Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2008, Article ID 406368, 8 pages
doi:10.1155/2008/406368

Research Article

A Generalisation of Contraction Principle in Metric Spaces

P. N. Dutta[1] and Binayak S. Choudhury[2]

[1] *Department of Mathematics, Government College of Engineering and Ceramic Technology, 73 A.C. Banerjee Lane, Kolkata, West Bengal 700010, India*
[2] *Department of Mathematics, Bengal Engineering and Science University, P.O. Botanical Garden, Shibpur, Howrah, West Bengal 711103, India*

Correspondence should be addressed to P. N. Dutta, prasanta_dutta1@yahoo.co.in

Received 28 March 2008; Revised 26 June 2008; Accepted 18 August 2008

Recommended by Górniewicz Lech

Here we introduce a generalisation of the Banach contraction mapping principle. We show that the result extends two existing generalisations of the same principle. We support our result by an example.

1. Introduction

Banach contraction mapping principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also its significance lies in its vast applicability in a number of branches of mathematics.

$T : X \to X$ where (X, d) is a complete metric space is said to be a contraction mapping if for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y), \quad \text{where } 0 < k < 1. \tag{1.1}$$

According to the contraction mapping principle, any mapping T satisfying (1.1) will have a unique fixed point.

Generalisation of the above principle has been a heavily investigated branch of research. The following are a few examples of such generalisations. In [1], Boyd and Wong proved that the constant k in (1.1) can be replaced by the use of an upper semicontinuous function. In [2, 3], generalised Banach contraction conjecture has been established. In [4], Suzuki has proved a generalisation of the same principle which characterises metric completeness. The contraction principle has also been extended to probabilistic metric spaces [5].

Here in this paper, we consider two such generalisations given by Khan et al. [6] and Alber and Guerre-Delabriere [7]. We prove a theorem which generalises both these results.

In [6], Khan et al. addressed a new category of fixed point problems with the help of a control function which they called an altering distance function.

Definition 1.1 (altering distance function [6]). A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(a) $\psi(0) = 0$,

(b) ψ is continuous and monotonically non-decreasing.

Theorem 1.2 (see [6]). *Let (X,d) be a complete metric space, let ψ be an altering distance function, and let $f : X \to X$ be a self-mapping which satisfies the following inequality:*

$$\psi(d(fx,fy)) \leq c\psi(d(x,y)) \tag{1.2}$$

for all $x, y \in X$ and for some $0 < c < 1$. Then f has a unique fixed point.

In fact Khan et al. proved a more general theorem [6, Theorem 2] of which the above result is a corollary.

Altering distance has been used in metric fixed point theory in a number of papers. Some of the works utilising the concept of altering distance function are noted in [8–11]. In [12], 2-variable and in [13] 3-variable altering distance functions have been introduced as generalisations of the concept of altering distance function. It has also been extended in the context of multivalued [14] and fuzzy mappings [15]. The concept of altering distance function has also been introduced in Menger spaces [16].

Another generalisation of the contraction principle was suggested by Alber and Guerre-Delabriere [7] in Hilbert Spaces. Rhoades [17] has shown that the result which Alber and Guerre-Delabriere have proved in [7] is also valid in complete metric spaces. We state the result of Rhoades in the following.

Definition 1.3 (weakly contractive mapping). A mapping $T : X \to X$, where (X,d) is a metric space, is said to be weakly contractive if

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)), \tag{1.3}$$

where $x, y \in X$ and $\phi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

If one takes $\phi(t) = kt$ where $0 < k < 1$, then (1.3) reduces to (1.1).

Theorem 1.4 (see [17]). *If $T : X \to X$ is a weakly contractive mapping, where (X,d) is a complete metric space, then T has a unique fixed point.*

In fact, Alber and Guerre-Delabriere assumed an additional condition on ϕ which is $\lim_{t\to\infty} \phi(t) = \infty$. But Rhoades [17] obtained the result noted in Theorem 1.4 without using this particular assumption.

