dense. Knight and Ma's system is decidedly not dense, being a discrete system, i.e. one where every time element has a unique predecessor and successor fundamental time element. Other systems, such as that of Allen and Hayes, permit a mixture of dense and discrete time elements.

Finally, there is a difference between systems in their ability to model the 'open' and 'closed' nature of intervals. Allen's system allows only intervals of indeterminate type: since points are not allowed, there is no definition of open\closed intervals. In Vilain's system, although both points and intervals are taken as primitive, it is still not possible to characterize the open and closed nature of intervals. However, Knight and Ma's temporal model allows modelling of open and closed intervals, and it can be shown that the characterization is in agreement with the conventional concepts of open, semi-open and closed intervals which are constructed out of points.

The importance of treating points and intervals as primitives on an equal footing lies in the need for the temporal theory to model the way things happen in time. Both Allen (1983, 1984) and McDermott (1982) give examples of properties defined over time, and many AI applications involve continuous change of variables in time. Galton (1990a) has shown that time-points are needed in order to accommodate the representation of facts concerned with continuous change and has proposed a revision of Allen's system to this effect.

It is the objective of this paper to provide a general axiomatic framework to serve as a unifying basis for these temporal systems. The axiomatization may be seen as an extension of Allen and Hayes' (1989) theory, to include points as primitive objects. A discussion of the implications of including points as primitive, and of distinguishing points from moments is given in Section 2. Here, a problem with Allen's interval based logic concerning reasoning about continuous change is examined. The discussion indicates that points are necessary as primitive objects for the correct modelling of continuous change. There follows a discussion of some limitations of Allen and Hayes' axiom, ⟨M6⟩, which states that moments never meet moments. It is shown that this axiom leads to the conclusion that we can have neither a completely discrete nor a completely dense system which contains moments. However, if we revise Allen's and Hayes' system to include points and limit ⟨M6⟩ to points, rather than moments, this objection does not apply.

We present the main body of the general axiomatization for a temporal frame based on both interval and points in Section 3. These axioms are independent of the specification of density and linearity. Additional axioms are provided in Section 4 to specify the linearity and density of time. Definitions are also given for the open and closed nature of an interval. A classification of all possible temporal relations over intervals and points is presented in Section 5. In Section 6 we give various models to illustrate the theory. We present a completely dense model and a completely discrete model of the theory. We further show how other temporal systems may be subsumed by the theory, with the appropriate denseness and linearity axioms. It is also shown that, assertions about the instantiation in time of properties and occurrences may naturally be expressed in the temporal frame.

2. ALLEN AND HAYES' AXIOMATIZATION OF TIME BASED ON INTERVALS

Allen and Hayes' theory of time is based on a non-empty class, **I**, of *time intervals*, and is axiomatized in terms of the single temporal relation '*meets*' between intervals. The set of axioms is proposed first in Allen (1985) and then revised in Allen (1989), as follows:

⟨M1⟩ $\forall i, j, k, l \in \mathbf{I}(meets(i, j) \wedge \text{meets}(i, k) \wedge meets(l, j) \Rightarrow meets(l, k))$

⟨M2⟩ $\forall i, j, k, l \in \mathbf{I}(meets(i, j) \wedge meets(k, l) \Rightarrow meets(i, l)$
$\quad \nabla\ \exists m \in I(meets(i, m) \wedge meets(m, l))$
$\quad \nabla\ \exists n \in I(meets(k, n) \wedge meets(n, j)))$

NB. In this paper, '∇' means exclusive disjunction.

⟨M3⟩ $\forall i \in \mathbf{I}\ \exists j, k \in \mathbf{I}(meets(j, i) \wedge meets(i, k))$

⟨M4⟩ $\forall j, k \in \mathbf{I}(\exists i, l \in \mathbf{I}(meets i, j) \wedge meets(j, l) \wedge meets(i, k) \wedge meets(k, l)) \Rightarrow j = k)$

NB. In this paper, we follow Allen and Hayes' notation that '$j = k$' means j and k represent the same time element.

⟨M5⟩ $\forall i, j \in \mathbf{I}(meets(i, j)$
$\quad \Rightarrow \exists k \in \mathbf{I}\ \forall m, n \in \mathbf{I}(meets(m, i) \wedge meets(j, n)$
$\quad \Rightarrow meets(m, k) \wedge meets(k, n))$

Axiom ⟨M1⟩ states that the 'place' where two intervals meet is unique and closely associated with the intervals. The role of ⟨M2⟩ is to ensure that meeting places are totally ordered. ⟨M3⟩ makes every interval have at least one neighbouring interval preceding it, and another succeeding. ⟨M4⟩ simply says that there is only one time interval between any two meeting places. Finally, ⟨M5⟩ states that if two meeting places are separated by a sequence of intervals, then there is an interval which connects these two meeting places. Hence, with axiom ⟨M4⟩ and the definition of equality, for any two adjacent intervals, i and j, the ordered union of i and j may be written as $i + j$.

A limitation of Allen and Hayes' theory, expressed by Tsang (1987), is that the axioms are not primitive enough for extensions. For example, linearity might be hoped to be removed from the axiomatization in order to address the issues such as **branching time** and **parallel time**. However, Tsang points out that it is difficult to see which axiom in Allen and Hayes' axiom set entails linearity. Allen and Hayes conclude that the linearity assumption is characterized by means of axiom ⟨M4⟩ in the revised version of the set of their axioms (1989).

