f(p')}{\mu^2-(p-p')^2-i\varepsilon} \qquad (11\cdot14)$$

and*)

	scalar	pseudo-scalar	vector (fa)	vector (ff)	axial-vector
λ^P/λ	1	−1	4	−4	−4
λ^A/λ	1	1	−2	2	−2
λ^T/λ	1	−1	0	0	0
λ^V/λ	1	1	2	−2	2
λ^S/λ	1	−1	−4	4	4.

Thus we have three decoupled systems ϕ^P, $(\phi_\mu^A, \phi_{\mu\nu}^T)$ and (ϕ_μ^V, ϕ^S).

The ϕ^P equation (11·9) is called the Goldstein equation. When $\mu=0$, for any $\lambda^P>0$ we have

$$\phi_{Llm}^P(p)=-i\mathcal{Z}_{Llm}(p)F(-\rho+1,\rho+L+2;\,L+2;\,p^2+i\varepsilon)$$
$$=-i\mathcal{Z}_{Llm}(p)\int_0^\infty d\alpha\frac{\varphi_L(\alpha)}{(\alpha+1-p^2-i\varepsilon)^{L+3}} \qquad (11\cdot15)$$

*) In the table, fa and ff are abbreviations of a fermion-antifermion system and a fermion-fermion one, respectively.

with

$$\varphi_L(\alpha)\equiv\alpha^{L+1}F(-\rho,\rho+L+1;\ L+2;\ -\alpha)/B(-\rho+1,\rho+L+2) \quad (11\cdot16)$$

and

$$\rho\equiv-\frac{1}{2}(L+1)+\left[\frac{1}{4}(L+1)^2-\lambda^P\right]^{1/2}, \quad (11\cdot17)$$

where F and B denote a hypergeometric function and a beta function, respectively. When $\mu\neq0$, as in §7(C), we can write

$$\phi^P_{Llm}(p)=-i\mathscr{Z}_{Llm}(p)\int_{-0}^{\infty}d\alpha\frac{\varphi_L^{[h]}(\alpha)}{(\alpha+1-p^2-i\varepsilon)^{h+1}} \quad (11\cdot18)$$

with

$$\varphi_L^{[h]}(\alpha)=h!\left(\int_{-0}^{\infty}d\alpha\right)^h\varphi_L^{[0]}(\alpha). \quad (11\cdot19)$$

The spectral function $\varphi_L^{[0]}(\alpha)$ satisfies

$$\varphi_L^{[0]}(\alpha)=\delta(\alpha)+\lambda^P\int_0^{\infty}d\alpha' K_L(\alpha,\alpha')\varphi_L^{[0]}(\alpha') \quad (11\cdot20)$$

together with

$$1=\lambda^P\int_0^{\infty}d\alpha\varphi_L^{[1]}(\alpha)\int_0^1 dx\frac{x^L(1-x)}{(1-x)\alpha+(1-x)^2+x\mu^2}, \quad (11\cdot21)$$

where

$$K_L(\alpha,\alpha')\equiv-\alpha^{-1}\int_0^1 dx\, x^L\theta(x(1-x)(\alpha+1)-(1-x)(\alpha'+1)-x\mu^2). \quad (11\cdot22)$$

It is noteworthy that (11·20) and (11·21) are well defined even for $\mu=0$; indeed they are satisfied by[N15)]

$$\varphi_L^{[0]}(\alpha)=[B(-\rho+1,\rho+L+2)/(L+1)!](d/d\alpha)^{L+2}[\varphi_L(\alpha)\theta(\alpha)], \quad (11\cdot23)$$

where $\varphi_L(\alpha)$ is given by (11·16).

The ϕ^T equations (11·11) become algebraic equations for vector and axialvector couplings because then $\lambda^T=0$. Hence it is straightforward to obtain

$$(1+p^2)\phi^T_{\mu\nu}=p_\mu\phi^A_\nu-p_\nu\phi^A_\mu. \quad (11\cdot24)$$

On substituting (11·24) for $\phi^T_{\mu\nu}$ in (11·10), we find

$$(1-p^2)[(1-p^2)\phi^A_\mu+2p_\mu p^\nu\phi^A_\nu]=\lambda^A(1+p^2)I[\phi^A_\mu]. \quad (11\cdot25)$$

Since it is still somewhat difficult to solve (11·25) generally, we consider*)

*) If ϕ^A_μ is proportional to p_μ, then (11·25) reduces to the Goldstein equation.