It may be observed that though the function ϕ has been defined in the same way as the altering distance function, the way it has been used in Theorem 1.4 is completely different from the use of altering distance function.

Weakly contractive mappings have been dealt with in a number of papers. Some of these works are noted in [17–20].

The purpose of this paper is to introduce a generalisation of Banach contraction mapping principle which includes the generalisations noted in Theorems 1.2 and 1.4. Lastly, we discuss an example.

2. Main results

Theorem 2.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)), \tag{2.1}$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$.

Then T has a unique fixed point.

Proof. For any $x_0 \in X$, we construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$.

Substituting $x = x_{n-1}$ and $y = x_n$ in (2.1), we obtain

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)), \tag{2.2}$$

which implies

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \quad \text{(using monotone property of } \psi\text{-function)}. \tag{2.3}$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing and consequently there exists $r \geq 0$ such that

$$d(x_n, x_{n+1}) \longrightarrow r \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

Letting $n \to \infty$ in (2.2) we obtain

$$\psi(r) \leq \psi(r) - \phi(r), \tag{2.5}$$

which is a contradiction unless $r = 0$.

Hence

$$d(x_n, x_{n+1}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.6}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon. \tag{2.7}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.7).

Then

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{2.8}$$

Then we have

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) < \epsilon + d(x_{n(k)-1}, x_{n(k)}). \tag{2.9}$$

Letting $k \to \infty$ and using (2.6),

$$\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.10}$$

Again,

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}), \\ d(x_{n(k)-1}, x_{m(k)-1}) &\leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}). \end{aligned} \tag{2.11}$$

Letting $k \to \infty$ in the above two inequalities and using (2.6), (2.10), we get

$$\lim_{k \to \infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon. \tag{2.12}$$

Setting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (2.1) and using (2.7), we obtain

$$\psi(\epsilon) \leq \psi(d(x_{m(k)}, x_{n(k)})) \leq \psi(d(x_{m(k)-1}, x_{n(k)-1})) - \phi(d(x_{m(k)-1}, x_{n(k)-1})). \tag{2.13}$$

Letting $k \to \infty$, utilising (2.10) and (2.12), we obtain

$$\psi(\epsilon) \leq \psi(\epsilon) - \Phi(\epsilon), \tag{2.14}$$

which is a contradiction if $\epsilon > 0$.

This shows that $\{x_n\}$ is a Cauchy sequence and hence is convergent in the complete metric space X.

Let

$$x_n \longrightarrow z \quad \text{(say) as } n \longrightarrow \infty. \tag{2.15}$$

Substituting $x = x_{n-1}$ and $y = z$ in (2.1), we obtain

$$\psi(d(x_n, Tz)) \leq \psi(d(x_{n-1}, z)) - \phi(d(x_{n-1}, z)). \tag{2.16}$$

Letting $n \to \infty$, using (2.15) and continuity of ϕ and ψ, we have

$$\psi(d(z, Tz)) \leq \psi(0) - \phi(0) = 0, \tag{2.17}$$

which implies $\psi(d(z, Tz)) = 0$, that is,

$$d(z, Tz) = 0 \quad \text{or} \quad z = Tz. \tag{2.18}$$

To prove the uniqueness of the fixed point, let us suppose that z_1 and z_2 are two fixed points of T.

Putting $x = z_1$ and $y = z_2$ in (2.1),

$$\begin{aligned} &\psi(d(Tz_1, Tz_2)) \leq \psi(d(z_1, z_2)) - \phi(d(z_1, z_2)) \\ \text{or} \quad &\psi(d(z_1, z_2)) \leq \psi(d(z_1, z_2)) - \phi(d(z_1, z_2)) \\ \text{or} \quad &\phi(d(z_1, z_2)) \leq 0, \end{aligned} \tag{2.19}$$

or equivalently $d(z_1, z_2) = 0$, that is, $z_1 = z_2$.