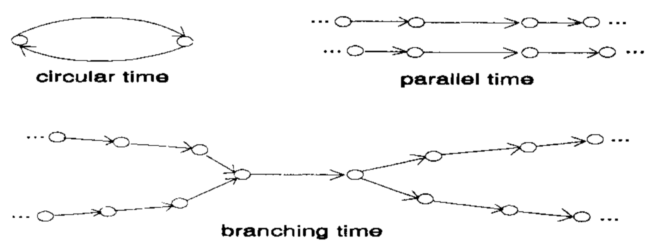

FIGURE 1.

However, it is indeed axiom ⟨M2⟩, rather than ⟨M4⟩, that entails the linearity of time. In fact, if we remove ⟨M2⟩ from the set of axioms, then the time may be **circular**, **parallel** or **branching**, as shown in Figure 1. In this graphical representation, the arcs of the graph represent time intervals, and the relation *meets*(i, j) is represented by i being in-arc and j being out-arc to a common node.

Another limitation of Allen and Hayes' time theory is that it takes only intervals, rather than, points, as primitive time elements, although points are later introduced as the 'meeting places' of intervals at a subsidiary status within the theory. Their contention is that nothing can be true at a point, for a point is not an entity at which things happen or are true. However, as Galton shows in his critical examination of Allen's interval logic (Galton, 1990), the theory of time based on intervals is not adequate, as it stands, for reasoning correctly about continuous change. We may illuminate the problem involved with reference to time points by means of the following example of a ball thrown vertically into the air: The motion may be described qualitatively by the use of two intervals, interval i where the ball is going up, and interval j where the ball is coming down. According to classical physics, there is a point p at which the ball is stationary. As Allen suggested, in the interval calculus we have two alternatives: we may assume that there is a very small interval where the ball is stationary, or we may assume that interval i '*meets*' interval j. The first alternative does not seem tenable, being inconsistent with the laws of physics, no matter how small the interval. The second alternative also gives problems, since the interval calculus allows us to combine two intervals which meet, that is, $i + j = k$ (see Allen, 1985; Allen and Hayes 1989): in Allen's logic, the formula HOLDS(*pro*, I) is used to say that the property *pro* holds *during* the interval I. More precisely, what it says is that *pro* holds *throughout* that interval (Galton, 1990a). However, although the property '*ball_in_motion*' holds throughout both of intervals i and j, that is:

HOLDS(*ball_in_motion*, i),
HOLDS(*ball_in_motion*, j)

we cannot assert that

HOLDS(*ball_in_motion*, i + j),

since the property '*ball_in_motion*' does not hold throughout the whole combined interval k, within which there is a point p at which the ball is stationary.

To characterize the times that some 'instant-like' events occupy, Allen and Hayes introduce the idea of *very short intervals*, called *moments*. A moment is simply a non-decomposable time interval. The important distinction between moments and points is: although being non-decomposable, moments are defined by having extent and by means of having distinct beginning and end points [just as for other intervals (Allen and Hayes (1989)], while points are defined by having no extent.

Relating to the meets relation, another obvious difference between points and moments is that moments can meet other intervals, and hence stand between them, while points are not treated as primitive objects and cannot meet anything. However, as Allen and Hayes themselves point out, a theory incorporating granularity involves introducing a '*tolerance relation*' that defines when two times are indistinguishable. For example, two intervals, i and j, might be indistinguishable if their beginning points are at most a moment apart, and likewise for their end points. To ensure that the tolerance relation is an equivalent relation, Allen and Hayes propose axiom ⟨M6⟩, which insists that moments never meet:

$$\langle M6\rangle \quad \forall m, n \in \mathbf{I}(\mathit{momentum}(m) \wedge \mathit{moment}(n) \Rightarrow \neg\, \mathit{meet}(m, n))$$

where *moment*(m) is defined by:

$$\forall m \in \mathbf{I}(\mathit{moment}(m) \Leftrightarrow \neg\, \exists i, j \in \mathbf{I}(m = i + j))$$

Allen and Hayes declare that their formulation permits either discrete or continuous time models, as well as more exotic models that may alternate between continuous and discrete stretches of time. Unfortunately, axiom ⟨M6⟩ leads to another limitation to the primitive time elements: for any interval, either it is non-decomposable, i.e. a moment, or it must be infinitely decomposable. For, if it is only finitely decomposable, then it must be the sum of a finite number of moments which would meet one another, contrary to ⟨M6⟩. This precludes discrete models from the theory containing axiom <M6>. In addition, dense models of the theory, i.e. where all intervals are infinitely decomposable, permit no moments at all, so that ⟨M6⟩ is only vacously true. Hence models of the theory including ⟨M6⟩ which contain moments can be neither dense nor discrete.

However, although ⟨M6⟩ appears to bring little benefit in the form that is presented here, dealing with moments, it is shown in the next section to play a critical role in a general theory if it is applied to '*time points*'. In this case the axiom does not limit the interval structure at all.

3. AN AXIOMATIZATION OF TIME BASED ON INTERVALS AND POINTS

As discussed in the above section, Allen and Hayes' time theory is not primitive enough for extensions and is not adequate for reasoning correctly about continuous change. Our objective is to develop and explore a first-order theory of time which should be more general as an underlying framework for most of representative temporal models in artificial intelligence. The new time theory may be seen as an extension of Allen and Hayes' axiomatization by means of some additional axioms relating to the inclusion of time points as primitive elements, and generalization of Allen and Hayes' axiomatization by removing the linearity of time in order to allow non-linear time structures such as branching time, parallel time, etc.

We start the formal theory by posing a non-empty set, T, of objects that we shall call **time-elements**, and a function d from $\mathbf{T}$ to $\mathbf{R}_0^+$, the set of non-negative real numbers. A time-element, t, is called a (time) interval if $d(t) > 0$, otherwise, t is called a (time) point. According to this classification, the set of time-elements, **T**, may be expressed as $\mathbf{T} = \mathbf{I} \cup \mathbf{P}$, where **I** is the set of intervals and **P** is the set of points. As in Allen and Hayes' approach, at this early stage we do not make any commitment as to whether all time intervals are decomposable or not. The density question will be addressed by some further axioms.