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only the case in which ϕ_μ^A satisfies the Lorentz condition

$$p^\mu \phi_\mu^A = 0 . \tag{11·26}$$

Then (11·25) reduces to

$$(1-p^2)^2 \phi_\mu^A(p) = \lambda^A (1+p^2) \frac{1}{\pi^2 i} \int \frac{d^4 p'}{-(p-p')^2 - i\varepsilon} \phi_\mu^A(p') \tag{11·27}$$

for $\mu=0$. It is remarkable that (11·27) has a discrete set of solutions for $\lambda^A > 0$, namely for the fermion-fermion, vector-coupling case. We have[K14),*)]

$$\phi_{NLlm}^A(p) = -i\mathcal{Z}_{Llm}(p) \frac{1+p^2}{(1-p^2-i\varepsilon)^{(\xi+3)/2}}$$
$$\times F(-N+L+1,\ -\xi+N+1;\ -\xi+1;\ 1-p^2),$$
$$(N-1 \geq L \geq l \geq |m|) \tag{11·28}$$

with

$$\xi \equiv \sqrt{8\lambda_{NL}^A + 1} > 2N+1 , \tag{11·29}$$

where the eigenvalues $\lambda^A = \lambda_{NL}^A$ are determined by

$$\sqrt{8\lambda_{NL}^A + 1} - \sqrt{4\lambda_{NL}^A + (L+1)^2} = 2N - L , \tag{11·30}$$

namely

$$\lambda_{NL}^A = 2\lambda_{NL} = 3N(N-L) + L\left(L+\frac{1}{2}\right)$$
$$+ \left[8N^2(N-L)^2 + N(N-L)(6L^2+4L+1) + L^2\left(L+\frac{1}{2}\right)^2\right]^{1/2} . \tag{11·31}$$

In particular, for $N=L+1$ (11·28) with (11·31) reduces to

$$\phi_{L+1,Llm}^A(p) = -i\mathcal{Z}_{Llm}(p) \frac{1+p^2}{(1-p^2-i\varepsilon)^{2L+5}} \tag{11·32}$$

with

$$\lambda_{L+1,L}^A = 2\lambda_{L+1,L} = (L+2)(2L+3). \tag{11·33}$$

For $L>0$, we can always construct ϕ_μ^A satisfying (11·26) by taking linear combinations of (11·28).

Though it is difficult to solve the (ϕ^V, ϕ^S) equations (11·12) and (11·13), we can find a particular solution from the fermion-fermion, vector-coupling equation

*) In Kummer's paper,[K14)] the argument of a hypergeometric function (2·31) has a wrong sign.

$$(1-\gamma^{\mu}p_{\mu})\phi^{C}(p,0)(1-\gamma^{\nu}p_{\nu})=\frac{\lambda}{\pi^{2}i}\int d^{4}p'\frac{\gamma^{\mu}\phi^{C}(p',0)\gamma_{\mu}}{-(p-p')^{2}-i\varepsilon}\,. \tag{11·34}$$

We have a solution[B8)]

$$\phi^{C}(p,0)=-(i/2)(\partial/\partial p)^{2}(1-\gamma p-i\varepsilon)^{-1}$$
$$=-i(1-\gamma p-i\varepsilon)^{-1}\gamma^{\mu}(1-\gamma p-i\varepsilon)^{-1}\gamma_{\mu}(1-\gamma p-i\varepsilon)^{-1} \tag{11·35}$$

with

$$\lambda=1/2\,, \tag{11·36}$$

as is easily verified by using

$$(\partial/\partial p)^{2}(-p^{2}-i\varepsilon)^{-1}=4\pi^{2}i\delta^{4}(p). \tag{11·37}$$

In terms of the notation of (11·8), (11·35) is rewritten as

$$\phi_{\mu}^{V}(p)=-ip_{\mu}\frac{6-2p^{2}}{(1-p^{2}-i\varepsilon)^{3}}\,, \tag{11·38}$$

$$\phi^{S}(p)=-i\frac{4}{(1-p^{2}-i\varepsilon)^{3}}\,. \tag{11·39}$$

It is quite plausible that the solution (11·35) belongs to a discrete spectrum.

The vector-coupling, $\mu=0$, equal-mass case is realized in the quantum electrodynamics. As is physically expected, near $s=4$ we have a discrete set of solutions for the fermion-antifermion system,[T7)] which corresponds to positronium, but not for the fermion-fermion system. According to the above results, however, this situation is reversed at $s=0$. It is unsolved as yet what happens in an intermediate energy. It is also interesting to investigate the gauge dependence of the solutions.