This proves the uniqueness of the fixed point. □

If we particularly take $\phi(t) = (1-k)\psi(t)$ $\forall t > 0$ where $0 < k < 1$, then we obtain the result noted in Theorem 1.2. Again, in particular, if we take $\psi(t) = t$ $\forall t \geq 0$, then the result noted in Theorem 1.4 is obtained.

Example 2.2. Let $X = [0,1] \cup \{2, 3, 4, \ldots\}$ and

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0,1],\ x \neq y, \\ x + y, & \text{if at least one of } x \text{ or } y \notin [0,1] \text{ and } x \neq y, \\ 0, & \text{if } x = y. \end{cases} \tag{2.20}$$

Then (X, d) is a complete metric space [1].

Let $\psi : [0, \infty) \to [0, \infty)$ be defined as

$$\psi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ t^2, & \text{if } t > 1, \end{cases} \tag{2.21}$$

and let $\phi : [0,\infty) \to [0,\infty)$ be defined as

$$\phi(t) = \begin{cases} \frac{1}{2}t^2, & \text{if } 0 \le t \le 1, \\ \frac{1}{2}, & \text{if } t > 1. \end{cases} \tag{2.22}$$

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} x - \frac{1}{2}x^2, & \text{if } 0 \le x \le 1, \\ x - 1, & \text{if } x \in \{2,3,\ldots\}. \end{cases} \tag{2.23}$$

Without loss of generality, we assume that $x > y$ and discuss the following cases.

Case 1 ($x \in [0,1]$). Then

$$\begin{aligned} \psi(d(Tx,Ty)) &= \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right) \\ &= (x-y) - \frac{1}{2}(x-y)(x+y) \le (x-y) - \frac{1}{2}(x-y)^2 \\ &= d(x,y) - \frac{1}{2}(d(x,y))^2 \\ &= \psi(d(x,y)) - \frac{1}{2}(d(x,y))^2 \\ &= \psi(d(x,y)) - \phi(d(x,y)) \quad (\text{since } x - y \le x + y). \end{aligned} \tag{2.24}$$

Case 2 ($x \in \{3,4,\ldots\}$). Then

$$\begin{aligned} & d(Tx,Ty) = d\left(x-1, y - \frac{1}{2}y^2\right) \quad \text{if } y \in [0,1] \\ \text{or} \quad & d(Tx,Ty) = x - 1 + y - \frac{1}{2}y^2 \le x + y - 1, \\ & d(Tx,Ty) = d(x-1,y-1) \quad \text{if } y \in \{2,3,4,\ldots\} \\ \text{or} \quad & d(Tx,Ty) = x + y - 2 < x + y - 1. \end{aligned} \tag{2.25}$$

Consequently,

$$\begin{aligned} \psi(d(Tx,Ty)) &= (d(Tx,Ty))^2 \le (x+y-1)^2 < (x+y-1)(x+y+1) \\ &= (x+y)^2 - 1 < (x+y)^2 - \frac{1}{2} \\ &= \psi(d(x,y)) - \phi(d(x,y)). \end{aligned} \tag{2.26}$$

Case 3 ($x = 2$). Then $y \in [0,1]$, $Tx = 1$, and $d(Tx,Ty) = 1 - (y - (1/2)y^2) \leq 1$.

So, we have $\psi(d(Tx,Ty)) \leq \psi(1) = 1$.

Again $d(x,y) = 2 + y$.

So,

$$\begin{aligned}\psi(d(x,y)) - \phi(d(x,y)) &= (2+y)^2 - \phi((2+y)^2)\\ &= (2+y)^2 - \frac{1}{2}\\ &= \frac{7}{2} + 4y + y^2 > 1\\ &= \psi(d(Tx,Ty)).\end{aligned} \tag{2.27}$$

Considering all the above cases, we conclude that inequality (2.1) remains valid for ϕ, ψ, and T constructed as above and consequently by an application of Theorem 2.1, T has a unique fixed point.