In order to define the primitive order over time elements, we adopt Allen and Hayes' axiomatization for the single relation '*meets*' between intervals while axiom ⟨M2⟩ will not be included in the first place. Since the time elements may now be not only intervals but also points, some critical axioms are necessary relating to the treatment of points. The whole set of axioms for the '*meets*' relation over **T** are listed below, where axioms ⟨A1⟩, ⟨A2⟩, ⟨A3⟩ and ⟨A4⟩ correspond to Allen and Hayes' ⟨M1⟩, ⟨M3⟩, ⟨M4⟩ and ⟨M5⟩ in the above section, respectively.

⟨A1⟩ $\forall t_1, t_2, t_3, t_4 \in \mathbf{T}(meets(t_1, t_2) \wedge meets(t_1, t_3) \wedge meets(t_4, t_2) \Rightarrow meets(t_4, t_3))$

⟨A2⟩ $\forall t \in \mathbf{T}\, \exists t', t'' \in \mathbf{T}(meets(t', t) \wedge meets(t, t''))$

⟨A3⟩ $\forall t_1, t_2 \in \mathbf{T}(\exists t', t'' \in \mathbf{T}(\ meets(t', t_1) \wedge meets(t_1, t'') \wedge meets(t', t_2) \wedge meets(t_2, t'')) \Rightarrow j = k)$

⟨A4⟩ $\forall t_1, t_2 \in \mathbf{T}(meets(t_1, t_2) \Rightarrow \exists t \in \mathbf{T}\, \forall t', t'' \in \mathbf{T}(\ meets(t', t_1) \wedge meets(t_2, t'') \Rightarrow meets(t', t) \wedge meets(t, t''))$

NB. For any two time elements, t_1 and t_2, such that $meets(t_1, t_2)$, axioms ⟨A4⟩ and ⟨A3⟩ ensure that there is a unique time element corresponding to the ordered union of t_1 and t_2. Following Allen and Hayes' notation, we shall still indicate it as i + j, which will always imply that $meets(i, j)$.

⟨A5⟩ $\forall t_1, t_2 \in \mathbf{T}(meets(t_1, t_2) \Rightarrow t_1 \in \mathbf{I} \vee t_2 \in \mathbf{I})$

⟨A6⟩ $\forall t_1, t_2 \in \mathbf{T}(meets(t_1, t_2) \Rightarrow d(t_1 + t_2) = d(t_1) + d(t_2))$

Axiom ⟨A5⟩ is based on the intuition that points will not meet other points, that is, between any two time points, there is a time interval. This is indeed very similar to Allen and Hayes' ⟨M6⟩ which states that moments never meet other moments. However, unlike ⟨M6⟩, ⟨M5⟩ does not imply the limitation that any decomposable interval must be **infinitely** decomposable. Additionally, axiom ⟨A5⟩ does not affect whether the set of points is dense or not. This issue will depend on a further assumption ensuring that 'within' any time interval, there is a time point (see Section 6). Axiom ⟨A6⟩ ensures that the additional operation '+', over time elements is consistent with the function d, which we shall call the **duration assignment** over **T**.

This is the complete fundamental set of axioms concerning the *meet* relation. We denote this set as **A**, and use a pair, (**T**, *meet*), to represent the temporal flame defined by the axiomatization.

4. SOME FURTHER ISSUES

The axiomatization proposed in the above section defines a general temporal frame based on both intervals and points as primitive objects. In this section, we address some further issues relating to the structure of the frame.

4.1. Open and closed nature of intervals

Although intervals are taken in the theory as primitive, that is there are no definitions about the ending-points for intervals, the axiomatization allows the expression of the 'open' and 'closed' nature of intervals. For example, to represent the quantity space for the motion of the ball described in Section 2, we may relate *ball_going_up*, *ball_stationary*, and *ball_coming_down* to interval i_1, point p, and interval i_2, respectively, where $meets(i_1, p)$, $meets(p, i_2)$. Intuitively, $t = p + i_2$ relates to *ball_stationary-ball_coming_down*. In Figure 2 (for clarity, we denote points with bold arcs), since i_1 has point p as its immediate successor, we may view i_1 as 'right-open' at p, and similarly, i_2 as 'left-open' at p. Since interval t (= p + i_2) and point p have the same immediate predecessor, i_1, we may view t as 'left-closed' at p.

Formally, the open and closed nature of primitive intervals may be defined as follows:

interval i is **left-open** at point p iff
meets(p, i);

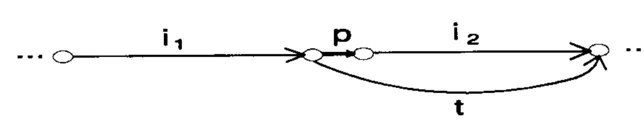

FIGURE 2.

interval i is **right-open** at point p iff
$meets(i, p)$;

interval i is **left-closed** at point p iff
$\exists i' \in \mathbf{I}(meets(i', i) \wedge meets(i', p))$;

interval i is **right-closed** at point p iff
$\exists i' \in \mathbf{I}(meets(i, i') \wedge meets(p, i'))$.