§12. Regge behavior

The scattering B–S equations provide convenient relativistic models for studying the Regge behavior in the t channel. For this reason, in 1962–1965 much work was done on the Reggeization of the scattering B–S equation.

Lee and Sawyer (1962)[L3),L4)] were the first to introduce the complex angular momentum l into the B–S formalism. They showed the meromorphy for $\mathrm{Re}\,l>-3/2$ in the scalar-meson-exchange ladder model by using the (unjustified) Wick rotation. This result is improved to $\mathrm{Re}\,l>-5/2$ by Tiktopoulos (1964)[T2)] and to all values of l by Abe, Konisi and Ogimoto (1964).[A1)] Domokos and Suranyi (1964)[D4)] classified the B–S kernels of various models according to their behaviors at the origin of the position space. Suranyi (1963, 1964)[S19),S20)] showed the existence of a fixed cut in the l

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plane in the models having a non-Fredholm-type kernel, namely, in the bubble-exchange*) model and in the vector-meson-exchange ladder model. Pac (1963)[P1)] and Kwiecinski and Suranyi (1964)[K15)] noted, however, that if chain sums are exchanged instead of single bubbles, we have a Fredholm-type kernel. This remark was extended to higher orders of the φ^4 theory by Kwiecinski and Suranyi (1965).[K16)] Martinis (1965)[M7)] and Contogouris (1965)[C14)] indicated the existence of the Gribov-Pomeranchuk essential singularity[31)] in the crossed-two-meson-exchange model. The Mandelstam-type moving cut[32)] was investigated by Wilkin (1964).[W6)] Martinis and Ahmed (1965)[M8)] considered another model having moving cuts.

The high-energy asymptotic behavior in the t channel can also be investigated directly. The normal absorptive part**) of the off-the-mass-shell scattering amplitude satisfies an integral equation quite analogous to the scattering B–S equation. This equation, which was called the multiperipheral model, was studied by Bertocchi, Fubini and Tonin (1962),[B11)] Ceolin, Duimio, Stroffolini and Fubini (1962),[C1)] Amati, Stanghellini and Fubini (1962)[A5)] and Lee and Swift (1963).[L5)] They showed that the Regge behavior is consistent with the multiperipheral model and that the Regge trajectory is determined by the continued partial-wave (homogeneous) B–S equation. Tiktopoulos and Treiman (1964, 1965)[T3),T5),T6)] calculated upper and lower bounds on the high-energy behavior of the forward scattering amplitudes in various models by using the positive definiteness of the kernel of the multiperipheral model [see also Rosner (1966)[R3)]]. Some further investigations based on the multiperipheral model were made by Simonov (1964)[S16)] and Dremin Roĭzen, White and Chernavskiĭ (1965).[D8),C3)]

The connection between the high-energy behavior and the continued partial-wave B–S equation can be established more directly by starting from the scattering B–S equation. This observation was made by Nakanishi (1964)[N10),N11)] by means of the perturbation-theoretical integral representation (PTIR). Nakanishi (1964, 1965)[N12)~N15)] obtained some exact solutions at $s=0$ of the scattering B–S equation in the Wick-Cutkosky case and in the Goldstein case (spinor-spinor model).

The exact high-energy asymptotic behavior of the forward scattering can be evaluated most easily by expanding the amplitude in terms of the four-dimensional spherical harmonics in the s channel. This approach was first developed by Bjorken (1964)[B16)] for the Fredholm-type kernel. Baker and Muzinich (1963)[B1)] applied his method to the bubble-exchange model and found the exact high-energy asymptotic behavior of the forward scattering. Their analysis was extended to the case of the higher-order kernels of the

*) The bubble exchange means the simultaneous exchange of two spinless mesons. This model is the lowest non-trivial approximation of the four-boson interaction, called the φ^4 theory.

**) The absorptive part due to anomalous thresholds is not taken into account.

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φ^4 theory by Banerjee, Kugler, Levinson and Muzinich (1965)[B3] and Nussinov and Rosner (1966).[N41] Swift and Lee (1964)[S22] investigated singular kernels in the position space.*) Cosenza, Sertorio and Toller (1964, 1965)[C15),C16] obtained the exact high-energy asymptotic behavior both in the bubble-exchange model and in the vector-meson-exchange ladder model and proposed a general theory**) of treating the non-Fredholm-type kernel on the basis of the theory of linear operators and of the operator-valued analytic functions. Restignoli, Sertorio and Toller (1965)[R2] calculated numerically the slope of the Regge trajectory at $s=0$ in the vector-meson-exchange ladder model. Willey (1967)[W7] studied the asymptotic behavior of the Goldstein model. Seto (1968)[S14] reduced the scattering B–S equation of the Wick-Cutkosky model for $s>0$ to that for $P_\mu=0$ by means of the stereographic projection method and solved it. Rosner[R4] made a numerical caculation of the high-energy behavior.