It is seen that "0" is the unique fixed point of T.

Note

The example discussed above cannot be covered by the result of Khan et al. noted in Theorem 1.2.

References

[1] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, no. 2, pp. 458–464, 1969.

[2] A. D. Arvanitakis, "A proof of the generalized Banach contraction conjecture," *Proceedings of the American Mathematical Society*, vol. 131, no. 12, pp. 3647–3656, 2003.

[3] J. Merryfield, B. Rothschild, and J. D. Stein Jr., "An application of Ramsey's theorem to the Banach contraction principle," *Proceedings of the American Mathematical Society*, vol. 130, no. 4, pp. 927–933, 2002.

[4] T. Suzuki, "A generalized Banach contraction principle that characterizes metric completeness," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.

[5] O. Hadžić and E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, vol. 536 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[6] M. S. Khan, M. Swaleh, and S. Sessa, "Fixed point theorems by altering distances between the points," *Bulletin of the Australian Mathematical Society*, vol. 30, no. 1, pp. 1–9, 1984.

[7] Ya. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," in *New Results in Operator Theory and Its Applications*, I. Gohberg and Y. Lyubich, Eds., vol. 98 of *Operator Theory: Advances and Applications*, pp. 7–22, Birkhäuser, Basel, Switzerland, 1997.

[8] G. V. R. Babu, B. Lalitha, and M. L. Sandhya, "Common fixed point theorems involving two generalized altering distance functions in four variables," *Proceedings of the Jangjeon Mathematical Society*, vol. 10, no. 1, pp. 83–93, 2007.

[9] S. V. R. Naidu, "Some fixed point theorems in metric spaces by altering distances," *Czechoslovak Mathematical Journal*, vol. 53, no. 1, pp. 205–212, 2003.

[10] K. P. R. Sastry and G. V. R. Babu, "Some fixed point theorems by altering distances between the points," *Indian Journal of Pure and Applied Mathematics*, vol. 30, no. 6, pp. 641–647, 1999.

[11] K. P. R. Sastry, S. V. R. Naidu, G. V. R. Babu, and G. A. Naidu, "Generalization of common fixed point theorems for weakly commuting map by altering distances," *Tamkang Journal of Mathematics*, vol. 31, no. 3, pp. 243–250, 2000.

[12] B. S. Choudhury and P. N. Dutta, "A unified fixed point result in metric spaces involving a two variable function," *Filomat*, no. 14, pp. 43–48, 2000.

[13] B. S. Choudhury, "A common unique fixed point result in metric spaces involving generalised altering distances," *Mathematical Communications*, vol. 10, no. 2, pp. 105–110, 2005.

[14] B. S. Choudhury and A. Upadhyay, "On unique common fixed point for a sequence of multi-valued mappings on metric spaces," *Bulletin of Pure & Applied Sciences. Section E*, vol. 19, no. 2, pp. 529–533, 2000.

[15] B. S. Choudhury and P. N. Dutta, "Common fixed points for fuzzy mappings using generalised altering distances," *Soochow Journal of Mathematics*, vol. 31, no. 1, pp. 71–81, 2005.

[16] B. S. Choudhury and K. Das, "A new contraction principle in Menger Spaces," *Acta Mathematica Sinica*, vol. 24, no. 8, pp. 1379–1386, 2008.

[17] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 4, pp. 2683–2693, 2001.

[18] I. Beg and M. Abbas, "Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition," *Fixed Point Theory and Applications*, vol. 2006, Article ID 74503, 7 pages, 2006.

[19] C. E. Chidume, H. Zegeye, and S. J. Aneke, "Approximation of fixed points of weakly contractive nonself maps in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 1, pp. 189–199, 2002.

[20] Q. Zhang and Y. Song, "Fixed point theory for generalized φ-weak contractions," *Applied Mathematics Letters*. In press.