It is easy to see that 'left-open' and 'left-closed' (symmetrically, 'right-open' and 'right-closed') are exclusive to each other under the axiomatization. In fact, if interval i is left-open at point p_1, and left-closed at point p_2, then by the above definition, we get:

$$meets(p_1, i) \wedge meets(i', i) \wedge meets(i', p_2), \text{ where } i' \in \mathbf{I}$$

Hence, by axiom ⟨A1⟩ we can infer that $meets(p_1, p_2)$, which is contradictory to axiom ⟨A5⟩.

The above interpretation of the 'open' and 'closed' nature of primitive intervals is in fact in line with the conventional meaning of the open and closed nature for point-based intervals. For instance, point-based interval $(p_1, p_2]$ is 'left-open' at point p_1, since intuitively p_1 is an immediate predecessor of interval $(p_1, p_2]$; similarly, $(p_1, p_2]$ is 'right-closed' at p_2, since both point p_2 and interval $(p_1, p_2]$ have the same immediate successor, $(p_2, \_\}$.

4.2. Linearity of time

Time is usually considered as having a *linear* structure. This corresponds to the classical physical model of time, where the structure is that of the real line, extending indefinitely in both directions.

The **(full) linearity** of a temporal frame (**T**, *meets*) can be characterized by adding an axiom, $\langle A_{Linear} \rangle$, to **A**, the set of axioms proposed in Section 3:

$\langle A_{Linear} \rangle$

$$\forall t_1, t_2, t_3, t_4 \in T(meets(t_1, t_2) \wedge meets(t_3, t_4) \Rightarrow meets(t_1, t_4)$$
$$\nabla \exists t' \in T(meets(t_1, t') \wedge meets(t', t_4))$$
$$\nabla \exists t'' \in T(meets(t_3, t'') \wedge meets(t'', t_2)))$$

NB. The axiom $\langle A_{Linear} \rangle$ is in fact the axiom ⟨M2⟩ (see Section 2) for Allen and Hayes' interval-based theory. The 'exclusive ors' in this axiom have some quite powerful consequences. In particular, they ensure that there can be no **circular**, **parallel** and **branching** times. The following lemma is straightforward (see Allen and Hayes, 1989):

LEMMA $\forall t \in \mathbf{T}(\neg meets(t, t))$

This lemma ensures that there is no possibility of circular time.

However, without $\langle A_{Linear} \rangle$, a temporal frame usually allows branching into both the past and the future. Branching temporal frames offer an attractive way to handle possible worlds, uncertainty about the past or the future and the effects of alternative actions when planning. A temporal frame which allows branching into the future but not into the past is called **left-linear** (see Figure 3). This may be characterized by adding to **A**, the axiom $\langle A_{L\text{-}Linear} \rangle$, rather than the stronger axiom $\langle A_{Linear} \rangle$:

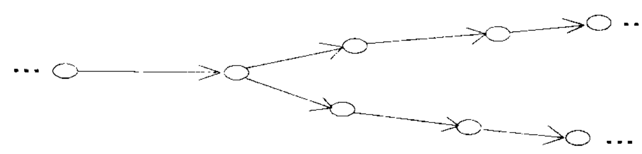

FIGURE 3.

$\langle A_{L\text{-}Linear} \rangle$

$$\forall t_1, t_2, t_3, t_4, t \in \mathbf{T}(meets(t_1, t_2) \wedge meets(t_2, t) \wedge meets(t_3, t_4) \wedge meets(t_4, t) \Rightarrow meets(t_1, t_4)$$
$$\nabla \exists t' \in T(meets(t_1, t') \wedge meets(t', t_4))$$
$$\nabla \exists t'' \in T(meets(t_3, t'') \wedge meets(t'', t_2)))$$

Analogously, **right-linearity** is defined by means axiom $\langle A_{R\text{-}Linear} \rangle$:

$\langle A_{R\text{-}Linear} \rangle$

$$\forall t, t_1, t_2, t_3, t_4 \in \mathbf{T}(meets(t, t_1) \wedge meets(t_1, t_2) \wedge meets(t, t_3) \wedge meets(t_3, t_4) \Rightarrow meets(t_1, t_4)$$
$$\nabla \exists t' \in T(meets(t_1, t') \wedge meets(t', t_4))$$
$$\nabla \exists t'' \in T(meets(t_3, t'') \wedge meets(t'', t_2)))$$

As Galton (1990b) puts it, it is interesting to note that **left-linearity** and **right-linearity** together just fail to imply **(full) linearity**, the exception being the case of parallel time lines as shown in Figure 4.

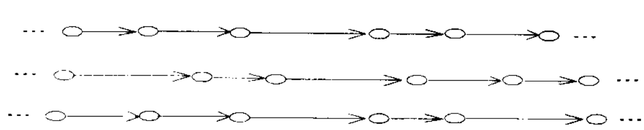

FIGURE 4.

Parallel temporal frames provide a way of modelling separate and asynchronous processes, and might prove useful in developing logics for reasoning about parallel computation and concurrent processes.

4.3. Dense and discrete time

According to axiom ⟨A2⟩, for each time-element t, there is a time-element which '*meets*' it, and another one which it '*meets*'.Therefore, in particular axiom ⟨A4⟩ and ⟨A5⟩ additionally ensure that, between any two distinct time points on the same time line, there is always a time interval. However, for time intervals, can we always assume that any interval can be decomposed into two distinct contiguous intervals? If so, we say that the set of time elements forms a dense system.