The so-called "leading term summation" method yields some information of the high-energy asymptotic behavior in the weak coupling limit, though this procedure is not justified mathematically. The scalar-scalar scalar-meson-exchange model was investigated by Polkinghorne (1963),[P4] Federbush and Grisaru (1963),[F1] Trueman and Yao (1963)[T8] and Polkinghorne (1964).[P5] The singular-kernel models were studied by Sawyer (1963),[S5] Swift and Lee (1963)[S21] and Nussinov (1965).[N40] See also, Halliday (1963),[H1] Bjorken and Wu (1963)[B15] and Martinis (1965).[M6]

The scalar-scalar scalar-meson-exchange ladder model has the Fredholm-type kernel when Wick-rotated or for Re $s\neq 0$, and the continued partial-wave amplitude is meromorphic in the l plane. Each Regge pole is the analytic continuation of a bound-state pole of the scattering Green's function. In the equal-mass case, the leading Regge behavior in the $\mu\neq 0$ model is majorized[T3] by that in the $\mu=0$ model, namely the Wick-Cutkosky model, in which the scattering amplitude behaves like $t^{\alpha(s)}$ with[N12]

$$\alpha(0)=-\frac{3}{2}+\left[\frac{1}{4}+\left(\frac{g}{4\pi m}\right)^2\right]^{1/2}, \qquad (12\cdot 1)$$

a result which can be confirmed also from $(6\cdot 21)$ with $\kappa=0$ and $n=l+1$, $[\lambda=g^2/(4\pi)^2]$. The weak-coupling limit of $(12\cdot 1)$ is of course $\lim_{g\to 0}\alpha(0)=-1$, as is seen from the inhomogeneous term of the B–S equation. The strong-coupling limit of $(12\cdot 1)$ becomes

$$\lim_{g\to\infty}\alpha(0)/(g/4\pi m)=1\,, \qquad (12\cdot 2)$$

a result which turns out to be true also for the non-ladder models whose

*) Unfortunately, their result on the bubble-exchange model seems to be wrong.
**) Unfortunately, it contains a wrong inequality.

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kernels have only one intermediate state in the t channel.[T6)]

In the ordinary renormalizable theories in which the coupling constant is dimensionless, we encounter fixd cuts in the l plane,[S20)] because the kernels are non-Fredholm-type but their non-Fredholm parts are independent of s. Indeed, in the bubble-exchange model, the scattering amplitude is shown to behave like[B1),T5),S5)]

$$t^{\alpha}(\log t)^{-1/2} \quad \text{for} \quad m=0,$$
$$t^{\alpha}(\log t)^{-3/2} \quad \text{for} \quad m\neq 0, \tag{12·3}$$

where*)

$$\alpha=-1+\left(1+\frac{g}{4\pi^2}\right)^{1/2} \tag{12·4}$$

If all internal masses are put equal to zero, the φ^4 theory involves no parameter having a dimension. Hence for $P_\mu=0$, when carried out the three angular integrations, the kernel depends only on the ratio, $|p|/|p'|$, of the magnitudes of momenta. Therefore, the scattering B-S equation is reduced to a linear algebraic equation by means of the Mellin transform. In this way, we can find the exponent α also for non-ladder models. It is interesting to note that in the strong coupling limit we always have[B3)]

$$\lim_{g\to\infty} \alpha/(\sqrt{g}/2\pi)=1 \tag{12·5}$$

for the models whose kernels have only one intermediate state in the t channel.

In the vector-meson-exchange ladder model, if the vector meson propagator is of the form

$$ig_{\mu\nu}/(\mu^2-k^2-i\varepsilon), \tag{12·6}$$

then the exponent α is given by[C15),T5)]

$$\alpha=-1+\left(1+\frac{g^2}{\pi^2}\right)^{1/2}. \tag{12·7}$$

In the spinor-spinor ladder model, the asymptotic behavior is easily calculated only for the Goldstein-type equation. The pseudoscalar part of the scattering amplitude at $P_\mu=0$ behaves like (12·3) with[N15),S21)]

$$\alpha=-1+(g/2\pi). \tag{12·8}$$

*) If the exchange term is taken into account (symmetrization), then g should be replaced by $\sqrt{2}g$. On the other hand, if we take account of a statistical fadtor $\frac{1}{2}$ for the identical-particle bubble, then g should be replaced by $g/\sqrt{2}$.