We may use the following axiom to characterize the density of a temporal from (**T**, *meets*):

$\langle A_{Dense} \rangle$

$$\forall i \in \mathbf{I}\, \exists t_1, t_2 \in \mathbf{T}(i = t_1 + t_2)$$

We can show that axiom $\langle A_{Dense} \rangle$ implies that each time interval can be decomposed into two distinct contiguous intervals. In fact, assume interval $i = t_1 + t_2$; if t_1 is a point, then by axiom $\langle A5 \rangle$, t_2 must be an interval; hence, by $\langle A_{Dense} \rangle$, $t_2 = t' + t''$, where $t', t'' \in \mathbf{T}$. By $\langle A4 \rangle$ and $\langle A3 \rangle$, we get $i = t_1 + t' + t''$. Since t_1 is a point, axiom $\langle A5 \rangle$ implies that t' must be an interval; hence $i_1 = t_1 + t'$ is an interval, and $i = i_1 + i_2$. Similar discussion applies to the case that t_2 is a point which implies that t_1 must be an interval.

The **discreteness** of a temporal frame ($\mathbf{T}$, *meets*) can be characterized by means of adding two axioms $\langle A_{L\text{-}Discrete} \rangle$ and $\langle A_{R\text{-}Discrete} \rangle$ to $\mathbf{A}$:

$\langle A_{L\text{-}Discrete} \rangle$

$$\forall t \in \mathbf{T}\, \exists t_1 \in \mathbf{T}\, (meets(t_1, t) \wedge \neg \exists t_2, t_3 \in \mathbf{T}(t_1 = t_2 + t_3))$$

$\langle A_{R\text{-}Discrete} \rangle$

$$\forall t \in \mathbf{T}\, \exists t_1 \in \mathbf{T}\, (meets(t, t_1) \wedge \neg \exists t_2, t_3 \in \mathbf{T}(t_1 = t_2 + t_3))$$

Axiom $\langle A_{L\text{-}Discrete} \rangle$ entails the **left-discreteness** and axiom $\langle A_{R\text{-}Discrete} \rangle$ entails the **right-discreteness** of a temporal frame. By taking t to be a non-decomposable interval (or moment, termed by Allen and Hayes) in the above axioms, since t_1 is by definition a moment, we see that $\langle A_{L\text{-}Discrete} \rangle$ or $\langle A_{R\text{-}Discrete} \rangle$ implies that each moment has a predecessor moment or successor moment respectively. Hence, Allen and Hayes' $\langle M6 \rangle$ is inconsistent with the discreteness axioms.

It is interesting to note that there may exist temporal frames which are neither dense nor discrete. In such a frame, there may be some intervals which are finite sums of moments. However, this case is axiomatically consistent with our axiom $\langle A5 \rangle$, but not consistent with Allen and Hayes' $\langle M6 \rangle$, which implies that each decomposable interval must be infinitely decomposable.

5. DERIVED TEMPORAL RELATIONS OVER TIME ELEMENTS

In terms of the primitive relation '*meets*', we may induce the complete set of possible relationships over time elements by means of the following definitions:

$EQUAL(t_1, t_2) \Leftrightarrow t_1 = t_2$,
$BEFORE(t_1, t_2) \Leftrightarrow \exists t \in \mathbf{T}\, (meets(t_1, t) \wedge meets(t, t_2))$,
$OVERLAPS(t_1, t_2) \Leftrightarrow$
$\quad \exists t, t', t'' \in \mathbf{T}(t_1 = t' + t \wedge t_2 = t + t'')$,
$START(t_1, t_2) \Leftrightarrow \exists t \in \mathbf{T}(t_2 = t_1 + t)$,
$DURING(t_1, t_2) \Leftrightarrow \exists t', t'' \in \mathbf{T}(t_2 = t' + t_1 + t'')$,
$FINISHES(t_1, t_2) \Leftrightarrow \exists t \in \mathbf{T}(t_2 = t + t_1)$,
$MEETS(t_1, t_2) \Leftrightarrow meets(t_1, t_2)$,
$AFTER(t_1, t_2) \Leftrightarrow BEFORE(t_2, t_1)$,
$OVERLAPPED\text{-}BY(t_1, t_2) \Leftrightarrow OVERLAPS(t_2, t_1)$,
$STARTED\text{-}BY(t_1, t_2) \Leftrightarrow STARTS(t_2, t_1)$,
$CONTAINS(t_1, t_2) \Leftrightarrow DURING(t_2, t_1)$,
$FINISHED\text{-}BY(t_1, t_2) \Leftrightarrow FINISHES(t_2, t_1)$,
$MET\text{-}BY(t_1, t_2) \Leftrightarrow MEETS(t_2, t_1)$,

NB. Since points are now allowed, the above 13 relations have a somewhat different 'feel' to Allen's 13 temporal relations between intervals. For instance, if i_1 and i_2 are open intervals separated by a point p, then we have $BEFORE(i_1, i_2)$, although this situation looks very like i_1 '*meets*' i_2 in Allen's system. Again, if i_1 is right-closed, and i_2 is left-closed at point p, respectively, according to the above definitions, we have $OVERLAPS(i_1, i_2)$, but again it 'looks' like the two intervals meeting. Additionally, from the above definitions, any open interval is '*DURING*' its closure. What all this means is that, taking both intervals and points as primitive time-elements, we have more than 13 significantly different relationships to be considered, because, for example, from almost any point of view, the first case mentioned above (i.e. $MEETS(i_1, p) \wedge MEETS(p, i_2)$) is no more similar to the case of two intervals separated by a third *interval* (a necessary condition of *BEFORE* in Allen's system) than it is to the case of two intervals strictly meeting.

As Allen and Hayes (1989) show, all the 13 relations may hold in the case that only intervals are taken as time elements. However, when we examine the general case where elements may also be points, some of these relationships hold and some do not hold.

For example, let $p \in \mathbf{P}$:

$MEETS(p, t_2)$ may hold for time elements $t_2 \in \mathbf{T}$ according to the axiomatization.