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§13. Daughter trajectories

As was discovered by Freedman and Wang,[F5)] a Regge trajectory (called a parent or a mother) has to be accompanied by a sequence of trajectories, called daughter trajectories, in order to cancel unphysical singularities of the parent at $s=0$ in the unequal-mass case. Though quite a large number of papers have appeared on the problem of daughter trajectories, here we are concerned only with the approach based on the B–S equation.

Prior to the work of Freedman and Wang, the existence of such a family of Regge trajectories was recognized by Domokos and Suranyi (1964)[D4)] and by Nakanishi (1964).[N14)] Freedman and Wang (1967)[F5)] verified the existence of daughter trajectories on the basis of the four-dimensional symmetry of the scattering B–S equation at $P_\mu=0$. Nakanishi (1967)[N22)] discussed the validity of the Regge formula in the B–S formalism by noting the change of the little group at $s=0$,*) and emphasized the importance of the presence of the B–S ghosts (see §9) in the Freedman-Wang cancellation. The existence of an infinite sequence of daughter trajectories was demonstrated by Chung and Wright (1967)[C6)] [see also Chang and Saxena (1968)[C2)]].

Domokos (1967)[D5)] derived a two-parameter formula**) for the slopes at $s=0$ of the Regge trajectories belonging to a family on the basis of the breakdown of the four-dimensional symmetry near $P_\mu=0$. His formula was extended to the unequal-mass case by Chung and Wright (1967),[C6)] and to that for the second derivatives of the trajectories and to the cases of non-zero-spin particles by means of group-theoretical techniques by Domokos and Suranyi (1968)[D6),D7)] and others.

In the Wick-Cutkosky model, Nakanishi's results[N14)] on the asymptotic expansion in t were confirmed by Seto (1968)[S14)] and by Green and Biswas (1968)[G15)] on the basis of the exact solution to the scattering B–S equation. Müller (1968)[M18)] explicitly calculated a few terms of the double power-series expansion in s and Δ^2 of the trajectory functions. Gatto and Menotti (1968)[G1)] investigated the behavior of Regge trajectories as $\lambda\to 0$ or as $s\to -\infty$ in the Wick-Cutkosky model.

In the $\mu\neq 0$ ladder model, exact results in the weak-coupling limit were obtained by Swift (1967)[S23),S24)] and Halliday and Landshoff (1968)[H2)] who used perturbation expansion, and by Fontannaz (1969),[F4)] who employed the Fredholm methed. Chung and Snider (1967)[C5)] made numerical calculation of daughter trajectories mainly in the equal-mass case [see also, Madan, Haymaker and Blankenbecler (1968)[M1)]]. Cutkosky and Deo (1967)[C20)] numerically calculated daughter trajectories in the unequal-mass case in detail by using an approximate kernel, and found a surprising result: A daughter trajectory turns back at a certain value of s and becomes another trajectory,

*) See also Breitenlohner.[B17)]

**) His formula is nothing but the Reggeized form of the Ciafaloni-Menotti formula (7·4).

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that is, a trajectory function can have branch points below the elastic threshold even for $\mathrm{Re}\, l > -1/2$. Swift (1968)[S25] confirmed this result by means of his perturbation method,[S24] and showed that such behavior would disappear if fourth-order corrections to the kernel was taken into account. A detailed numerical calculation of trajectories was performed by Linden (1969).[L12]

Regge trajectories in the B–S formalism are obtained by Reggeizing the bound-state poles of the scattering Green's function. Since the angular momentum l is defined on the basis of the $O(3)$ symmetry, we first consider the $s>0$ case and take the rest frame $P_\mu = (\sqrt{s}, 0, 0, 0)$. We suppose that the scalar-scalar Green's function G has a bound-state pole

$$i\epsilon_{\nu k}(P)\sum_{m=-l}^{l}\phi_{\nu klm}(p, P)\bar{\phi}_{\nu klm}(q, P)/(s-s_{\nu kl}) \tag{13·1}$$

with $k=L-l$. Here $\epsilon_{\nu k}(P)$ denotes a norm factor (see §9), and

$$\phi_{\nu klm}(p, P) = \mathcal{Y}_{lm}(\boldsymbol{p})\Phi_{\nu k}(v, w, s;\, l),$$
$$\bar{\phi}_{\nu klm}(q, P) = \mathcal{Y}^*_{lm}(\boldsymbol{q})\bar{\Phi}_{\nu k}(v_0, w_0, s;\, l), \tag{13·2}$$

where $\Phi_{\nu k}(v, w, s;\, l)$ is a solution to the partial-wave B–S equation. The summation over m in (13·1) is easily carried out by means of (4·32):

$$\sum_{m=-l}^{l}\mathcal{Y}_{lm}(\boldsymbol{p})\mathcal{Y}^*_{lm}(\boldsymbol{q}) = \frac{2l+1}{4\pi}|\boldsymbol{p}|^l|\boldsymbol{q}|^l P_l(z) \tag{13·3}$$

with

$$z = \boldsymbol{pq}/|\boldsymbol{p}|\cdot|\boldsymbol{q}|. \tag{13·4}$$

The partial-wave Green's function G_l is defined by the expansion

$$G = \sum_{l=0}^{\infty}(2l+1)P_l(z)G_l. \tag{13·5}$$

We now consider the analytic continuation of G_l in l and the Watson transform[33] of (13·5).*) The partial-wave B–S equation is also Reggeized, and then it determines a Regge trajectory $l=\alpha_{\nu k}(s)$, which is nothing but the inverse function of $s=s_{\nu kl}$ with respect to l. The continued partial-wave Green's function G_l has a Regge pole at $l=\alpha_{\nu k}(s)$, whose residue is given by

$$i(4\pi)^{-1}(|\boldsymbol{p}|\cdot|\boldsymbol{q}|)^{\alpha_{\nu k}(s)}\alpha'_{\nu k}(s)R_{\nu k}(v, w, v_0, w_0, s), \tag{13·6}$$

where

$$R_{\nu k}(v, w, v_0, w_0, s) \equiv -\epsilon_{\nu k}(P)\Phi_{\nu k}(v, w, s;\, \alpha_{\nu k}(s))\bar{\Phi}_{\nu k}(v_0, w_0, s;\, \alpha_{\nu k}(s)). \tag{13·7}$$

*) If the exchange force is taken into account, we should consider the Regge amplitudes for l even and for l odd separately.

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The on-the-mass-shell residue of $R_{\nu k}$ has the sign equal to $\epsilon_{\nu k}(P)$ if $s < (m_1+m_2)^2$ and if $\alpha_{\nu k}(s)$ is real.

We can make a similar consideration for $s<0$ if we take a Lorentz frame $P_\mu = (0,0,0,\sqrt{-s})$. We then find[N22] that the resulting formula is exactly the analytic continuation of (13·6) to $s<0$ through a complex neighborhood of $s=0$. Since $\alpha_{\nu k}(s)$ is a function of an invariant s, it is expected to be holomorphic also at $s=0$, but $R_{\nu k}$ becomes in general singular at $s=0$ because it has nothing to do with the little group $E(2)$. To see this more explicitly, we consider the $s\to 0$ limit of (13·2). Then $\phi_{\nu k l m}(p, P)$ should be proportional to $\mathcal{Z}_{l+k,l,m}(p)$. Since $\mathcal{Z}_{l+k,l,m}(p)/\mathcal{Y}_{lm}(\boldsymbol{p})$ is a polynomial in

$$p_0 = (v-w)/2\sqrt{s} \qquad (13\cdot 8)$$

of degree k and in

$$p^2 = \frac{1}{2}(v+w) - \frac{1}{4}s\,, \qquad (13\cdot 9)$$

$\Phi_{\nu k}$ behaves like $s^{-k/2}$ near $s=0$ for $v\neq w$, whence $R_{\nu k}$ has a pole of order k at $s=0$ for $v\neq w$ and $v_0\neq w_0$.

Because of the four-dimensional symmetry at $P_\mu=0$, we can define the four-dimensional partial-wave Green's function in the complex L plane. If it has a pole at $L=\alpha_\nu$, this pole corresponds to an infinite sequence of Regge poles at $s=0$. The Regge trajectories $\alpha_{\nu 0}(s)$ and $\alpha_{\nu k}(s)$ are called a parent and a daughter trajectory of order $k(\geq 1)$, respectively, where

$$\alpha_{\nu k}(0) = \alpha_\nu - k. \quad (k=0,1,2,\cdots) \qquad (13\cdot 10)$$