However, consider the following case:
$OVERLAPS(p, t_2) \Leftrightarrow \exists t, t', t'' \in \mathbf{T}(p = t' + t \wedge t_2 = t + t'')$,

On the one hand, by axiom $\langle A6 \rangle$, $d(p) = d(t') + d(t)$; and the assumption that p is a point gives:

$$d(t') + d(t) = d(p) = 0 \quad (1)$$

On the other hand, axiom $\langle A5 \rangle$ ensures that at least one of t' and t is an interval, hence:

$$d(t') + d(t) > 0 \quad (2)$$

(1) and (2) show that $OVERLAPS(p, t_2)$ cannot hold.

It is straightforward to prove in a similar fashion that all the possible relations over intervals and points may be classified into the following four groups:

Point–Point:
{*EQUAL*, *BEFORE*, *AFTER*}
which relate points to other points;

Interval–Interval:
{*EQUAL*, *BEFORE*, *MEETS*, *OVERLAPS*, *STARTS*, *DURING*, *FINISHES*, *FINISHED-BY*, *CONTAINS*, *STARTED-BY*, *OVERLAPPED-BY*, *MET-BY*, *AFTER*}
which relate intervals to intervals;

Point–Interval:
{*BEFORE*, *MEETS*, *STARTS*, *DURING*,

FINISHES, *MET-BY*, *AFTER*}
which relate points to intervals;

Interval–Point:
{*BEFORE*, *MEETS*, *FINISHED-BY*, *CONTAINS*, *STARTED-BY*, *MET-BY*, *AFTER*}
which relate intervals to points.

According to the above classification, there are in total 30 possible temporal relations over time-elements which may be both intervals and points. It is interesting to note that, however, in Vilain's (1982) interval and point based system, only 26 of these 30 temporal relations are addressed. There is a critical omission from the primitive relations between points and intervals in Vilain's system, for the '*MEETS*' relation is defined only between intervals and is not allowed between points and intervals. This omission leads to some difficulties in modelling the 'open' and 'closed' nature of intervals (see Section 4).

6. MODELS OF THE THEORY

Since the time theory itself characterizes a very general temporal structure, we may interpret the axiomatization in various temporal models: dense or discrete, linear or branching, etc.

As an example of dense and linear models of the axiomatization, consider the interpretation in which the set of time points, **P**, is the set of all real numbers; and the set of time intervals, **I**, is the set of periods which are constructions over all possible point-pairs, $p_1, p_2 \in \mathbf{P}$ such that $p_1 < p_2$, with the following structures:

$$(p_1, p_2, \text{open}, \text{open}) =_{def} \{r \in \mathbf{R} \mid p_1 < r < p_2\},$$
$$(p_1, p_2, \text{open}, \text{closed}) =_{def} \{r \in \mathbf{R} \mid p_1 < r \leqslant p_2\},$$
$$(p_1, p_2, \text{closed}, \text{open}) =_{def} \{r \in \mathbf{R} \mid p_1 \leqslant r < p_2\},$$
$$(p_1, p_2, \text{closed}, \text{closed}) =_{def} \{r \in \mathbf{R} \mid p_1 \leqslant r \leqslant p_2\},$$

where '$<$' and '$\leqslant$' are the ordinary ordering relations over the set, **R**, of real numbers.

NB. Here, we represent the interval structure by means of the extra primitives: left-type, l, and right-type, r, which take values from a set $\mathbf{Type} =_{def} \{\text{open}, \text{closed}\}$. There are thus four types of intervals based on points. For convenience of expression, we may denote a point, p, as (p, p, closed, closed), i.e. a special period whose left ending point and right ending point are identical, with 'closed' type for both left-type and right-type.

The duration assignment function d is simply defined by:

$$d((p_1, p_2, \_, \_)) = p_2 - p_1.$$

We may define the *meets* relation over $\mathbf{T} = \mathbf{P} \cup \mathbf{I}$ as following:

$$meets((p_{11}, p_{12}, l_1, r_1), (p_{21}, p_{22}, l_2, r_2)) \Leftrightarrow$$
$$p_{12} = p_{21} \wedge r_1 = \text{open} \wedge l_2 = \text{closed}$$
$$\vee\ p_{12} = p_{21} \wedge r_1 = \text{closed} \wedge l_2 = \text{open}$$

It is easy to see that this model satisfies axioms ⟨A1⟩–⟨A6⟩. Additionally, the (full) linearity axiom, $\langle A_{\text{Linear}} \rangle$, and the dense axiom, $\langle A_{\text{Dense}} \rangle$, are also satisfied. Hence, the above structure forms a dense and linear temporal model of the theory.

A discrete model satisfying axioms ⟨A1⟩–⟨A6⟩, $\langle A_{\text{Linear}} \rangle$, $\langle A_{\text{L-Discrete}} \rangle$ and $\langle A_{\text{R-Discrete}} \rangle$ can be constructed by simply limiting all elements of **P** to be integers in the above model, although the internal points of intervals are still reals. It is interesting to note that in such a discrete model, although points never meet each other, intervals are not necessarily infinitely decomposable. For instance, according to our axiomatization, interval (6, 8, open, closed) can be only decomposed into at most four (non-decomposable) time elements:

$$(6, 8, \text{open}, \text{closed}) = (6, 7, \text{open}, \text{open}) + (7, 7, \text{closed}, \text{closed}) + (7, 8, \text{open}, \text{open}) + (8, 8, \text{closed}, \text{closed})$$

However, this model will not be valid for Allen and Hayes' axiomatization including ⟨M6⟩ (see Section 2), which implies that if an interval is decomposable then it must be infinitely decomposable. (Otherwise, if it is only finitely decomposable, then it must be the sum of a finite number of moments which would meet one another, contrary to ⟨M6⟩.)