As shown above, the reduced residue*) of the daughter trajectory of order k behaves like s^{-k} near $s=0$. The singularities of the Khuri satellite poles[34] of a parent in the Khuri ($\tilde{l}$) plane**) are canceled by the singularities of its daughters (and those of their Khuri satellite poles). This mechanism is called the Freedman-Wang cancellation, which is quite analogous to the situation discussed in §10(A) (λ and s there correspond to s and $\tilde{l}$ here, respectively). As long as the slopes $\alpha'_{\nu k}(0)$ $(k=0,1,\cdots,K)$ are not equal to each other, we have a Khuri multiple pole of order $K+1$ at $\tilde{l}=\alpha_\nu - K$, $s=0$. Furthermore, analogously to §10(C), the consistency of the cancellation conditions yields

$$\alpha'_{\nu k}(0) = A_{\nu k} + B_{\nu k}(\alpha_\nu - k)(\alpha_\nu - k + 1), \qquad (13\cdot 11)$$

*) The reduced residue is defined by (13·6) with omission of the factor $(|\boldsymbol{p}|\cdot|\boldsymbol{q}|)^{\alpha_{\nu k}}$.

**) By the analytic continuation of the exponent $\tilde{l}$ of a power series expansion in t, we can define Khuri poles just as done for Regge poles. A Regge pole at $l=\alpha(s)$ is transcribed into a Khuri pole at $\tilde{l}=\alpha(s)$ and Khuri satellite poles at $\tilde{l}=\alpha(s)-j$ $(j=1,2,\cdots)$.

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etc. The formula (13·11) is nothing but the Reggeized form of (7·4).*)

In the Wick-Cutkosky model, we have detailed information on the daughter trajectories. In the equal-mass case, the four-dimensional angular-momentum quantum number L is not unambiguously expressed in terms of κ, n, l because all solutions for $P_\mu=0$ are not obtained as straightforward limits of those for $s>0$ [see (6·60)]. On the contrary, in the unequal-mass case, L is uniquely identified with $n-1$ because of (6·46). Hence we have[N22)]

$$k=n-l-1 \tag{13·12}$$

in the unequal-mass Wick-Cutkosky model. According to (9·14), therefore, the norm of the daughter trajectory of order k is $(-1)^k$ in $0<s<(m_a-m_b)^2$, that is, the odd-order daughter trajectories are ghosts. This fact is the very reason why the Freedman-Wang cancellation can take place, because it could not occur if all reduced residues had the same sign. Furthermore, if we agree to adopting the definition of the mass shell presented in (6·61), ghost daughter trajectories do not vanish even on the mass shell in accordance with the fact that the Freedman-Wang cancellation has to occur in the S-matrix. For $(m_a-m_b)^2<s<(m_a+m_b)^2$, however, they change into positive-norm trajectories by giving their negative norm to the odd κ trajectories, which vanish on the mass shell, at the pseudothreshold. Thus the existence of the multiple poles at $s=(m_a-m_b)^2$ [see §10(C)], plays an important role for eliminating ghosts near the elastic threshold on the mass shell.

In the unequal-mass Wick-Cutkosky model, we also note that all daughter trajectories are parallel to their parent because of the $O(3,1)$ symmetry for s arbitrary. Hence, in this model, there are no Khuri multiple poles at $s=0$. This result is of course a speciality of the Wick-Cutkosky model.

In the $\mu\neq0$ ladder model, the detailed behavior of daughter trajectories is unknown. In the equal-mass case, it is quite natural to identify L with $\kappa+l$ on account of the p_0-parity, whence

$$k=\kappa\,. \tag{13·13}$$

This identification is different from (13·12), i.e., identification is model-dependent.**) Since the odd κ solutions vanish on the mass shell, there is no ghost difficulty in the equal-mass case. In the unequal-mass case, however, if $\alpha'_{\nu k}(0)>0$, ghost daughter trajectories have to be present on the mass shell because of the Freedman-Wang cancellation. Since ghosts should not exist near the elastic threshold because of the unitarity of the S-matrix,

*) Note that if $\lambda=F(s,\alpha)$ we have $\alpha'=-\lambda'/(\partial\lambda/\partial\alpha)_{s=0}$ from the formula for the derivative of an implicit function.

**) This fact does not contradict the factorizability of the Regge residues because the equal-mass-to-unequal-mass process cannot be realized on the mass shell in the ladder model.

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one may expect[N24)] that multiple poles are present at some points between $s=0$ and $s=(m_a+m_b)^2$ in order to convert on-the-mass-shell ghost trajectories into positive-norm ones.