NB. As mentioned in Section 2, in order to interpret Allen and Hayes' axioms in discrete models, their axiom ⟨M6⟩ must be excluded. In other words, axiom ⟨M6⟩ is inconsistent with discrete times. However, the above example shows that the axiom ⟨A5⟩ in our axiomatization can be satisfied by discrete models.

In what follows, we shall show that our axiomatization is powerful enough to subsume many representative temporal systems, such as: the point based systems of Bruce, of McDermott, Allen's logic of intervals and Galton's revised theory, and the point and interval based theories of Vilain, and of Knight and Ma.

6.6. Bruce's point based system

Bruce's *time-system* is simply a set of time points with a partial order (see Bruce, 1972). In our theory, we may define a partial order, '$\leqslant$', over the set of points, **P**, as:

$$p_1 \leqslant p_2 \Leftrightarrow EQUAL(p_1, p_2) \vee BEFORE(p_1, p_2),$$

where *EQUAL* and *BEFORE* are introduced as in Section 5. Hence, the sub-frame, (**P**, $\leqslant$), of the temporal frame (T, *meets*) defined by the axiomatization, forms a temporal system of Bruce.

In a similar way, we may define Bruce's 7 binary relations over *time-segments* (see Bruce, 1972), in terms of the temporal relations over intervals introduced in Section 5.

NB. As discussed in the introduction, the temporal theories of Ladkin (1987, 1992), of Dechter *et al.* (1991), and of Maiocchi (1992) are similar to that of Bruce in the sense that intervals are defined to be constructed out of points. Hence, in a similar way, we may induce the corresponding time model for each of these temporal frameworks.

6.2. McDermott's temporal logic

McDermott develops a first-order temporal logic to provide a versatile 'common-sense' model for temporal reasoning. The theory assumes '*no later than*' ordering relation over a dense collection of states (points), which is axiomatized to give rise to a left linear (branching into future) time structure. That is, there are many possible futures branching forward in time from the present. Each single branch, called a '*Chronicle*', consists of a dense set of states and is isomorphic to the real line (see McDermott, 1982). Consider the temporal frame axiomatized by axioms $\langle A1\rangle$–$\langle A6\rangle$, $\langle A_{L\text{-}Linear}\rangle$, and the following additional axioms $\langle A_{P\text{-}Dense}\rangle$ which states that there is always a time point during any time interval.

$\langle A_{P\text{-}Dense}\rangle$
$$\forall i \in \mathbf{I}\ \exists p \in \mathbf{P}\ \exists i_1, i_2 \in \mathbf{I}(i = i_1 + p + i_2)$$

By consideration of axioms $\langle A2\rangle$ and $\langle A5\rangle$, we can infer that axiom $\langle A_{P\text{-}Dense}\rangle$ ensures that between any two distinct time points on the same time line, there is a third. In fact, axiom $\langle A_{P\text{-}Dense}\rangle$ is stronger than axiom $\langle A_{Dense}\rangle$ (see Section 4), since it is clear that $\langle A_{P\text{-}Dense}\rangle$ implies $\langle A_{Dense}\rangle$.

In the same way as for Bruce's partial order, we may also define the 'no later than' relation over time points in terms of relations *EQUAL* and *BEFORE*. In this way, we may take McDermott's time structure as a model of the above theory by addressing only time points and the 'no later than' relation.

6.3. Allen and Hayes' interval based theory

Since the axiomatization proposed in this paper may be seen as an extension of Allen and Hayes' (1989) interval based temporal theory, it is straightforward to subsume Allen and Hayes' theory by taking the set of time points to be empty, and including the linearity axiom $\langle A_{Linear}\rangle$ in the fundamental axiomatization. Of course, in this case, axiom $\langle A5\rangle$ will become vacuous.

NB. Allen and Hayes' (1989) temporal theory only handles time as a pure abstraction, although Allen's interval based temporal logic is originally supposed to be set up as a framework on which to hang assertions about the instantiation in time of properties and occurrence (Allen, 1984). In Allen's interval based logic, there are a small number of predicates among which HOLDS is one of the most important. To secure the interpretation of HOLDS (see Section 2), Allen introduces the following axiom:

$$\mathrm{HOLDS}(pro, i) \Leftrightarrow \forall i' \in \mathbf{I}(IN(i', i) \Rightarrow \mathrm{HOLDS}(pro, i'))$$

where *IN* is defined in terms of the temporal relations over intervals, as below:

$$IN(i', i) \Leftrightarrow DURING(i', i) \wedge STARTS(i', i) \vee FINISHES(i', i)$$

The negation of a property is then characerized by the axiom

$$\mathrm{HOLDS}(\neg pro, i) \Leftrightarrow \forall i' \in \mathbf{I}(IN(i', i) \Rightarrow \neg \mathrm{HOLDS}(pro, i'))$$

However, Galton (1990) has shown that there are some problems with reasoning correctly about continuous change in Allen's logic (in particular, with Allen's property-negation), and suggested the way out: instantaneous property-ascriptions.

As Galton puts it, the problems with Allen's system can be traced to the assumption that all properties should receive a uniform treatment with respect to the logic of their temporal incidence. Galton's starting point is then to distinguish sharply between two kinds of properties, i.e. *states of position* and *states of motion*, which have different temporal logics. States of position can hold at isolated points; and if a state of position holds throughout an interval, then it must hold at the limits of that interval. States of motion cannot hold at isolated points, i.e. if a state of motion holds at a point then it must hold throughout some interval within which that point falls. Additionally, Galton defines three types of statement by the forms

$$\mathrm{HOLDS\text{-}ON}(pro, i),\ \mathrm{HOLDS\text{-}IN}(pro, i), \text{ and } \mathrm{HOLDS\text{-}AT}(pro, p),$$

which assert that a property, *pro*, holds *throughout* an interval, *during* an interval (i.e. at some time *DURING* an interval, not necessarily through all of it), and *at* a point, respectively, while in Allen's logic, there is only one way, HOLDS, of ascribing properties to times, that is, HOLDS-ON.

Since our general temporal theory allows both intervals and points, it is straightforward to form Galton's revised temporal theory. For example, we may formally characterize a state of position s_p by:

$$\begin{aligned}\forall i \in \mathbf{I}\ \forall p \in \mathbf{P}(\ &\mathrm{HOLDS\text{-}ON}(s_p, i)\\ &\wedge (MET\text{-}BY(i, p) \vee\\ &\quad MEETS(i, p) \vee\\ &\quad STARTED\text{-}BY(i, p) \vee\\ &\quad FINISHED\text{-}BY(i, p))\\ &\Rightarrow \mathrm{HOLDS\text{-}AT}(s_p, p))\end{aligned}$$

and a state of motion s_m by:

$$\forall p \in \mathbf{P}(\mathrm{HOLDS\text{-}AT}(s_m, p) \Rightarrow \exists i \in \mathbf{I}(DURING(p, I) \wedge \mathrm{HOLDS\text{-}ON}(s_m, i)))$$

It is interesting to note that, the definitions relating to the open and closed nature of intervals given in Section 4 provide another formal and intuitive characterization for the distinction between *states of position* and *states of motion*: states of position can hold at isolated points; and if a states of position holds on an interval, then it must hold on the closure of that interval. States of motion hold only on open intervals. For instance, in the example of a ball thrown vertically into the air described in Section 2, the property *ball_stationary* is a state of position, while *ball_going_up* and *ball_coming_down* are states of motion.

In a similar way, we may axiomatize other results for general properties (see Allen, 1984; Galton, 1990a), as well as other issues such as processes and events, etc. However, since the main objective of this paper is to present a general time theory at some abstract level, here, we will not go further on addressing these broader issues.

6.4. Vilain's interval and point based system

Noting that intervals are not the only mechanism by which human beings understand time, another common construct being that of time points, Vilain (1982, 1986) proposes a system which handles time points in much the same way that it handles intervals. This system is arrived at by expanding Allen's 13 temporal relations over intervals to 26, which are primitively defined to relate points to points, intervals to intervals, intervals to points, and points to intervals. It is interesting to note that all Vilain's 26 temporal relations form a subset of the set of those 30 relations we introduced in Section 5. The excluded four relations in Vilain's system are: *MEETS*, *MET_BY* that relate points to intervals, and *MEETS*, *MET_BY* that relate intervals to points (see Section 5). Hence, if we employ the following more strict axiom instead of ⟨A5⟩:

$$\forall t_1, t_2 \in \mathbf{T}(meets(t_1, t_2) \Rightarrow t_1 \in \mathbf{I} \wedge t_2 \in \mathbf{I})$$

then we get Vilain's temporal system. The above axiom ensures that if two time elements meet each other, then both of them must be intervals.

6.5. Knight and Ma's temporal model

Knight and Ma (1993) have proposed a temporal model akin to that presented here, taking both points and intervals with duration assignments as primitive time elements. However, this model addresses only finite linear sets of time elements. Hence, it is possible to consider it as a specialization of the time theory to a finite set of time elements. In fact:

Assume (**T**, *meets*) is the temporal frame defined by axioms ⟨A1⟩–⟨A6⟩, A_{Linear}, $\langle A_{L\text{-}Discrete}\rangle$ and $\langle A_{R\text{-}Discrete}\rangle$. The discreteness property of the temporal frame allows us to form a non-empty finite set $\mathbf{T}_f \subset \mathbf{T} = \mathbf{I} \cup \mathbf{P}$, such that:

$$\mathbf{T}_f = \{t_1, t_2, \ldots, t_n\};$$
$$meets(t_i, t_{i+1}),\ i = 1, 2, \ldots, n-1;$$
$$meets(t_i, t_{i+1}) \Rightarrow t_i \in \mathbf{I} \vee t_{i+1} \in \mathbf{I}.$$

These theorems are precisely the axioms for Knight and Ma's set **E**, of '*fundamental time elements*' [which may be thought as Allen and Hayes' (1989) '*moments*']. Additionally, it is easy to see that the limitation of axioms ⟨A4⟩, ⟨A5⟩ and ⟨A6⟩ onto $\mathbf{T}_f$ precisely gives the definition of the *closure* of E, under the binary operations of combining adjacent time elements and corresponding addition of duration, that is, the so-called temporal system.

It is interesting to note that, in computer-based modelling approach, a database consists of only a finite (discrete) set of elements, that is, the database models only a finite subset of the fundamental (dense or discrete) set of primitive elements. The existence of complete set of primitive elements is a belief which may be used to test the consistency of the database. Hence, with this meaning, the consistency checker provided in Knight and Ma (1989) may be used for any finite temporal subframe defined by the axiomatization.

7. CONCLUSIONS

In this paper, we have proposed a general time theory which may be seen as an extension of Allen and Hayes' axiomatization by the addition of axioms relating to the inclusion of time points as primitive elements. This theory allows other first order temporal systems as models. It unifies a variety of temporal concepts into a single framework. We have attempted to define key concepts and terms with respect to the axiomatic system. And in addition, we have separated axioms for linearity and for density from the main body of axioms, since these appear to be most 'user dependent'. The resulting theory presents a unified view of what is currently a disparate field.

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