Finally, we note that contrary to the reality of the eigenvalue $\lambda_{\nu Ll}(s)$ the Regge trajectory $l=\alpha_{\nu k}(s)$ can become complex. In this case, we always have the complex conjugate trajectory $l=\alpha^*_{\nu k}(s)$, because as is seen from (7·9) the Fredholm determinant is real analytic in s and l ($\mathrm{Re}\, l>-3/2$). A numerical calculation[C20)] indeed indicates the existence of complex trajectories in a certain intervl of s (see Fig. 6). One should note that such a phenomenon has nothing to do with the above-mentioned ghost problem.

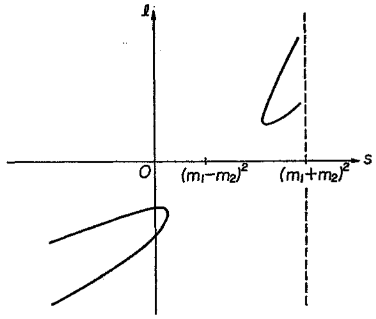

Fig. 6. The $k=1$ daughter trajectory calculated by Cutkosky and Deo.

§14. Miscellany

There are a number of topics related to the B–S equation which are not mentioned in the previous sections. We summarize miscellaneous problems very briefly in this section.

Eden (1952, 1953)[E1),E2),E4)] and Eden and Rickayzen (1953)[E3)] discussed various processes involving bound states and the energy levels of unstable bound states (resonances). More general formulations of the former problem were presented by Nishijima (1953, 1954, 1955)[N34),N35),N36)] and Mandelstam (1955)[M4)] (see §3).

The vertex function of a bound state can be calculated according to Mandelstam's prescription. Its properties, mainly its high-energy asymptotic behaviors, are investigated by Ciafaloni and Menotti (1966, 1968),[C8),C11)] Ciafaloni (1968),[C12)] Amati, Caneschi and Jengo (1968)[A6)] and Yamada (1968).[Y1)] Barbieri (1969)[B5)] considered the photoproduction process of a bound state.

The pion-nucleon B–S equation was analyzed by Deser and Martin (1953),[D3)] Nieland and Tjon (1968)[N33)] and Rothe (1968).[R5)] Alpers (1968)[A4)] decomposed the general B–S amplitude into standard covariant amplitudes.

The B–S equation for unrenormalizable interaction were discussed by a number of authors. Feinberg and Pais (1963, 1964)[F2),F3)] proposed the so-called "peratization" method, by which they extracted a finite result from a term-by-term-divergent series. Their theory was further investigated by Pwu and Wu (1964).[P8),P9)] Sawyer (1964)[S6)] found a finite solution for a particular B–S equation having a very singular kernel. A general way of solving the B–S equation for unrenormalizable interaction was discussed in the

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Euclidean position space by Güttinger, Penzl and Pfaffelhuber (1965).[G18),G19)]

The B–S equation was encountered in the superconductor model of elementary particles proposed by Nambu and Jona-Lasinio (1961).[N28)] In the quantum electrodynamics based on the self-consistent equations proposed by Johnson, Baker and Willey,[35)] the relevance of the Goldstone theorem[36)] was discussed by means of the B–S equation by Baker, Johnson and Lee (1964),[B2)] Nambu (1964)[N29)] and Willey (1967).[W7)]

The bootstrap theory based on the B–S equation was studied by Rowe (1964),[R6)] Lin and Cutkosky (1965),[L10)] Harte (1966),[H3)] Kaufmann (1968)[K3)] and Golowich (1968).[G5)]

Kita and Wakano (1957)[K10)] proved that the exact B–S equation cannot determine the energy levels of the excited states of a hydrogen atom. Schweber (1962)[S11)] discussed the B–S equation in the non-relativistic theory. Nakanishi (1966)[N19),N21)] investigated the vertex pole and the bound-state propagator in the B–S formalism. Nishijima and Saffouri (1965)[N38)] treated a decay process by means of the unequal-mass Wick-Cutkosky model (see §6). Symanzik (1954)[S26)] and Zimmermann (1954)[Z1)] made some theoretical considerations on the B–S equation. Enflo (1965)[E5)] discussed the connection between the vertex function and the B–S amplitude.

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Note added in Proof:

In the $P_\mu=0$ spinor-spinor ladder model, by introducing modified generalized $O(4)$ harmonics, Seto has succeeded in reproducing Kummer's equation (11·27) for helicity amplitudes without imposing Lorentz condition (11·26), and calculated the normalization integral (K. Seto, private communication).

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